Lecture 011  (October 17, 2014)

26  \textit{K}-theory of finite fields

We will sketch a proof of the following well-known result of Quillen [5]:

\textbf{Theorem 26.1.} Suppose that $\mathbb{F}_q$ is the field with $q = p^n$ elements. Then there are isomorphisms

$$K_i(\mathbb{F}_q) = \begin{cases} \mathbb{Z}/(q^j - 1) & \text{if } i = 2j - 1, j > 0, \text{ and} \\ 0 & \text{if } i = 2j, j > 0. \end{cases}$$

The proof that is given here appears in [4].

The first step is to completely compute the groups $K_n(\mathbb{F}_q)$ for the algebraic closure $\overline{\mathbb{F}}_q$. For this, we use the following form of the Gabber rigidity theorem [3]:

\textbf{Theorem 26.2.} Suppose that $\mathcal{O}$ is a henselian local ring with residue field $k$, and that $1/n \in \mathcal{O}$. Then the residue map $\pi : \mathcal{O} \to k$ induces isomorphisms

$$\pi_* : K_*(\mathcal{O}, \mathbb{Z}/n) \xrightarrow{\cong} K_*(k, \mathbb{Z}/n).$$

Examples of henselian local rings $\mathcal{O}$ include the Witt ring $W(\overline{\mathbb{F}}_q)$ of the field $\overline{\mathbb{F}}_q$. The residue map
for $W(\overline{F}_q)$ has the form $W(\overline{F}_q) \to \overline{F}_q$, and $W(\overline{F}_q)$ is a complete DVR with a quotient field $K$ of characteristic 0, since $\overline{F}_q$ is perfect.

It is a consequence of Gabber rigidity that the ring homomorphisms

$$\overline{F}_q \leftarrow W(\overline{F}_q) \to K \leftarrow \overline{Q} \to \mathbb{C}$$

induce isomorphisms

$$K_*(\overline{F}_q, \mathbb{Z}/n) \cong K_*(W(\overline{F}_q), \mathbb{Z}/n) \cong K_*(\overline{K}, \mathbb{Z}/n) \cong K_*(\overline{Q}, \mathbb{Z}/n) \cong K_*(\mathbb{C}, \mathbb{Z}/n)$$

for $(n, p) = 1$.

Some comments:

1) The map

$$H^*_\text{et}(B\text{Gl}_{W(\overline{F}_q)}, \mathbb{Z}/n) \to H^*_\text{et}(B\text{Gl}_{K}, \mathbb{Z}/n)$$

is an isomorphism by smooth proper base change for algebraic groups [2], and the map

$$H^*_\text{et}(B\text{Gl}_{W(\overline{F}_q)}, \mathbb{Z}/n) \to H^*(B\text{Gl}(W(\overline{F}_q)), \mathbb{Z}/n)$$

is an isomorphism by the Gabber theorem (the homology sheaves $\tilde{H}_n(B\text{Gl}_{W(\overline{F}_q)}, \mathbb{Z}/n)$ are constant) and the fact that strict local hensel rings $\mathcal{O}$ have no étale cohomology (global sections on $\textbf{Shv}(\text{Sch} \mid \mathcal{O})_{\text{et}}$).
is exact). From the diagram

\[
\begin{array}{ccc}
H^*_\text{et}(BGl_K, \mathbb{Z}/n) & \xrightarrow{\cong} & H^*(BGl(K), \mathbb{Z}/n) \\
\cong & & \downarrow \\
& H^*(BGl(W(\overline{F}_q)), \mathbb{Z}/n)
\end{array}
\]

we see that the map

\[H^*(BGl(K), \mathbb{Z}/n) \rightarrow H^*(BGl(W(\overline{F}_q), \mathbb{Z}/n))\]

is an isomorphism.

2) The map

\[BGl(\mathbb{C}) \rightarrow BGl(\mathbb{C})^{\text{top}} \cong BU\]

induces a monomorphism

\[H^*(BU, \mathbb{Z}/\ell) \rightarrow H^*(BGl(\mathbb{C}), \mathbb{Z}/\ell).\]

In effect, the comparison

\[BT(\mathbb{C}) \rightarrow BT(\mathbb{C})^{\text{top}}\]

induces an isomorphism

\[H^*(BT(\mathbb{C})^{\text{top}}, \mathbb{Z}/\ell) \cong H^*(BT(\mathbb{C}), \mathbb{Z}/\ell)\]

for any complex torus \(T\) (exercise), and the map

\[H^*(BGl_n(\mathbb{C})^{\text{top}}, \mathbb{Z}/\ell) \rightarrow H^*(BT(\mathbb{C})^{\text{top}}, \mathbb{Z}/\ell)\]

which is induced by the inclusion \(T \subset Gl_n\) of a maximal torus induces a monomorphism.
It follows that the map

$$H^p(BU, \mathbb{Z}/\ell) \to H^p(BGl(C), \mathbb{Z}/\ell)$$

is a monomorphism of finite dimensional $\mathbb{Z}/\ell$-vector spaces of the same dimension (by Gabber rigidity), and is therefore an isomorphism, for all $p \geq 0$. 

3) In complex $K$-theory, there is an isomorphism $\pi_2 BU \cong \pi_1(U) \cong \mathbb{Z}$, with generator $\beta$, and complex Bott periodicity says that cup product (induced by tensor product) with the generator $\beta$ induces a map

$$\beta_* : ku[2] \cong S^2 \wedge ku \to ku$$

which is an isomorphism in stable homotopy groups $\pi_i$ for $i \geq 2$. Here, $ku$ is connective (topological) complex $K$-theory, which is formed by group-completing the monoid

$$\bigsqcup_{n \geq 0} BU_n \cong \bigsqcup_{n \geq 0} BGl_n(C)^{top},$$

or rather by taking a fibrant model of the spectrum associated to the $\Gamma$-space which arises from direct sum of matrices. In particular, there is a weak equivalence

$$ku^0 \cong \mathbb{Z} \times BU.$$
The map of monoids
\[
\bigsqcup_{n \geq 0} BGl_n(\mathbb{C}) \to \bigsqcup_{n \geq 0} BGl_n(\mathbb{C})^{\text{top}}
\]
induces a map of (symmetric) spectra
\[
\epsilon : K(\mathbb{C}) \to ku.
\]
Both spectra are ring spectra with ring structure induced by tensor product, so that \(\epsilon\) is a map of ring spectra.

The map
\[
BGl(\mathbb{C}) \to BU
\]
induces an isomorphism
\[
H_*(BGl(\mathbb{C}), \mathbb{Z}/\ell) \cong H_*(BU, \mathbb{Z}/\ell).
\]
In effect, the map is dual to an isomorphism in \(\text{mod } \ell\) cohomology.

It follows that the induced map
\[
\epsilon : K(\mathbb{C})/\ell \to ku/\ell
\]
is a stable equivalence. To see this, one shows that the homotopy fibre \(F\) of the map \(\epsilon\) has uniquely \(\ell\)-divisible homotopy groups, by an argument similar to that for Theorem 20.4 in Lecture 008.

It follows in particular that the map
\[
\pi_2 K(\mathbb{C})/\ell \to \pi_2 ku/\ell
\]
is an isomorphism, and therefore takes Bott element to Bott element.

The Bott element $\beta \in \pi_2 ku/\ell$ is the image of the Bott element in $\pi_2 ku$ under the map

$$\pi_2 ku \to \pi_2 ku/\ell.$$ 

There is a comparison of cofibre sequences

$$
\begin{array}{ccc}
S^2 \wedge ku & \longrightarrow & S^2 \wedge ku \\
\beta_* \downarrow & & \downarrow \beta_* \\
ku & \longrightarrow & ku/\ell
\end{array}
$$

in which all vertical maps are defined by (left) multiplication by the Bott element. It follows that the map

$$\beta_* : S^2 \wedge ku/\ell \to ku/\ell$$

is an isomorphism in $\pi_i$ for $i \geq 2$.

If $E$ is a ring spectrum and the Moore spectrum $S/n$ has a ring spectrum structure, then the composite

$$E \wedge S/n \wedge E \wedge S/n \xrightarrow{1 \wedge 1 \wedge 1} E \wedge E \wedge S/n \wedge S/n \xrightarrow{m \wedge m} E \wedge S/n$$

defines a ring spectrum structure on

$$E/n = E \wedge S/n.$$
The Moore spectrum $S/n$ has such a ring spectrum structure if $n = \ell^\nu$ where $\ell > 3$, $\nu \geq 2$ if $\ell = 3$, and $\nu \geq 4$ if $\ell = 2$ [7, p.544]. Assume that $n$ is such a prime power henceforth.

There is a commutative diagram

$$
\begin{array}{ccc}
S^2 \wedge ku/n & \xrightarrow{\beta \wedge 1} & ku \wedge ku/n \\
\downarrow_{1 \wedge \epsilon} \simeq & \downarrow & \downarrow_{\epsilon \wedge \epsilon} \simeq \\
S^2 \wedge K(\mathbb{C})/n & \xrightarrow{\beta \wedge 1} & K(\mathbb{C})/n \wedge K(\mathbb{C})/n \\
\end{array}
$$

The top composite $(m \wedge 1)(\beta \wedge 1)$ is the composite $\beta_*$ above, and is an isomorphism in $\pi_i$ for $i \geq 2$. It follows that the composite

$$
S^2 \wedge K(\mathbb{C})/n \xrightarrow{\beta \wedge 1} K(\mathbb{C})/n \wedge K(\mathbb{C})/n \xrightarrow{m} K(\mathbb{C})/n
$$

is an isomorphism in $\pi_i$ for $i \geq 2$.

We have proved:

**Lemma 26.3.** Suppose that $n = \ell^\nu$, subject to the constraints on the prime $\ell$ and the power $\nu$ listed above. Then all powers of the Bott element $\beta^k \in K_{2k}(\mathbb{C}, \mathbb{Z}/n)$ are non-trivial, and there is an isomorphism

$$
K_*(\mathbb{C}, \mathbb{Z}/n) \cong \mathbb{Z}/n[\beta].
$$
Corollary 26.4. Suppose that $n = \ell^n$, subject to the constraints on the prime $\ell$ and the power $\nu$ listed above. Suppose that $k$ is an algebraically closed field with $(n, \text{char}(k)) = 1$. Then there is an isomorphism

$$K_*(k, \mathbb{Z}/n) \cong \mathbb{Z}/n[\beta].$$

For the record, the following is proved with a transfer argument:

Lemma 26.5. Suppose that $L$ is a separably closed field and that $(n, \text{char}(L)) = 1$. Then the inclusion $L \subset \overline{L}$ induces an isomorphism

$$K_*(L, \mathbb{Z}/n) \cong K_*(\overline{L}, \mathbb{Z}/n).$$

Now let’s talk about finite fields.

Lemma 26.6. $K_i(\mathbb{F}_q)$ is a finite abelian group of order prime to $p$ if $i \geq 0$.

Proof. Homological stability [6] says that there is an isomorphism

$$H_i(BGl(\mathbb{F}_q), \mathbb{Z}) \cong H_i(BGl_n(\mathbb{F}_q), \mathbb{Z})$$

for $n$ sufficiently large. It follows that the reduced homology $\tilde{H}_*(BGl(\mathbb{F}_q), \mathbb{Z})$ consists of finite groups.
It is known [5] that $\tilde{H}_*(BGl(\mathbb{F}_q), \mathbb{Z}/p) = 0$. Thus $\tilde{H}_*(BGl(\mathbb{F}_q), \mathbb{Z})$ consists of uniquely $p$-divisible finite abelian groups. \(\square\)

**Lemma 26.7.** There are isomorphisms

$$K_i(\mathbb{F}_q) \cong \begin{cases} \mathbb{Q}(p)/\mathbb{Z} & \text{if } i = 2j - 1, j \geq 1 \text{ and } \\ 0 & \text{if } i = 2j, j \geq 1. \end{cases}$$

**Proof.** $K_i(\mathbb{F}_q) = \varprojlim K_i(\mathbb{F}_{q'})$ is a torsion abelian group with no $p$-torsion by the last Lemma.

$K_{2j+1}(\mathbb{F}_q, \mathbb{Z}/n) = 0$ for all $j \geq 1$ if $(n, p) = 1$. It follows that $K_{2j}(\mathbb{F}_q)$ is a torsion group with $\text{Tor}(\mathbb{Z}/n, K_{2j}(\mathbb{F}_q)) = 0$ for $(n, p) = 1$, so that $K_{2j}(\mathbb{F}_q) = 0$ for all $j \geq 1$.

It follows that there are isomorphisms

$$\mathbb{Z}/n \cong K_{2j}(\mathbb{F}_q, \mathbb{Z}/n) \cong \text{Tor}(\mathbb{Z}/n, K_{2j-1}(\mathbb{F}_q))$$

for $j \geq 1$. These isomorphisms are functorial in the poset of numbers $n$ with $(n, p) = 1$ and with $n \leq m$ if $n|m$.

An element of $\mathbb{Q}(p)$ is a fraction $\frac{m}{n}$ such that $p$ does not divide $n$, and this element is in $\mathbb{Z}$ if $n = 1$. The maps

$$\mathbb{Z}/n \to \mathbb{Q}(p)/\mathbb{Z}$$
defined by \( 1 \mapsto \frac{1}{n} \) define the isomorphism
\[
\lim_{n \to \infty} \mathbb{Z}/n \cong \mathbb{Q}_p/\mathbb{Z}.
\]

\[\square\]

Note that
\[K_1(\overline{\mathbb{F}}_q) \cong \mathbb{F}_q^* \cong \mathbb{Q}_p/\mathbb{Z}.
\]

**Corollary 26.8.** The group \( K_{2j-1}(\overline{\mathbb{F}}_q) \) is \( n \)-divisible for all \( n \) with \( (n,p) = 1 \), if \( j \geq 1 \).

Now recall that the Frobenius automorphism \( \phi : \mathbb{F}_q \to \mathbb{F}_q \) is defined by \( \phi(\alpha) = \alpha^q \). Recall that \( \phi \) is the identity on \( \mathbb{F}_q \), since \( \mathbb{F}_q \) is the splitting field of the polynomial \( x^q - x \).

The Frobenius induces a morphism of spectra \( \phi : K(\mathbb{F}_q) \to K(\mathbb{F}_q), \) and there is a commutative diagram of spectra
\[
\begin{array}{ccc}
K(\mathbb{F}_q) & \xrightarrow{i} & K(\overline{\mathbb{F}}_q) \\
\downarrow{i} & & \downarrow{\Delta} \\
K(F_q) & \xrightarrow{(\phi, 1)} & K(\mathbb{F}_q) \times K(\overline{\mathbb{F}}_q)
\end{array}
\]

where \( i \) is induced by the inclusion \( \mathbb{F}_q \subset \overline{\mathbb{F}}_q \).

Here’s the main theorem:
Theorem 26.9. The square

\[
\begin{array}{ccc}
K(\mathbb{F}_q) & \rightarrow & K(\overline{\mathbb{F}}_q) \\
\downarrow i & & \downarrow \Delta \\
K(\overline{\mathbb{F}}_q) & \rightarrow & K(\overline{\mathbb{F}}_q) \times K(\overline{\mathbb{F}}_q)
\end{array}
\]

is homotopy cartesian.

This result is often paraphrased by saying that the spectrum \( K(\mathbb{F}_q) \) is the homotopy fixed points of the Frobenius.

Corollary 26.10. There are isomorphisms

\[
K_i(\mathbb{F}_q) = \begin{cases} 
\mathbb{Z}/(q^j - 1) & \text{if } i = 2j - 1, j > 0, \text{ and} \\
0 & \text{if } i = 2j, j > 0.
\end{cases}
\]

Proof. The squares in the diagram

\[
\begin{array}{ccc}
K(\mathbb{F}_q) & \rightarrow & K(\overline{\mathbb{F}}_q) \\
\downarrow i & & \downarrow \Delta \\
K(\overline{\mathbb{F}}_q) & \rightarrow & K(\overline{\mathbb{F}}_q) \times K(\overline{\mathbb{F}}_q)
\end{array}
\]

are homotopy cartesian, so that there is a fibre sequence

\[
K(\mathbb{F}_q) \rightarrow K(\overline{\mathbb{F}}_q) \rightarrow K(\overline{\mathbb{F}}_q) \rightarrow K(\overline{\mathbb{F}}_q).
\]

(1)

It follows from the Suslin calculations that the map

\[
\phi_* : K_{2j}(\overline{\mathbb{F}}_q) \rightarrow K_{2j}(\overline{\mathbb{F}}_q)
\]
is multiplication by $q^j$. In effect,

$$\phi_* : K_2(\overline{F}_q, \mathbb{Z}/n) \rightarrow K_2(\overline{F}_q, \mathbb{Z}/n)$$

is multiplication by $q$ (ie. $\beta \mapsto q\beta$) since

$$K_2(\overline{F}_q, \mathbb{Z}/n) \cong \text{Tor}(\mathbb{Z}/n, \overline{F}_q^*)$$.

It follows that the map

$$\phi_* : K_{2j}(\overline{F}_q, \mathbb{Z}/n) \rightarrow K_{2j}(\overline{F}_q, \mathbb{Z}/n)$$

is multiplication by $q^j$ ($\beta^j \mapsto (q\beta)^j$). Thus, the map

$$\phi_* : \text{Tor}(\mathbb{Z}/n, K_{2j-1}(\overline{F}_q)) \rightarrow \text{Tor}(\mathbb{Z}/n, K_{2j-1}(\overline{F}_q))$$

is also multiplication by $q^j$. But $K_{2j-1}(\overline{F}_q)$ consists of torsion prime to $p$, so that $\phi_* : K_{2j-1}(\overline{F}_q) \rightarrow K_{2j-1}(\overline{F}_q)$ is multiplication by $q^j$.

It follows that the map

$$\phi_* - 1 : K_{2j-1}(\overline{F}_q) \rightarrow K_{2j-1}(\overline{F}_q)$$

is multiplication by $q^j - 1$. This number is prime to $p$, so that the map $\phi_* - 1$ is surjective. It follows from the long exact sequence for the fibre sequence (1) that

$$K_{2j-1}(\overline{F}_q) = \text{Tor}(\mathbb{Z}/(q^j-1), K_{2j-1}(\overline{F}_q)) \cong \mathbb{Z}/(q^j-1).$$

and $K_{2j}(\overline{F}_q) = 0$. $\square$
To prove Theorem 26.9, form the homotopy pullback diagram

\[
\begin{array}{ccc}
F_\phi(\overline{F}_q) & \rightarrow & K(\overline{F}_q) \\
\downarrow & \downarrow & \downarrow \\
K(\overline{F}_q) & \rightarrow & K(\overline{F}_q) \times K(\overline{F}_q)
\end{array}
\]

\[\Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta \quad \Delta
\]

The game is to show that the induced map

\[K(\overline{F}_q) \rightarrow F_\phi(\overline{F}_q)\]

is a stable equivalence.

It suffices to do this at the level of 1-connected covers. In effect, if \(\tilde{K}(R) \rightarrow K(R)\) denotes the 1-connected cover (ie. fibre of the map \(K(R) \rightarrow P_1K(R)\)), form the homotopy pullback

\[
\begin{array}{ccc}
\tilde{F}_\phi(\overline{F}_q) & \rightarrow & \tilde{K}(\overline{F}_q) \\
\downarrow & \downarrow & \downarrow \\
\tilde{K}(\overline{F}_q) & \rightarrow & \tilde{K}(\overline{F}_q) \times \tilde{K}(\overline{F}_q)
\end{array}
\]

Then \(\tilde{F}_\phi(\overline{F}_q) \rightarrow F_\phi(\overline{F}_q)\) is the 1-connected cover, while we already know that the map \(\pi_iK(\overline{F}_q) \rightarrow \pi_iF_\phi\) is an isomorphism for \(i = 0, 1\).

Note that the map \(\overline{F}_q^* \rightarrow \overline{F}_q^*\) defined by \(\alpha \mapsto \alpha/\phi(\alpha)\) is an isomorphism (Lang isomorphism) so that \(\tilde{F}_\phi\) is simply-connected.
We therefore want to show that the map
\[ \tilde{K}(\mathbb{F}_q) \to \tilde{F}_\phi(\overline{\mathbb{F}}_q) \]
is a stable equivalence. Both spectra are 1-connected, so it suffices to show that the map
\[ \tilde{K}^0(\mathbb{F}_q) \to \tilde{F}^0_\phi(\overline{\mathbb{F}}_q) \]
of pointed simplicial sets is a weak equivalence. Both spaces are simply-connected and have homotopy groups which are finite and of order prime to \( p \), so it is enough to show that the maps
\[ H_*(\tilde{K}^0(\mathbb{F}_q), \mathbb{Z}/\ell) \to H_*(\tilde{F}^0_\phi(\overline{\mathbb{F}}_q), \mathbb{Z}/\ell) \]
are isomorphisms for all primes \( \ell \) with \( (\ell, p) = 1 \).

Suppose that \( E(R) \subset \text{Gl}(R) \) is the subgroup of elementary transformations, and recall that \( E(R) = [\text{Gl}(R), \text{Gl}(R)] \), naturally in rings \( R \). There is map
\[ BE \to \tilde{K}^0 \]
of simplicial presheaves on the big étale site \( (\text{Sch}|_{\mathbb{F}_q})_{et} \) which is an \( H_*(\ , \mathbb{Z}) \)-isomorphism on affine patches \( (\tilde{K}^0(R) = BE(R)^+) \). It therefore suffices to show that the composition map
\[ BE(\mathbb{F}_q) \to \tilde{K}^0(\mathbb{F}_q) \to \tilde{F}^0_\phi(\overline{\mathbb{F}}_q) \]
is an \( H_*(\ , \mathbb{Z}/\ell) \)-isomorphism for \( (\ell, p) = 1 \).
The Frobenius homomorphism induces a natural map \( \phi : E \to E \) for presheaves of spectra on \( (Sch|\mathbb{F}_q)_{et} \), and we can form the sectionwise homotopy cartesian diagram

\[
\begin{array}{ccc}
\tilde{F}_\phi & \longrightarrow & \tilde{K} \\
\downarrow & & \downarrow \Delta \\
\tilde{K} & \longrightarrow & \tilde{K} \times \tilde{K} \\
\end{array}
\]

The diagram (2) is global sections of this diagram of presheaves of spectra. There is a corresponding pointwise homotopy cartesian diagram

\[
\begin{array}{ccc}
\tilde{F}_\phi^0 & \longrightarrow & \tilde{K}^0 \\
\downarrow & & \downarrow \Delta \\
\tilde{K}^0 & \longrightarrow & \tilde{K}^0 \times \tilde{K}^0 \\
\end{array}
\]

of pointed simplicial presheaves.

**Lemma 26.11.** *The simplicial presheaf \( \tilde{F}_\phi^0 \) is rigid in the sense that the map*

\[
\Gamma^* \tilde{F}_\phi^0(\mathbb{F}_q) \to \tilde{F}_\phi^0
\]

*induces an isomorphism in homology sheaves \( \tilde{H}_* \) for all \( (\ell,p) = 1 \).*

**Proof.** The \( K \)-theory presheaf is rigid in mod \( \ell \) stable homotopy groups (Gabber rigidity), and \( P_1K \)
is rigid (calculation), so that $\tilde{K}$ is rigid and then $\tilde{F}_\phi$ is rigid, as presheaves of spectra. Extract the homology statement in the usual way. \qed

In particular, there is an isomorphism

$$H^*_\text{et}(\tilde{F}_0^0, \mathbb{Z}/\ell) \cong H^*(\tilde{F}_0^0(\overline{F}_q), \mathbb{Z}/\ell),$$

and it suffices to show that the maps

$$H^*_\text{et}(\tilde{F}_0^0, \mathbb{Z}/\ell) \to H^*(\Gamma^*BE(\overline{F}_q), \mathbb{Z}/\ell) \cong H^*(BE(\overline{F}_q), \mathbb{Z}/\ell)$$

are isomorphisms.

The natural inclusion $E_n(R) \subset Sl_n(R)$ induces local weak equivalences

$$BE_n \to BS_l n, \quad BE \to BS_l$$

of simplicial presheaves on $(\text{Sch}|_{\overline{F}_q})_{\text{et}}$ since the groups in question coincide on local rings.

**Lemma 26.12.** There is a homotopy cartesian diagrams of simplicial presheaves

$$\begin{array}{ccc}
\Gamma^*BSl_n(\overline{F}_q) & \longrightarrow & BSl_n \\
\downarrow & & \downarrow \Delta \\
BSl_n & \longrightarrow & BSl_n \times BSl_n \\
(\phi,1) & & \\
\end{array}$$

**Proof.** Any inclusion $G \subset H$ of groups determines a homotopy fibre sequence

$$EG \times_G H \to BG \to BH$$
and all homotopy groups $\pi_i(EG \times_G H)$ are trivial for $i \geq 1$. Checking that the diagram above is homotopy cartesian amounts to showing that the induced map on homotopy fibres is a local equivalence, but this amounts to showing that the induced map

$$Sl_n/\Gamma^*Sl_n(F_q) \to Sl_n$$

defined by $A \mapsto \phi(A)A^{-1}$ is an isomorphism. The fact that the displayed map is an isomorphism is well known — this map is called the Lang isomorphism [1, Prop. 2].

Form the comparisons of homotopy cartesian diagrams

$$\begin{array}{ccc}
\Gamma^*BE(F_q) & \xrightarrow{\zeta} & BE \\
\downarrow & & \downarrow \\
F^0 & \xrightarrow{\Delta} & K^0 \\
\downarrow & & \downarrow \\
BE & \xrightarrow{(\phi,1)} & BE \times BE \\
\downarrow & & \downarrow \\
K^0 & \xrightarrow{(\phi,1)} & K^0 \times K^0
\end{array}$$
We want to show that the map $\zeta$ induces an isomorphism in étale cohomology, and we do this by showing that the maps $\epsilon_*$ and $\zeta_*$ are isomorphisms in étale cohomology.

1) The map $\epsilon_*$ is a mod $\ell$ homology sheaf isomorphism, because the object $\tilde{K}^0$ is rigid. Use a comparison of Serre spectral sequences in stalks to see this.

2) There is a weak equivalence

$$Y_1 \simeq ES\ell(\overline{F}_q) \times_{Sl(\overline{F}_q)} Sl \times (Sl(\overline{F}_q) \times ES\ell(\overline{F}_q)).$$

The idea is to show that

$$\tilde{H}_{et}^*(ES\ell(\overline{F}_q) \times_{Sl(\overline{F}_q)} Sl, \mathbb{Z}/\ell) = 0$$

But there is an isomorphism

$$H_{et}^*(ES\ell(\overline{F}_q) \times_{Sl(\overline{F}_q)} Sl, \mathbb{Z}/\ell) \cong H^*(ES\ell(\mathbb{C}) \times_{Sl(\mathbb{C}) Sl_{top}}, \mathbb{Z}/\ell)$$
by rigidity (and GAGA), and $ESl(C) \times_{Sl(C)} Sl^{top}$ is the homotopy fibre of $BSl(C) \to BSl^{top}$, which fibration is a mod $\ell$ cohomology isomorphism (by rigidity) with simply connected base. The fibre is therefore mod $\ell$ cohomologically acyclic.

3) To show that $\gamma$ is an étale cohomology isomorphism, it suffices to show that the map $E \to \Omega \hat{K}^0$ is a mod $\ell$ étale cohomology isomorphism. One uses a comparison of Serre-type spectral sequences, eg.

$$H^p(\hat{K}^0(\mathbb{F}_q), H^q_{et}(\Omega \hat{K}^0, \mathbb{Z}/\ell)) \Rightarrow H^{p+q}_{et}(X_2, \mathbb{Z}/\ell)$$

to see this. But again the question about $E \to \Omega \hat{K}^0$ base changes to topology, where there is an actual weak equivalence.

References


