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4 Spaces and homotopy groups

Some definitions

CGWH is the category of compactly generated weak Hausdorff spaces.

A space X is *compactly generated* if a subset Z is closed if and only if $Z \cap K$ is closed for all maps $K \to X$ with K compact.

A compactly generated space X is *weakly Hausdorff* if and only if the image of the diagonal $\Delta : X \to X \times X$ is closed in $X \times X$, where the product is in the category of compactly generated spaces.

CGWH is the "convenient category" for homotopy theory, because it's cartesian closed, as well as complete and cocomplete.

The product $X \times Y$ in **CGWH** has the underlying point set that you expect, but it's topologized as a colimit of all products $C \times D$ where $C \to X$ and $D \to Y$ are maps such that C and D are compact.

If X and Y are compact, this definition doesn't affect the topology on $X \times Y$.

All CW-complexes (spaces inductively built from cells) are members of **CGWH**.

See the preprint

N.P. Strickland. The category of CGWH spaces. Preprint, Sheffield `http://www.neil-strickland.staff.shef.ac.uk/courses/homotopy/cgwh.pdf`, 2009

Examples that we care about

The *topological standard n-simplex* is the space $|\Delta^n|$ defined by

$$|\Delta^n| = \{(t_0, \dots, t_n) \in \mathbb{R}^{n+1} \mid \sum t_i = 1, t_i \geq 0 \}.$$

$|\Delta^0|$ is a point, $|\Delta^1|$ is a copy of the unit interval, $|\Delta^2|$ is a triangle, etc.

$\mathbf{n} = \{0, 1, \dots, n\}$, $n \geq 0$, with the obvious poset structure — this is the *finite ordinal number* $\mathbf{n}$.

The finite ordinal numbers $\mathbf{n}$, $n \geq 0$, form a category Δ, whose morphisms are the order-preserving functions (aka. poset morphisms) $\theta : \mathbf{m} \to \mathbf{n}$.

The monomorphisms $d^i : \mathbf{n-1} \to \mathbf{n}$ have the form

$$d^i(j) = \begin{cases} j & \text{if } j < i, \text{ and} \\ j+1 & \text{if } j \geq i. \end{cases}$$

with $0 \leq i \leq n$. The map d^i misses the element $i \in \mathbf{n}$.

$s^j : \mathbf{n}+\mathbf{1} \to \mathbf{n}$, $0 \leq j \leq n$, is the unique poset epimorphism such that $s^j(j) = s^j(j+1) = j$.

The s^j, $0 \leq j \leq n$ form a complete list of epimorphisms $\mathbf{n}+\mathbf{1} \to \mathbf{n}$ in Δ.

The singular set

There is a functor

$$|\Delta| : \Delta \to \mathbf{CGWH}$$

with $\mathbf{n} \mapsto |\Delta^n|$. The morphism $\theta : \mathbf{m} \to \mathbf{n}$ induces the continuous map $\theta_* : |\Delta^m| \to |\Delta^n|$, with

$$\theta_*(t_0, \ldots, t_m) = (s_0, \ldots, s_n),$$

and

$$s_i = \sum_{j \in \theta^{-1}(i)} t_j.$$

An *n-simplex* of a space X is a continuous map $\sigma : |\Delta^n| \to X$.

The i^{th} *face* $d_i(\sigma)$ of the n-simplex σ is the composite

$$|\Delta^{n-1}| \xrightarrow{d^i} |\Delta^n| \xrightarrow{\sigma} X.$$

The *vertex* v_j of σ is the composite

$$|\Delta^0| \xrightarrow{j} |\Delta^n| \xrightarrow{\sigma} X,$$

(an element of X), where $j : \mathbf{0} \to \mathbf{n}$ is defined by $j(0) = j \in \mathbf{n}$.

v_j is the vertex opposite the face $d_j(\sigma)$.

Example: Suppose $\sigma : |\Delta^2| \to X$ is a 2-simplex. Here's the picture:

$$\begin{array}{ccc} v_0 & \xrightarrow{d_2(\sigma)} & v_1 \\ & {\scriptstyle d_1(\sigma)}\searrow & \downarrow{\scriptstyle d_0\sigma} \\ & & v_2 \end{array} \qquad\qquad \begin{array}{ccc} 0 & \xrightarrow{[0,1]} & 1 \\ & {\scriptstyle [0,2]}\searrow & \downarrow{\scriptstyle [1,2]} \\ & & 2 \end{array}$$

Some language:

$$S(X)_n = \hom(|\Delta^n|, X)$$

is the set of *n-simplices* of X.

An ordinal number map $\theta : \mathbf{m} \to \mathbf{n}$ induces a function $\theta^* : S_n(X) \to S_m(X)$ by precomposition with $\theta : |\Delta^m| \to |\Delta^n|$.

The composite

$$|\Delta^m| \xrightarrow{\theta} |\Delta^n| \xrightarrow{\sigma} X$$

is $\theta^*(\sigma) \in S_m(X)$.

The simplices and precompositions define a (contravariant) functor

$$S(X) : \Delta^{op} \to \mathbf{Set}$$

taking values in sets. $S(X)$ is a *simplicial set*, called the *singular set* for the space X.

Path components

A *path* in X is a 1-simplex $\omega : |\Delta^1| \to X$ of X, while a vertex is an element $x : |\Delta^0| \to X$.

A path has a natural orientation:

$$x = d_1(\omega) \xrightarrow{\omega} d_0(\omega) = y$$

reflects

$$d^1(0) = 0 \to 1 = d^0(0)$$

in **1**.

The set of *path components* $\pi_0|X|$ of X is defined by a coequalizer

$$S(X)_1 \underset{d_1}{\overset{d_0}{\rightrightarrows}} S(X)_0 \longrightarrow \pi_0|X|$$

in the set category.

Fundamental groupoid

Suppose that the paths $\omega, \omega' : |\Delta^1| \to X$ start at x and end at y in the sense that $d_1(\omega) = d_1(\omega') = x$ and $d_0(\omega) = d_0(\omega') = y$.

Alternate notation:

$$\partial(\omega) = \partial(\omega') = (y, x).$$

Say that ω is *homotopic to* ω' *rel. end points* if there is a commutative diagram

$$\begin{array}{ccc}
(|\Delta^0| \sqcup |\Delta^0|) \times I & \xrightarrow{pr} & (|\Delta^0| \sqcup |\Delta^0|) \\
{\scriptstyle (d^1,d^0)\times I}\downarrow & & \downarrow{\scriptstyle (x,y)} \\
|\Delta^1| \times I & \xrightarrow{\quad h \quad} & X \\
{\scriptstyle (d^1,d^0)}\uparrow & \nearrow{\scriptstyle (\omega,\omega')} & \\
|\Delta^1| \sqcup |\Delta^1| & &
\end{array}$$

Here, $I = [0,1]$, or some homeomorphic copy of it, like $|\Delta^1|$.

The map h is a *homotopy* from ω to ω' (note the direction). One represents h by the following picture:

$$\begin{array}{ccc}
x & \xrightarrow{\omega} & y \\
{\scriptstyle x}\downarrow & \Downarrow & \downarrow{\scriptstyle y} \\
x & \xrightarrow[\omega']{} & y
\end{array}$$

Homotopy of paths rel end points in a space X is an equivalence relation (exercise), and the set of homotopy classes of paths rel end points from x to y is denoted by $\pi(X)(x,y)$. This is the set of morphisms from x to y in the *fundamental groupoid* $\pi(X)$ of the space X.

There's a law of composition for $\pi(X)$, but we need more notation to describe it.

Nice little spaces

1) $|\partial\Delta^n|$ is the *topological boundary* of $|\Delta^n|$: it is the union of the faces $d^i : |\Delta^{n-1}| \to |\Delta^n|$. Any two such faces intersect in a lower dimensional face $|\Delta^{n-2}|$, and there is a coequalizer picture

$$\bigsqcup_{i<j,0\leq i,j\leq n} |\Delta^{n-2}| \rightrightarrows \bigsqcup_{0\leq i\leq n} |\Delta^{n-1}| \longrightarrow |\partial\Delta^n|$$

in spaces, which is defined by the identities $d^j d^i = d^i d^{j-1}$ for $i < j$.

2) $|\Lambda^n_k| \subset |\partial\Delta^n|$ is obtained by throwing away the the k^{th} face $d^k : |\Delta^{n-1}| \to |\Delta^n|$. There is a coequalizer

$$\bigsqcup_{i<j,i,j\neq k} |\Delta^{n-2}| \rightrightarrows \bigsqcup_{0\leq i\leq n, i\neq k} |\Delta^{n-1}| \longrightarrow |\Lambda^n_k|$$

defined by the identities $d^j d^i = d^i d^{j-1}$ for $i < j$. $|\Lambda^n_k|$ is the k^{th} *horn* of $|\Delta^n|$.

The inclusion $i : |\Lambda^n_k| \subset |\Delta^n|$ is a *strong deformation retraction*.

There is a map $r : |\Delta^n| \to |\Lambda^n_k|$ (projection along the normal to the missing simplex) such that $r \cdot i = 1$ and $i \cdot r$ is homotopic to the identity on $|\Delta^n|$ rel $|\Lambda^n_k|$.

It follows that the dotted arrow exists, making the diagram commute in all solid arrow pictures

$$\begin{array}{ccc} |\Lambda^n_k| & \xrightarrow{\alpha} & X \\ {\scriptstyle i}\downarrow & \nearrow & \downarrow \\ |\Delta^n| & \longrightarrow & * \end{array}$$

The lift is given by the composite $\alpha \cdot r$, and $*$ is the one-point space (aka $|\Delta^0|$), which is terminal in **CGWH**.

Here's some other inclusions which admit strong deformation retractions (secretly made up of instances of inclusions of horns in simplices):

- $(|\Delta^n| \times \{\varepsilon\}) \cup (|\partial\Delta^n| \times I) \subset |\Delta^n| \times I$ where $\varepsilon = 0, 1$,
- $(|\Delta^n| \times \{0,1\}) \cup (|\Lambda^n_k| \times I) \subset |\Delta^n| \times I$.

$I = [0,1]$ is the unit interval, and $I \cong |\Delta^1|$.

Any space X has the right lifting property for these inclusions, as for the maps $|\Lambda^n_k| \subset |\Delta^n|$.

Composition law

A map $|\Lambda^2_1| \to X$ is a string of paths $x \xrightarrow{\omega} y \xrightarrow{\gamma} z$ in X, and there is an extension

$$\begin{array}{ccc} |\Lambda^2_1| & \xrightarrow{(\gamma,?,\omega)} & X \\ \downarrow & \nearrow_{\sigma} & \\ |\Delta^2| & & \end{array}$$

The face $d_1\sigma$ represents a well defined element $[d_1\sigma]$ of $\pi(X)(x,z)$ which is independent of the classes of ω and γ, by an argument involving an extension

$$\begin{array}{ccc} (|\Delta^2| \times \{0,1\}) \cup (|\Lambda^2_1| \times I) & \longrightarrow & X \\ \downarrow & \nearrow & \\ |\Delta^2| \times I & & \end{array}$$

[This is a prototypical "prismatic" argument. Flling in the labels on the diagram is an exercise.]

We therefore have the composition law

$$[\gamma] * [\omega] = [d_1(\sigma)]$$

defined for the fundamental groupoid $\pi(X)$.

Associativity

Suppose given a string of paths

$$x_0 \xrightarrow{\omega_1} x_1 \xrightarrow{\omega_2} x_2 \xrightarrow{\omega_3} x_3$$

in X. There is a corresponding string of paths

$$0 \xrightarrow{[0,1]} 1 \xrightarrow{[1,2]} 2 \xrightarrow{[2,3]} 3$$

which defines a subspace $P \subset |\Delta^3|$, while the string of paths in X can be represented as a map $\omega : P \to X$. The map $\omega : P \to X$ extends to a map $\sigma : |\Delta^3| \to X$, in the sense that the diagram

$$\begin{array}{ccc} P & \xrightarrow{\omega} & X \\ \downarrow & \nearrow_{\sigma} & \\ |\Delta^3| & & \end{array}$$

commutes.

[Fill in $[0,1,2]$, $[1,2,3]$, $[0,1,3]$, then $[0,1,2,3]$.]

The image of the path $[0,3]$ in $|\Delta^3|$ represents both $[\omega_3] * ([\omega_2] * [\omega_1])$ and $([\omega_3] * [\omega_2]) * [\omega_1]$, so the composition law in $\pi(X)$ is associative.

Identities

Write x for the *constant path*

$$|\Delta^1| \xrightarrow{s^0} |\Delta^0| \xrightarrow{x} X$$

at an element x of X.

Suppose that $\omega : x \to y$ is a path of X. Then

$$\partial s_0(\omega) = (\omega, \omega, x) \text{ and } \partial s_1(\omega) = (y, \omega, \omega).$$

The constant paths are 2-sided identities for the composition law.

Inverses

Again, suppose that $\omega : x \to y$ is a path of X. Then there are extensions

$$\begin{array}{ccc} |\Lambda^2_0| & \xrightarrow{(?,x,\omega)} & X \\ \downarrow & \nearrow & \\ |\Delta^2| & & \end{array} \quad \text{and} \quad \begin{array}{ccc} |\Lambda^2_2| & \xrightarrow{(\omega,y,?)} & X \\ \downarrow & \nearrow & \\ |\Delta^2| & & \end{array}$$

so that the composition law on $\pi(X)$ is invertible.

We have therefore shown that the fundamental groupoid $\pi(X)$ of a space X is a groupoid.

Fundamental groups

The *fundamental group* $\pi_1(X,x)$ of X based at the element x is the set of homomorphisms (isomorphisms) $\pi(X)(x,x)$ from x to itself in $\pi(X)$.

Explicitly, $\pi_1(X,x)$ is the group of homotopy classes of loops $x \to x$ rel end points in X, with composition law defined by extensions

$$\begin{array}{ccc} |\Lambda^2_1| & \xrightarrow{(\omega_2,?,\omega_1)} & X \\ \downarrow & \nearrow & \\ |\Delta^2| & & \end{array}$$

with identity defined by the constant path at x.

Higher homotopy groups

Suppose that x is a vertex (aka. element) of X. The members of $\pi_n(X,x)$ are homotopy classes

$$[(|\Delta^n|,|\partial\Delta^n|),(X,x)]$$

of simplices with boundary mapping to x, rel boundary. These classes are represented by diagrams

$$\begin{array}{ccc} |\partial\Delta^n| & \xrightarrow{x} & X \\ \downarrow & \nearrow_{\alpha} & \\ |\Delta^n| & & \end{array}$$

which one tends to refer to by the name of the simplex, in this case α.

Here's a cheat: one can show inductively (or by an explicit homeomorphism of pairs) that the set

$$[(|\Delta^n|,|\partial\Delta^n|),(X,x)]$$

is in bijective correspondence with the set

$$[(I^{\times n},\partial I^{\times n}),(X,*)].$$

One starts the induction by using using extensions

$$\begin{array}{cccc} & & (|\Delta^n|\times\{0,1\})\cup(|\Lambda^n_0|\times I) & \xrightarrow{((\alpha,x),x)} X \\ & & \downarrow & \nearrow \!\!\cdots \\ |\Delta^{n-1}|\times I & \xrightarrow[d^0\times I]{} & |\Delta^n|\times I & \end{array}$$

to show that there is a bijection

$$[(|\Delta^n|, |\partial\Delta^n|), (X,x)] \cong [(|\Delta^{n-1}| \times I, \partial(|\Delta^{n-1}| \times I)), (X,x)].$$

Homotopy classes of maps $(I^{\times n}, \partial I^{\times n}) \to (X,x)$ can be composed in multiple directions, potentially giving n different group structures according to the description given above (recall that $I = |\Delta^1|$).

These multiplications have a common identity, namely the constant cell at x, and they satisfy *interchange laws*

$$(a_1 *_i a_2) *_j (b_1 *_i b_2) = (a_1 *_j b_1) *_i (a_2 *_j b_2).$$

The multiplications therefore coincide and are abelian if $n \geq 2$ (exercise).

The interchange laws follow from the existence of solutions to lifting problems

$$\begin{array}{ccc} |\Lambda^2_1| \times |\Lambda^2_1| & \longrightarrow & X \\ \downarrow & \nearrow & \\ |\Delta^2| \times |\Delta^2| & & \end{array}$$

(exercise again).

Homotopy equivalences, weak equivalences

The construction of $\pi_0(X)$, $\pi(X)$ and all $\pi_n(X,x)$ are functorial: every map $f : X \to Y$ induces

- $f_* : \pi_0(X) \to \pi_0(Y)$ (function between sets),
- $f_* : \pi(X) \to \pi(Y)$ (functor between groupoids),
- $f_* : \pi_n(X,x) \to \pi_n(Y, f(x))$, $n \geq 1$, $x \in X$ (group homomorphisms).

A) A map $f : X \to Y$ is said to be a *homotopy equivalence* if there is a map $g : Y \to X$ such that $g \cdot f \simeq 1_X$ (homotopic to the identity on X) and $f \cdot g \simeq 1_Y$.

B) $f : X \to Y$ is a *weak equivalence* if

1) $f_* : \pi_0(X) \to \pi_0(Y)$ is a bijection, and
2) $f_* : \pi_n(X,x) \to \pi_n(Y, f(x))$ is an isomorphism for all $n \geq 1$ and all $x \in X$.

Exercises

1) Show that every homotopy equivalence is a weak equivalence.
2) Show that every weak equivalence $f : X \to Y$ induces an equivalence of groupoids $f_* : \pi(X) \to \pi(Y)$.

Group objects

Write X_0 for the set of points underlying X (the vertices of $S(X)$). Every base point $x \in X_0$ has an associated homotopy group $\pi_n(X,x)$, and we can collect all such homotopy groups together to define a function

$$\pi_n(X) = \bigsqcup_{x \in X_0} \pi_n(X,x) \to \bigsqcup_{x \in X_0} * = X_0.$$

The function $\pi_n(X) \to X_0$ defines a group object over the set X_0 for $n \geq 1$ which is abelian if $n \geq 2$.

Fact: A map $f : X \to Y$ is a weak equivalence if and only if

1) the function $f_* : \pi_0(X) \to \pi_0(Y)$ is a bijection, and

2) the induced diagrams

$$\begin{array}{ccc} \pi_n(X) & \xrightarrow{f_*} & \pi_n(Y) \\ \downarrow & & \downarrow \\ X_0 & \xrightarrow[f_*]{} & Y_0 \end{array}$$

are pullbacks (in **Set**) for $n \geq 1$.

5 Serre fibrations and the model structure for CGWH

A map $p : X \to Y$ is said to be a *Serre fibration* if it has the RLP wrt all $|\Lambda^n_k| \subset |\Delta^n|$, $n \geq 1$.

All spaces X are fibrant: the map $X \to *$ is a Serre fibration.

Main formal properties of Serre fibrations:

Lemma 5.1. *A map $p : X \to Y$ is a Serre fibration and a weak equivalence if and only if it has the right lifting property with respect to all inclusions $|\partial\Delta^n| \subset |\Delta^n|$, $n \geq 0$.*

Here, $|\partial\Delta^0| = \emptyset$.

Lemma 5.2. *Suppose that $p : X \to Y$ is a Serre fibration, and that $F = p^{-1}(y)$ is the fibre over an element $y \in Y$. Then we have the following:*

1) For each $x \in F$ there is a sequence of pointed sets

$$\dots \pi_n(F,x) \xrightarrow{i_*} \pi_n(X,x) \xrightarrow{p_*} \pi_n(Y,y) \xrightarrow{\partial} \pi_{n-1}(F,x) \to \dots$$
$$\dots \pi_1(Y,y) \xrightarrow{\partial} \pi_0(F) \xrightarrow{i_*} \pi_0(X) \xrightarrow{p_*} \pi_0(Y)$$

which is exact in the sense that $\ker = \operatorname{im}$ *everywhere.*

2) There is a group action

$$* : \pi_1(Y,y) \times \pi_0(F) \to \pi_0(F)$$

such that $\partial([\alpha]) = [\alpha] * [x]$, and such that $i_*[z] = i_*[w]$ if and only if there is an element $[\beta] \in \pi_1(Y,y)$ such that $[\beta] * [z] = [w]$.

The boundary map

$$\partial : \pi_n(Y, p(x)) \to \pi_{n-1}(F, x)$$

is defined by $\partial([\alpha]) = [d_0\theta]$, where θ is a choice of lifting in the following diagram

$$\begin{array}{ccc} |\Lambda^n_0| & \xrightarrow{x} & X \\ \downarrow & \overset{\theta}{\nearrow} & \downarrow p \\ |\Delta^n| & \xrightarrow[\alpha]{} & Y \end{array}$$

Lemma 5.1 is needed for the following result, while Lemma 5.2 is needed for almost all calculations of homotopy groups.

The proof of Lemma 5.1 is sketched below, and the proof of Lemma 5.2 is an exercise.

A map $i : A \to B$ is said to be a *cofibration* if it has the LLP wrt all trivial Serre fibrations.

Lemma 5.1 implies that all inclusions $|\partial\Delta^n| \subset |\Delta^n|$ are cofibrations. All CW-complexes (spaces built inductively by attaching cells) are *cofibrant*.

Theorem 5.3. *The weak equivalences, Serre fibrations and cofibrations as defined above give* **CGWH** *the structure of a closed model category.*

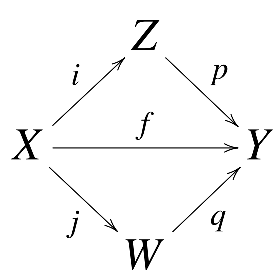
Proof. $p : X \to Y$ is a Serre fibration if and only if it has the RLP wrt all $|\Lambda^n_k| \subset |\Delta^n|$.

p is a trivial Serre fibration if and only if it has the RLP wrt all $|\partial\Delta^n| \subset |\Delta^n|$ by Lemma 5.1.

All inclusions $|\Lambda^n_k| \subset |\Delta^n|$ are strong deformation retractions, as are all of their pushouts.

Pushouts of monomorphisms are monomorphisms.

A small object argument (which depends on an observation of J.H.C. Whitehead that a compact subset of a CW-complex meets only finitely many cells) shows that every continuous map $f : X \to Y$ has factorizations



such that i is a trivial cofibration which has the LLP wrt all fibrations and p is a Serre fibration, and j is a cofibration and a monomorphism and q is a trivial Serre fibration. This gives **CM5**.

Suppose that $j : A \to B$ is a trivial cofibration. j has a factorization

$$\begin{array}{ccc} A & \xrightarrow{i} & C \\ & {\scriptstyle j}\searrow & \downarrow{\scriptstyle p} \\ & & B \end{array}$$

such that i is a trivial cofibration which has the LLP wrt all Serre fibrations, and p is a Serre fibration. Then p is a trivial Serre fibration, so the lift exists in the diagram

$$\begin{array}{ccc} A & \xrightarrow{i} & C \\ {\scriptstyle j}\downarrow & \overset{\alpha}{\nearrow} & \downarrow{\scriptstyle p} \\ B & \xrightarrow[1]{} & B \end{array}$$

Then j is a retract of i, so j has the LLP wrt all Serre fibrations.

For **CM4**, suppose given a diagram (lifting problem)

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle i}\downarrow & \nearrow & \downarrow{\scriptstyle p} \\ B & \longrightarrow & Y \end{array}$$

where i is a cofibration and p is a Serre fibration. The lift exists if p is trivial (definition of cofibration), and we just showed that every trivial cofibration has the LLP wrt all Serre fibrations.

The other model axioms are exercises. □

We need the following for the proof of Lemma 5.1:

Lemma 5.4. *A map $\alpha : (\Delta^n, \partial\Delta^n) \to (X, x)$ represents the identity element of $\pi_n(X, x)$ if and only if the lifting problem*

$$\begin{array}{ccc} |\partial\Delta^{n+1}| & \xrightarrow{(\alpha,x,\ldots,x)} & X \\ \downarrow & \nearrow & \\ |\Delta^{n+1}| & & \end{array}$$

can be solved.

Proof. Exercise. □

Proof of Lemma 5.1. 1) Suppose $p : X \to Y$ is a trivial Serre fibration, and suppose given a lifting problem

$$\begin{array}{ccc} |\partial\Delta^n| & \xrightarrow{\alpha} & X \\ \downarrow & \nearrow & \downarrow p \\ |\Delta^n| & \xrightarrow[\beta]{} & Y \end{array}$$

Suppose $x = \alpha(0)$. There is a homotopy of diagrams

$$\begin{array}{ccc} |\partial\Delta^n| \times I & \longrightarrow & X \\ \downarrow & & \downarrow \\ |\Delta^n| \times I & \longrightarrow & Y \end{array}$$

from the original diagram to one of the form

$$\begin{array}{ccc} |\partial\Delta^n| & \xrightarrow{(\alpha_0,x,\ldots,x)} & X \\ \downarrow & \nearrow & \downarrow{\scriptstyle p} \\ |\Delta^n| & \xrightarrow[\beta']{} & Y \end{array}$$

so the two lifting problems are equivalent.

$p_*([\alpha_0]) = 0$ so $[\alpha_0] = 0 \in \pi_{n-1}(X,x)$, and it follows from a second homotopy of diagrams that the original lifting problem is equivalent to one of the form

$$\begin{array}{ccc} |\partial\Delta^n| & \xrightarrow{x} & X \\ \downarrow & \nearrow & \downarrow{\scriptstyle p} \\ |\Delta^n| & \xrightarrow[\beta'']{} & Y \end{array}$$

Since $p_* : \pi_n(X,x) \to \pi_n(Y, p(x))$ is surjective, β'' lifts up to homotopy rel $|\partial\Delta^n|$ to a simplex of X, so that this last diagram is homotopic to a diagram for which the lifting problem is solved.

2) Suppose $p : X \to Y$ has the RLP wrt all $|\partial\Delta^n| \subset |\Delta^n|$.

Then p has the right lifting property with respect to all $|\Lambda^n_k| \subset |\Delta^n|$ (exercise), so p is a Serre fibration.

Suppose that $[\alpha] \in \pi_n(X,x)$ such that $p_*([\alpha]) = 0$. Then there is a commutative diagram

$$\begin{array}{ccc} |\partial\Delta^{n+1}| & \xrightarrow{(\alpha,x,\ldots,x)} & X \\ \downarrow & \nearrow & \downarrow p \\ |\Delta^{n+1}| & \xrightarrow[\beta]{} & Y \end{array}$$

and the existence of the indicated lift implies that $[\alpha] = 0 \in \pi_n(X,x)$. Thus, p_* is a monomorphism.

The existence of liftings

$$\begin{array}{ccc} |\partial\Delta^{n}| & \xrightarrow{x} & X \\ \downarrow & \overset{\theta}{\nearrow} & \downarrow p \\ |\Delta^{n}| & \xrightarrow[\beta]{} & Y \end{array}$$

means that p_* is surjective: $p_*([\theta]) = [\beta]$. $\square$