Stacks and homotopy theory

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The Borel construction

First appeared during seminar at IAS 1958-59: [1], 1960.

$G \times M \to M$: Lie group $G$, manifold $M$. $EG = \text{contractible space with free } G\text{-action}$, then

$$EG \times_G M := (EG \times M)/G \text{ (diagonal action)}$$

defines the **Borel construction** for the $G$-space $M$ (= $M_G$ in [1]).

The $G$-equiv. maps $M \to \ast$, $EG \to \ast$ a standard natural picture

$$
\begin{array}{ccc}
M & \longrightarrow & EG \times_G M \\
\downarrow & & \downarrow \pi \\
M/G & \longrightarrow & BG = EG/G
\end{array}
$$

Horizontal row is fibre sequence. $p$ may not be a homotopy equiv.

$EG \times_G M$ is the space of **homotopy coinvariants**.
Suppose $H \times F \to F$ is action of a discrete group $H$ on a set $F$.

Swan (1982): the **translation groupoid** $E_H F$ has objects $x \in F$ and morphisms $g : x \to g \cdot x$. All morphisms are invertible.

Each category $C$ has a **nerve** $BC$. $BC$ is a simplicial set with $n$-simplices $BC_n$ consisting of strings of morphisms

$$a_0 \xrightarrow{f_1} a_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} a_n$$

Simplicial structure def. by composition and insertion of identities.

**Example:** $B(E_H F)_n = H^{\times n} \times F$ consists of strings

$$x_0 \xrightarrow{g_1} g_1 \cdot x_0 \xrightarrow{g_2} \cdots \xrightarrow{g_n} (g_n \cdots g_1) \cdot x_0.$$
**FACT**: Every natural trans. of functors \( f, g : C \to D \) induces a homotopy \( BC \times \Delta^1 \to BD \).

The groupoid \( E_H H \) corr. to \( H \times H \to H \) has initial object \( e \), with \( e \overset{h}{\to} h \), so \( B(E_H H) =: EH \) is contractible.

There is an isomorphism

\[
B(E_H F) \cong EH \times_H F = (EH \times F)/F \text{ (diagonal action)}
\]

There is a natural picture

\[
\begin{array}{ccc}
F & \longrightarrow & EH \times_H F \\
\downarrow & & \downarrow \pi \\
F/H & \longrightarrow & BH
\end{array}
\]

in simplicial sets. Horiz. row is fibre sequence, and \( p \) may not be a weak equivalence.
More properties

1) \( E_H F = \sqcup_{F_i \in F/H} E_H F_i \), and
2) \( E_H F_i \cong H_x \) as groupoids, \( x \in F_i \) (homework)

so

\[
E H \times_H F \cong \sqcup_{[x] \in F/H} B H x.
\]

**Moral**: The map

\[
p : E H \times_H F \cong \sqcup_{[x] \in F/H} B H x \to \sqcup_{[x] \in F/H} * = F / H
\]

is a weak equivalence if and only if \( H \) acts freely on \( F \).

The construction \( E G \times_G X \) generalizes to simplicial groups \( G \) acting on simplicial sets \( X \) — captures the topological const.

**Fact**: \( X \to Y \) \( G \)-equivariant weak equiv. Then
\( E G \times_G X \to E G \times_G Y \) is a weak equivalence (formal nonsense).

**Remark**: \( X / G \to Y / G \) may not be a weak equivalence. Example:
\( E G \to * \).
Group homology, equivariant homology

$EG$ is contractible and $G$ acts freely on $EG$.

Apply the free abelian group functor $F \mapsto \mathbb{Z}(F)$ ...

$\mathbb{Z}(EG)$ is a simplicial abelian group with associated chain complex $\mathbb{Z}(EG)$, having boundaries

$$\mathbb{Z}(EG)_n \xrightarrow{\sum_{i=0}^n (-1)^i d_i} \mathbb{Z}(EG)_{n-1}.$$  

Then $\mathbb{Z}(EG) \rightarrow \mathbb{Z}[0]$ is a homology isomorphism, so $\mathbb{Z}(EG)$ is a $G$-free resolution of the trivial $G$-module $\mathbb{Z}$.

Then

$$H_n(G, M) = \text{Tor}_n(\mathbb{Z}, M) = H_n(\mathbb{Z}(EG) \otimes_G M).$$

$EG \times_G X$ is the non-abelian version of $\mathbb{Z}(EG) \otimes_G M$.

(Cartan-Eilenberg, 1956; also Eilenberg-Mac Lane, 1946?).

$H_*(EG \times_G X, A)$ is one of the flavours of equivariant homology theory for a $G$-space $X$. 

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Stacks and homotopy theory
The best (and first) examples of stacks are categories of principal $G$-bundles, in topology and geometry.

$G$ sheaf of groups: a $G$-bundle ($G$-torsor) is a sheaf $F$ with action $G \times F \to F$, which is free (principal) and locally transitive.

$G \text{ tors}$ is the category of $G$-bundles and $G$-equiv maps, actually a groupoid — see below.

**Basic definition:** $G = \text{sheaf of groups}$

$$\pi_0(G \text{ tors}) = \{\text{Iso classes of } G\text{-torsors}\} =: H^1(\mathcal{E}, G)$$

defines non-abelian $H^1$ with coeffs in $G$, where $\mathcal{E}$ is the underlying category of sheaves.
Repackaging the definition

The simplicial sheaf $EG \times_G F$ is defined in sections by

$$(EG \times_G F)(U) = EG(U) \times_{G(U)} F(U).$$

($G(U)$-action on set $F(U)$, eg. $U$ open subset of top. space)

**locally transitive:** $\pi_0(EG \times_G F) = G/F$ has trivial associated sheaf.

**free:** all stabilizer subgroups $G(U)_x$ for $G(U) \times F(U) \to F(U)$ are trivial.

$G(U)_x = \pi_1(EG(U) \times_{G(U)} F(U), x)$ is trivial for all $x \in F(U)$

so $EG \times_G F \to F/G$ is (sectionwise) weak equivalence

Put these together: $F$ is a $G$-bundle ($G$-torsor) iff $EG \times_G F \to *$

is a stalkwise (local) weak equivalence.

$* = $ one-pt (terminal) sheaf.
Examples

1) $L/k$ finite Galois extension with Galois group $G$.

$EG \times_G Sp(L) \to \ast$ is a local weak equiv for the étale topology (looks like $EG$ locally), so $Sp(L)$ is a $G$-bundle ($G$-torsor) for the étale topology on $Sp(K)$.

2) $P =$ projective module on a ring $R$ of rank $n$:

$P$ is locally free of rank $n$ (Zariski topology), or a vector bundle of rank $n$ over $Sp(R)$.

$\text{Iso}(P_n)$ is the groupoid of isomorphisms of vector bundles of rank $n$ over $R$.

$\text{Gl}_n(R)$ is the group of automorphisms of $R^n$. There is an isomorphism

$$\pi_0(\text{Gl}_n - \text{tors}) \to \pi_0(\text{Iso}(P_n)).$$
3) Suppose $A$ is an $n \times n$ invertible symmetric matrix (non-deg sym bil form of rank $n$) over a field $K$ ($\text{char}(K) \neq 2$).

For the étale topology, $A$ is locally trivial: there is an invertible $n \times n$ matrix defined on $L/K$ (finite Galois extension) such that $B^tAB = I_n$.

These things ”are” the $O_n$-torsors for the étale topology on $K$.

There is an isomorphism

$$H^1_{et}(K, O_n) \cong \{\text{iso. classes of non-deg symm. bil. forms}/K \text{ of rank } n\}.$$ 

$O_n$ is the standard orthogonal group, the group of automorphisms of the trivial form of rank $n$. 
**Fact:** Every morphism $F \to F'$ of $G$-torsors is an isomorphism.

\[
\begin{array}{ccc}
F & \longrightarrow & EG \times_G F \\
\downarrow & & \downarrow \pi \\
F' & \longrightarrow & EG \times_G F'
\end{array}
\]

\[
\begin{array}{ccc}
& & BG \\
\pi & \longrightarrow & \\
& & 1
\end{array}
\]

$F, F'$ are simplicial sheaves as well as just sheaves.

$F \to F'$ stalkwise equivalence since “total spaces” are contractible, so is an isomorphism of sheaves.
A **cocycle** (from \(*\) to \(BG\)) is a picture ("span")

\[
\begin{array}{ccc}
* & \xleftarrow{\sim} & U \\
\downarrow & & \downarrow \\
& BG & \\
\end{array}
\]

A morphism of cocycles is a picture

The category of cocycles is \(h(\ast, BG)\).

**Examples:**
1) standard cocycles \(* \xleftarrow{\sim} \check{C}(U) \rightarrow BG\) defined on Čech resolutions for coverings
2) “twisted” cocycles \(* \xleftarrow{\sim} EG \times_G Sp(L) \rightarrow BH\) in algebraic groups \(H\), for the étale topology on \(K\).
Theorem ("Hammock localization") There is an isomorphism

\[ \pi_0 h(\ast, BG) \cong [\ast, BG] \]

\([\ast, BG]\) is morphisms in homotopy category of simplicial sheaves.

1) Every \(G\)-torsor \(F\) determines a \textbf{canonical cocycle}

\[ * \overset{\sim}{\leftarrow} EG \times_G F \to BG \]

2) Every cocycle \( \ast \overset{\sim}{\leftarrow} U \to BG \) determines a “pullback” torsor \( \pi_0(EG \times_{BG} U) \) by pullback over \( EG \to BG \).

**Theorem**: The canonical cocycle and pullback functors

\[ h(\ast, BG) \leftrightarrow G - \text{tors} \]

are adjoint, so that there are isos

\[ H^1(\mathcal{E}, G) = \pi_0(G - \text{tors}) \cong \pi_0 h(\ast, BG) \cong [\ast, BG]. \]

This is the \textbf{homotopy classification} of \(G\)-torsors.
Symmetric bilinear forms, again

Every non-deg symm bilinear form $\beta$ over $K$ (char $\neq 2$) determines a homotopy class of maps $[\beta] : * \to BO_n$ for the etale topology, and an induced map

$$H^*_\text{Gal}(K, \mathbb{Z}/2)[\text{HW}_1, \ldots, \text{HW}_n] \cong H^*_\text{et}(BO_n, \mathbb{Z}/2) \xrightarrow{\beta^*} H^*_\text{Gal}(K, \mathbb{Z}/2).$$

$\text{deg}(\text{HW}_i) = i$, like Stiefel-Whitney classes.

$\text{HW}_i \mapsto \beta^*(\text{HW}_i) = \text{HW}_i(\beta)$, higher Hasse-Witt invariants of $\beta$ (formerly Delzant Stiefel-Whitney classes).

**Examples** $\text{HW}_1(\beta) = \text{det}(\beta)$, $\text{HW}_2(\beta) = \text{Hasse-Witt invariant}$.

This where the homotopy classification of torsors and the homotopy theoretic approach to stacks began (1989).
Stacks

Origins: Grothendieck “effective descent” (1959); Giraud “champ” (1966, 1971); Deligne-Mumford “stack” (1969)

Most compact definition: a **stack** is a sheaf of groupoids $H$ for which the simplicial sheaf $BH$ **satisfies descent**

ie. there is a fibrant model $BH \to Z$ which is a sectionwise equivalence, ie. all $BH(U) \to Z(U)$ are weak equivs.

Alternative: A **stack** is a sheaf of groupoids $H$ which satisfies **effective descent**, ie. any covering $R \subset \text{hom}(, U)$ induces an equivalence of groupoids

$$H(U) \to \lim_{\phi: V \to U} H(V).$$

NB: $H$ is a **sheaf** of groupoids, so only need show that

$$\pi_0 H(U) \to \pi_0 \left( \lim_{\phi: V \to U} H(V) \right)$$

is surjective.
We have defined $G - \text{tors}$ only in global sections.

If $X = \text{scheme}$ and $G = \text{alg. group}$, we have $G - \text{tors}/X$ for any decent topology on $X$.

Given $f : Y \rightarrow X$, inverse image $f^* : \text{Shv}/X \rightarrow \text{Shv}/Y$ is exact, hence induces a functor $f^* : G - \text{tors}/X \rightarrow G - \text{tors}/Y$.

$U \subset X \mapsto G - \text{tors}/U$ is only a pseudo-functor (in groupoids): there are natural isomorphisms

$$(\beta \alpha)^* \cong \alpha^* \beta^*, \quad \eta : 1 \cong 1^*$$

which satisfy standard coherence conditions.
Suppose \( i \mapsto G(i) \) is a pseudo-functor in groupoids on \( i \in I \).

The **Grothendieck construction** \( E_I G \) has objects \((i, x)\) with \( x \in G(i) \), and morphisms \((\alpha, f) : (i, x) \to (j, y)\) with \( \alpha : i \to j \) in \( I \) and \( f : \alpha_* x \to y \) in \( G(j) \).

The composite

\[
(i, x) \xrightarrow{(\alpha, f)} (j, y) \xrightarrow{(\beta, g)} (k, z)
\]

is defined by \( \beta \alpha \) and the composite

\[
(\beta \alpha)_*(x) \xrightarrow{\omega} \beta_* \alpha_*(x) \xrightarrow{\beta_* f} \beta_*(y) \xrightarrow{g} z.
\]

There is a **canonical functor**

\[
\pi : E_I G \to I \text{ with } (i, x) \mapsto i.
\]
The slice categories $\pi/i$ have objects $\pi(j, y) \to i$, and define a functor $I \to \textbf{Cat}$ with $i \mapsto \pi/i$.

There is a homotopy equivalence of categories $\pi/i \to G(i)$ defined by flowing objects into $G(i)$.

By applying fundamental groupoid, we have equivalences of groupoids $G(\pi/i) \simeq G(i)$.

The assignment $i \mapsto G(\pi/i)$ defines a functor in groupoids, sectionwise equivalent to the pseudo-functor $i \mapsto G(i)$.

**Remark:** Effective descent was originally defined for the pseudo-functor $U \mapsto G - \text{tors}/U$, with a description equivalent to that given above for the equivalent diagram (sheaf) of groupoids $U \mapsto G(\pi/U)$. 
There is a homotopy theory for sheaves of groupoids (Joyal-Tierney, 1990; Hollander, 2008), for which $G \to H$ is a weak equivalence (resp. fibration) if the induced map $BG \to BH$ is a local weak equivalence (resp. fibration) of simplicial sheaves.

A stack is a sheaf (or presheaf) of groupoids which satisfies descent in this homotopy theory

Equiv.: $G$ is a stack if every fibrant model $G \to H$ is a sectionwise equivalence.

Every fibrant sheaf of groupoids is a stack (formal nonsense).

**Slogan:** Stacks are homotopy types of sheaves of groupoids.
Torsors are stacks

Suppose that $j : G \to H$ is a fibrant model (stack completion) for group object $G$ in sheaves of groupoids. Form the diagram

$$
\begin{array}{ccc}
BG & \xrightarrow{j} & BH \\
\downarrow & & \downarrow \\
B(G - \text{tors}) & \xrightarrow{\sim} & B(H - \text{tors}) \\
\downarrow & & \downarrow \\
B\mathbb{H}(\ast, BG) & \xrightarrow{\sim} & B\mathbb{H}(\ast, BH)
\end{array}
$$

All displayed weak equivs are sectionwise, $j : BG \to BH$ is a local weak equiv.

So $j : BG \to B(G - \text{tors})$ is a local weak equiv, and $B(G - \text{tors})$ satisfies descent.
Example: quotient stacks

$X$ is a scheme (sheaf) with $G$-action.

The **quotient stack** $[X/G]$ is the groupoid with objects all $G$-equivariant maps $P \to X$ with $P$ a $G$-torsor, and all $G$-equivariant pictures

\[
\begin{align*}
P & \quad \rightarrow X \\
\downarrow R & \quad \downarrow \\
\downarrow Q & \quad \rightarrow
\end{align*}
\]

as morphisms.

**Fact:** There is an isomorphism

\[
\pi_0([X/G]) \cong [\ast, EG \times_G X].
\]

Every $P \to X$ in $[X/G]$ determines a cocycle

\[
\ast \overset{\sim}{\leftarrow} EG \times_G P \to EG \times_G X.
\]
Given a cocycle $* \leftarrow U \rightarrow EG \times_G X$, pull back over $EG \rightarrow BG$ to form $P = \pi_0(EG \times_{BG} U) \rightarrow X$ in homotopy fibres:

$$
\begin{array}{ccc}
EG \times_{BG} U & \longrightarrow & U \\
\downarrow & & \downarrow \\
EG \times_G X & \longrightarrow & X \\
\downarrow & & \downarrow \\
EG & \longrightarrow & BG
\end{array}
$$

$EG \times_{BG} U$ is homotopy fibre of $U \rightarrow BG$, so weakly equivalent to

$$
P := \tilde{\pi}_0(EG \times_{BG} U).
$$

General nonsense: $EG \times_G (EG \times_{BG} U) \rightarrow U \simeq *$ is a weak equivalence.
Examples

1) Borel constructions $EG \times_G F = B(E_G F)$ are quotient stacks. The stack completion is the functor $\phi : E_G F \to [F/G]$, defined in global sections by

$$x \in F \mapsto G \ltimes x F$$

$\phi$ is a local weak equivalence, and there is a sect. equiv.

$$B([F/G]) \simeq B\mathbb{H}(\ast, EG \times_G F).$$

2) A **gerbe** $H$ is a locally connected stack, i.e. $\pi_0 BH$ is the trivial sheaf.

Gerbes are souped up sheaves of groups.

**Fact**: If $H$ is an ordinary connected groupoid and $x \in \text{Ob}(H)$ then the inclusion functor $H(x, x) \subset H$ is an equivalence of groupoids, so $H$ is a group.
1) There are model structures for sheaves of 2-groupoids, presheaves of $n$-groupoids for $n \geq 2$. The homotopy types are 2-stacks, $n$-stacks etc. See [3] — other people define them differently.

2) Weak equivalence classes of gerbes (with structure) are classified homotopy theoretically by cocycles in 2-stacks. This is Giraud’s non-abelian $H^2$ [2].

e.g. Gerbes locally equivalent to a fixed sheaf of groups $H$ are classified by the 2-groupoid $\text{Aut}(H)$, which has automorphisms of $H$ as 1-cells, and 2-cells given by homotopies.

Opinions: a) Stacks and higher stacks should have geometric content, like groupoids enriched in simplicial sets.

b) Simpson: a stack is a homotopy type of simplicial presheaves (non-abelian Hodge theory).
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