Strict Rezk completions of models of HoTT and homotopy canonicity

Rafaël Bocquet

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Motivation: metatheory up to homotopy

- Metatheory up to equality:
  - Canonicity: the set of closed boolean terms is isomorphic to $\{\mathsf{true}, \mathsf{false}\}$.
  - Normalization: related to decidability of equality.
- Metatheory up to homotopy:
  - Homotopy canonicity: the ∞-groupoid of closed boolean terms is equivalent to $\{\mathsf{true}, \mathsf{false}\}$.
  - Homotopy normalization (?): related to 0-truncatedness.

Homotopy canonicity

In HoTT, univalence is an axiom blocking computation.

Homotopy canonicity: conjectured by Voevodsky in 2010.

Statement (for booleans): for every closed boolean term b, we can find either an identification in $\mathsf{Id}_{\mathsf{Bool}}(b, \mathsf{true})$, or in $\mathsf{Id}_{\mathsf{Bool}}(b, \mathsf{false})$.

Models (simplicial sets), and especially constructive models (cubical sets) provide a computational explanation of univalence.

Is the syntax HoTT "complete" with respect to this computational explanation?

Semantics of type theory

Models: categories with families + additional structure.
Generalized algebraic structure.

A model $\mathcal{M}$ has:

- an underlying 1-category $\mathcal{M}$.
- sets of types $\mathcal{M}.\mathsf{Ty}(\Gamma)$ for $\Gamma \in \mathcal{M}$.
- sets of terms $\mathcal{M}.\mathsf{Tm}(\Gamma, A)$ for $A \in \mathcal{M}.\mathsf{Ty}(\Gamma)$.
- operations: type and terms formers.
- strict equalities: substitution laws and computation rules.

Category of models $\mathbf{Alg}_{\mathsf{HoTT}}$.
Syntax = initial model $\mathbf{0}_{\mathsf{HoTT}}$.

Strict canonicity: logical relations and sconing

Model $\mathcal{G}$ constructed by gluing over the global sections functor:

$$\mathsf{Hom}(1, -) : \mathbf{0}_{\mathsf{MLTT}} \to \mathbf{Set}.$$

Initiality of $\mathbf{0}_{\mathsf{MLTT}}$ $\rightsquigarrow$ interpretation $[\![-]\!] : \mathbf{0}_{\mathsf{MLTT}} \to \mathcal{G}$.

- Logical predicates: $[\![A]\!] : \mathsf{Tm}(1, A) \to \mathrm{Set}$ for $A : \mathsf{Ty}(1)$.
- $[\![x]\!] : [\![A]\!](x)$ for $x : \mathsf{Tm}(1, A)$.
- $[\![\mathsf{Bool}]\!] = \lambda b \mapsto (b = \mathsf{true}) + (b = \mathsf{false})$.

∞-groupoid-valued logical relations

∞-groupoidal global sections functor

$$\mathsf{Hom}_{\infty}(1, -) : \mathbf{0}_{\mathsf{MLTT}} \to \infty\text{-}\mathbf{Grp}.$$

∞-groupoid-valued logical predicates.

$$[\![A]\!] : \mathsf{Tm}_{\infty}(1, A) \to \infty\text{-}\mathrm{Grp}.$$

Previous approaches

[Shulman]: 1-truncated homotopy canonicity:
Use a global sections functor valued in 1-groupoids.

[Kapulkin, Sattler]: present the global sections functor by a span.

$$\mathbf{0}_{\mathsf{HoTT}} \overset{\sim}{\twoheadleftarrow} [\Delta_{+}^{\mathsf{op}}, \mathbf{0}_{\mathsf{HoTT}}] \to \widehat{\Delta_{+}} \to \widehat{\Delta} \to \widehat{\Box}.$$

For intuition: Higher models and ∞-type theories

[Kraus]: ∞-categories with families (in two-level type theory).
[Nguyen,Uemura]: ∞-type theories (∞-categories of models).

Untruncated judgemental "equalities" $(- \sim -)$.

- 1-HoTT is 0-truncated.

$$\frac{p, q : x \sim y}{p \sim q}$$

- ∞-HoTT has completeness / higher extensionality.

$$\frac{\mathsf{Id}_A(x, y)}{x \sim y}$$

For intuition: Higher models and ∞-type theories

Likely: ∞-HoTT satisfies homotopy canonicity.
(Using a ∞-categorical sconing construction)

But ∞-HoTT $\neq$ 1-HoTT!

The 1-category $\mathbf{Alg}_{1\text{-HoTT}}$ has a homotopy theory (left semi-model structure).
Hard coherence conjecture: this presents the ∞-category $\mathbf{Alg}_{\infty\text{-HoTT}}$.

Problem: ∞-HoTT and higher models lose the strictness of 1-HoTT.

The homotopy theory of type theories: classes of maps

[Kapulkin, Lumsdaine]: Left semi-model structures on categories of models.

Definition

A morphism $F : \mathcal{M} \to \mathcal{N}$ in $\mathbf{Alg}_{\mathsf{HoTT}}$ is a **weak equivalence** if:

- (weak term lifting): given $a : \mathcal{N}.\mathsf{Tm}(F(\Gamma), F(A))$, there is $a_0 : \mathcal{M}.\mathsf{Tm}(\Gamma, A)$ and $p : \mathcal{N}.\mathsf{Tm}(F(\Gamma), \mathsf{Id}(F(a_0), a))$.

Fibrations satisfy path lifting.
Trivial fibrations satisfy strict term lifting.

Constructively: use split weak equivalences.

Two-level type theory (2LTT)

Universes Set of sets, with strict equality $(- = -)$.
Models extensional type theory.

Universes $\mathrm{Set}_{\mathsf{fib}}$ of fibrant sets, with paths $(- \sim -)$.
Models homotopy type theory.

Fibrant sets can be seen as ∞-groupoids.

Idea: use logical relations valued in fibrant sets.

Models of two-level type theory: $\widehat{\boldsymbol{\Delta}}$, $\widehat{\square}$, $\widehat{\mathcal{C} \times \square}$, etc.

Let's use cubical sets ($\mathbf{cSet} = \widehat{\square}$).

Internal/cubical models

Consider models of HoTT in the internal language of $\mathbf{cSet}$.
Equivalently: cubical models (cubical set of contexts, etc.).
Equivalently: cubical presheaves of models $[\square^{\mathrm{op}}, \mathbf{Alg}_{\mathsf{HoTT}}]$.

Main idea: Cubical models can be seen as higher models.
(With fibrant components)

Initial cubical model $\mathbf{0}_{\mathsf{HoTT}}$ (coincides with the external syntax).

Internal/cubical models

The components of $\mathbf{0}_{\mathsf{HoTT}}$ are fibrant.

Wrong higher dimensional structure (discrete cubical sets).

The correct homotopical structure is available:
use $\mathsf{Tm}(\Gamma, \mathsf{Id}_A(x, y))$ instead of $(x \sim y)$, etc.

$\rightsquigarrow$ reconstruct fibrant sets with the correct higher dimensional structure?

The Rezk completion in HoTT does the same for categories!

Rezk completion of categories in HoTT

Category, functors in HoTT: laws up to identifications.

Definition

A Rezk completion of a category $\mathcal{C}$ is a category $\overline{\mathcal{C}}$ with a functor $i : \mathcal{C} \to \overline{\mathcal{C}}$ such that:

- the functor $i : \mathcal{C} \to \overline{\mathcal{C}}$ is a weak equivalence.
- $\overline{\mathcal{C}}$ is complete/univalent:

$$(x : \overline{\mathcal{C}}.\mathrm{Ob}) \to \text{is-contr}((y : \overline{\mathcal{C}}.\mathrm{Ob}) \times (x \cong y)).$$

Rezk completions of categories are known to exist (construction using presheaves).

In $\overline{\mathcal{C}}$, the objects have the correct higher dimensional structure.

Rezk completions of models of HoTT in HoTT?

Cannot consider untruncated models of type theory in pure HoTT.

Choose one:

- Infinitely many components (e.g. a semi-simplicial type of contexts).
- 0-truncated components.
- Strict equalities. This requires 2LTT.

Let's work in 2LTT, and specify a stricter Rezk completion.

Strict Rezk completion of categories in 2LTT/cubical sets

Categories, functors in 2LTT: strict laws.

Definition

A strict Rezk completion of a category $\mathcal{C}$ is a category $\overline{\mathcal{C}}$ with a functor $i : \mathcal{C} \to \overline{\mathcal{C}}$ such that:

- the functor $1^*_\square(i) : 1^*_\square(\mathcal{C}) \to 1^*_\square(\overline{\mathcal{C}})$ is a split weak equivalence.
- the components of $\overline{\mathcal{C}}$ are fibrant.
- $\overline{\mathcal{C}}$ is complete/univalent:

$$(x : \overline{\mathcal{C}}.\mathrm{Ob}) \to \mathsf{is\text{-}contr}((y : \overline{\mathcal{C}}.\mathrm{Ob}) \times (x \cong y)).$$

Strict Rezk completion of models of HoTT

Definition

A strict Rezk completion of a model $\mathcal{M}$ is a model $\overline{\mathcal{M}}$ with a morphism $i : \mathcal{C} \to \overline{\mathcal{M}}$ such that:

- the morphism $1^*_\square(i) : 1^*_\square(\mathcal{M}) \to 1^*_\square(\overline{\mathcal{M}})$ is a split weak equivalence.
- the components of $\overline{\mathcal{M}}$ are fibrant.
- $\overline{\mathcal{M}}$ is complete/univalent:

  $(x : \overline{\mathcal{M}}.\mathsf{Tm}(\Gamma, A)) \to \text{is-contr}((y : \overline{\mathcal{M}}.\mathsf{Tm}(\Gamma, A)) \times \overline{\mathcal{M}}.\mathsf{Tm}(\Gamma, \mathsf{Id}_A(x, y)))$.

  Equivalently:

  $$\text{is-equiv}((x \sim y) \xrightarrow{\text{path-to-id}} \overline{\mathcal{M}}.\mathsf{Tm}(\Gamma, \mathsf{Id}_A(x, y)))$$

Existence of strict Rezk completion

Theorem (Strict Rezk completions of categories)

In cartesian cubical sets, any global, algebraically cofibrant category with fibrant components admits a strict Rezk completion.

Theorem (Strict Rezk completions of models of HoTT)

In cartesian cubical sets, any global, algebraically cofibrant model of HoTT with fibrant components admits a strict Rezk completion.

Main ideas of the construction at the end of the talk!

Generalizations?

The constructions should generalize to generalized algebraic theories with a left semi-model category of algebras satisfying some assumptions.

Christian Sattler has some notes with constructions for “generalized algebraic homotopy theories”: generalized algebraic theories equipped with a notion of trivially fibrant sorts.

Proof of homotopy canonicity

Theorem (Homotopy canonicity for HoTT)

The syntax of HoTT with ω univalent universes, Id, Π, Σ, $\mathbf{0}$, $\mathbf{1}$, Bool, W -types and axiomatic homotopy pushouts satisfies homotopy canonicity.

Consider the strict Rezk completion $i : \mathbf{0}_{\mathsf{HoTT}} \to \overline{\mathbf{0}_{\mathsf{HoTT}}}$.

Construct $\mathcal{G}$ by gluing over the functor

$$\overline{\mathbf{0}_{\mathsf{HoTT}}}(1, -) : \overline{\mathbf{0}_{\mathsf{HoTT}}} \to \mathbf{Set}_{\mathsf{fib}}.$$

$$\begin{tikzcd} & \mathcal{G} \arrow[d, two heads] \\ \mathbf{0}_{\mathsf{HoTT}} \arrow[ur, bend left, "{[\![-]\!]}"] \arrow[r, dashed, "i"] & \overline{\mathbf{0}_{\mathsf{HoTT}}} \end{tikzcd}$$

Proof of homotopy canonicity

- $[\![A]\!] : \overline{\mathbf{0}_{\mathsf{HoTT}}}.\mathsf{Tm}(1, i(A)) \to (\mathrm{Set}_{\mathsf{fib}})_i$ for $A : \mathbf{0}_{\mathsf{HoTT}}.\mathsf{Ty}_i(1)$.
- $[\![x]\!] : [\![A]\!](i(x))$ for $x : \mathbf{0}_{\mathsf{HoTT}}.\mathsf{Tm}(1, A)$.
- $[\![\mathsf{Bool}]\!](b) = (b \sim \mathsf{true}) + (b \sim \mathsf{false})$.
- $[\![\mathcal{U}_i]\!](A) = \overline{\mathbf{0}_{\mathsf{HoTT}}}.\mathsf{Tm}(1, A) \to (\mathrm{Set}_{\mathsf{fib}})_i$.

Completeness of $\overline{\mathbf{0}_{\mathsf{HoTT}}}$ is used to interpret the univalence axiom.
(together with univalence for $\mathrm{Set}_{\mathsf{fib}}$.)

Fibrancy of the components is used for some type formers.

Proof of homotopy canonicity

Now take any global $b : \mathbf{0}_{\mathsf{HoTT}}.\mathsf{Tm}(1, \mathsf{Bool})$.
We have $[\![b]\!] : (i(b) \sim \mathsf{true}) + (i(b) \sim \mathsf{false})$.
By transport, we have:
$\overline{\mathbf{0}_{\mathsf{HoTT}}}.\mathsf{Tm}(1, \mathsf{Id}(i(b), \mathsf{true})) + \overline{\mathbf{0}_{\mathsf{HoTT}}}.\mathsf{Tm}(1, \mathsf{Id}(i(b), \mathsf{false}))$.

But $1^*_{\square}(i)$ is a **split weak equivalence**, i.e. i satisfies a weak lifting property for global elements.
So we have $\mathbf{0}_{\mathsf{HoTT}}.\mathsf{Tm}(1, \mathsf{Id}(b, \mathsf{true})) + \mathbf{0}_{\mathsf{HoTT}}.\mathsf{Tm}(1, \mathsf{Id}(b, \mathsf{false}))$.

$\square$

Construction of strict Rezk completions

Cubical definition of contractibility (trivial fibration):

$$\mathsf{has\text{-}ext}(X) = \forall(\alpha : \mathsf{Cof}), (x : [\alpha] \to X) \to \{X \mid \alpha \mapsto x\}.$$

[Cherubini, Coquand, Hutzler]: description of the propositional truncation of a set X as freely generated by:

$$\begin{aligned} &i : X \to \| X \|, \\ &\mathsf{ext} : \| X \| \to \mathsf{has\text{-}ext}(\| X \|) \end{aligned}$$

This can be seen as a strict Rezk completion for propositions.
(Propositions = categories with trivial hom-sets).

Construction of strict Rezk completions

Strict Rezk completion of a category $\mathcal{C}$ generated by:

$$
\begin{aligned}
& i : \mathcal{C} \to \overline{\mathcal{C}}, \\
& \mathsf{ext} : (x : \overline{\mathcal{C}}.\mathsf{Ob}) \to \mathsf{has\text{-}ext}(\Sigma_y (x \cong y)).
\end{aligned}
$$

- Holds by definition: completeness of $\overline{\mathcal{C}}$.
- To be proven: $1^*_\square(i)$ is a weak equivalence.
- To be proven: Fibrancy of the components of $\overline{\mathcal{C}}$.

Construction of strict Rezk completions

Cofibrations are levelwise decidable, $1^*_\square(\mathsf{Cof}) \cong \{\top, \bot\}$.

$\rightsquigarrow$ the category $1^*_\square(\overline{\mathcal{C}})$ is generated by:

$$1^*_\square(i) : 1^*_\square(\mathcal{C}) \to 1^*_\square(\overline{CC}),$$
$$\mathsf{ext} : (x : \overline{\mathcal{C}}.\mathsf{Ob}) \to \Sigma_y (x \cong y)$$

$\rightsquigarrow$ the category $1^*_\square(\overline{CC})$ is the algebraic fibrant replacement of $1^*_\square(\mathcal{C})$.
$\rightsquigarrow$ $1^*_\square(i)$ is an algebraic trivial cofibration (with a cofibrant source).
$\rightsquigarrow$ $1^*_\square(i)$ is a weak equivalence.

Construction of strict Rezk completions

The operation

$$\mathsf{ext} : (x : \overline{\mathcal{C}}.\mathsf{Ob}) \to \mathsf{has\text{-}ext}(\Sigma_y (x \cong y))$$

can be seen as an "isomorphism extension structure".

Similar to the equivalence extension structure

$$\langle \mathsf{Glue}, \mathsf{glue} \rangle : (A : \mathrm{Set}_{\mathsf{fib}}) \to \mathsf{has\text{-}ext}(\Sigma_B (A \simeq B))$$

which is used to prove the fibrancy of $\mathrm{Set}_{\mathsf{fib}}$.

Fibrancy from pseudo-reflexive graphs

A pseudo-reflexive graph consists of:

$$V_A : \mathrm{Set}, \qquad \text{(vertices)}$$
$$E_A : V_A \to V_A \to \mathrm{Set}, \qquad \text{(edges)}$$
$$R_A : (x : V_A) \to E_A(x, x) \to \mathrm{Set}. \qquad \text{(reflexive loops)}$$

A weak coercion operation consists, for $a : \mathbb{I} \to V_A$, of:

$$\mathsf{wcoe}^{r \to s}(a) : E_A(a(r), a(s)),$$
$$\mathsf{wcoh}^{r}(a) : R_A(a(r), a(s)),$$

Fibrancy from pseudo-reflexive graphs

Lemma

$$\begin{tikzcd} R_B \arrow[r] \arrow[d] & E_B \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & V_B \arrow[d] \\ R_A \arrow[r] & E_A \arrow[r, shift left] \arrow[r, shift right] & V_A \end{tikzcd}$$

- (R_B, E_B, V_B) *is homotopical:* $R_B \to R_A \times_{V_A} V_B$ *and* $\pi_1, \pi_2 : E_B \to E_A \times_{V_A} V_B$ *are trivial fibrations.*
- (R_A, E_A, V_A) *and* (R_B, E_B, V_B) *have compatible weak coercion operations.*

Then $V_B \to V_A$ *is a fibration.*

Fibrancy from pseudo-reflexive graphs

Pseudo-reflexive graph object in **Cat**.

$$\mathsf{ReflLoop}(\overline{\mathcal{C}}) \to \mathsf{Path}(\overline{\mathcal{C}}) \rightrightarrows \overline{\mathcal{C}}.$$

Every sort (equivalently generating cofibration) induces a dependent pseudo-reflexive graph.

$$\begin{tikzcd}
\mathsf{ReflLoop}(\overline{\mathcal{C}}).\mathsf{Hom} \arrow[r] \arrow[d] & \mathsf{Path}(\overline{\mathcal{C}}).\mathsf{Hom} \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & \overline{\mathcal{C}}.\mathsf{Hom} \arrow[d] \\
\mathsf{ReflLoop}(\overline{\mathcal{C}}).\mathsf{Ob}^2 \arrow[r] & \mathsf{Path}(\overline{\mathcal{C}}).\mathsf{Ob}^2 \arrow[r, shift left] \arrow[r, shift right] & \overline{\mathcal{C}}.\mathsf{Ob}^2.
\end{tikzcd}$$