Fuzzy sets and presheaves

J.F. Jardine*

Department of Mathematics
University of Western Ontario
London, Ontario, Canada

jardine@uwo.ca

August 30, 2019

Abstract
This paper presents a presheaf theoretic approach to the construction of fuzzy sets, which builds on Barr’s description of fuzzy sets as sheaves of monomorphisms on a locale. A presheaf method is used to show that the category of fuzzy sets is complete and co-complete, and is used to give explicit descriptions of classical fuzzy sets that arise as limits and colimits. The Boolean localization construction for sheaves and presheaves on a locale \( L \) specializes to a theory of stalks for sheaves and presheaves on \( L \), provided that \( L \) approximates the structure of a closed interval in the real line.

The system \( V(X) \) of Vietoris-Rips complexes for a data set \( X \) becomes both a simplicial fuzzy set and a simplicial sheaf in this general framework. This example is explicitly discussed in this paper, in stages through a series of examples.

Introduction

Fuzzy sets were originally defined to be functions

\[ \psi : X \to [0, 1] \]

which take values in the unit interval [2].

Michael Barr changed the game in his paper [1]; he replaced the unit interval by a more general well-behaved poset, or locale \( L \), and redefined fuzzy sets to be functions

\[ \psi : X \to L. \]

*Supported by NSERC.
These functions are fuzzy sets over $L$, and form a category $\text{Fuzz}(L)$ that can be described in various ways.

Locales are complete lattices in which finite intersections distribute over all unions. The unit interval $[0, 1]$ qualifies, but so does the poset of open subsets of a topological space, and locales are categorical models for spaces. Every locale $L$ has a Grothendieck topology, with coverings defined by joins, and so one is entitled to a presheaf category $\text{Pre}(L)$ and a sheaf category $\text{Shv}(L)$ for these objects. Presheaves on $L$ are contravariant set-valued functors on $L$, and sheaves are presheaves that satisfy a patching condition with respect to the Grothendieck topology on $L$.

Barr showed that, starting with a fuzzy set $\psi : X \to L$ over $L$, one can pull back over subobjects to define a sheaf $T(\psi)$, which is a sheaf of monomorphisms in the sense that all restriction maps are injections. Technically, one needs to adjoin a new zero object to $L$ to make this work, giving a new (but not so different) locale $L_+$. The resulting object $T(\psi)$ is a sheaf of monomorphisms on $L_+$.

Write $\text{Mon}(L_+)$ for the category of sheaves of monomorphisms on $L_+$. Barr showed [1] that his functor $T : \text{Fuzz}(L) \to \text{Mon}(L_+)$ is part of a categorical equivalence. His result appears as Theorem 12 in the first section of this paper, along with a proof.

The inverse functor for $T$ is constructed for a sheaf $F$ by taking the generic fibre $F(i)$, and constructing a function $\psi_F : F(i) \to L$. The set $F(i)$ is the set of sections corresponding to the initial object $i$ of $L$. Given an element $x \in F(i)$, there is a maximum $s_x \in L$ such that $x$ is in the image of the monomorphism $F(s_x) \to F(i)$, and one defines $\psi_F(x)$ to be this element $s_x$.

The first section of this paper is largely expository and presents a quick synthesis of a modern approach to fuzzy set theory, with a particular view to applications in topological data analysis [8]. We establish notation and introduce our main examples, and present a proof of the the Barr result (Theorem 12). Examples 6 and Example 10 show, respectively, that the Vietoris-Rips filtration corresponding to a data set has the structure of a simplicial fuzzy set and (through the Barr theorem) a simplicial sheaf.

To proceed with such applications, for example if you want to shearify persistent homology theory or clustering and use fuzzy sets to do so, or if you want to investigate the homotopy types of simplicial fuzzy sets, it is helpful to have more explicit information about how fuzzy sets are constructed. One needs, in particular, straightforward descriptions of basic constructions such as limits, colimits and stalks in the (simplicial) fuzzy set category, or rather in the associated category of (simplicial) sheaves of monomorphisms. The difficulties, such as they are, arise from the fact that the category $\text{Mon}(L_+)$ is not quite a sheaf category, and constructing the fuzzy set $\psi_F : F(i) \to L$ from a sheaf $F$ can be a bit interesting.
These issues are dealt with in Sections 2 and 3 of this paper. There is a perfectly good category $\text{Mon}_p(L_+)$ of presheaves of monomorphisms, and it turns out that if $L$ is sufficiently well behaved (like the unit interval $[0,1]$), then the associated sheaf functor is easily described and preserves presheaves of monomorphisms. The upshot is that one can make constructions on the presheaf category as a geometer or topologist would, and then sheafify.

Limits of presheaves of monomorphisms are formed as in the ambient sheaf category, but colimits are more involved. The inclusion $\text{Mon}(L_+) \subset \text{Shv}(L_+)$ of sheaves of monomorphisms in all sheaves has a left adjoint $F \mapsto \text{Im}(F)$ that I call the image functor, which is defined by taking images of sections in the generic fibre — see Lemma 26. This observation allows one to define colimits of diagrams $A(j)$ in $\text{Mon}(L_+)$: take the presheaf theoretic colimit $\varinjlim_j A(j)$, and then apply the image functor (suitably sheafified) to get the colimit in $\text{Mon}(L_+)$. The image functor and colimit constructions are described in the second section. That section also contains the formal definitions and properties around presheaves of monomorphisms.

One has the nicest form of the associated sheaf functor for presheaves on a locale $L$ when one assumes that $L$ is an interval (Lemma 22). The interval assumption on $L$ is consistent with both the classical theory of fuzzy sets and intended applications in topological data analysis.

The general theory of Boolean localization for sheaves and presheaves on a locale $L$ is relatively straightforward to describe, and is the starting point for the discussion of stalks for sheaves and presheaves on $L$ that is the subject of the third section of this paper.

Every locale $L$ has a standard imbedding $\omega : L \to B$ into a complete Boolean algebra, by a rather transparent construction that is displayed here (see also [7], [6], for example). Technically, this is the easier part of the general Boolean localization construction — the more interesting bit is the formation of the Diaconescu cover, which faithfully imbeds a Grothendieck topos in the topos of sheaves on a locale.

If the locale $L$ is an interval, then the corresponding Boolean algebra $B$ is the set of subsets of some set, so that the sheaf category for $L$ has enough points, and hence a theory of stalks. The same is true for finite products of intervals. Stalks for sheaves of monomorphisms on an interval that arises from the general theory have a fairly simple description, which can be used as a starting point for a result (Lemma 32) that describes the standard behaviour of stalks in that case, and has a relatively non-technical proof.

The third section finishes with a description of stalks for presheaves on an interval $L$. Stalks for presheaves are constructed with a left Kan extension, just like stalks for presheaves on a topological space, or on the étale site of a scheme.

This paper was written to clear the air about the sheaf theoretic properties of fuzzy sets, and to set the stage for potential applications of the local homotopy theory of simplicial sheaves in topological data analysis from a fuzzy sets point of view.
of view.

So yes, in Example 10 we see that the system of Vietoris-Rips complexes $s \mapsto V_s(X)$ which is associated to a data set $X \subset \mathbb{R}^n$ does form a simplicial fuzzy set, or a simplicial sheaf (of monomorphisms) on the locale $[0, R]^{op}_+$, where $R$ is larger than all distances between points of $X$. But we also see in Example 34 that this simplicial sheaf has a rather awkward collection of stalks, which includes the full data set $X$ sitting as a discrete simplicial set stalks corresponding to small values of the distance parameter. It follows, for example, that an inclusion $X \subset Y \subset \mathbb{R}^n$ induces a stalkwise weak equivalence of sheaves on $[0, R]^{op}_+$ if and only if $X = Y$.

This is quite like the situation that was encountered in the first attempt to give a sheaf theoretic context for topological data analysis [5]. That early paper uses a different topology on the underlying interval of distance parameters, but the resulting sheaf theory has essentially the same stalks and therefore has the same problem with local weak equivalences that are too tightly defined to be useful.

The facile conclusion is that the local homotopy theory of simplicial sheaves does not yet have a model in topological data analysis that has good applications, and a more subtle approach is required.

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1 Fuzzy sets and sheaves

A frame $L$ is a complete lattice in which finite meets distribute over all joins. Examples include the poset $op|X$ of open subsets of a topological space $X$.

According to this definition, $L$ has a terminal object 1 (empty meet) and an initial object 0 (empty join) — see [7, p.471].

The completeness assumption means that every set of elements $a_i \in L$ has a least upper bound $\bigvee a_i$. The set $a_i$ also has a greatest lower bound $\bigwedge a_i$, which is the least upper bound $\bigvee x \leq a_i$ of the elements $x$ which are smaller than all $a_i$.

A morphism $L_1 \to L_2$ of frames is a poset morphism which preserves meets and joins, and hence preserves initial and terminal objects. The category of locales is the opposite of the frame category, and one tends to use the terms “frame” and “locale” interchangeably, and I shall refer to these objects as locales henceforth.
Example 1. The interval $[0, 1]$ of real numbers $s$ with $0 \leq s \leq 1$, with the standard ordering, is a locale. The poset $[0, 1]$ imbeds in $\text{op}][0, 1]$ by the assignment $a \mapsto [0, a)$. Then $s \land t$ corresponds to the interval 

$$[0, s) \cap [0, t) = [0, s \land t),$$

so that $s \land t = \min\{s, t\}$. Similarly,

$$\cup_i [0, s_i) = [0, \lor_i s_i),$$

and $\lor_i s_i$ is the least upper bound of the numbers $s_i$.

Example 2. The interval $[0, 1]$ with the opposite ordering $[0, 1]^\text{op}$ is also a locale. In other words, $s \leq t$ in $[0, 1]^\text{op}$ if and only if $t \leq s$ in $[0, 1]$.

In this case, $[0, 1]^\text{op}$ imbeds in $\text{op}][0, 1]$ by the assignment $s \mapsto (s, 1]$. Then $\lor_i s_i$ is the greatest lower bound of the $s_i$ and $s \land t = \max\{s, t\}$.

Observe that the posets $[0, 1]$ and $[0, 1]^\text{op}$ both have infinite meets, given by greatest lower bound and least upper bound, respectively.

Example 3. The closed interval $[a, b]$ and its opposite $[a, b]^\text{op}$ are locales, and there are linear scaling isomorphisms $[a, b] \cong [0, 1]$ and $[a, b]^\text{op} \cong [0, 1]^\text{op}$, respectively.

Example 4. Suppose that $L_1, \ldots, L_k$ are locales. Then the product poset

$$L_1 \times \cdots \times L_k$$

is also a locale.

Suppose that $L$ is a locale. Following [1], a function $\psi : X \to L$ is a fuzzy set over $L$. These are the objects of a category $\text{Fuzz}(L)$, called the category of fuzzy sets over $L$.

Suppose that $\phi : Y \to L$ is another such function. A morphism $(f, h) : \psi \to \phi$ of $\text{Fuzz}(L)$ consists of a function $f : X \to Y$ and a relation $h : \psi \leq \phi \cdot f$ of functions taking values in the poset $L$. The existence of the relation $h$ means precisely that $\psi(x) \leq \phi(f(x))$ in the poset $L$ for all $x \in X$.

There is a poset $L^X$ whose objects are the functions $\psi : X \to L$. If $\gamma : X \to L$ is another such function, then there is a relation $\psi \leq \gamma$ if $\psi(x) \leq \gamma(x)$ for all $x \in X$. Every function $f : X \to Y$ determines a restriction functor $L^Y \to L^X$, so that there is a contravariant functor $\text{Set} \to \text{cat}$ which is defined by associating the poset $L^X$ to the set $X$.

Following Quillen (see [3], for example), a homotopy $h : \psi \to \phi \cdot f$ is a natural transformation between functors $L^Y \to L^X$, which in the case at hand is given by the relations $h(x) \leq \phi(f(x)), x \in X$.

From this point of view, a morphism of $\text{Fuzz}(L)$ is a morphism

$$(f, h) : \psi \to \phi$$

in the Grothendieck construction associated to the diagram of restriction functors, and the fuzzy set category $\text{Fuzz}(L)$ is that Grothendieck construction.
**Example 5.** All commutative diagrams

\[
\begin{array}{ccc}
X & \xrightarrow{g} & Y \\
\downarrow & & \downarrow \\
L & \xleftarrow{} & L
\end{array}
\]

correspond to morphisms \((g, 1)\) of \(\text{Fuzz}(L)\) with identity homotopies in \(L\), but the full collection of fuzzy set morphisms \(X \rightarrow Y\) is larger — these are the homotopy commutative diagrams.

**Example 6.** Suppose that a finite set \(X \subset \mathbb{R}^n\) is a data set, and suppose that \(X \xrightarrow{\cong} \mathbb{N}\) is a listing of the members of \(X\), where \(\mathbb{N} = \{0, 1, \ldots, N\}\). Choose a number \(R\) such that \(d(x, y) < R\) for all \(x, y \in X\).

Here, \(d(x, y)\) is the distance between the points \(x\) and \(y\) in \(\mathbb{R}^n\).

Let \(\sigma\) be an ordered set of points \(\sigma = \{x_0, x_1, \ldots, x_k\}\) in \(X\). Write

\[
\phi(\sigma) = \max_{i,j} d(x_i, x_j)
\]

Suppose that \(\theta : r \rightarrow k\) is an ordinal number map. Then \(\phi(\theta^*(\sigma)) \leq \phi(\sigma)\), with equality if \(\theta\) is surjective, or if \(\theta^*(\sigma)\) is a degeneracy of \(\sigma\). Further, \(\phi(\sigma) = 0\) if and only if \(\sigma\) is a degeneracy of a vertex.

The assignment \(\sigma \mapsto \phi(\sigma)\) defines a function

\[
\phi : \Delta^N_k \rightarrow [0, R].
\]

on the set \(\Delta^N_k\) of \(k\)-simplices of the simplicial set \(\Delta^N\).

If \(\theta : r \rightarrow k\) is an ordinal number map, then the relation \(\phi(\theta^*(\sigma)) \leq \phi(\sigma)\) defines a homotopy commutative diagram

\[
\begin{array}{ccc}
\Delta^N_k & \xrightarrow{\theta^*} & \Delta^N_r \\
\downarrow{\phi} & & \downarrow{\phi} \\
[0, R]^\text{op} & & [0, R]^\text{op}
\end{array}
\]

or equivalently a morphism of fuzzy sets with values in the locale \([0, R]^\text{op}\).

The ordering \(X \cong \mathbb{N}\) on the elements of the data set \(X\) and the ambient distance function on \(\mathbb{R}^n\) combine to give the simplicial set \(\Delta^X := \Delta^N\) the structure of a simplicial fuzzy set \(\phi : \Delta^X \rightarrow [0, R]^\text{op}\), with coefficients in the locale \(L = [0, R]^\text{op}\).

A simplicial fuzzy set \(X\) is a simplicial object in \(\text{Fuzz}(L)\), meaning a contravariant functor \(X : \Delta^\text{op} \rightarrow \text{Fuzz}(L)\) on the category of finite ordinal numbers. This usage is standard: a simplicial object in a category \(\mathcal{A}\) is a functor \(\Delta^\text{op} \rightarrow \mathcal{A}\). See [3].

The first appearance of simplicial fuzzy sets in the literature may be in Spivak’s preprint [9] of 2009, where these objects are called fuzzy simplicial sets.
The explicit interpretation of the Vietoris-Rips filtration as a simplicial fuzzy set that is presented in this paper seems to be new, but see the Healy-McInnes paper [8].

Suppose that \( L \) is a locale. Then \( L_+ = L \sqcup \{0\} \) is also a locale, where \( 0 \) is a new initial element.

**Remark 7.** If \( L = [0, R]^{op} \) then the object \( R \) is no longer initial in \( L_+ \). I normally write \( i \) for the number \( R \) (the original initial object of \( [0, R]^{op} \)) to distinguish this element from the initial object \( 0 \) of \( L_+ \). Clearly, \( 0 < i \) in \( L_+ \).

Any locale \( L \) has a Grothendieck topology, for which the covering families of \( a \in L \) are sets of objects \( b_i \leq a \) such that \( \lor_i b_i = a \). This relation is equivalent to the assertion that \( a \) is the least upper bound in \( L \) for all elements \( b_i \).

Given a family of elements \( b_i \leq a \), the associated sieve \( R \) is the set of all elements \( s \) such that \( s \leq b_i \) for some \( i \). The sieve \( R \) is covering if \( \{b_i\} \) is a covering family.

Equivalently, an arbitrary sieve \( R \), i.e. a subset of the collection of elements \( s \leq a \) which is closed under taking subobjects, is covering if \( \lor_{s \in R} s = a \).

Since \( L \) has a Grothendieck topology, it has associated categories \( \text{Pre}(L) \) and \( \text{Shv}(L) \) of presheaves and sheaves on \( L \), respectively.

A presheaf is a functor \( F : L^{op} \to \text{Set} \), and a morphism of presheaves is a natural transformation.

One says that the presheaf \( F \) is a sheaf if the map
\[
F(a) \to \lim_{b \in R} F(b)
\]
is an isomorphism for all covering sieves \( R \) of all objects \( a \). This is equivalent to requiring that the diagram
\[
F(a) \to \prod_i F(b_i) \rightrightarrows \prod_{i,j} F(b_i \land b_j)
\]
is an equalizer for all covering families \( \{b_i\} \) of all objects \( a \). In other words, \( F(a) \) should be recovered from the values of \( F(b_i) \) by patching, for all coverings \( \{b_i\} \) of \( a \).

**Remark 8.** If \( i \) is an initial object of \( L \) and \( F \) is a sheaf, then \( F(i) \) must be the one-point set. I write \( F(i) = * \) to express this.

In effect, the empty sieve \( \emptyset \subset \text{hom}(, i) \) is covering, because \( i \) is an empty join. It follows that there is an isomorphism
\[
F(i) \cong \text{hom}(\text{hom}(, i), F) \cong \text{hom}(\emptyset, F) = *
\]
for any sheaf \( F \). Compare with [6, p.35].

We are therefore entitled to categories \( \text{Pre}(L_+) \) and \( \text{Shv}(L_+) \) of presheaves and sheaves, respectively for the locale \( L_+ \), and these are the examples that we will focus on.
Write $\textbf{Mon}(L_+)$ for the full subcategory of the sheaf category $\textbf{Shv}(L_+)$, whose objects are the sheaves $F$ such that all restriction maps $F(b) \to F(a)$ associated to relations $a \leq b$ in $L$ are monomorphisms. The requirement that the relation $a \leq b$ is in $L$ is important, because $F(0) = \ast$.

Barr constructs a functor $[1]$

$$T : \textbf{Fuzz}(L) \to \textbf{Mon}(L_+)$$

which defines an equivalence of categories. The existence of this equivalence of categories is the main result of [1], and it appears as Theorem 12 below.

Explicitly, define

$$L_{\geq a} = \{ x \in L \mid x \geq a \}.$$ 

If $\psi : X \to L$ is a member of $\textbf{Fuzz}(L)$, define a presheaf $T(\psi)$ by

$$T(\psi)(a) = \psi^{-1}(L_{\geq a}).$$

for $a \in L$, and set $T(\psi)(0) = \ast$. Then the assignment $a \mapsto T(\psi)(a)$ defines a presheaf on $L_+$ such that every relation $s \leq t$ induces a monomorphism $T(\psi)(t) \to T(\psi)(s)$. The presheaf $T(\psi)$ is a sheaf because $x \in T(\psi)(a)$ if and only if $x \in T(\psi(b_i))$ for any covering family $\{b_i\}$ of $a$.

If $(f,h) : \psi \to \phi$ is a morphism of fuzzy sets (as above), then the relations $\psi(x) \leq \phi(f(x))$ in $L$ (i.e. the homotopy $h$) imply that if $\psi(x) \in L_{\geq a}$ then $\phi(f(x)) \in L_{\geq a}$, and so the function $f$ restricts to functions $f : \psi^{-1}L_{\geq a} \to \phi^{-1}L_{\geq a}$ that are natural in $a$, so that we have a sheaf homomorphism

$$f_* : T(\psi) \to T(\phi).$$

**Remark 9.** $T(X)$ is sometimes called the level cut description of the fuzzy set $\psi : X \to L$ — see [2].

**Example 10.** Suppose that the finite set $X \subset \mathbb{R}^n$ is a data set, with ordering $X \xrightarrow{\sim} \mathbb{N}$ as in Example 6. Again, choose $R > d(x,y)$ for all pairs of points $x,y \in X$.

Recall that the simplicial fuzzy set $\phi : \Delta^N \to [0,R]^\text{op}$ is defined for a simplex $\sigma = \{x_0,x_1,\ldots,x_k\}$ by

$$\phi(\sigma) = \max_{i,j} d(x_i,x_j).$$

Then, for $s \in [0,R]$,

$$T(\phi)_s = \phi^{-1}[0,s],$$

which is the set of $k$-simplices $\sigma = \{x_0,\ldots,x_k\}$ of $\Delta^N$ such that $d(x_i,x_j) \leq s$. It follows that

$$T(\phi)_s = V_s(X)_k$$

is the set of $k$-simplices of the Vietoris-Rips complex $V_s(X)$ for the data set $X$.

The Vietoris-Rips complex functor $s \mapsto V_s(X)$ is the simplicial sheaf that Barr’s construction associates to the simplicial fuzzy set $\phi : \Delta^X \to [0,R]^\text{op}$. 

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For any sheaf $F \in \text{Mon}(L_+)$ there is an isomorphism

$$\lim_{a \in L} F(a) \xrightarrow{\cong} F(i),$$

since $i$ is initial in $L$. This colimit is filtered, and the canonical maps $F(a) \to F(i)$ are monomorphisms.

I say that $F(i)$ is the generic fibre of the object $F$.

**Lemma 11.** Suppose that $F$ is a sheaf of monomorphisms on $L_+$ and that $x \in F(i)$. Then there is a unique maximum element $s_x$ such that $x \in F(s_x)$.

**Proof.** Consider all $c$ in $L$ such that $x \in F(c)$, and let

$$s_x = \vee_{x \in F(c)} c.$$ 

Then $s_x$ is covered by the elements $c$, and so $x \in F(s_x)$.

Suppose again that $F \in \text{Mon}(L_+)$. By Lemma 11, for each $x \in F(i)$, there is a unique maximum $s_x$ such that $x \in F(s_x)$. Define $\psi(F) : F(i) \to L$ by setting $\psi_F(x) = s_x$. Then we have a function

$$\psi(F) : F(i) \to L,$$

which is a fuzzy set.

To put it a slightly different way, the fuzzy set $\psi_F : F(i) \to L$ is defined by

$$\psi_F(x) = \sup \{ b \mid x \in F(b) \}$$

for $x \in F(i)$, and $F \in \text{Mon}(L_+)$.  

**Theorem 12** (Barr). The assignments $\psi$ and $\phi$ define an equivalence of categories

$$\psi : \text{Mon}(L_+) \cong \text{Fuzz}(L) : T$$

**Proof.** Suppose that $F$ is a sheaf of monomorphisms, and that $b \in L$, and let $\psi_F : F(i) \to L$ be the corresponding fuzzy set. Then

$$F(b) = \psi(F)^{-1}(L \geq b)$$

as subsets of $F(i)$, so that there is a natural sheaf isomorphism

$$F \xrightarrow{\cong} T(\psi(F)).$$

Suppose that $f : X \to L$ is a fuzzy set. Then $\psi(f)(i) = f^{-1}(L) = X$ and $x \in f^{-1}(L \geq f(x))$. If $x \in f^{-1}(L \geq b)$ for some $b$, then $f(x) \geq b$. It follows that $f(x) = \psi(T(f))(x).$  

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Example 13. The representable functor $\text{hom}(\cdot, s)$ on $L_+$ has the form

$$\text{hom}(\cdot, s)(t) = \begin{cases} \ast & \text{if } t \leq s, \text{ and} \\ \emptyset & \text{otherwise.} \end{cases}$$

Here, $\ast$ is the one-point set.

This presheaf is a sheaf, so one says that the topology on $L_+$ is sub-canonical. The sheaf $\text{hom}(\cdot, s)$ is a sheaf of monomorphisms. The corresponding fuzzy set, for $s \in L$, is the function $s : \ast \rightarrow L$ which picks out the element $s \in L$.

The constant presheaf $\ast$ is defined to be a one-point set $\ast(a)$ for all $a \in L_+$, with identity maps associated to all relations $a \leq b$. This presheaf is a sheaf, and is a member of $\text{Mon}(L_+)$. This sheaf is represented by the terminal object $t : \ast \rightarrow L$.

If $s \leq t$ in $L$, then the induced sheaf map $\text{hom}(\cdot, s) \rightarrow \text{hom}(\cdot, t)$ corresponds to the fuzzy set map from $s : \ast \rightarrow L$ to $t : \ast \rightarrow L$ which is given by the identity function on $\ast$ and the relation $s \leq t$.

Example 14. Suppose that $K$ is a simplicial set. The simplicial presheaf $L_s(K)$ that is defined by

$$L_s(k) = \text{hom}(\cdot, s) \times K$$

is a simplicial sheaf of monomorphisms, and therefore represents a simplicial fuzzy set. Observe that there is a natural isomorphism

$$\text{hom}(L_s(K), X) \cong \text{hom}(K, X(s))$$

for all simplicial sheaves (or simplicial presheaves) $X$ on $L_+$. See also [6, Sec. 2.3].

Lemma 15. The category $\text{Mon}(L_+)$ is complete. Limits are formed in the ambient sheaf category $\text{Shv}(L_+)$. 

Proof. This result follows from the fact that an inverse limit of monomorphisms is a monomorphism. 

Example 16. Form the pullback diagram

$$\begin{array}{ccc} Z & \longrightarrow & F \\ \downarrow & & \downarrow q \\ E & \longrightarrow & X \end{array}$$

of sheaves on $L_+$, with $E, F, X$ all in $\text{Mon}(L_+)$. Take

$$(x, y) \in Z(i) = E(i) \times_{X(i)} F(i)$$

and suppose that $(x, y) \in Z(a)$. Then $x \in E(a)$ and $y \in F(a)$ so that $a \leq \psi_E(x)$ and $a \leq \psi_F(y)$. It follows that $a \leq \psi_E(x) \wedge \psi_F(y)$. 

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On the other hand, if \( b \leq \psi \) \( E(x) \land \psi_F(y) \), then there is a \( v \in E(b) \) which restricts to \( x \) and a \( u \in F(b) \) which restricts to \( y \). Also, \( p(v) \) and \( q(u) \) in \( X(b) \) restrict to the same element of \( X(i) \), so that \( p(v) = q(u) \), in \( X(b) \) and \((u,v) \in Z(b)\).

It follows that
\[
\psi_Z((x,y)) = \psi_E(x) \land \psi_F(y)
\]
for all \((x,y) \in Z(i)\).

Another way of saying this is to assert that \( \psi_Z((x,y)) \) is the greatest lower bound of \( \psi_E(x) \) and \( \psi_E(y) \).

**Example 17.** Suppose that \( X : J \to \text{Mon}(L_+) \) is a small diagram. For a fixed object \( a \in L_+ \), the \( a \)-sections of \( Z = \lim \leftarrow_j X(j) \) are the \( J \)-compatible families \( \{x_j\} \) of elements in the various sets \( X(j)(a) \).

One can use the methods of the pullback case in Example 16 to show that \( \psi_Z(\{x_j\}) \) is the greatest lower bound in \( L_+ \) of the elements \( \psi_{X(j)}(x_j) \).

## 2 Presheaves of monomorphisms

Suppose that \( L \) is a locale.

We now consider presheaves \( F : (L_+)^{op} \to \text{Set} \) such that \( F(0) = * \) and all morphisms \( a \leq b \) of \( L \) induce monomorphisms \( F(b) \to F(a) \). Such a presheaf is called a presheaf of monomorphisms.

Write \( \text{Mon}_p(L_+) \) for the category of presheaves of this form.

Most of the results of this section depend on the assumption that the locale \( L \) is an interval in the sense that

1) \( L \) has a total ordering, and

2) the ordering is dense, meaning that if \( a < b \) in \( L \), there is an \( s \in L \) such that \( a < s < b \).

The locales of immediate practical interest, such as a closed interval \([c, d] \subset \mathbb{R}\) and its opposite, are intervals in this sense.

**Lemma 18.** Suppose that the locale \( L \) is totally ordered. Then the covering sieves for \( a \in L \) are defined by the families of all \( b \) such that \( b < a \) or such that \( b \leq a \).

**Proof.** Suppose that a covering sieve \( R \subset \text{hom}(), a \) is generated by a set of elements \( b_i \), so that \( a = \vee_i b_i \). Suppose that \( R \neq \text{hom}(), a \).

Suppose that \( c < a \). If \( c \) is not bounded above by some \( b_i \), then \( b_i < c \) for all \( i \) since \( L \) is totally ordered, so that
\[
a = \vee_i b_i \leq c < a,
\]
and we have a contradiction. It follows that \( c \leq b_i \) for some \( i \), and so the relation \( c < a \) is in \( R \). \( \square \)
Remark 19. The collection of all \( b \) such that \( b \leq a \) is the trivial covering sieve for \( a \), because it includes the identity relation on the object \( a \). Lemma 18 says that an element \( a \in L \) has at most two covering sieves if \( L \) is totally ordered.

In order to be assured that \( a \in L \) has a non-trivial covering, or that the elements \( b < a \) cover \( a \), we need to know that \( L \) satisfies condition 2) above, so that \( L \) is an interval.

Example 20. The total ordering on \( L \) is necessary for the conclusion of Lemma 18.

The elements \((1, 0)\) and \((0, 1)\) define a covering of \((1, 1)\) in \([0, 1] \times 2\), and the element \((\frac{1}{2}, \frac{1}{2})\) is not bounded above by either \((1, 0)\) or \((0, 1)\).

We shall assume that the locale \( L \) is an interval for the rest of this section.

It follows from Lemma 18 that a presheaf \( F \) on \( L^+ \) is a sheaf if and only if \( F(0) = * \) and the map

\[
\eta : F(a) \to \lim_{0 < b < a} F(b) =: LF(a)
\]

is an isomorphism for all \( a \in L \) with \( a \) not initial. There is no condition on \( F(i) \) if \( i \in L \) is initial, and \( LF(i) = F(i) \).

The assignment \( a \mapsto LF(a) \) defines a presheaf \( LF \) on \( L_+ \). Because \( L \) has a total ordering and there are so few covering sieves for elements of \( L \), the presheaf \( LF \) is the universal separated presheaf associated to \( F \) [6, Lem 3.13].

In general, there is a canonical natural map \( \eta : E \to LE \) for all presheaves \( E \), and \( LE \) is a sheaf if \( E \) is separated. A presheaf \( E \) is separated if the map \( \eta : E \to LE \) is a sectionwise monomorphism.

Corollary 21. If \( F \in \text{Mon}_p(L^+) \), then the map \( \eta \) is a sectionwise monomorphism, so that \( F \) is a separated presheaf and \( LF \) is its associated sheaf.

Lemma 22. If \( F \in \text{Mon}_p(L^+) \), then \( LF \in \text{Mon}(L^+) \). In particular, the associated sheaf functor

\[
L^2 : \text{Pre}(L^+) \to \text{Shv}(L^+)
\]

restricts to a functor \( \text{Mon}_p(L^+) \to \text{Mon}(L^+) \).

Proof. Suppose that \( b \leq c \) in \( L \). We show the restriction map

\[
LF(c) \to LF(b)
\]

is a monomorphism.

Given compatible families \( \{x_s\} \) and \( \{y_s\} \) for \( s < c \), if \( x_s = y_s \) for \( s < b \), then \( x_s \) and \( y_s \) have the same image in \( F(t) \) for some \( t < b \), and so \( x_s = y_s \). \( \square \)

Example 23. Suppose that \( L = [0, 1] \), let \( A \) be a pointed set with base point \( * \). Define a presheaf \( F_A : (L_+)^{op} \to \text{Set} \) by

\[
F_A(s) = \begin{cases} * & \text{if } s = 1, \\ A & \text{if } s < 1 \text{ in } [0, 1]. \end{cases}
\]
Set $F_A(0) = \ast$, where 0 is the new initial object of $[0, 1]^{+}$.

If $s < 1$ the induced map $F_A(1) \to F_A(s)$ is the inclusion of the base point of $A$, and if $s \leq t < 1$ in $L$ then $F_A(t) \to F_A(s)$ is the identity on $A$. Then $\lim_{s \prec 1} F_A(s)$ is the set $A$ and not the base point $\ast$ in general, so that $F_A$ is a presheaf of monomorphisms, and is not a sheaf.

**Example 24.** Suppose that $F_i, i \in I$ is a list of objects in $\text{Mon}_p(L_+)$. Then the disjoint union $\sqcup_i F_i$ is in $\text{Mon}_p(L_+)$. Note that we must set $(\sqcup_i F_i)(0) = \ast$ for this to work.

**Example 25.** Suppose that $A_i \subset F$ are subobjects of a fixed object $F \in \text{Mon}_p(L_+)$, so that all $A_i$ are in $\text{Mon}_p(L_+)$. Then the (sectionwise) union $\sqcup_i A_i$ is a subobject of $F$, and is also in $\text{Mon}_p(L_+)$. It follows that the category $\text{Sub}(F)$ of subobjects of an object $F \in \text{Mon}_p(L_+)$ is a locale.

Suppose that $E$ is a presheaf on $L_+$. The epi-monic factorizations of the maps $E(s) \to E(i)$ for $s \in L$ determine subobjects $\text{Im}(E)(s) \subset E(i)$ with commutative diagrams

\[
\begin{array}{ccc}
E(t) & \longrightarrow & \text{Im}(E)(t) \\
\downarrow & & \downarrow \\
E(s) & \longrightarrow & \text{Im}(E)(s)
\end{array}
\]

for $s \leq t$. Set $\text{Im}(E)(0) = \ast$.

If $E$ is in $\text{Mon}_p(L_+)$, then the maps $E(t) \to \text{Im}(E)(t)$ are isomorphisms. These constructions are functorial in presheaves $E$.

We therefore have the following:

**Lemma 26.** There is a natural presheaf map $E \to \text{Im}(E)$ such that $\text{Im}(E)$ is in $\text{Mon}_p(L_+)$ and that this map is initial among all maps $E \to F$ with $F \in \text{Mon}_p(L_+)$. In other words, there is a natural bijection

\[
\text{hom}_{\text{Mon}_p(L_+)}(\text{Im}(E), F) \cong \text{hom}(E, F),
\]

so that the functor $E \mapsto \text{Im}(E)$ is left adjoint to the inclusion of $\text{Mon}_p(L_+)$ in the presheaf category on $L_+$.

**Corollary 27.** Suppose that $E$ is a presheaf on $L_+$. Then the morphism $E \to L(\text{Im}(E))$ is initial among all presheaf maps $E \to F$ such that $F$ is in $\text{Mon}(L_+)$. 

**Proof.** The object $L(\text{Im}(E))$ is the sheaf associated to $\text{Im}(E)$ and it is a sheaf of monomorphisms by Lemma 22. Use also the adjointness assertion of Lemma 26. \qed

Colimits of fuzzy sets can be described by the following result:
Lemma 28. Suppose that $A : J \to \text{Mon}(L_+)$ is a small diagram in the category of sheaves with monomorphisms on $L_+$. Form the colimit

$$X = L(\text{Im}(\lim_{j \in J} A(j)))$$

in $\text{Mon}(L_+)$, and let $\psi_X : X(i) \to L$ be the corresponding fuzzy set. Then

$$\psi_X(x) = \bigvee_{j,y} \psi_{A(j)}(y),$$

where the index is over all pairs $j, y$ such that $y \mapsto x$ under a composite of the form

$$A(j)(s) \to \lim_{j} A(j)(s) \to \text{Im}(\lim_{j} A(j))(s) \to X(i).$$

(3)

Proof. We have

$$X(i) = \lim_{j} A(j)(i).$$

and it follows that every $x \in X(i)$ is in the image of some composite (3).

Suppose that $y \in A(j)(s) \mapsto x$ under the composite (3). Then

$$\psi_{A(j)}(y) \leq \psi_X(x).$$

This is true for all such pairs $(y, j)$ so that

$$\bigvee_{j,y} \psi_{A(j)}(y) \leq \psi_X(x).$$

Suppose that $x \in X(i)$ lifts to $x' \in X(t)$, where $t$ is maximal. The element $x'$ is in the image of some composite

$$A(i')(s) \to \lim_{j} A(j)(t) \to \text{Im}(\lim_{j} A(j))(s) \to X(i)$$

for all $s < t$. This means that there is an element $y' \in A(i')(t)$ which maps to $x'$ under the composite above, and so $s \leq \psi_{A(i')}(y')$ for all $s < t$. It follows that

$$t = \psi_X(x) \leq \bigvee_{i,y} \psi_{A(i)}(y).$$

Remark 29. Colimits of fuzzy sets and the left adjoint of the inclusion functor

$$\text{Fuzz}(L) = \text{Mon}(L_+) \subset \text{Shv}(L_+)$$

are described in Lemma 1.3 of Spivak’s preprint [9]. In the present notation, that left adjoint is the functor $F \mapsto L^2 \text{Im}(F)$. The cocompleteness of the category of fuzzy sets follows from the cocompleteness of the sheaf category and the existence of the left adjoint of the inclusion.
Example 30. Form the union $A \cup B$ of two subsheaves $A, B \subset F$ of a sheaf $F \in \text{Mon}(L_+)$. Then there is a pushout diagram

$$
\begin{array}{ccc}
A \cap B & \rightarrow & B \\
\downarrow & & \downarrow \\
A & \rightarrow & A \cup B
\end{array}
$$

in $\text{Mon}(L_+)$. Here, $A \cup B$ is the sheaf $L(A \cup B)$ that is associated to the presheaf union $A \cup B$, which is in $\text{Mon}_p(L_+)$. Note that

$$A(i) \cup B(i) = (A \cup B)(i) = L(A \cup B)(i),$$

by the construction of $L(A \cup B)$. It follows from Lemma 28 that

$$\psi_{A \cup B}(y) = \max\{\psi_A(y), \psi_B(y)\}.$$

3 Stalks

Suppose generally that $L$ is a locale.

Following [6, p.51] and [7], for $x \in L$, write

$$\neg x = \bigvee_{x \land y = 0} y.$$  

The subobject $\neg \neg L$ of $L$ is defined to be the set of all $x \in L$ such that $\neg \neg x = x$. There is a frame morphism $\gamma : L \to \neg \neg L$ which is defined by $x \mapsto \neg \neg x$, since $\neg x = \neg \neg \neg x$ for all $x$.

For $x \in L$, write $L_{\geq x}$ (as above) for the sublocale of objects $y$ with $y \geq x$. There is a homomorphism $\phi_x : L \to L_{\geq x}$ which is defined by $y \mapsto y \lor x$.

Let $\omega$ denote the composite frame morphism

$$L \xrightarrow{(\phi_x)} \prod_{x \in L} L_{\geq x} \xrightarrow{(\gamma)} \prod_{x \in L} \neg \neg L_{\geq x}.$$ (4)

Then one knows (see, for example, [6, p.52]) that $\omega$ is a monomorphism and that

$$B := \prod_{x \in L} \neg \neg L_{\geq x}$$

is a complete Boolean algebra.

Note that $L_{\geq 0} = L$ and that $\phi_0 : L \to L_{\geq 0}$ is the identity.

The corresponding geometric morphism

$$\omega : \text{Shv}(B) \to \text{Shv}(L)$$

is a Boolean localization of $\text{Shv}(L)$.

This means that the inverse image functor

$$\omega^* : \text{Shv}(L) \to \text{Shv}(B)$$
is faithful, and hence is a “fat point” for the topos $\text{Shv}(L)$ in that it preserves and reflects monomorphisms, epimorphisms and isomorphisms.

The fat point assertion means that a map $E \to F$ of sheaves on $L$ is an monomorphism (respectively epimorphism, isomorphism) if and only if the induced map $\omega^* E \to \omega^* F$ is a monomorphism (respectively epimorphism, isomorphism) of sheaves on $B$. See [6, Sec. 3.4] or [4].

**Example 31.** Suppose that $L$ is totally ordered. If $x \in L$ and $x \neq 0$, then $x \wedge y = \min\{x, y\} = 0$ forces $y = 0$. Thus,

$$\neg x = \begin{cases} 0 & \text{if } x \neq 0, \\
1 & \text{if } x = 0. \end{cases}$$

The corresponding Boolean algebra

$$B = \prod_{x \in L} \{x, 1\} \cong \prod_{x \in L - \{1\}} \{x, 1\}$$

is isomorphic to the power set $\mathcal{P}(L - \{1\})$ of $L - \{1\}$, so that the sheaf category $\text{Shv}(L)$ has enough points.

The poset map $\phi_x : L \to L \geq x$ takes $y$ to $x$ if $y < x$ and takes $y$ to $y$ if $y \geq x$. It follows that the composite

$$L \xrightarrow{\phi_x} L \geq x \xrightarrow{\gamma} \{x, 1\}$$

takes $y$ to $x$ if $y \leq x$ and takes $y$ to $1$ if $y > x$. It follows that the poset map $\omega : L \to \mathcal{P}(L - \{1\})$ has the form

$$\omega(y) = L \leq y$$

for $y \neq 0, 1$.

It follows that, for $x \in L - \{1\}$, the stalk $F_x$ of a sheaf $F$ on $L$ is defined by

$$F_x = \lim_{x < s} F(s). \quad (5)$$

This colimit corresponds to the category of inclusions $\{x\} \subset L < s$, so $F_x$ is the evaluation of the sheaf $\omega^*(F)$ at the set $\{x\}$.

The locale $L_+$ has a total order if $L$ has a total order. For the object 0, the stalk $F_0$ is isomorphic to the generic fibre:

$$F_0 = \lim_{s \in L} F(s) \cong F(i),$$

since $i$ is the initial object of $L$. The stalk

$$F_i = \lim_{i < s} F(s)$$

is more “conventional”.

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The following result is true by formal nonsense, given that we have a theory of stalks for sheaves on \( L_+ \) for a totally ordered locale \( L \) in Example 31. The point of the following result (and its proof) is that it is easier to show directly that these stalks have the right properties in cases that we care about for data science applications.

**Lemma 32.** Suppose that the locale \( L \) is an interval and that the map \( E \to F \) is a stalkwise epimorphism (respectively stalkwise monomorphism, stalkwise isomorphism) of sheaves of monomorphisms on \( L_+ \). Then the map \( E \to F \) is an epimorphism (respectively monomorphism, isomorphism) of sheaves.

One says that a map \( E \to F \) of sheaves on \( L_+ \) is a stalkwise epimorphism if the induced functions \( E_x \to F_x \) are surjective for all \( x \in L \setminus \{1\} \). Similarly, stalkwise monomorphisms (respectively stalkwise isomorphisms) are defined by the requirement that all induced functions in stalks are injective (respectively bijective).

**Proof of Lemma 32.** Suppose that the map \( E \to F \) is a stalkwise epimorphism.

By the observation in Example 31, there is a natural isomorphism \( F_0 \cong F(i) \) for all sheaves \( F \) on \( L_+ \), so that the map \( E(i) \to F(i) \) is an epimorphism.

Suppose that \( x \in L \setminus \{1\} \). Then the collection of relations \( t < x \) is a covering family. Take \( u \in F(x) \) and let \( u_t \in F(t) \) be its image under the restriction map \( F(x) \to F(t) \), where \( t < x \). Then \( u_t \) represents an element of \( F_t \), and the map \( E_t \to F_t \) is an isomorphism. It follows that there is an element \( v_t \in E(t) \) such that \( v_t \to u_t \in F(t) \). This means that the sheaf map \( E \to F \) is a local epimorphism [6, Sec. 3.2], and it is therefore an epimorphism of sheaves.

If \( E \to F \) is a stalkwise monomorphism, then the map \( E(i) \to F(i) \) is a monomorphism. For each \( x \in L \) there is a commutative diagram

\[
\begin{array}{ccc}
E(x) & \to & E(i) \\
\downarrow & & \downarrow \cong \\
F(x) & \to & F(i)
\end{array}
\]

The horizontal morphisms are monomorphisms since \( E \) and \( F \) are sheaves of monomorphisms, so the map \( E(x) \to F(x) \) is a monomorphism. This is true for all \( x \in L \), and so the sheaf map \( E \to F \) is a monomorphism.

If the map \( E \to F \) is a stalkwise isomorphism, then it is both a stalkwise epimorphism and a stalkwise monomorphism, so that \( E \to F \) is an epimorphism and a monomorphism of sheaves by the previous paragraphs. It follows that \( E \to F \) is an isomorphism of sheaves.

\[\Box\]

**Example 33.** Suppose that the locale

\[L = L_1 \times \cdots \times L_k\]

is a product of intervals \( L_i \).
The construction of the locale morphism (4) preserves products, so that the sheaf category \( \text{Shv}(L) \) again has enough points.

The poset map \( \omega \) has the form

\[
L = L_1 \times \cdots \times L_k \xrightarrow{\omega \times \cdots \times \omega} \mathcal{P}(L_1 - \{1\}) \times \cdots \times \mathcal{P}(L_k - \{1\}) = \mathcal{P}((L_1 - \{1\}) \sqcup \cdots \sqcup (L_k - \{1\})),
\]

and takes \((y_1, \ldots, y_k)\) to the disjoint union \((L_1)_{<y_1} \sqcup \cdots \sqcup (L_k)_{<y_k}\).

If \( x \in L_1 \) and \( F \) is a sheaf on \( L = L_1 \times \cdots \times L_k \), then

\[
F_x = \lim_{s > x} F(s, 0, \ldots, 0).
\]

In effect, the collection of all \( k \)-tuples \((s, 0, \ldots, 0)\) with \( s > x \) is cofinal in the collection of all \( k \)-tuples \((s_1, \ldots, s_k)\) with \( s_1 > x \).

In other words, \( F_x \) is the stalk at \( x \) of the restriction of \( F \) along the poset morphism

\[i_1 : L_1 \to L_1 \times \cdots \times L_k\]

which is defined by \( s \mapsto (s, 0, \ldots, 0)\).

Suppose that \( T \) is a topological space, \( x \in T \) and that \( E \) is a sheaf on \( T \). Then the stalk \( E_x \) at \( x \) is defined by the filtered colimit

\[
E_x = \lim_{x \in U} E(U),
\]

which is indexed over the open subsets \( U \) of \( T \) which contain \( x \). If \( F \) is a presheaf on \( T \), then one can define a stalk at \( x \) for \( F \) by analogy: set

\[
F_x = \lim_{x \in U} F(U).
\]

Every presheaf \( F \) has an associated sheaf \( \check{F} = L^2(F) \) and a canonical associated sheaf map \( \eta : F \to \check{F} \). It is well known that the associated sheaf map induces a bijection

\[
\eta_* : F_x \cong \check{F}_x
\]

for all \( x \in T \).

There is an analogue of the result expressed in (6) for sheaves and presheaves on an interval, by essentially the same argument.

Explicitly, suppose that \( L \) is an interval, and write

\[
I = L - \{1\}
\]

so that the poset morphism \( \omega \) of Example 31 has the form

\[
\omega : L \to \mathcal{P}(I).
\]
The direct image $\omega_*$ and inclusion functors $i$ for the various presheaf and sheaf categories fit into a commutative diagram

$$
\text{Pre}(\mathcal{P}(I)) \xrightarrow{\omega_*} \text{Pre}(L) \\
i \downarrow \quad \downarrow i \\
\text{Shv}(\mathcal{P}(I)) \xrightarrow{\omega_*} \text{Shv}(L)
$$

If $L^2$ is the associated sheaf functor, then there is a natural isomorphism of left adjoint functors

$$L^2 \omega^p(F) \cong L^2 \omega^p L^2(F) = \omega^p L^2(F)$$

that is induced by the associated sheaf map $\eta : F \to L^2(F)$. Here, $\omega^p$ is the left Kan extension of the restriction functor $\omega_*$ on the presheaf level.

For $x \in I$ and a presheaf $E$ on the power set $\mathcal{P}(I)$, the associated sheaf map induces a bijection

$$E(\{x\}) \cong L^2 E(\{x\}).$$

It follows that the functions

$$\omega^p F(\{x\}) \xrightarrow{\eta} \omega^p L^2 F(\{x\}) = (L^2 F)_x$$

are bijections for all presheaves $F$ on $L$ and $x \in L - \{1\}$.

Finally, since $\omega^p$ is a left Kan extension of $\omega_*$, we have an isomorphism

$$\omega^p F(\{x\}) \cong \lim_{s < x} F(s)$$

for all $x \in L - \{1\}$, and we can define

$$F_x = \omega^p F(\{x\}) \cong \lim_{s < x} F(s)$$

to be the stalk of the presheaf $F$ at $x \in L - \{1\}$, as in the definition of stalk at $x$ for any sheaf that one sees in Example 31.

It is a consequence of the definitions that a map $F \to F'$ of presheaves on an interval $L$ is a stalkwise isomorphism if and only if the induced map $L^2 F \to L^2 F'$ is an isomorphism of associated sheaves.

**Example 34.** Suppose, as in Example 10, that $X \subset \mathbb{R}^n$ is a data set, with ordering $X \subset \mathbb{N}$, and choose $R > d(x, y)$ for all pairs of points $x, y \in X$.

The association $s \mapsto V_s(X)$ for $s \in [0, R]$ defines a simplicial sheaf $V(X)$ (of Vietoris-Rips complexes) of monomorphisms on the totally ordered locale $[0, R]_+$. The stalk $V(X)_t$ for $t \in (0, R]$ is defined by

$$V(X)_t = \lim_{s < t} V_s(X),$$

where the indicated ordering is that of $[0, R]$. 

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Note that

\[ V(X)_i = \lim_{s \to R} V_s(X) = \Delta^N, \]

because we have chosen \( R > d(x, y) \) for all pairs of points \( x, y \in X \).

Observe as well that for small numbers \( t \), the stalk \( V(X)_x \) is the discrete space on the set \( X \).

Suppose that \( X \subset Y \subset \mathbb{R}^n \) are data sets and \( R > d(x, y) \) for all pairs of points \( x, y \in Y \) (hence in \( X \)). Then the inclusion \( X \subset Y \) defines a map of simplicial sheaves (of monomorphisms) \( V(X) \to V(Y) \). This map is a local weak equivalence if and only if \( X = Y \), because \( V(X)_t \) and \( V(Y)_t \) are discrete for small numbers \( t \).

References


