Fuzzy sets and presheaves

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Abstract

This note presents a presheaf theoretic approach to the construction of fuzzy sets, which builds on Barr's description of fuzzy sets as sheaves of monomorphisms on a locale. A presheaf-theoretic method is used to show that the category of fuzzy sets is complete and co-complete, and to present explicit descriptions of classical fuzzy sets that arise as limits and colimits. The Boolean localization construction for sheaves and presheaves on a locale L specializes to a theory of stalks if L approximates the structure of a closed interval in the real line.

The system $V(X)$ of Vietoris-Rips complexes for a data cloud X becomes both a simplicial fuzzy set and a simplicial sheaf in this general framework. This example is explicitly discussed in this paper, in stages.

Introduction

Fuzzy sets were originally defined to be functions

$$\psi : X \to [0, 1]$$

which take values in the unit interval [2]. Michael Barr changed the game in his paper [1]: he replaced the unit interval by a more general well-behaved poset, or locale L, and redefined fuzzy sets to be functions

$$\psi : X \to L.$$

These are fuzzy sets over L, and form a category $\mathbf{Fuzz}(L)$, which is described in various ways below.

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Locales are complete lattices in which finite intersections distribute over all unions. The unit interval $[0,1]$ qualifies, but so does the poset of open subsets of a topological space, and locales are models for spaces in this sense. Every locale L has a Grothendieck topology, with coverings defined by joins, and so one is entitled to a presheaf category $\mathbf{Pre}(L)$ and a sheaf category $\mathbf{Shv}(L)$ for such objects. Presheaves are contravariant set-valued functors $L^{op} \to \mathbf{Set}$, and sheaves are presheaves that satisfy a patching condition with respect to the Grothendieck topology on L.

Barr showed that, starting with a function $\psi : X \to L$ (i.e. a fuzzy set defined over the locale L), one can pull back over suitable intervals to define a sheaf $T(\psi)$ on L, which is a sheaf of monomorphisms in the sense that all restriction maps are injections. Technically, one needs to adjoin a new zero object to L to make this work, giving a new (but not so different) locale L_+.

Write $\mathbf{Mon}(L_+)$ for the category of sheaves of monomorphisms on L_+. Barr showed [1] that the functor

$$T : \mathbf{Fuzz}(L) \to \mathbf{Mon}(L_+)$$

is part of a categorical equivalence. The best way to describe the inverse functor for T on a sheaf F is to take the *generic fibre* $F(i)$, and construct a function $\psi_F : F(i) \to L$. The set $F(i)$ is the set of sections corresponding to the initial object i of L. Given an element $x \in F(i)$, there is a maximum $s \in L$ such that x is in the image of the canonical monomorphism $F(s) \to F(i)$. One defines $\psi_F(x)$ to be this maximum element s.

To go further with applications, for example if you want to sheafify peristent homology theory or clustering and use fuzzy sets to do it, you need more explicit information about how fuzzy sets are constructed. One needs, in particular, straightforward descriptions of of basic constructions such as limits, colimits and stalks in the fuzzy set category, or rather in the associated category of sheaves of monomorphisms. The "difficulties", such as they are, arise from the fact that $\mathbf{Mon}(L_+)$ is not quite a sheaf category, and constructing the fuzzy set $\psi_F : F(i) \to L$ from a sheaf F can be a bit interesting.

These issues are dealt with in this paper. There is a perfectly good category of presheaves of monomorphisms $\mathbf{Mon}_p(L_+)$, and it turns out that if L is sufficiently well behaved (like the unit interval $[0,1]$), then the associated sheaf functor is easily described and preserves presheaves of monomorphisms. The upshot is that one can make constructions on the presheaf category, as a geometer or topologist would expect, and then sheafify. It follows that the fuzzy set category $\mathbf{Fuzz}(L)$ has all limits and colimits.

Limits are formed as in the ambient sheaf category, i.e. sectionwise, but colimits are more interesting. The inclusion $\mathbf{Mon}(L_+) \subset \mathbf{Shv}(L_+)$ of sheaves of monomorphisms in all sheaves has a left adjoint $F \mapsto \mathrm{Im}(F)$, the *image functor*, which is defined by taking images of sections in the generic fibre. This observation allows one to define colimits of diagrams $A(i)$ in $\mathbf{Mon}(L_+)$: take the presheaf theoretic colimit $\varinjlim_i A(i)$, and then apply the image functor (suitably sheafified) to get the colimit in $\mathbf{Mon}(L_+)$.

These last constructions are described in the second section. That section also contains the formal definitions and properties around presheaves of monomorphisms. The first section sets the stage for the Barr result, albeit in more modern language. One has the nicest form of the associated sheaf functor when one assumes that the locale L is an interval. The interval assumption on L is consistent with both the classical theory of fuzzy sets and with intended applications in topological data analysis — the Vietoris-Rips complex for a data cloud is a simplicial fuzzy set (or simplicial sheaf) on an interval.

Defining the fuzzy set structure function $\psi_F : F(i) \to L$ for a sheaf F can be interesting in practice, and this is done here for limits and colimits. Roughly speaking the functor $\psi_?$ is defined on inverse limits by greatest lower bounds of the functions on consituent objects (end of Section 1), and is defined on colimits by least upper bounds (end of Section 2). Some of the standard constructions for fuzzy sets, such as taking a pointwise product of two functions $X \to [0,1]$, do not yet have a topos theoretic meaning.

The general theory of Boolean localization for sheaves and presheaves on a locale L is fairly easy to describe. Every locale L has an imbedding $\omega : L \to \mathcal{B}$ into a complete Boolean algebra, by a rather transparent construction that is displayed in the third section. This is the easy part of the general Boolean localization construction — the more interesting bit is the construction of the Diaconescu cover, which faithfully imbeds a Grothendieck topos in the topos of sheaves on a locale.

If the locale L is an interval, then the corresponding Boolean algebra $\mathcal{B}$ is the set of subsets of some set, so that the sheaf category that is defined by L has enough points, and hence a theory of stalks. The same is true for finite products of locales.

This paper was written to clear the air about the sheaf theoretic properties of fuzzy sets, and to set the stage for potential applications of the local homotopy theory of simplicial presheaves in topological data analysis.

So yes, in Example 10 we see that the system of Vietoris-Rips complexes $s \mapsto V_s(X)$ which is associated to a data cloud $X \subset \mathbb{R}^n$ does form a simplicial fuzzy set, or a simplicial sheaf (of monomorphisms) on locale $[0,R]^{op}$. But we also see in Example 32 that this simplicial sheaf on $[0,R]^{op}$ has a rather awkward collection of stalks, which includes the full data cloud X sitting as a discrete simplicial set in the generic fibre. It follows, for example, that an inclusion $X \subset Y \subset \mathbb{R}^n$ induces a stalkwise weak equivalence if and only if $X = Y$.

The facile conclusion is that the local homotopy theory of simplicial sheaves on the locale $[0,R]^{op}$ does not naive applications in topological data analysis, at least according to the present construction, and must be replaced by a more clever approach.

Contents

1 Fuzzy sets and sheaves

A *frame* L is a complete lattice in which finite meets distribute over all joins. Examples include the poset $op|_X$ of open subsets of a topological space X.

According to this definition, L has a terminal object 1 (empty meet) and an initial object 0 (empty join) — see [5, p.471].

The completeness assumption means that every set of elements $a_i \in L$ has a least upper bound $\vee a_i$. The set a_i also has a greatest lower bound $\wedge a_i$, which is the least upper bound $\vee_{x \leq a_i} x$ of the elements x which are smaller than all a_i.

A morphism $L_1 \to L_2$ of frames is a poset morphism which preserves meets and joins, and hence preserves initial and terminal objects. The category of locales is the opposite of the frame category, and one tends to use the terms "frame" and "locale" interchangeably, and I shall refer to these objects as locales.

Example 1. The interval $[0, 1]$ of numbers s with $0 \leq s \leq 1$, with the standard ordering, is a locale.

The poset $[0, 1]$ imbeds in $op|_{[0,1]}$ by the assignment $a \mapsto [0, a)$. Then $s \wedge t$ corresponds to the interval

$$[0, s) \cap [0, t) = [0, s \wedge t),$$

so that $s \wedge t = \min\{s, t\}$. Similarly,

$$\cup_i \ [0, s_i) = [0, \vee_i \ s_i),$$

and $\vee_i \ s_i$ is the least upper bound of the numbers s_i.

Example 2. The interval $[0, 1]$ with the opposite ordering $[0, 1]^{op}$ is also a locale. In other words, $s \leq t$ in $[0, 1]^{op}$ if and only if $t \leq s$ in $[0, 1]$.

In this case, $[0, 1]^{op}$ imbeds in $op|_{[0,1]}$ by the assignment $s \mapsto (s, 1]$. Then $\vee_i \ s_i$ is the greatest lower bound of the s_i and $s \wedge t = \max\{s, t\}$.

Observe that the posets $[0, 1]$ and $[0, 1]^{op}$ both have infinite meets, given by greatest lower bound and least upper bound, respectively.

Example 3. The closed interval $[a, b]$ and its opposite $[a, b]^{op}$ are locales, and there are linear scaling isomorphisms $[a, b] \cong [0, 1]$ and $[a, b]^{op} \cong [0, 1]^{op}$, respectively.

Example 4. Suppose that $L_1, \ldots, L_k$ are locales. Then the product poset

$$L_1 \times \cdots \times L_k$$

is also a locale.

Suppose that L is a locale. Following [1], the functions $\psi : X \to L$ are the objects of a category $\mathbf{Fuzz}(L)$, called the category of *fuzzy sets* over L.

Suppose that $\phi : Y \to L$ is another such function. A morphism $(f, h) : \psi \to \phi$ of $\mathbf{Fuzz}(L)$ consists of a function $f : X \to Y$ and a relation (homotopy) $h : \psi \leq \phi \cdot f$ of functions taking values in the poset L. The existence of the relation h means precisely that $\psi(x) \leq \phi(f(x))$ in the poset L for all $x \in X$.

There is a poset L^X whose objects are the functions $\psi : X \to L$. If $\gamma : X \to L$ is another such function, then there is a relation $\psi \leq \gamma$ if $\psi(x) \leq \gamma(x)$ for all $x \in X$. Every function $f : X \to Y$ determines a restriction functor $L^Y \to L^X$, so that there is a contravariant functor $\mathbf{Set} \to \mathbf{cat}$ which is defined by associating the poset L^X to the set X.

From this point of view, a morphism of $\mathbf{Fuzz}(L)$ is a morphism $(f, \leq) : \psi \to \phi$ in the Grothendieck construction associated to the diagram of restriction functors, and the fuzzy set category $\mathbf{Fuzz}(L)$ is that Grothendieck construction.

The relation $\psi \leq \phi \cdot f$ can also be viewed as a homotopy of functors.

Example 5. All commutative diagrams

$$\begin{tikzcd} X \arrow[rr, "g"] \arrow[dr] & & Y \arrow[dl] \\ & L & \end{tikzcd}$$

correspond to morphisms $(g, 1)$ of $\mathbf{Fuzz}(L)$ with identity homotopies in L, but the full collection of fuzzy set morphisms $X \to Y$ is larger — these are the homotopy commutative diagrams.

Example 6. Suppose that a finite set $X \subset \mathbb{R}^n$ is a data cloud, and suppose that $X \xrightarrow{\cong} \mathbf{N}$ is a listing of the members of X, where $\mathbf{N} = \{0, 1, \ldots, N\}$. Choose a number R such that $d(x, y) < R$ for all $x, y \in X$.

Here, $d(x, y)$ is the distance between the points x and y in $\mathbb{R}^n$.

Let σ be an ordered set of points $\sigma = \{x_0, x_1, \ldots, x_k\}$ in X. Write

$$\phi(\sigma) = \max_{i,j} \; d(x_i, x_j)$$

Suppose that $\theta : \mathbf{r} \to \mathbf{k}$ is an ordinal number map. then $\phi(\theta^*(\sigma)) \leq \phi(\sigma)$, with equality if θ is surjective, or if $\theta^*(\sigma)$ is a degeneracy of σ. Further, $\phi(\sigma) = 0$ if and only if σ is a degeneracy of a vertex.

The assignment $\sigma \mapsto \phi(\sigma)$ defines a function

$$\phi : \Delta_k^N \to [0, R].$$

If $\theta : \mathbf{r} \to \mathbf{k}$ is an ordinal number map, then the relation $\phi(\theta^*(\sigma)) \leq \phi(\sigma)$ defines a homotopy commutative diagram

$$\begin{tikzcd} \Delta_k^N \arrow[rr, "\theta^*"] \arrow[dr, "\phi"'] & & \Delta_r^N \arrow[dl, "\phi"] \\ & {[0,R]^{op}} & \end{tikzcd} \tag{1}$$

or equivalently a morphisms of fuzzy sets with values in the locale $[0, R]^{op}$.

The ordering $X \cong \mathbf{N}$ on the elements of the data cloud X and the ambient distance function on $\mathbb{R}^n$ combine to give the simplicial set $\Delta^X := \Delta^N$ the structure of a simplicial fuzzy set $\phi : \Delta^X \to [0, R]^{op}$, with coefficients in the locale $[0, R]^{op}$.

Suppose that L is a locale. Then $L_+ = L \sqcup \{0\}$, is also a locale, where 0 is a new initial element.

Remark 7. If $L = [0, R]^{op}$ then the object R is no longer initial in L_+. I normally write i for the number R (the original initial object of $[0, R]^{op}$) to distinguish this element from the new initial object 0 of L_+. Clearly, $0 < i$ in L_+.

Any locale L has a Grothendieck topology, for which the *covering families* of $a \in L$ are sets of objects $b_i \leq a$ such that $\cup_i b_i = a$. This relation is equivalent to the assertion that a is the least upper bound in L for all elements b_i.

Given a family of elements $b_i \leq a$, the associated *sieve* R is the set of all elements s such that $s \leq b_i$ for some i. The sieve R is *covering* if $\{b_i\}$ is a covering family.

Independently, an arbitrary sieve S, i.e. a subset of the collection of elements $s \leq a$ which is closed under subobjects, is *covering* if $\vee_{s \in R}\ s = a$.

Since L has a Grothendieck topology, it has associated categories $\mathbf{Pre}(L)$ and $\mathbf{Shv}(L)$ of presheaves and sheaves on L, respectively. A *presheaf* is a functor $F : L^{op} \to \mathbf{Set}$, and a morphism of presheaves is a natural transformation.

One says that the presheaf F is a *sheaf* if the map

$$F(a) \to \varprojlim_{b \in R}\ F(b)$$

is an isomorphism for all covering sieves R of all objects a. This is equivalent to requiring that the diagram

$$F(a) \to \prod_i\ F(b_i) \rightrightarrows \prod_{i,j}\ F(b_i \wedge b_j)$$

is an equalizer for all covering families $\{b_i\}$ of all objects a. In other words, $F(a)$ should be recovered from the values of $F(b_i)$ by patching, for all coverings $\{b_i\}$ of a.

Remark 8. If i is an initial object of L and F is a sheaf, then $F(i)$ must be the one-point set. I write $F(i) = *$ to express this.

In effect, the empty sieve $\emptyset \subset \hom(\ ,i)$ is covering, because i is an empty join. It follows that there is an isomorphism

$$F(i) \cong \hom(\hom(\ ,i), F) \cong \hom(\emptyset, F) = *$$

for any sheaf F. Compare with [4, p.35].

We are therefore entitled to categories $\mathbf{Pre}(L_+)$ and $\mathbf{Shv}(L_+)$ of presheaves and sheaves, respectively for the locale L_+, and these are the examples that we will focus on.

Write $\mathbf{Mon}(L_+)$ for the full subcategory of the sheaf category $\mathbf{Shv}(L_+)$, whose objects are the sheaves F such that all restriction maps $F(b) \to F(a)$ associated to $a \leq b$ in L are monomorphisms. The requirement that the relation $a \leq b$ is in L is important, because $F(0) = *$.

It is a result of Barr [1, Th. 3] that there is a functor

$$T : \mathbf{Fuzz}(L) \to \mathbf{Mon}(L_+)$$

which defines an equivalence of categories.

More explicitly, define

$$L_{\geq a} = \{\ x \in L \mid x \geq a\ \}.$$

If $\psi : X \to L$ is a member of $\mathbf{Fuzz}(L)$, define a presheaf $T(\psi)$ by

$$T(\psi)(a) = \psi^{-1}(L_{\geq a}).$$

Set $T(\psi)(0) = *$. Then the assignment $a \mapsto T(\psi)(a)$ defines a presheaf on L_+ such that every relation $s \leq t$ induces a monomorphism $T(\psi)(t) \to T(\psi)(s)$. The presheaf $T(\psi)$ is a sheaf because $x \in T(\psi)(a)$ if and only if $x \in T(\psi(b_i))$ for any covering family $\{b_i\}$ of a.

If $(f, h) : \psi \to \phi$ is a morphism of fuzzy sets (as above), then the relations $\psi(x) \leq \phi(f(x))$ in L (i.e. the homotopy h) imply that if $\psi(x) \in L_{\geq a}$ then $\phi(f(x)) \in L_{\geq a}$, and so the function f restricts to functions $f : \psi^{-1}L_{\geq a} \to \phi^{-1}L_{\geq a}$ that are natural in a, and hence defines a sheaf homomorphism $f_* : T(\psi) \to T(\phi)$.

Remark 9. $T(\psi)$ is the *level cut* description of the fuzzy set $\psi : X \to L$ — see [2].

Example 10. Suppose that the finite set $X \subset \mathbb{R}^n$ is a data cloud, with ordering $X \xrightarrow{\cong} \mathbf{N}$ as in Example 6. Again, choose $R > d(x, y)$ for all pairs of points $x, y \in X$.

Recall that the simplicial fuzzy set $\phi : \Delta_k^N \to [0, R]^{op}$ is defined for a simplex $\sigma = \{x_0, x_1, \ldots, x_k\}$ by

$$\phi(\sigma) = \max_{i,j}\ d(x_i, x_j).$$

Then, for $s \in [0, R]$,

$$T(\phi)_s = \phi^{-1}[0, s],$$

which is the set of k-simplices $\sigma = \{x_0, \ldots, x_k\}$ of Δ^N such that $d(x_i, x_j) \leq s$. It follows that

$$T(\phi)_s = V_s(X)_k$$

is the set of k-simplices of the Vietoris-Rips complex $V_s(X)$ for the data cloud X.

The Vietoris-Rips complex $s \mapsto V_s(X)$ is the simplicial sheaf $V(X)$ that is associated to the simplicial fuzzy set $\phi : \Delta^X \to [0, R]^{op}$.

For any sheaf $F \in \mathbf{Mon}(L_+)$ there is an isomorphism

$$\varinjlim_{a \in L} F(a) \xrightarrow{\cong} F(i), \tag{2}$$

since i is initial in L. This colimit is filtered, and the canonical maps $F(a) \to F(i)$ are monomorphisms.

I say that $F(i)$ is the *generic fibre* of the object F. The same construction and terminology applies to more general sheaves and presheaves on L_+.

Lemma 11. *Suppose that F is a sheaf of monomorphisms on L_+ and that $x \in F(i)$. Then there is a unique maximum element b such that $x \in F(b)$.*

Proof. Consider all c in L such that $x \in F(c)$, and let

$$b = \cup_{x \in F(c)} c.$$

Then b is covered by the elements c, and so $x \in F(b)$. □

Suppose again that $F \in \mathbf{Mon}(L_+)$. By Lemma 11, for each $x \in F(i)$ there is a unique maximum b such that $x \in F(b)$. Define $\psi_F : F(i) \to L$ by setting $\psi_F(x) = b$. Then we have a function

$$\psi_F : F(i) \to L,$$

which is a fuzzy set.

Remark 12. The preimage $\psi_F^{-1}(a)$ of an element $a \in L$ is the subset $F[a] \subset F(a)$ which consists of elements not in the image of any restriction map $F(c) \to F(a)$ with $c > a$. This is important for readers of Barr's paper [1], but it will not be used here.

The assignment which takes a sheaf F to the function $\psi_F : F(i) \to L$ defines the inverse of the functor T.

To put it a slightly different way, the fuzzy set $\psi_F : F(i) \to L$ is defined by

$$\psi_F(x) = sup \ \{ \ b \mid x \in F(b) \ \}$$

for $x \in F(i)$, and $F \in \mathbf{Mon}(L_+)$.

Example 13. The representable functor $\hom(\ ,s)$ for $s \in L$ has the form

$$\hom(\ ,s)(t) = \begin{cases} * & \text{if } t \leq s, \text{ and} \\ \emptyset & \text{otherwise.} \end{cases}$$

Here, $*$ is the one-point set.

This presheaf is a sheaf, so the topology on L is sub-canonical. The corresponding fuzzy set is the function $s : * \to L$ which picks out the element $s \in L$.

If $s \leq t$ the induced sheaf map $\hom(\ ,s) \to \hom(\ ,t)$ corresponds ot the fuzzy set map from $s : * \to L$ to $t : * \to L$ which is given by the identity function on $*$ and the relation $s \leq t$.

Example 14. The constant presheaf $*$ is defined to be a one-point set $*(a)$ for all $a \in L_+$, with identity maps associated to all relations $a \leq b$. This presheaf is a sheaf, and is a member of $\mathbf{Mon}(L_+)$. This sheaf is represented by the terminal object $1 \in L$, and the corresponding fuzzy set is the function $1 : * \to L$.

Lemma 15. *The category* $\mathbf{Mon}(L_+)$ *is complete. Limits are formed in the ambient sheaf category* $\mathbf{Shv}(L_+)$.

Proof. This result is essentially a triviality. It follows from the fact that an inverse limit of monomorphisms is a monomorphism. □

Example 16. Form the pullback diagram

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & F \arrow[d, "q"] \\ E \arrow[r, "p"'] & X \end{tikzcd}$$

of sheaves on L_+, with E, F, X all in $\mathbf{Mon}(L_+)$.

Take

$$(x, y) \in Z(i) = E(i) \times_{X(i)} F(i)$$

and suppose that $(x, y) \in Z(a)$.

Then $x \in E(a)$ and $y \in F(a)$ so that $a \leq \psi_E(x)$ and $a \leq \psi_F(y)$. It follows that $a \leq \psi_E(x) \wedge \psi_F(y)$.

On the other hand, if $b \leq \psi_E(x) \wedge \psi_F(y)$, then there is a $v \in E(b)$ which restricts to x and a $u \in F(b)$ which restricts to y. Also, $p(v)$ and $q(u)$ in $X(b)$ restrict to the same element of $X(i)$, so that $p(v) = q(u)$ in $X(b)$, and $(u, v) \in Z(b)$.

It follows that

$$\psi_Z((x, y)) = \psi_E(x) \wedge \psi_F(y)$$

for all $(x, y) \in Z(i)$.

Another way of saying this is to assert that $\psi_Z((x, y))$ is the greatest lower bound of $\psi_E(x)$ and $\psi_F(y)$.

Example 17. Suppose that $X : J \to \mathbf{Mon}(L_+)$ is a small diagram. For a fixed object $a \in L_+$, the a-sections of $\varprojlim_j X(j)$ are the J-compatible families $\{x_j\}$ of elements in the various sets $X(j)(a)$.

One can use the methods of the pullback case in Example 16 to show that $\psi_Z(\{x_j\})$ is the greatest lower bound in L_+ of the elements $\psi_{X(j)}(x_j)$.

2 Presheaves of monomorphisms

We now consider presheaves $F : (L_+)^{op} \to \mathbf{Set}$ such that $F(0) = *$ and all morphisms $a \leq b$ of L induce monomorphisms $F(b) \to F(a)$. Such a presheaf will be called a *presheaf of monomorphisms*. Write $\mathbf{Mon}_p(L_+)$ for the category of presheaves of this form.

Most of the results of this section depend on the assumption that the locale L is an *interval* in the senses that

1) L has a total ordering, and

2) if $a < b$ in L, there is an $s \in L$ such that $a < s < b$.

The locales of immediate practical interest, namely the closed interval $[c, d] \subset \mathbb{R}$ and its opposite, are intervals in this sense.

Lemma 18. *Suppose that the locale L is totally ordered. Then the covering sieves for $a \in L$ are defined by the families of all b such that $b < a$ or such that $b \leq a$.*

Proof. Suppose that a covering sieve $R \subset \hom(\ , a)$ is generated by a set of elements b_i, so that $a = \vee_i\ b_i$. Suppose that $R \neq \hom(\ , a)$.

Suppose that $c < a$. If c is not bounded above by some b_i then $b_i < c$ for all i since L is totally ordered, so that

$$a = \vee_i\ b_i \leq c < a,$$

and we have a contradiction. It follows that $c \leq b_i$ for some i, and so the relation $c < a$ is in R. □

Remark 19. The collection of all b such that $b \leq a$ is the trivial covering sieve for a, because it includes the object a. Lemma 18 says that an element $a \in L$ has at most two covering sieves if L is totally ordered.

In order to be assured that $a \in L$ has a non-trivial covering, or that the elements $b < a$ cover a, we need to know that L satisfies condition 2) above, so that L is an interval.

Example 20. The total ordering on L is necessary for the conclusion of Lemma 18.

The elements $(1, 0)$ and $(0, 1)$ define a covering of $(1, 1)$ in $[0, 1]^{\times 2}$, and the element $(\frac{1}{2}, \frac{1}{2})$ is not bounded above by either $(1, 0)$ or $(0, 1)$.

We shall assume that the locale L is an interval for the rest of this section.

It follows from Lemma 18 that a presheaf F on L_+ is a sheaf if and only if $F(0) = *$ and the map

$$\eta : F(a) \to \varprojlim_{0<b<a} F(b) =: LF(a)$$

is an isomorphism for all $a \in L$ with a not initial. There is no condition on $F(i)$ if $i \in L$ is initial, and $LF(i) = F(i)$.

The assignment $a \mapsto LF(a)$ defines a presheaf LF on L_+. Because L has a total ordering and there are so few covering sieves for elements of L, the presheaf LF is the universal separated presheaf associated to F [4, Lem 3.13].

In general, there is a canonical natural map $\eta : E \to LE$ for all presheaves E, and LE is a sheaf if E is separated. Recall that a presheaf E is *separated* if the map $\eta : E \to LE$ is a sectionwise monomorphism.

Corollary 21. *If $F \in \mathbf{Mon}_p(L_+)$ then the map η is a monomorphism, so that F is a separated presheaf and LF is its associated sheaf.*

Lemma 22. *If $F \in \mathbf{Mon}_p(L_+)$, then $LF \in \mathbf{Mon}(L_+)$. In particular, the associated sheaf functor*

$$L^2 : \mathbf{Pre}(L_+) \to \mathbf{Shv}(L_+)$$

restricts to a functor $\mathbf{Mon}_p(L_+) \to \mathbf{Mon}(L_+)$.

Proof. Suppose that $b \leq c$ in L. We show the restriction map

$$LF(c) \to LF(b)$$

is a monomorphism.

Given compatible families $\{x_s\}$ and $\{y_s\}$ for $s < c$, if $x_s = y_s$ for $s \leq b$, then x_s and y_s have the same image in $F(t)$ for some $t \leq b$, and so $x_s = y_s$. □

Example 23. Suppose that $L = [0, 1]$, let A be a pointed set with base point $*$. Define a presheaf $F_A : (L_+)^{op} \to \mathbf{Set}$ by

$$F_A(s) = \begin{cases} * & \text{if } s = 1, \text{ and} \\ A & \text{if } s < 1 \text{ in } [0, 1]. \end{cases}$$

Set $F_A(0) = *$, where 0 is the new initial object of $[0, 1]_+$.

If $s < 1$ the induced map $F_A(1) \to F_A(s)$ is the inclusion of the base point of A, and if $s \leq t < 1$ in L then $F_A(t) \to F_A(s)$ is the identity on A. Then $\varprojlim_{s<1} F_A(s)$ is the set A and not the base point $*$ in general, so that F_A is a presheaf of monomorphisms, and is not a sheaf.

Example 24. Suppose that $F_i,\ i \in I$ is a list of objects in $\mathbf{Mon}_p(L_+)$. Then the disjoint union $\sqcup_i F_i$ is in $\mathbf{Mon}_p(L_+)$. Note that we must set $(\sqcup_i F_i)(0) = *$ for this to work.

Example 25. Suppose that $A_i \subset F$ are subobjects of a fixed object $F \in \mathbf{Mon}_p(L_+)$, so that all A_i are in $\mathbf{Mon}_p(L_+)$. Then the (sectionwise) union $\cup_i A_i$ is a subobject of F, and is also in $\mathbf{Mon}_p(L_+)$.

It follows that the category $\mathbf{Sub}(F)$ of subobjects of an object $F \in \mathbf{Mon}_p(L_+)$ is a locale.

Suppose that E is a presheaf on L_+. The epi-monic factorizations of the maps $E(s) \to E(i)$ for $s \in L$ determine subobjects $\mathrm{Im}(E)(s) \subset E(i)$ with commutative diagrams

$$\begin{tikzcd} E(t) \arrow[r] \arrow[d] & \mathrm{Im}(E)(t) \arrow[d] \arrow[dr] & \\ & & E(i) \\ E(s) \arrow[r] & \mathrm{Im}(E)(s) \arrow[ur] & \end{tikzcd}$$

for $s \leq t$. Set $\mathrm{Im}(E)(0) = *$.

If E is in $\mathbf{Mon}_p(L_+)$, then the maps $E(t) \to \mathrm{Im}(E)(t)$ are isomorphisms. These constructions are functorial in presheaves E.

The functor $E \mapsto \mathrm{Im}(E)$ preserves monomorphisms of presheaves, since the generic fibre functor $E \mapsto E(i)$ preserves monomorphisms.

It follows that there is a natural presheaf map $E \to \mathrm{Im}(E)$ such that $\mathrm{Im}(E)$ is in $\mathbf{Mon}_p(L_+)$ and that this map is initial among all maps $E \to F$ with $F \in \mathbf{Mon}_p(L_+)$. In other words, there is a natural bijection

$$\hom_{\mathbf{Mon}_p(L_+)}(\mathrm{Im}(E), F) \cong \hom(E, F),$$

so that the functor $E \mapsto \mathrm{Im}(E)$ is left adjoint to the inclusion of $\mathbf{Mon}_p(L_+)$ in the presheaf category on L_+.

The sheaf $L(\mathrm{Im}(F))$ is initial among all presheaf maps $F \to E$ such that E is in $\mathbf{Mon}(L_+)$, since L is an interval.

Suppose that $A : J \to \mathbf{Mon}(L_+)$ is a small diagram in the category of sheaves of monomorphisms on L_+. Form the colimit

$$X = L(\mathrm{Im}(\varinjlim_{j \in J} A(j)))$$

in $\mathbf{Mon}(L_+)$. Then

$$X(i) = \varinjlim_j A(j)(i),$$

and it follows that every $x \in X(i)$ is in the image of some composite

$$A(j)(s) \to \varinjlim_j A(j)(s) \to \mathrm{Im}(\varinjlim_j A(j))(s) \to X(i). \tag{3}$$

I claim that

$$\psi_X(x) = \vee_{j,y} \psi_{A(j)}(y), \tag{4}$$

where the index is over all pairs j, y such that $y \mapsto x$ under a composite of the form (3).

Suppose that $y \in A(j)(s) \mapsto x$ under the composite (3). Then $\psi_{A(j)}(y) \leq \psi_X(x)$. This is true for all such pairs (y, j) so that

$$\cup_{j,y}\ \psi_{A(j)}(y) \leq \psi_X(x).$$

Suppose that $x \in X(i)$ lifts to $x' \in X(t)$, where t is maximal. The element x' is in the image of some composite

$$A(i')(s) \to \varinjlim_j\ A(j)(t) \to \mathrm{Im}(\varinjlim_j\ A(j))(s) \to X(i)$$

for all $s < t$. This means that there is an element $y' \in A(i')(t)$ which maps to x' under the composite above, and so $s \leq \psi_{A(i')}(y')$ for all $s < t$. It follows that

$$t = \psi_X(x) \leq \cup_{i,y}\ \psi_{A(i)}(y).$$

Example 26. Form the union $A \cup B$ of two subsheaves $A, B \subset F$ of a sheaf $F \in \mathbf{Mon}(L_+)$. Then there is a pushout diagram

$$\begin{tikzcd} A \cap B \arrow[r] \arrow[d] & B \arrow[d] \\ A \arrow[r] & A \cup B \end{tikzcd}$$

in $\mathbf{Mon}(L_+)$. Here, $A \cup B$ is the sheaf $L(A \cup B)$ that is associated to the presheaf union $A \cup B$, which is in $\mathbf{Mon}_p(L_+)$. Note that

$$A(i) \cup B(i) = (A \cup B)(i) = L(A \cup B)(i),$$

by Corollary 21. We know that $\psi_{A \cap B}(x) \leq \psi_A(x), \psi_B(x)$ for all $x \in (A \cap B)(i)$. It follows from the relation (4) that

$$\psi_{A \cup B}(y) = \max\{\psi_A(y), \psi_B(y)\},$$

appropriately interpreted: one sets $\psi_A(y) = 0$ if y is not a member of a section of A.

3 Stalks

Suppose generally that L is a locale. Following [4, p.51] and [5], for $x \in L$ write

$$\neg x = \vee_{x \wedge y = 0}\ y.$$

The subobject $\neg\neg L$ of L is defined to be the set of all $x \in L$ such that $\neg\neg x = x$. There is a frame morphism $\gamma : L \to \neg\neg L$ which is defined by $x \mapsto \neg\neg x$.

For $x \in L$, write $L_{\geq x}$ (as above) for the sublocale of objects z with $z \geq x$. There is a homomorphism $\phi_x : L \to L_{\geq x}$ which is defined by $y \mapsto y \vee x$.

Let ω denote the composite frame morphism

$$L \xrightarrow{(\phi_x)} \prod_{x\in L} L_{\geq x} \xrightarrow{(\gamma)} \prod_{x\in L} \neg\neg L_{\geq x}. \tag{5}$$

Then one knows (see, for example, [4, p.52]) that ω is a monomorphism and that

$$\mathcal{B} := \prod_{x\in L} \neg\neg L_{\geq x}$$

is a complete Boolean algebra.

Note that $L_{\geq 0} = L$ and that $\phi_0 : L \to L_{\geq 0}$ is the identity.

The corresponding geometric morphism

$$\omega : \mathbf{Shv}(\mathcal{B}) \to \mathbf{Shv}(L)$$

is a Boolean localization of $\mathbf{Shv}(L)$. This means in particular that the inverse image functor $\omega^* : \mathbf{Shv}(L) \to \mathbf{Shv}(\mathcal{B})$ is faithful, and is a "fat point" for the topos $\mathbf{Shv}(L)$.

Example 27. Suppose that L is totally ordered. If $x \in L$ and $x \neq 0$, then $x \wedge y = \min\{x, y\} = 0$ forces $y = 0$. Thus,

$$\neg x = \begin{cases} 0 & \text{if } x \neq 0, \text{ and} \\ 1 & \text{if } x = 0. \end{cases}$$

The corresponding Boolean algebra

$$\mathcal{B} = \prod_{x\in L} \{x, 1\} \cong \prod_{x\in L-\{1\}} \{x, 1\}$$

is isomorphic to the power set $\mathcal{P}(L - \{1\})$ of $L - \{1\}$, so that the sheaf category $\mathbf{Shv}(L)$ has enough points.

The poset map $\phi_x : L \to L_{\geq x}$ takes y to x if $y < x$ and takes y to y if $y \geq x$. It follows that the composite

$$L \xrightarrow{\phi_x} L_{\geq x} \xrightarrow{\gamma} \{x, 1\}$$

takes y to x if $y \leq x$ and takes y to 1 if $y > x$.

The poset map $\omega : L \to \mathcal{P}(L - \{1\})$ takes y to the set $L_{<y}$ of elements x such that $x < y$ if $y \neq 1$, takes 1 to the set $L - \{1\}$, and takes 0 to the empty set.

It follows that, for $x \in L - \{1\}$, the *stalk* F_x of a sheaf F on L is defined by

$$F_x = \varinjlim_{x<s} F(s).$$

This colimit corresponds to the category of inclusions $\{x\} \subset L_{<s}$, so F_x is the evaluation of the sheaf $\omega^*(F)$ at the set $\{x\}$.

Example 28. The locale L_+ has a total order if L has a total order. In that case, the stalk

$$F_0 = \varinjlim_{s \in L} F(s) \cong F(i),$$

where i is the initial object of L, so the stalk F_0 is the generic fibre $F(i)$ of F. The stalk

$$F_i = \varinjlim_{s > i} F(s)$$

is more conventional.

Example 29. In general, if L is a locale then $\neg\neg L_+ = \{0, 1\}$. In effect, $x \wedge y = 0$ then $y = 0$, for otherwise $x \wedge y \in L$. It follows that $\neg x = 0$ for all $x \in L$, while $\neg 0 = 1$.

It follows that the map ω

$$L_+ \xrightarrow{(\phi_x)} \prod_{x \in L_+} (L_+)_{\geq x} \xrightarrow{(\gamma)} \prod_{x \in L_+} \neg\neg(L_+)_{\geq x}$$

is the composite

$$L \sqcup \{0\} \to L_+ \times \prod_{x \in L} L_{\geq x} \to \{0, 1\} \times \prod_{x \in L} \neg\neg L_{\geq x}$$

which takes elements $y \in L$ to pairs $(1, \omega(y))$, where $\omega : L \to \prod_{x \in L} \neg\neg L_{\geq x}$ is the imbedding for L.

Example 30. Suppose that the locale

$$L = L_1 \times \cdots \times L_k$$

is a product of intervals L_i.

The construction of the locale morphism (5) preserves products, so that the sheaf category $\mathbf{Shv}(L)$ again has enough points.

The poset map ω has the form

$$\begin{aligned} L = L_1 \times \cdots \times L_k \xrightarrow{\omega \times \cdots \times \omega} & \mathcal{P}(L_1 - \{1\}) \times \cdots \times \mathcal{P}(L_k - \{1\}) \\ & = \mathcal{P}((L_1 - \{1\}) \sqcup \cdots \sqcup (L_k - \{1\})), \end{aligned}$$

and takes $(y_1, \ldots, y_k)$ to the disjoint union $(L_1)_{<y_1} \sqcup \cdots \sqcup (L_k)_{<y_k}$.

If $x \in L_1$ and F is a sheaf on $L = L_1 \times \cdots \times L_k$, then

$$F_x = \varinjlim_{s > x} F(s, 0, \ldots, 0).$$

In effect, the collection of all k-tuples $(s, 0, \ldots, 0)$ with $s > x$ is cofinal in the collection of all k-tuples $(s_1, \ldots, s_k)$ with $s_1 > x$.

In other words, F_x is the stalk at x of the restriction of F along the poset morphism

$$i_1 : L_1 \to L_1 \times \cdots \times L_k$$

which is defined by $s \mapsto (s, 0, \ldots, 0)$.

Write

$$I = (L_1 - \{1\}) \sqcup \cdots \sqcup (L_k - \{1\}),$$

so that the corresponding poset morphism ω corresponding to $L = L_1 \times \cdots \times L_k$ has the form

$$\omega : L \to \mathcal{P}(I).$$

The direct image functors (restriction) for the presheaf and sheaf categories fit into a commutative diagram

$$\begin{tikzcd}
\mathbf{Pre}(\mathcal{P}(I)) \arrow[r, "\omega_*"] & \mathbf{Pre}(L) \\
\mathbf{Shv}(\mathcal{P}(I)) \arrow[u, "i"] \arrow[r, "\omega_*"'] & \mathbf{Shv}(L) \arrow[u, "i"']
\end{tikzcd}$$

where the vertical functors are inclusions of categories. If L^2 is the associated sheaf functor, then there is a natural isomorphism of left adjoint functors

$$L^2 \omega^p \cong \omega^* L^2.$$

Here, ω^p is the left Kan extension of the restriction functor ω_* on the presheaf level.

For $x \in I$ and a presheaf F on $\mathcal{P}(I)$, there is an isomorphism

$$F(\{x\}) \cong L^2 F(\{x\}),$$

and it follows that there are isomorphisms

$$\omega^p G(\{x\}) \cong \omega^* L^2 G(\{x\})$$

for all presheaves G on L.

In other words, the stalks $\omega^p G(\{x\})$ of the presheaf G coincide up to natural isomorphism with the stalks $\omega^* L^2 G(\{x\})$ of the associated sheaf $L^2 G$.

Example 31. Suppose again that $L = L_1 \times \cdots \times L_k$ is a product of totally ordered locales.

It follows from Example 29 and Example 30 that $\mathbf{Shv}(L_+)$ has enough points. The corresponding index set is $\{0\} \sqcup I$, where

$$I = (L_1 - \{1\}) \sqcup \cdots \sqcup (L_k - \{1\}),$$

is defined in Example 30. The poset map

$$\omega : L_+ \to \mathcal{P}(\{0\} \sqcup I)$$

takes 0 to the subset $\{0\}$, and takes $(y_1, \ldots, y_k)$ to the subset

$$\{0\} \sqcup (L_1)_{<y_1} \sqcup \cdots \sqcup (L_k)_{<y_k}.$$

Example 32. Suppose, as in Example 10, that $X \subset \mathbb{R}^n$ is a data cloud, with ordering $X \subset \mathbf{N}$, and choose $R > d(x, y)$ for all pairs of points $x, y \in X$.

The association $s \mapsto V_s(X)$ for $s \in [0, R]$ defines a simplicial sheaf $V(X)$ (of Vietoris-Rips complexes) of monomorphisms on the totally ordered locale $[0, R]_+^{op}$. The stalk $V(X)_t$ for $t \in (0, R]$ is defined by

$$V(X)_t = \varinjlim_{s<t} V_s(X),$$

where the indicated ordering is that of $[0, R]$.

Note that

$$V(X)_i = \varinjlim_{s<R} V_s(X) = \Delta^N,$$

because we have chosen $R > d(x, y)$ for all pairs of points $x, y \in X$.

Observe as well that for small numbers t, the stalk $V(X)_t$ is the discrete space on the set X.

Suppose that $X \subset Y \subset \mathbb{R}^n$ are data clouds and $R > d(x, y)$ for all pairs of points $x, y \in Y$ (hence in X). Then the inclusion $X \subset Y$ defines a map of simplicial sheaves (of monomorphisms) $V(X) \to V(Y)$. This map is a local weak equivalence if and only if $X = Y$, because $V(X)_t$ and $V(Y)_t$ are discrete for small numbers t.

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