Uniform Kan fibrations in simplicial sets (jww Eric Faber)

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Section 1

Motivation

Voevodsky

Voevodsky

Interpretation of univalent type theory in simplicial sets. The idea is: types are Kan complexes and dependent types are modelled by Kan fibrations.

This interpretation relies on the existence of the Kan-Quillen model structure on simplicial sets.

The metatheory: **ZFC** plus some inaccessibles.

Key question

Can we also prove this constructively? Say, in Aczel's set theory **CZF** plus some inaccessibles?

Obstruction

Theorem (Bezem-Coquand-Parmann)

The classical result which says that if A and B are Kan, then so is A^B, is not constructively valid.

So in particular the following result cannot be shown constructively:

Theorem (classical)

If $f : Y \to X$ and $g : Z \to Y$ are Kan fibrations, then so is $\Pi_f(g)$. (With Π_f being the right adjoint to pulling back along f.)

In response Coquand and many others have ...

- switched to cubical sets.
- added uniformity conditions to the notion of a fibration.

Today

We remain in simplicial sets.

A possible approach (not ours)

One possibility would be to take the text book definitions of a Kan fibration (map having the RLP wrt to Horn inclusions) and trivial Kan fibration (RLP wrt to boundary inclusions) and see how far one gets.

Theorem (Henry)

One can use the standard definitions of a (trivial) Kan fibration to show *constructively* that there is a model structure on simplicial sets.

Gambino-Henry-Sattler-Szumilo show how this can be extended to a model of univalent type theory as well, modulo some issues:

- They only have a weak form of Π (constructively) and stability is an issue.
- An appropriate coherence theorem to turn this into a genuine model of type theory is (so far) missing.

Our approach

We try to define a notion of a *uniform Kan fibration* in simplicial sets. By that we mean two things:

- We think of the existence of lifts as *structure* on a uniform Kan fibration (and not a property).
- We believe these lifts should satisfy certain compatibility ("uniformity") conditions.

Our aim

Define a notion of a uniform Kan fibration in simplicial sets such that ...

- they are closed under Π, constructively.
- uniform Kan fibration have the RLP wrt Horn inclusions, constructively.
- every map which has the RLP wrt Horn inclusions can be equipped with the structure of a uniform Kan fibration, classically.

A notion of fibred structure

Objection

Wait, wasn't that already done by Gambino & Sattler in their paper "The Frobenius condition, right properness, and uniform fibrations"?

Response

True, but they were unable to show that their notion of uniform Kan fibration was *local*.

Definition

Let $\mathcal{E}$ be some category and write $\mathcal{E}^{\rightarrow}_{\mathrm{cart}}$ for the category of arrows in $\mathcal{E}$ and pullback squares between them. Then we can call a functor

$$\mathrm{Fib} : (\mathcal{E}^{\rightarrow}_{\mathrm{cart}})^{op} \rightarrow \mathcal{S}ets$$

a notion of fibred structure.

Local notion of fibred structure

Definition

Suppose $\mathcal{E}$ is the category of simplicial sets, and let $\mathcal{R}$ be the full subcategory of $\mathcal{E}^{\to}_{\mathrm{cart}}$ consisting of those arrows with representable codomain. Let us say that a notion of fibred structure is *local (or: locally presentable)* if for any $f : Y \to X$ in $\mathcal{E}$ we have the following: given a family $(\, t_\sigma \in \mathrm{Fib}(g) \,:\, g \in \mathcal{R}, \sigma : g \to f \,)$ satisfying $\mathrm{Fib}(\tau)(t_\sigma) = t_{\sigma \circ \tau}$ for any pair $\sigma \in \mathcal{R} \downarrow f, \tau \in \mathrm{Ar}(\mathcal{R})$, there exists a unique element $t \in \mathrm{Fib}(f)$ such that $\mathrm{Fib}(\sigma)(t) = t_\sigma$ for any $\sigma \in \mathcal{R} \downarrow f$.

$$
\begin{array}{ccccc}
Y_{x \cdot \alpha} & \longrightarrow & Y_x & \longrightarrow & Y \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle f} \\
\Delta^m & \xrightarrow[\alpha]{} & \Delta^n & \xrightarrow[x]{} & X
\end{array}
$$

Compare: Sattler ("The equivalence extension property and model structures"), Shulman ("All $(\infty, 1)$-toposes have strict univalent universes").

Our aim, again

Because they were unable to show that their notion of a uniform Kan fibration is local, Gambino & Sattler were unable to show *constructively* that universal uniform Kan fibrations exist.

Our aim

Define a notion of a uniform Kan fibration in simplicial sets such that ...

- they are closed under Π, constructively.
- uniform Kan fibration have the RLP with respect to Horn inclusions, constructively.
- every map which has the RLP with respect to Horn inclusions can be equipped with the structure of a uniform Kan fibration, classically.
- the notion of a uniform Kan fibration is a local notion of fibred structure.

Today, I will explain our definition as a modification of the one by Gambino & Sattler (which follows the cubical sets approach). So let's first recall that one.

Section 2

The Gambino-Sattler approach

Cofibrations *à la* Gambino-Sattler

Cofibrations (G & S)

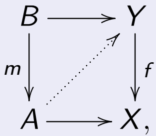
A map $m : B \to A$ of simplicial sets is a *cofibration* if each $m_n : B_n \to A_n$ is a complemented monomorphism (in the subobject lattice of A_n).

Note that cofibrations are stable under pullback.

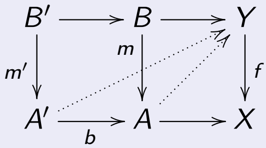
Uniform trivial fibrations *à la* Gambino-Sattler

Uniform trivial fibration (G & S)

A map $f : Y \to X$ is a *uniform trivial fibration* if it comes equipped with a choice of filler for any cofibration m and commutative square



in such a way that for any commutative diagram



in which m and m' are cofibrations and the left hand square is a pullback, the chosen fillers commute with each other.

Uniform Kan fibration *à la* Gambino-Sattler

Uniform Kan fibration (G & S)

A morphism $f : Y \to X$ is a *uniform Kan fibration* if $(Y^{\delta_i}, f^I) : Y^I \to Y \times_X X^I$ is a uniform trivial fibration. (Here $I = \Delta^1$ and $\delta^i : 1 = \Delta^0 \to \Delta^1$ with $i \in \{0,1\}$ chooses an end point of the interval.)

Ignoring the uniformity condition, this says: given

- $i \in \{0,1\}$,
- a point $y \in Y_n$,
- a path $\pi : \Delta^n \times \mathbb{I} \to X$ with $\pi(1_{[n]}, \delta_i) = f(y)$,
- a cofibrant sieve $S \subseteq \Delta^n$ and path $\rho : S \times \mathbb{I} \to Y$ with $\rho(\alpha, \delta_i) = y \cdot \alpha$ for any $\alpha \in S$ and $f \circ \rho = \pi \upharpoonright S \times \mathbb{I}$,

we get a path $\tau : \Delta^n \times \mathbb{I} \to Y$ satisfying:

- $\tau(1_{[n]}, \delta_i) = y$,
- $f \circ \tau = \pi$,
- $\tau \upharpoonright S \times \mathbb{I} = \rho$.

Section 3

Our approach

Simplicial Moore path object

In an earlier paper with Richard Garner, I defined a *simplicial Moore path functor*.

The idea is that there is an endofunctor M on simplicial sets together with natural transformations

$$r : X \to MX$$
$$s, t : MX \to X$$
$$\mu : MX \times_X MX \to MX$$

so that (X, MX, r, s, t, μ) becomes a simplicial category for any simplicial set X.

We will call the n-simplices in MX *Moore paths*.

First definition of M

Let $\mathbb{T}_0$ be the simplicial set whose n-simplices are zigzags (*traversals*) of the form

$$\bullet \xleftarrow{p_1} \bullet \xrightarrow{p_2} \bullet \xrightarrow{p_3} \bullet \xleftarrow{p_4} \bullet \xrightarrow{p_5} \bullet$$

with $p_i \in [n]$. With the final segment ordering this can be seen as a poset internal to simplicial sets (simplicial poset). Then M can be defined as the polynomial functor associated to the map $\mathrm{cod} : \mathbb{T}_1 \to \mathbb{T}_0$.

This makes M an instance of a *polynomial comonad* (more about that in Richard Garner's talk in a few weeks).

Second definition of M

Alternatively, we can define for any such n-dimensional traversal θ its *geometric realisation* $\widehat{\theta}$ (often just written θ) as the colimit of the diagram

$$\Delta^n \xrightarrow{d_{p_1^s}} \Delta^{n+1} \xleftarrow{d_{p_1^t}} \Delta^n \xrightarrow{d_{p_2^s}} \Delta^{n+1} \xleftarrow{d_{p_2^t}} \Delta^n \xrightarrow{d_{p_3^s}} \Delta^{n+1} \xleftarrow{d_{p_3^t}} \cdots \xrightarrow{d_{p_k^s}} \Delta^{n+1} \xleftarrow{d_{p_k^t}} \Delta^n$$

where p_i^s is $p_i + 1$ if edge i point to the right, and p_i^s is p_i if edge i points to the left and *vice versa* for p_i^t. Then

$$(MX)_n = \sum_{\theta \in (\mathbb{T}_0)_n} \mathrm{Hom}(\widehat{\theta}, X).$$

By considering only those n-dimensional traversals of the form

$$\bullet \xrightarrow{n} \bullet \xrightarrow{n-1} \dots \xrightarrow{1} \bullet \xrightarrow{0} \bullet$$

one can show that $X^{\mathbb{I}} \subseteq MX$.

Cofibrations

Cofibrations

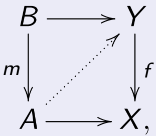
A map $m : B \to A$ of simplicial sets is a *cofibration* if each $m_n : B_n \to A_n$ is a complemented monomorphism (in the subobject lattice of A_n).

Note that cofibrations are stable under pullback *and closed under composition*.

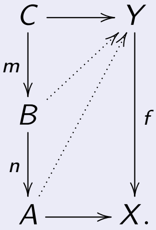
Uniform trivial fibrations

Uniform trivial fibration

A map $f : Y \to X$ is a *uniform trivial fibration* if it comes equipped with a chosen filler for any cofibration m and any commutative square



in a way which respects both pullbacks of cofibrations (as before) and composition of cofibrations:



Uniform Kan fibration (first attempt)

Uniform Kan fibration (G & S)

A morphism $f : Y \to X$ is a *uniform Kan fibration* if $(Y^{\delta_i}, f^I) : Y^I \to Y \times_X X^I$ is a uniform trivial fibration for any $i \in \{0, 1\}$.

Uniform Kan fibration (first attempt)

A morphism $f : Y \to X$ is a *uniform Kan fibration* if $(t, Mf) : MY \to Y \times_X MX$ is a uniform trivial fibration.

Ignoring the uniformity condition, this says: given

- a point $y \in Y_n$,
- an n-dimensional traversal and Moore path $\pi : \theta \to X$ with $t(\pi) = f(y)$,
- a cofibrant sieve $S \subseteq \Delta^n$ and Moore path $\rho : \theta \cdot S \to Y$ with $s(\rho) = y \cdot S$ and $f \circ \rho = \pi \restriction (\theta \cdot S)$,

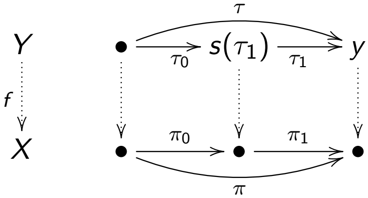
we get a Moore path $\tau : \theta \to Y$ satisfying $t(\tau) = y, f \circ \tau = \pi$ and $\tau \restriction (\theta \cdot S) = \rho$.

One more condition

To make the definition local, we add one more condition:

Final condition (ignoring the sieves)

If $\pi = \pi_1 \pi_0$ is a composition of Moore paths and y lies over the target of π, then the lift τ for π given y coincides with the composition of the lift τ_1 of π_1 given y and the lift τ_0 of π_0 given $s(\tau_1)$.



Section 4

Results

Goal achieved

We claim to have defined a notion of a uniform Kan fibration in simplicial sets such that ...

- they are closed under Π, constructively.
- uniform Kan fibration have the RLP with respect to Horn inclusions, constructively.
- every map which has the RLP with respect to Horn inclusions can be equipped with the structure of a uniform Kan fibration, classically.
- the notion of a uniform Kan fibration is a local notion of fibred structure.

We can also show that every uniform Kan fibration in our sense is also a uniform Kan fibration in the sense of Gambino-Sattler, but we expect the converse to be unprovable constructively.

Towards an algebraic model structure

The main motivation for our work was to give constructive proofs of:

- the existence of an algebraic model structure on simplicial sets.
- the existence of a model of univalent type theory in simplicial sets.

Currently we have constructive proofs/proof sketches for:

- the existence of a model structure on the simplicial sets, when restricted to those that are uniformly Kan.
- the existence of a model of type theory with $\Pi, \Sigma, \mathbb{N}, 0, 1, +, \mathrm{Id}, \times$.

Future work

What remains to be proven (constructively!):

- We can show that universal uniform Kan fibration exist, but we haven't shown they are univalent.
- We haven't shown that universes are uniformly Kan.
- And we haven't shown that there exists an algebraic model structure on the entire category of simplicial sets based on our notion of a uniform Kan fibration.

THANK YOU!