Fields Lectures: Simplicial presheaves

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January 31, 2007

This is a cleaned up and expanded version of the lecture notes for a short course that I gave at the Fields Institute in late January, 2007. I expect the cleanup and expansion processes to continue for a while yet, so the interested reader should check the web site `http://www.math.uwo.ca/~jardine` periodically for updates.

Contents

*This research was supported by NSERC.

1 Simplicial presheaves and sheaves

In all that follows, $\mathcal{C}$ will be a small Grothendieck site.

Examples include the site $op|_X$ of open subsets and open covers of a topological space X, the site $Zar|_S$ of Zariski open subschemes and open covers of a scheme S, or the étale site $et|_S$, again of a scheme S.

All of these sites have "big" analogues, like the big sites $\mathbf{Top}_{op}$, $(Sch|_S)_{Zar}$ and $(Sch|_S)_{et}$ of "all" topological spaces with open covers, and "all" S-schemes $T \to S$ with Zariski and étale covers, respectivley. Of course, there are many more examples.

Warning: All of the "big" sites in question have (infinite) cardinality bounds on the objects which define them so that the sites are small, and we don't talk about these bounds. The idea, ultimately, is that these bounds don't matter homotopy theoretically.

A simplicial presheaf on $\mathcal{C}$ is a contravariant functor $X : \mathcal{C}^{op} \to s\mathbf{Set}$ taking values in simplicial sets. One can alternatively think of it as a simplicial object in the category of presheaves. Morphisms of simplicial presheaves are natural transformations of functors, and $s\,\mathrm{Pre}(\mathcal{C})$ denotes the category of simplicial presheaves.

A simplicial sheaf is a simplicial object in the category of sheaves on $\mathcal{C}$ and the corresponding category is denoted by $s\,\mathrm{Shv}(\mathcal{C})$. Every simplicial sheaf is a simplicial presheaf, and the inclusion functor

$$s\,\mathrm{Shv}(\mathcal{C}) \subset s\,\mathrm{Pre}(\mathcal{C})$$

has a left adjoint

$$L^2 : s\,\mathrm{Pre}(\mathcal{C}) \to s\,\mathrm{Shv}(\mathcal{C})$$

which is defined by putting in the appropriate limits over covering sieves twice. I shall occasionally lapse and write $\tilde{X} = L^2X$ for the simplicial sheaf associated to a simplicial presheaf X.

Some examples:

1) Every simplicial set X determines a constant simplicial presheaf $X = \Gamma^*X$, which is defined by

$$\Gamma^*X(U) = X$$

for $U \in \mathcal{C}$, with morphisms $U \to V$ in $\mathcal{C}$ inducing identity maps on X. The functor $X \mapsto \Gamma^*X$ is left adjoint to the inverse limit functor (aka. global sections).

2) Every object $U \in \mathcal{C}$ represents a presheaf (or sheaf for subcanonical topologies)

$$U(V) = \hom(V, U)$$

Thus, every simplicial object of $\mathcal{C}$ represents a simplicial presheaf on $\mathcal{C}$. In particular, simplicial schemes represent simplicial presheaves (commonly simplicial sheaves) on the algebraic geometric sites.

3) If A is a presheaf of groups (or categories), applying the nerve construction sectionwise gives a simplicial presheaf BA.

a) Any algebraic group G (like Gl_n) represents a sheaf of groups on the geometric sites:

$$G(U) = \hom(U, G).$$

We are therefore entitled, for example, to the simplicial sheaves BGl_n on all such sites. The object

$$BGl = \varinjlim_n BGl_n,$$

on the other hand, is best thought of as a simplicial presheaf. K-theorists like this gadget.

b) Generally, if a presheaf of groups G acts on a presheaf X, then this action consists of group actions

$$G(U) \times X(U) \to X(U)$$

in sections which are natural for morphisms of $\mathcal{C}$. Since $G(U)$ is a group, there is a translation groupoid $E_{G(U)}X(U)$ with objects $x \in X(U)$ and morphisms $g : x \to gx$ for each $x \in X(U)$ and $g \in G(U)$. We therefore obtain a presheaf of categories $E_G X$ whose associated nerve is quite special:

$$B(E_G X) \cong EG \times_G X,$$

otherwise known as the Borel construction. We'll see later that this construction is the source of all quotient stacks.

c) Suppose that $p : Y \to X$ is a local epimorphism of presheaves (which means that every section of X lifts to Y after refinement along some covering sieve — if Y and X are sheaves this is a sheaf epimorphism). The nerve $BG(p)$ of the presheaf of groupoids $G(p)$ is the Čech resolution for this cover.

d) Suppose that L/k is a finite Galois extension of fields with Galois group G. The corresponding map of schemes $p : \mathrm{Sp}(L) \to \mathrm{Sp}(k)$ is a covering for the étale topology, and so we are entitled to the Čech resolution $BG(p)$ in this case. By Galois theory, there is an isomorphism of rings

$$L \otimes_k L \cong \prod_{g \in G} L.$$

The object $BG(p)$ is represented by a simplicial affine scheme, with n-simplices

$$\mathrm{Sp}(L \otimes_k \cdots \otimes_k L) \quad (n\text{-fold tensor power}).$$

It follows (after some fiddling) that there is an isomorphism of simplicial schemes

$$BG(p) \cong EG \times_G \mathrm{Sp}(L)$$

4) Suppose that A is a presheaf of abelian groups and let $A[-n]$ be the presheaf of chain complexes consisting of A concentrated in degree n. The Dold-Kan correspondence is an adjoint equivalence

$$N : s\mathbf{Ab} \leftrightarrows \mathbf{Ch}_+ : \Gamma$$

determined by the normalized chains functor N. Here, $s\mathbf{Ab}$ is simplicial abelian groups and $\mathbf{Ch}_+$ is ordinary chain complexes. The Eilenberg-Mac Lane object $K(A, n)$ is defined by

$$K(A, n) = \Gamma(A[-n])$$

where we apply Γ to the presheaf of chain complexes $A[-n]$ in each section.

2 Local weak equivalences

Simplicial sets have homotopy groups. If X is a simplicial set and x is a vertex of X, then

$$\pi_n(X, x) = \pi_n(|X|, x).$$

is the corresponding homotopy group of the realization $|X|$. The set of path components $\pi_0(X) = \pi_0(|X|)$ has a combinatorial description: $\pi_0(X)$ is defined by the coequalizer

$$X_1 \underset{d_1}{\overset{d_0}{\rightrightarrows}} X_0 \longrightarrow \pi_0(X).$$

A simplicial set map $f : X \to Y$ is a weak equivalence if and only if

1) the function $\pi_0 X \to \pi_0 Y$ is a bijection, and
2) the maps $\pi_n(X, x) \to \pi_n(Y, f(x))$ are isomorphisms of groups for $n \geq 1$ and all $x \in X_0$.

There's a "base point free" way to organize this: write

$$\pi_n(X) = \bigsqcup_{x \in X_0} \pi_n(X, x),$$

and observe that there is a canonical map $\pi_n(X) \to X_0$ which is a group object over the set X_0. This group object is abelian if $n \geq 2$. Any simplicial set map $f : X \to Y$ induces a commutative diagram

$$\begin{array}{ccc} \pi_n X & \longrightarrow & \pi_n Y \\ \downarrow & & \downarrow \\ X_0 & \longrightarrow & Y_0 \end{array} \tag{1}$$

Then the map f is a weak equivalence if

1) the function $\pi_0 X \to \pi_0 Y$ is a bijection, and

2) the diagram (1) is a pullback if $n \geq 1$.

This is all perfectly functorial, so we're entitled to a presheaf $\pi_0 X$ and presheaf maps $\pi_n X \to X_0$ for all $n \geq 1$, and any simplicial presheaf map induces presheaf morphisms $\pi_0 X \to \pi_0 Y$ and diagrams of presheaves of the form (1). Write $\tilde{\pi}_n X$ for the sheaf associated to the presheaf $\pi_n X$. Now here's the local definition of weak equivalence:

A map $f : X \to Y$ of simplicial presheaves is a *local weak equivalence* if and only if

1) the map $\tilde{\pi}_0 X \to \tilde{\pi}_0 Y$ is an isomorphism of sheaves, and

2) the diagram of sheaf morphisms

$$\begin{array}{ccc} \tilde{\pi}_n X & \longrightarrow & \tilde{\pi}_n Y \\ \downarrow & & \downarrow \\ \tilde{X}_0 & \longrightarrow & \tilde{Y}_0 \end{array}$$

is a pullback for all $n \geq 1$.

Example: Every sectionwise weak equivalence is a local weak equivalence, since the two conditions are satisfied at the presheaf level, and hence on the sheaf level.

Every $U \in \mathcal{C}$ determines a site $\mathcal{C}/U$ whose all objects are all morphisms $V \to U$, and such that a family

$$V_i \to V \to U$$

is covering if and only if the family $V_i \to V$ covers V. Precomposition with the canonical functor $\mathcal{C}/U \to \mathcal{C}$ determines a restricted simplicial presheaf $X|_U$ on $\mathcal{C}/U$ for every simplicial presheaf X on $\mathcal{C}$, and of course this construction is functorial. A vertex $x \in X(U)$ determines a global section on $X|_U$ (in degree 0), and every simplicial presheaf map $X \to Y$ determines a presheaf map

$$\pi_n(X|_U, x) \to \pi_n(Y|_U, f(x))$$

for all choices of n and x. Then the pullback condition for the definition of a local weak equivalence $f : X \to Y$ is equivalent to the requirement that all induced sheaf maps

$$\tilde{\pi}_n(X|_U, x) \to \tilde{\pi}_n(Y|_U, f(x)) \tag{2}$$

are isomorphisms for all $x \in X_0(U)$, all $U \in \mathcal{C}$ and all $n \geq 0$. Thus, somewhat perjoratively, a map is a local weak equivalence if and only if it induces an isomorphism in all possible sheaves of homotopy groups (at all local choices of base points).

Example: The adjunction map $\eta : X \to \tilde{X}$ is a local weak equivalence. This is because it has the local right lifting property with respect to all inclusions $\partial\Delta^n \to \Delta^n$, $n \geq 0$ since it induces an isomorphism of associated sheaves.

To say that a map $p : Z \to W$ has the *local right lifting property* with respect to $\partial\Delta^n \subset \Delta^n$ means that all lifting problems

$$\begin{array}{ccc} \partial\Delta^n & \longrightarrow & Z(U) \\ \downarrow & & \downarrow \\ \Delta^n & \longrightarrow & W(U) \end{array}$$

can be solved after refinement along a covering of U. Equivalently, this means that all presheaf maps

$$Z_n \to \operatorname{cosk}_{n-1} Z_n \times_{\operatorname{cosk}_{n-1} W_n} W_n$$

are local epimorphisms (ie. sections lift after refinement along covers).

If you look at the old Artin-Mazur book [1], you will recognize this as their definition of hypercover, at least for the étale topology where $p : Z \to W$ is a morphism of simplicial schemes. It is common, now, to say that a map of simplicial presheaves which has this local right lifting property is a hypercover.

3 First local model structure

We're going to need a standard construction from ordinary simplicial homotopy theory. Suppose that $K \subset \Delta^n$ is a (polyhedral) subcomplex, and write $P(K)$ for the poset of non-degenerate simplices of K. Then the nerve $BP(K)$ (sometimes called the "order complex") is what one calls the subdivision $\operatorname{sd} K$, because its realization is the barycentric subdivision of $|K|$. There is a map $\operatorname{sd} K \to K$, defined by taking last vertices of ascending chains of simplices. This map is natural in ordinal number maps $\Delta^m \to \Delta^m$. It follows that a simplicial set X has an associated simplicial set $\operatorname{Ex} X$ defined by

$$\operatorname{Ex} X_n = \hom(\operatorname{sd} \Delta^n, X),$$

and a natural map $X \to \operatorname{Ex} X$. The object $\operatorname{Ex}^\infty X$ is the colimit of the string

$$X \to \operatorname{Ex} X \to \operatorname{Ex}^2 X \to \dots$$

The salient features of the object $\operatorname{Ex}^\infty X$ are the following:

1) $\operatorname{Ex}^\infty X$ is a Kan complex.

2) the map $X \to \operatorname{Ex}^\infty X$ is a natural weak equivalence.

Statement 1) is best proved with simplicial approximation arguments, while Statement 2) essentially follows from a properness argument [19].

The basic point of the construction, for us, is that the simplices of $\operatorname{Ex} X$ are finite limits of simplices of X, and such finite limits are preserved by the associated sheaf functor and inverse image morphisms.

Lemma 1. *All hypercovers are local weak equivalences.*

In effect, if $p : Z \to W$ is a hypercover then so is the induced map $p_* : \operatorname{Ex}^\infty Z \to \operatorname{Ex}^\infty W$ of presheaves of Kan complexes. But then the (local) presheaves of homotopy groups for presheaves of Kan complexes can be defined combinatorially using simplicial homotopy groups, and then one shows that p_* is a local weak equivalence by playing with homotopy groups. Finally there is a canonical diagram

$$\begin{array}{ccc} Z & \xrightarrow{\eta} & \operatorname{Ex}^\infty Z \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p_*} \\ W & \xrightarrow[\eta]{} & \operatorname{Ex}^\infty W \end{array}$$

in which the maps η are sectionwise weak equivalences.

Example: Suppose that $p : X \to Y$ is a local epimorphism of presheaves. The path component functor induces a simplicial presheaf map $BG(p) \to Y$, which is a hypercover.

To see this, let Z be the presheaf image of p in Y. Then the map $BG(p) \to Z$ is a trivial Kan fibration in each section, and the map $Z \subset Y$ is a local epi as well as a monomorphism, and is therefore a hypercover.

Now here's the original model structure for simplicial presheaves on a small site $\mathcal{C}$: the cofibrations are the monomorphisms, the weak equivalences are the local fibrations, and the fibrations are what they are, in other words determined by a lifting property with respect to trivial cofibrations. I am in the habit of saying that these fibrations are global fibrations, following an ancient paper of Brown and Gersten [5], where the concept was first introduced for the Zariski topology.

Theorem 2. *With these definitions, the category* $s\operatorname{Pre}(\mathcal{C})$ *of simplicial presheaves on a small Grothendieck site* $\mathcal{C}$ *has the structure of a proper closed simplicial model category.*

The tensor product $X \otimes K$ of a simplicial presheaf X with a simplicial set K is specified in sections by

$$X \otimes K(U) = X(U) \times K$$

Alternatively, $X \otimes K = X \times K$, is just the product of X and the constant simplicial presheaf associated to K.

The function complex $\mathbf{hom}(X, Y)$ is the simplicial set whose n-simplices are all simplicial presheaf maps $X \times \Delta^n \to Y$.

There's essentially one interesting step in the proof of the Theorem, namely that we must show that every map $f : X \to Y$ has a factorization

$$\begin{array}{ccc} X & \xrightarrow{i} & W \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle p} \\ & & Y \end{array}$$

where p is a global fibration and i is a trivial cofibration. This involves two bits of technology:

1) **Boolean localization** (Barr, Diaconescu, et. al. — see [33])

For every small site $\mathcal{C}$ there is a geometric morphism

$$p : \mathrm{Shv}(\mathcal{B}) \to \mathrm{Shv}(\mathcal{C})$$

where $\mathcal{B}$ is a complete Boolean algebra, and such that the inverse image functor

$$p^* : \mathrm{Shv}(\mathcal{C}) \to \mathrm{Shv}(\mathcal{B})$$

is faithful.

You are to think of $\mathrm{Shv}(\mathcal{B})$ as a big fat point, because it satisfies the Axiom of Choice: every sheaf epimorphism has a section in $\mathrm{Shv}(\mathcal{B})$. This means that simplicial sheaves which are locally fibrant are actually Kan complexes in each section, and a map $f : X \to Y$ of locally fibrant simplicial sheaves is a local weak equivalence if and only if it is a sectionwise weak equivalence.

The inverse image morphism p^* preserves local weak equivalences of simplicial sheaves (as does every inverse image morphism). In effect, p^* preserves the sheaf-theoretic Ex^∞ construction so we can assume that the simplicial sheaves in question are locally fibrant. Then every map $f : X \to Y$ of locally fibrant simplicial sheaves has a (the) canonical factorization

$$\begin{array}{ccc} X & \xrightarrow{j} & X \times_Y Y^I \\ & \searrow^{f} & \downarrow{\scriptstyle \pi} \\ & & Y \end{array}$$

where π is a local Kan fibration and j is a section of a hypercover. Thus f is a local weak equivalence if and only if π is a hypercover, and hypercovers are preserved by inverse image functors.

Sheaf epimorphisms are reflected by p^* since p^* is faithful. Thus $f : X \to Y$ is a local weak equivalence if and only if $p^*\tilde{f} : p^*\tilde{X} \to p^*\tilde{Y}$ is a local weak equivalence.

Exercise: Suppose given a pushout diagram

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i_*} \\ B & \longrightarrow & B \cup_A X \end{array}$$

of simplicial sheaves on a complete Boolean algebra $\mathcal{B}$ such that i is a cofibration and a local weak equivalence. Show that i_* is a local weak equivalence.

The exercise is the central step in showing that trivial cofibrations of simplicial presheaves are closed under pushout.

2) **The bounded cofibration condition**

Suppose α is an infinite cardinal which is larger than the cardinality of the set $\mathrm{Mor}(\mathcal{C})$ of morphisms of the site $\mathcal{C}$. Say that a simplicial presheaf B is α-bounded if all sets $B_n(U)$ of all sections in all degrees satisfy $|B_n(U)| < \alpha$.

Lemma 3. *Suppose that α is an infinite cardinal which is an upper bound for the cardinality of the set of morphisms of $\mathcal{C}$. Suppose given cofibrations*

$$\begin{array}{ccc} & & X \\ & & \downarrow{\scriptstyle i} \\ B & \longrightarrow & Y \end{array}$$

that a simplicial presheaf B is α-bounded and i is a local equivalence. Then there is an α-bounded subobject $C \subset Y$ such that $B \subset C$ and $C \cap X \to C$ is a trivial cofibration.

I'm not going to prove this. You do so by playing around with relative homotopy groups in a sufficiently clever way. Lemma 3 is a familiar statement — you find it all over localization theory.

It follows that a map $p : X \to Y$ is a global fibration if and only if it has the right lifting property with respect to a (the) set of all α-bounded trivial cofibrations. Since we know that trivial cofibrations are closed under pushout by the Boolean localization trick, a transfinite small object argument of sufficient size proves the desired factorization result.

Corollary 4. *The model structure on $sPre(\mathcal{C})$ is cofibrantly generated.*

The α-bounded trivial cofibrations generate all trivial cofibrations. The class of cofibrations is generated by the "set" of α-bounded cofibrations.

Say that a map of simplicial sheaves on $\mathcal{C}$ is a local weak equivalence if the underlying map of simplicial presheaves is a local weak equivalence. A cofibration of simplicial sheaves is a monomorphism, and a global fibration is a map of simplicial sheaves which has the right lifting property with respect to all trivial cofibrations.

Theorem 5. *1) (Joyal) With these definitions, the category $s\,\mathrm{Shv}(\mathcal{C})$ of simplicial presheaves on a small Grothendieck site $\mathcal{C}$ has the structure of a proper closed simplicial model category. This model category is cofibrantly generated.*

2) The inclusion $i : s\,\mathrm{Shv}(\mathcal{C}) \subset s\,\mathrm{Pre}(\mathcal{C})$ and the associated sheaf functor $L^2 : s\,\mathrm{Pre}(\mathcal{C}) \to s\,\mathrm{Shv}(\mathcal{C})$ together define a Quillen equivalence between the respective model structures. In particular, these functors induce an adjoint equivalence

$$i : \mathrm{Ho}(s\,\mathrm{Shv}(\mathcal{C})) \leftrightarrows \mathrm{Ho}(s\,\mathrm{Pre}(\mathcal{C})) : L^2$$

between the associated homotopy categories.

Proof. Exercise. Use the fact that the associated sheaf functor preserves monomorphisms (aka. cofibrations) and local weak equivalences, and do a transfinite induction based on the factorization axioms for simplicial presheaves to prove the factorization axioms for simplicial sheaves. □

Little Facts:

1) Suppose that $p : X \to Y$ is a global fibration. Then all maps $p : X(U) \to Y(U)$ in sections are Kan fibrations.

Proof. p has the right lifting property with respect to all inclusions $\Lambda^n_k \times U \to \Delta^n \times U$. Local weak equivalences are closed under finite products by a Boolean localization argument, since sheafification preserves finite products. □

Remark: The converse is wildly false.

2) Every global fibration of simplicial sheaves is a global fibration of simplicial presheaves, since the associated sheaf functor preserves trivial cofibrations.

3) Every map of sheaves (thought of as discrete simplicial sheaves) is a global fibration.

4) Suppose that $f : X \to Y$ is a local weak equivalence of globally fibrant objects. Then f is a sectionwise weak equivalence, meaning that all maps $X(U) \to Y(U)$ are weak equivalences of simplicial sets.

X and Y are cofibrant and fibrant, so that $f : X \to Y$ is a homotopy equivalence. $Z \times \Delta^1$ is a cylinder object for any simplicial presheaf Z, so that f is a simplicial homotopy equivalence. It follows that $f : X(U) \to Y(U)$ is a simplicial homotopy equivalence for all $U \in \mathcal{C}$.

Say that a map $p : X \to Y$ is a *local fibration* if it has the local right lifting property with respect all $\Lambda^n_k \to \Delta^n$. Such a map p therefore has the local right lifting property with respect to all finite anodyne extensions. Every global fibration is a local fibration, but not conversely. Every presheaf of Kan complexes is locally fibrant.

5) Suppose that the functor $u : \mathcal{C} \to \mathcal{D}$ is a site morphism. This means that the functor

$$u_* : \mathrm{Pre}(\mathcal{D}) \to \mathrm{Pre}(\mathcal{C})$$

preserves sheaves, and that the left adjoint

$$u^* : \mathrm{Pre}(\mathcal{C}) \to \mathrm{Pre}(\mathcal{D})$$

(defined by left Kan extension) is left exact in the sense that it preserves finite limits. Then u^* preserves monics, local epimorphisms, and the Ex^∞ construction, and hence preserves local fibrations and local trivial fibrations.

It follows that u^* preserves local weak equivalences. In effect, if $f : X \to Y$ is a map between locally fibrant simplicial presheaves, form the diagram

$$\begin{array}{ccccc} X \times_Y \mathbf{hom}(\Delta^1, Y) & \xrightarrow{f_*} & \mathbf{hom}(\Delta^1, Y) & \xrightarrow{d_1} & Y \\ {\scriptstyle d_{0*}} \downarrow & & \downarrow {\scriptstyle d_0} & & \\ X & \xrightarrow[f]{} & Y & & \end{array}$$

The map d_0 is a trivial local fibration, as is d_1. The maps d_1 and d_0 have a common section

$$s : Y \to \mathbf{hom}(\Delta^1, Y),$$

so that the trivial local fibration d_{0*} has a section s_*. The composite map $\pi = d_{1*}f_*$ is a local fibration (see the next paragraph), and $\pi s_* = f$. Thus the local weak equivalence f has a factorization $f = \pi s_*$ where π is a trivial local fibration and s is a section of a trivial local fibration.

To see that π is a local fibration, look at the diagram

$$\begin{array}{ccc} X \times_Y Y^I & \xrightarrow{f_*} & Y^I \\ {\scriptstyle (d_{0*}, d_1 f_*)} \downarrow & & \downarrow {\scriptstyle (d_0, d_1)} \\ X \times Y & \xrightarrow[f \times 1]{} & Y \times Y \\ {\scriptstyle pr_L} \downarrow & & \downarrow {\scriptstyle pr_L} \\ X & \xrightarrow[f]{} & Y \end{array}$$

The outer and bottom squares are pullbacks so the top square is a pullback. Thus $(d_{0*}, d_1 f_*)$ is a local fibration, and X is locally fibrant so the projection $X \times Y \to Y$ is a local fibration. $d_1 f_*$ is then a composite of local fibrations.

This means that u induces a Quillen adjunction

$$u^* : s\,\mathrm{Pre}(\mathcal{C}) \leftrightarrows s\,\mathrm{Pre}(\mathcal{D}) : u_*.$$

In particular, u_* preserves global fibrations. u_* *does not* preserve local weak equivalences in general, but it has a right derived functor:

$$Ru_* X = u_* GX$$

where $j : X \to GX$ is a globally fibrant model of X, meaning that j is a local weak equivalence and GX is globally fibrant. It's nice to know that the choice can be made functorially (because of the cofibrant generation of the model structure), and the homotopy type (even sectionwise) is independent of the choice of globally fibrant model. The homotopy groups $\pi_* Ru_* X$ are the higher right derived functors of the homotopy groups of X.

4 Other model structures

Simplicial presheaves on a site $\mathcal{C}$ are just diagrams $\mathcal{C}^{op} \to s\mathbf{Set}$. There is a model structure due to Bousfield and Kan on the ambient category of $\mathcal{C}^{op}$-diagrams in simplicial sets which ignores all topologies on $\mathcal{C}$. Explicitly, a natural transformation $X \to Y$ of $\mathcal{C}^{op}$-diagrams is a fibration (respectively weak equivalence) for this model structure if and only if all maps $X(U) \to Y(U)$ in sections are Kan fibrations (respectively cofibrations) of simplicial sets. Cofibrations are defined by a left lifting property with respect to trivial fibrations. This is now called the projective structure for $\mathcal{C}^{op}$-diagrams, and one decorates the cofibrations, fibrations and weak equivalences for this theory with the adjective "projective".

Explicitly, a map $p : X \to Y$ of $\mathcal{C}^{op}$-diagrams is a projective fibration if and only if it has the right lifting property with respect to all morphisms

$$\Lambda^n_k \times U \to \Delta^n \times U, \tag{3}$$

and $p : X \to Y$ is a projective trivial fibration if and only if it has the right lifting property with respect to all maps

$$\partial\Delta^n \times U \to \Delta^n \times U. \tag{4}$$

It is then an easy exercise to show that the model axioms are satisfied, and to observe that the resulting projective model structure is cofibrantly generated, by the two lists of cofibrations displayed above.

Now switch back to the simplicial presheaf world, and write S_0 for the set of cofibrations displayed in (4). Choose a set S of (α-bounded) cofibrations containing S_0, and let $\mathbf{C}_S$ denote the saturation of the set of all cofibrations

$$(B \times \partial\Delta^n) \cup (A \times \Delta^n) \subset B \times \Delta^n$$

which are induced by cofibrations $A \to B$ appearing in the set S. $\mathbf{C}_S$ is called the set of S-cofibrations.

Say that a map $p : Z \to W$ is an S-fibration if it has the right lifting property with respect to all S-cofibrations which are local weak equivalences.

Theorem 6. *The category $s\,\mathrm{Pre}(\mathcal{C})$, together with the S-cofibrations, local weak equivalences and the S-fibrations, satisfies the axioms for a proper closed simplicial model category.*

Proof. Every map $f : X \to Y$ has a factorization

$$\begin{array}{ccc} X & \xrightarrow{j} & W \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle p} \\ & & Y \end{array}$$

where p is a global fibration and j is a cofibration and a local weak equivalence. The map p is also an S-fibration. The map j has a factorization

$$\begin{array}{ccc} X & \xrightarrow{i} & V \\ & \searrow^{j} & \downarrow{\scriptstyle \pi} \\ & & W \end{array}$$

where i is an S-cofibration and π has the right lifting property with respect to all S-cofibrations (this is the other part of the factorization axiom, which is easy to verify). The map π therefore has the right lifting property with respect to all projective cofibrations, and is therefore a sectionwise weak equivalence. The composite $p \cdot \pi$ is an S-fibration, and the factorization axiom is proved. $\square$

Fact: For extra credit, prove that this model structure is cofibrantly generated. This involves a lovely trick (some would call it a solution set condition verification) that we don't have time to discuss here. This appears in [22], and is explained in some detail in the online notes [27].

Remark: The case $S = S_0$ gives the local projective model structure of Blander. All others are intermediate structures between the projective structure and the standard structure (now sometimes called the injective structure).

The overall moral is that there are many different model structures for simplicial presheaves with the same weak equivalences, namely local weak equivalences.

Some people are actually using the intermediate structures, in complex analytic geometry in particular. Finnur Lárusson uses these structures on a suitable site of Stein spaces to study the Oka principle which says roughly that analytic problems of a cohomological nature on a Stein manifold have only topological obstructions [32]. See also [10].

5 Cocycle categories

Suppose that X and Y are objects in some model category. The *cocycle category* $H(X, Y)$ is the category whose objects are all pairs of maps (f, g)

$$X \xleftarrow{f} Z \xrightarrow{g} Y$$

where f is a weak equivalence. A morphism $\alpha : (f, g) \to (f', g')$ of $H(X, Y)$ is a commutative diagram

$$\begin{array}{ccccc} & {\scriptstyle f} \swarrow & Z & \searrow {\scriptstyle g} & \\ X & & \downarrow{\scriptstyle \alpha} & & Y \\ & {\scriptstyle f'} \nwarrow & Z' & \nearrow {\scriptstyle g'} & \end{array}$$

Example: $U_\alpha \subset S$ an open cover of some space S, G a topological group.

A (normalized) cocyle on $\{U_\alpha\}$ with values in G consists of elements $g_{\alpha\beta} \in G(U_\alpha \cap U_\beta)$ s.t.

$$g_{\alpha\alpha} = e,$$
$$g_{\alpha\beta}g_{\beta\gamma} = g_{\alpha\gamma} \in G(U_\alpha \cap U_\beta \cap U_\gamma).$$

Equivalently, the $g_{\alpha\beta}$ define a simplicial sheaf map

$$f : C(U_\bullet) \to BG$$

Here, $C(U_\bullet)$ is the Čech resolution associated to the covering $\{U_\alpha\}$. The map $C(U_\bullet) \to *$ is a local weak equivalence of simplicial sheaves on S, so the picture

$$* \xleftarrow{\simeq} C(U_\bullet) \xrightarrow{f} BG$$

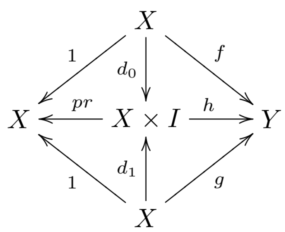
is a member of the cocycle category $H(*, BG)$.

Write $\pi_0 H(X,Y)$ for the class of path components of $H(X,Y)$. There is a function

$$\phi : \pi_0 H(X,Y) \to [X,Y] \quad (f,g) \mapsto g \cdot f^{-1}$$

Theorem 7. *The canonical map $\phi : \pi_0 H(X,Y) \to [X,Y]$ is a bijection for all X and Y.*

Lest you think that I've done away with the homotopy theory in this statement, suppose that $f \simeq g : X \to Y$. Then there is a picture



where h is the homotopy. Then

$$(1_X, f) \sim (pr, h) \sim (1_X, g)$$

Thus $f \mapsto [(1_X, f)]$ defines a function

$$\psi : \pi(X,Y) \to \pi_0 H(X,Y)$$

Here, $\pi(X,Y)$ denotes naive simplicial homotopy classes of maps.

For the proof of Theorem 7, there are a couple of things to show:

Lemma 8. *Suppose $\alpha : X \to X'$ and $\beta : Y \to Y'$ are weak equivalences. Then*

$$(\alpha,\beta)_* : \pi_0 H(X,Y) \to \pi_0 H(X',Y')$$

is a bijection.

Lemma 9. *Suppose that Y is fibrant and X is cofibrant. Then the canonical map*

$$\phi : \pi_0 H(X,Y) \to [X,Y]$$

is a bijection.

Theorem 7 is a formal consequence. The result holds in extreme generality, specifically in any model category which is right proper (weak equivalences pull back to weak equivalences along fibrations), and such that weak equivalences are closed under finite products.

Examples: spaces, simplicial sets, simplicial presheaves, spectra, presheaves of spectra, right proper localizations such as the motivic model structures.

Proof of Lemma 8. $(f,g) \in H(X',Y')$ is a map $(f,g) : Z \to X' \times Y'$ s.t. f is a weak equivalence.

There is a factorization

$$\begin{array}{ccc} Z & \xrightarrow{j} & W \\ & \searrow^{(f,g)} & \downarrow{\scriptstyle (p_{X'},p_{Y'})} \\ & & X' \times Y' \end{array}$$

s.t. j is a triv. cofibration and $(p_{X'}, p_{Y'})$ is a fibration. $p_{X'}$ is a weak equivalence.

Form the pullback

$$\begin{array}{ccc} W_* & \xrightarrow{(\alpha\times\beta)_*} & W \\ {\scriptstyle (p_X^*,p_Y^*)}\downarrow & & \downarrow{\scriptstyle (p_{X'},p_{Y'})} \\ X \times Y & \xrightarrow[\alpha\times\beta]{} & X' \times Y' \end{array}$$

(p_X^*, p_Y^*) is a fibration and $(\alpha \times \beta)_*$ is a weak equivalence (since $\alpha \times \beta$ is a weak equivalence, and by right properness). p_X^* is also a weak equivalence.

$(f,g) \mapsto (p_X^*, p_Y^*)$ defines a function

$$\pi_0 H(X',Y') \to \pi_0 H(X,Y)$$

which is inverse to $(\alpha,\beta)_*$. □

Proof of Lemma 9. The function $\pi(X,Y) \to [X,Y]$ is a bijection since X is cofibrant and Y is fibrant.

We have seen that the assignment $f \mapsto [(1_X, f)]$ defines a function

$$\psi : \pi(X,Y) \to \pi_0 H(X,Y)$$

and there is a diagram

$$\begin{array}{ccc} \pi(X,Y) & \xrightarrow{\psi} & \pi_0 H(X,Y) \\ & \searrow^{\cong} & \downarrow{\scriptstyle \phi} \\ & & [X,Y] \end{array}$$

It suffices to show that ψ is surjective, or that any object $X \xleftarrow{f} Z \xrightarrow{g} Y$ is in the path component of some a pair $X \xleftarrow{1} X \xrightarrow{k} Y$ for some map k.

Form the diagram

$$\begin{array}{ccccc} & \overset{f}{\swarrow} & Z & \overset{g}{\searrow} & \\ X & & \downarrow{\scriptstyle j} & & Y \\ & \underset{p}{\nwarrow} & V & \underset{\theta}{\nearrow} & \end{array}$$

where j is a triv. cofibration and p is a fibration; θ exists because Y is fibrant.

X is cofibrant, so the trivial fibration p has a section σ, and so there is a commutative diagram

$$\begin{array}{ccccc} & \overset{1}{\swarrow} & X & \overset{\theta\sigma}{\searrow} & \\ X & & \downarrow{\scriptstyle \sigma} & & Y \\ & \underset{p}{\nwarrow} & V & \underset{\theta}{\nearrow} & \end{array}$$

The composite $\theta\sigma$ is the required map k. □

6 Sheaf cohomology

Lemma 10 (Van Osdol [37])**.** *Suppose that $f : X \to Y$ is a local weak equivalence of simplicial presheaves. Then the induced map $f_* : \mathbb{Z}X \to \mathbb{Z}Y$ of simplicial abelian presheaves is also a local weak equivalence.*

Proof. It's enough to show that if $f : X \to Y$ is a local equivalence of locally fibrant simplicial sheaves, then $f_* : \tilde{\mathbb{Z}}X \to \tilde{\mathbb{Z}}Y$ is a local equivalence of simplicial abelian sheaves.

It's also enough to assume that the map $f : X \to Y$ is a morphism of locally fibrant simplicial sheaves on a complete Boolean algebra $\mathcal{B}$, since the inverse image functor p^* for a Boolean localization $p : \mathrm{Shv}(\mathcal{B}) \to \mathrm{Shv}(\mathcal{C})$ commutes with the free abelian sheaf functor (p_* preserves abelian group structures).

But then $f : X \to Y$ is a sectionwise weak equivalence, so $f_* : \mathbb{Z}X \to \mathbb{Z}Y$ is a sectionwise weak equivalence of associated free abelian presheaves, so that $f_* : \tilde{\mathbb{Z}}X \to \tilde{\mathbb{Z}}Y$ is a local weak equivalence. □

Remark: Once upon a time, the statement of Lemma 10 was called the Illusie conjecture.

Suppose that A is a simplicial abelian group. Then A is a Kan complex, and we know that there is a natural isomorphism

$$\pi_n(A,0) \cong H_n(NA).$$

There is a canonical isomorphism

$$\pi_n(A,0) \xrightarrow{\cong} \pi_n(A,a)$$

which is defined for any $a \in A_0$ by $[\alpha] \mapsto [\alpha + a]$ where we have written a for the composite

$$\Delta^n \to \Delta^0 \xrightarrow{a} A$$

The collection of these isomorphisms, taken together, define isomorphisms

$$\begin{array}{ccc} \pi_n(A,0) \times A_0 & \xrightarrow{\cong} & \pi_n A \\ & {\scriptstyle pr}\searrow \quad \swarrow & \\ & A_0 & \end{array}$$

of abelian groups fibred over A_0, and these isomorphisms are natural in simplicial abelian group homomorphisms.

Lemma 11. *A map $A \to B$ of presheaves of simplicial abelian groups is a local weak equivalence if and only if the presheaf of chain complex maps $NA \to NB$ induces an isomorphism in all homology sheaves.*

Proof. If $NA \to NB$ induces an isomorphism in all homology sheaves, then the map $\tilde{\pi}_0(A) \to \tilde{\pi}_0(B)$ and all maps $\tilde{\pi}_n(A,0) \to \tilde{\pi}_n(B,0)$ are isomorphisms of sheaves. The diagram of sheaves associated to

$$\begin{array}{ccc} \pi_n(A,0) \times A_0 & \longrightarrow & \pi_n(B,0) \times B_0 \\ \downarrow & & \downarrow \\ A_0 & \longrightarrow & B_0 \end{array}$$

coincides with the diagram of sheaves associated to the picture

$$\begin{array}{ccc} \tilde{\pi}_n(A,0) \times A_0 & \longrightarrow & \tilde{\pi}_n(B,0) \times B_0 \\ \downarrow & & \downarrow \\ A_0 & \longrightarrow & B_0 \end{array}$$

which is a pullback. □

Suppose that A is a sheaf of abelian groups, and let $A \to J$ be an injective resolution of A, thought of as a $\mathbb{Z}$-graded chain complex, concentrated in negative degrees. Write $A[-n]$ for the chain complex consisting of A concentrated in degree n, and consider the chain map $A[-n] \to J[-n]$. Recall that $K(A,n) = \Gamma A[-n]$ defines the Eilenberg-Mac Lane sheaf associated to A. Let $K(J,n) = \Gamma T(J[-n])$ where $T(J[-n])$ is the good truncation of $J[-n]$ in non-negative degrees (ie. $T(J[-n])_0 = \ker(J_{-n} \to J_{-n-1})$) .

Lemma 12. *Every local weak equivalence $f : X \to Y$ induces an isomorphism*

$$\pi_{ch}(N\tilde{\mathbb{Z}}Y, J[-n]) \xrightarrow{\cong} \pi_{ch}(N\tilde{\mathbb{Z}}X, J[-n])$$

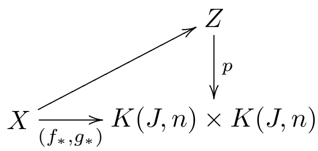
in chain homotopy classes for all $n \geq 0$.

Proof. The map f induces a homology sheaf isomorphism $N\tilde{\mathbb{Z}}X \to N\tilde{\mathbb{Z}}Y$, and then a comparison of spectral sequences

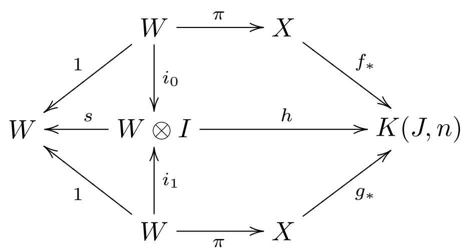
$$E_2^{p,q} = \operatorname{Ext}^q(\tilde{H}_p(X), A) \Rightarrow \pi_{ch}(N\tilde{\mathbb{Z}}X, J[-p-q])$$

gives the desired result. The spectral sequence comes from the bicomplex $\hom(N\tilde{\mathbb{Z}}X_m, J_n)$. □

If two chain maps $f, g : N\tilde{\mathbb{Z}}X \to J[-n]$ are chain homotopic, then there is a right homotopy



for some path object Z over $K(J,n)$ in the projective model structure for $\mathcal{C}^{op}$-diagrams. Choose a sectionwise trivial fibration $\pi : W \to X$ such that W is projective cofibrant. Then $f_*\pi$ is left homotopic to $g_*\pi$ for some choice of cylinder object $W \otimes I$ for W, again in the projective structure. This means that there is a diagram



where the maps s, i_0, i_1 are all part of the cylinder object structure for $W \otimes I$, and are sectionwise weak equivalences. It follows that

$$(1, f_*) \sim (\pi, f_*\pi) \sim (\pi s, h) \sim (\pi, g_*\pi) \sim (1, g_*)$$

in $\pi_0 H(X, K(J,n))$. It follows that there is a well defined abelian group homomorphism

$$\phi : \pi_{ch}(N\tilde{\mathbb{Z}}X, J[-n]) \to \pi_0 H(X, K(J,n)).$$

Lemma 13. *ϕ is an isomorphism.*

Proof. Suppose that $X \xleftarrow{f} Z \xrightarrow{g} K(J,n)$ is an object of $H(X, K(J,n))$. Then there is a unique chain homotopy class $[v] : N\tilde{\mathbb{Z}}X \to J[-n]$ such that $[v_* f] = [g]$ since f is a local weak equivalence. This chain homotopy class $[v]$ is also independent of the component of (f,g). We therefore have a well defined function

$$\psi : \pi_0 H(X, K(J,n)) \to \pi_{ch}(N\tilde{\mathbb{Z}}X, J[-n]).$$

Then the composites $\psi \cdot \phi$ and $\phi \cdot \psi$ are identity morphisms. $\square$

Corollary 14. *Suppose that A is a sheaf of abelian groups on $\mathcal{C}$, and let $A \to J$ be an injective resolution of A in the category of abelian sheaves. Let X be a simplicial presheaf on $\mathcal{C}$. Then there is an isomorphism*

$$\pi_{ch}(N\tilde{\mathbb{Z}}X, J[-n]) \cong [X, K(A,n)].$$

This isomorphism is natural in X.

Suppose that A is an abelian (pre)sheaf on $\mathcal{C}$ and that X is a simplicial presheaf. Write

$$H^n(X, A) = [X, K(A,n)],$$

and say that this group is the n^{th} *cohomology group* of X with coeffients in A.

Remarks:

1) One can (and does) define sheaf cohomology $H^n(\mathcal{C}, B)$ for an abelian sheaf B by

$$H^n(\mathcal{C}, B) = H_{-n}(\Gamma_* J)$$

where $B \to J$ is an injective resolution of B concentrated in negative degrees and Γ_* is global sections (ie. inverse limit). But $\Gamma_* Y = \hom(*, Y)$ for any Y, and so

$$H^n(\mathcal{C}, B) \cong \pi_{ch}(\tilde{\mathbb{Z}}*, J[-n]) \cong [*, K(B,n)].$$

2) There is a *universal coefficients spectral sequence*

$$E_2^{p,q} = \mathrm{Ext}^q(\tilde{H}_p(X), \tilde{A}) \Rightarrow H^{p+q}(X, A)$$

3) Suppose that

$$X \xleftarrow{\simeq} X' \to K(A,n), \quad Y \xleftarrow{\simeq} Y' \to K(B,m)$$

are cocycles. Then the adjoint simplicial abelian presheaf maps

$$\mathbb{Z}X' \to K(A,n), \quad \mathbb{Z}Y' \to K(B,n)$$

have a (simplicial abelian group) tensor product

$$\mathbb{Z}(X' \times Y') \cong \mathbb{Z}X' \otimes \mathbb{Z}Y' \to K(A,n) \otimes K(B,n)$$

and there is a natural weak equivalence

$$K(A,n) \otimes K(B,m) \simeq K(A \otimes B, n+m).$$

in simplicial abelian groups, hence in simplicial abelian presheaves (Exercise: suppose first that $A = B = \mathbb{Z}$). The composite

$$X \times Y \xleftarrow{\simeq} X' \times Y' \to K(A \otimes B, n+m)$$

represents the external cup product of the classes represented by the two cocycles. We have defined the *external cup product*

$$H^n(X,A) \times H^m(Y,B) \to H^{n+m}(X \times Y, A \otimes B).$$

If A happens to be a presheaf of rings this construction specializes to the cup product pairing

$$\begin{aligned} H^n(X,A) \times H^m(X,A) &\to H^{n+m}(X \times X, A) \\ &\xrightarrow{\Delta^*} H^{n+m}(X,A). \end{aligned}$$

where $\Delta : X \to X \times X$ is the diagonal map.

Up the the early 1980s, cup products in sheaf cohomology could only be constructed in the presence of enough points on the underlying topos. It was not even known, for example, how to construct cup products for flat cohomology.

Theorem 15. *Suppose that $f : X \to Y$ induces an isomorphism $\tilde{H}_*(X) \cong \tilde{H}_*(Y)$ in all homology sheaves. Then the induced map in cohomology*

$$H^*(Y,A) \to H^*(X,A)$$

is an isomorphism for all coefficient presheaves A.

Proof. Compare universal coefficients spectral sequences. □

There is a torsion coeffients version:

Theorem 16. *If $f : X \to Y$ induces a homology sheaf isomorphism*

$$\tilde{H}_*(X, \mathbb{Z}/n) \cong \tilde{H}_*(Y, \mathbb{Z}/n)$$

then f induces an isomorphism

$$H^*(Y,A) \to H^*(X,A)$$

for all n-torsion presheaves A.

Proof. Construct the universal coefficients spectral sequence in the category of n-torsion sheaves. □

Example: [11] Suppose that k is an algebraically closed field and ℓ is a prime $\neq char(k)$. Let $\mathcal{C} = (Sm|_k)_{et}$ = smooth schemes over k with the étale topology. The *Gabber rigidity* theorem asserts, in this case, that the map $\epsilon : \Gamma^* BGl(k) \to BGl$ of simplicial presheaves on $(Sm|_k)_{et}$ induces an isomorphism

$$\tilde{H}_*(\Gamma^* BGl(k), \mathbb{Z}/\ell) \cong \tilde{H}_*(BGl, \mathbb{Z}/\ell)$$

on mod ℓ homology sheaves. It follows that the induced map

$$H^*_{et}(BGl, \mathbb{Z}/\ell) \to H^*(\Gamma^* BGl(k)\mathbb{Z}/\ell) \cong H^*(BGl(k), \mathbb{Z}/\ell)$$

is an isomorphism.

There are several consequences:

a) $H^*(BGl(k), \mathbb{Z}/\ell) \cong H^*(BU, \mathbb{Z}/\ell) \cong \mathbb{Z}/\ell[c_1, c_2, \dots]$ is a polynomial ring in Chern classes.

b) Any inclusion $k \subset L$ of algebraically closed fields induces isomorphisms

$$H^*(BGl(L), \mathbb{Z}/\ell) \cong H^*(BGl(k), \mathbb{Z}/\ell)$$
$$K_*(k, \mathbb{Z}/\ell) \cong K_*(L, \mathbb{Z}/\ell)$$

c) $K_*(k, \mathbb{Z}/\ell) \cong \mathbb{Z}/\ell[\beta]$ is a polynomial ring on $\beta \in K_2(k, \mathbb{Z}/\ell)$. Here, the Bott element $\beta \mapsto \zeta_\ell$ where ζ_l is a primitive ℓ^{th} root of unity under the isomorphism

$$K_2(k, \mathbb{Z}/\ell) \cong \mathrm{Tor}(\mathbb{Z}/\ell, K_1(k)) \cong \mathrm{Tor}(\mathbb{Z}/\ell, k^*).$$

d) The simplicial presheaf map

$$\epsilon : \Gamma^* K(k, \mathbb{Z}/\ell) \to K(\ , \mathbb{Z}/\ell)$$

is a local weak equivalence on $(Sm|_k)_{et}$. In other words, the mod ℓ étale K-theory sheaves $\tilde{\pi}_i K(\ , \mathbb{Z}/\ell)$ are constant.

5) A *cohomology operation* is a map

$$K(A, n) \to K(B, m)$$

in the homotopy category.

The *Steenrod operation* Sq^i is a morphism $K(\mathbb{Z}/2, n) \to K(\mathbb{Z}/2, n+i)$ in the ordinary homotopy category. The constant presheaf functor preserves weak equivalences, and so Sq^i induces a morphism $K(\mathbb{Z}/2, n) \to K(\mathbb{Z}/2, n+i)$ in the homotopy category of simplicial presheaves on an arbitrary small site $\mathcal{C}$. It therefore induces a homomorphism

$$\mathrm{Sq}^i : H^n(X, \mathbb{Z}/2) \to H^{n+i}(X, \mathbb{Z}/2)$$

which is natural in simplicial presheaves X. The collection of Steenrod operations $\{\mathrm{Sq}^i\}$ for simplicial presheaves has the same basic list of properties as the Steenrod operations for ordinary spaces.

Steenrod operations were first introduced at this level of generality in [14] — Breen's definition for the mod 2 étale cohomology of schemes appeared about ten years previously in [4].

The first calculational application of the simplicial presheaves definition arose from questions concerning Hasse-Witt classes for non-degenerate symmetric bilinear forms in the mod 2 Galois cohomology of fields. Suppose that k is a field such that $char(k) \neq 2$. A non-degenerate symmetric bilinear form α on k represents an element of

$$H^1_{et}(k, O_n) \cong [*, BO_n]$$

where the homotopy classes of maps are in the homotopy category of simplicial presheaves on $(Sch|_k)_{et}$ (this remains to be explained).

There are isomorphisms

$$\begin{aligned} H^*_{et}(BO_n, \mathbb{Z}/2) &\cong H^*(BO_n, \mathbb{Z}/2) \\ &\cong H^*_{Gal}(k, \mathbb{Z}/2)[HW_1, \dots, HW_n] \end{aligned}$$

where the polynomial generator HW_i has degree i. In fact HW_i is characterized by mapping to the i^{th} elementary symmetric polynomial $\sigma_i(x_1, \dots, x_n)$ under the isomorphism

$$\begin{aligned} H^*(BO_n, \mathbb{Z}/2) &\cong H^*(\Gamma^* B\mathbb{Z}/2^{\times n}, \mathbb{Z}/2)^{\Sigma_n} \\ &\cong H^*_{Gal}(k, \mathbb{Z}/2)[x_1, \dots, x_n]^{\Sigma_n}. \end{aligned}$$

where $(\ )^{\Sigma_n}$ denotes invariants for the symmetric group Σ_n

Every symmtric bilinear form α determines a map $\alpha : * \to BO_n$ in the simplicial presheaf homotopy category, and therefore induces a map

$$\alpha^* : H^*_{et}(BO_n, \mathbb{Z}/2) \to H^*_{Gal}(k, \mathbb{Z}/2),$$

and $HW_i(\alpha) = \alpha^*(HW_i)$ is the i^{th} Hasse-Witt class of α. $HW_1(\alpha)$ is the pullback of the determinant $BO_n \to B\mathbb{Z}/2$, and $HW_2(\alpha)$ is the classical Hasse-Witt invariant of α.

The application of the Steenrod algebra is about calculating the relation between Hasse-Witt and Stiefel-Whitney classes for Galois representations, and depends on knowing the Wu formulas for the action of the Steenrod algebra on elementary symmetric polynomials. See [13] and [15].

7 Descent

Proposition 17. *Suppose that A is a presheaf of abelian groups and suppose that $GK(A, n)$ is a globally fibrant model of $K(A, n)$. Then there are isomorphisms*

$$\pi_j GK(A, n)(U) \cong \begin{cases} H^{n-j}(\mathcal{C}/U, \tilde{A}|_U) & 0 \leq j \leq n \\ 0 & j > n. \end{cases}$$

for all $U \in \mathcal{C}$.

Lemma 18. *Suppose that $U \in \mathcal{C}$ and write $X|_U$ for restriction of X along the functor $\mathcal{C}/U \to \mathcal{C}$. Then the restriction functor $X \mapsto X|_U$ preserves global fibrations and local weak equivalences.*

Proof. The restriction functor $X \mapsto X|_U$ has a left adjoint j_U^* where

$$j_U^*(Y)(V) = \bigsqcup_{V \to U} Y(V).$$

j_U^* clearly preserves cofibrations and sectionwise weak equivalences. j_U^* also preserves local trivial fibrations (exercise) and therefore preserves local weak equivalences.

Restriction preserves sectionwise equivalences and local trivial fibrations, and therefore preserves local weak equivalences. □

Proof of Proposition 17. It's enough to suppose that A is a sheaf.

There is a sectionwise fibre sequence

$$\begin{aligned} K(A, n-1) \to & WK(A, n-1) \\ & \to \overline{W}K(A, n-1) = K(A, n) \end{aligned}$$

where $WK(A, n-1)$ is sectionwise contractible. Take a globally fibrant model

$$\begin{array}{ccc} WK(A, n-1) & \xrightarrow{j} & GWK(A, n-1) \\ \downarrow & & \downarrow p \\ K(A, n) & \xrightarrow[j]{} & GK(A, n) \end{array}$$

where the maps labelled j are local weak equivalences, $GK(A, n)$ is globally fibrant and p is a global fibration. Let $F = p^{-1}(0)$. Then F is globally fibrant and the induced map

$$K(A, n-1) \to F$$

is a local weak equivalence, at worst by a Boolean localization argument. Write $GK(A, n-1)$ for F.

We have sectionwise fibre sequences

$$\begin{aligned} GK(A, n-1)(U) \to & GWK(A, n-1)(U) \\ & \to GK(A, n)(U) \end{aligned}$$

for all $U \in \mathcal{C}$. The map

$$GWK(A, n-1) \to *$$

is a trivial global fibration, and is therefore a sectionwise trivial fibration. It follows that

$$\pi_j GK(A, n)(U) \cong \pi_{j-1} GK(A, n-1)(U)$$

for $1 \leq j \leq n$, so that

$$\pi_j GK(A,n)(U) \cong H^{n-j}(\mathcal{C}/U, \tilde{A}|_U)$$

for $1 \leq j \leq n$ by induction on n. Finally

$$\begin{aligned} \pi_0 GK(A,n)(U) &\cong [*, GK(A,n)(U)]_{\mathbf{S}} \\ &\cong [*, GK(A,n)|_U]_U \\ &\cong [*, GK(A|_U, n)]_U. \end{aligned}$$

$GK(A,n)|_U$ globally fibrant by the Lemma, giving the second isomorphism; the other isomorphisms are formal. □

Example: Suppose $\mathcal{C}$ is the big site $(Sch|_S)_{et}$ for a scheme S with the étale topology. Then $\mathcal{C}/U$ is isomorphic to the site $(Sch|_U)_{et}$. If A is a sheaf on the big étale site for S, and if $K(A,n) \to GK(A,n)$ is a globally fibrant model for $K(A,n)$, then the presheaves of homotopy groups for $GK(A,n)$ have the form

$$\pi_j GK(A,n)(U) \cong \begin{cases} H^{n-j}_{et}(U, \tilde{A}|_U) & 0 \leq j \leq n \\ 0 & j > n. \end{cases}$$

for all $U \in \mathcal{C}$.

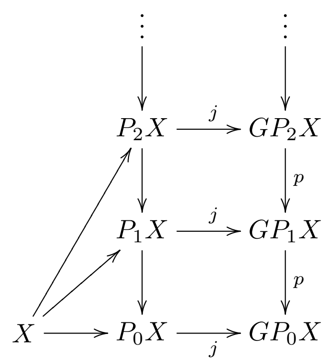
Similar statements obtain for all other geometric topologies on categories of S-schemes.

Say that a simplicial presheaf X *satisfies descent* if some (hence any) globally fibrant model $j : X \to Z$ induces weak equivalences $X(U) \to Z(U)$ of simplicial sets in all sections.

Simplicial presheaves which satisfy descent are not common: Eilenberg-Mac Lane objects $K(A,n)$, for example, almost never satisfy descent.

This concept is, however, at the heart of the applications of the homotopy theory of simplicial presheaves, and was the primary reason for the introduction of that theory. The basic idea is that if a simplicial presheaf satisfies descent for some topology, then its homotopy groups can (or should) be computed with a cohomological *descent spectral sequence* for that same topology.

Suppose that X is a presheaf of pointed Kan complexes, and form the Postnikov tower



where all maps labelled j are globally fibrant models and the maps p are global fibrations.

The fibre of $GP_nX \to GP_{n-1}X$ over the base point is sectionwise equivalent to $GK(\tilde{\pi}_nX, n)$, where

$$\tilde{\pi}_nX = \tilde{\pi}_n(X, *)$$

is the n^{th} homotopy group sheaf, based at the global base point.

Now take $U \in \mathcal{C}$ and consider the tower of fibrations

$$GP_0X(U) \leftarrow GP_1X(U) \leftarrow GP_2X(U) \leftarrow \dots$$

The fibre $GK(\tilde{\pi}_nX, n)(U)$ of the map

$$GP_nX(U) \to GP_{n-1}X(U)$$

has homotopy groups

$$\begin{aligned} &\pi_j GK(\tilde{\pi}_nX, n)(U) \\ &\quad \cong \begin{cases} H^{n-j}(\mathcal{C}/U, \tilde{\pi}_nX|_U) & 0 \leq j \leq n \\ 0 & j > n. \end{cases} \end{aligned}$$

and so the tower of fibrations spectral sequence (with the Thomason re-indexing trick) determines a spectral sequence with

$$E_2^{s,t}(U) = H^s(\mathcal{C}/U, \tilde{\pi}_sX|_U)$$

This is "the" descent spectral sequence — it is actually a presheaf of spectral sequences.

There are two issues:

1) The spectral sequence might not converge to

$$\pi_{t-s} \varprojlim GP_nX(U).$$

2) It can be a bit of work to show that the map $X \to \varprojlim_n GP_nX$ is a local weak equivalence.

Both issues can be resolved (ie. the spectral sequence converges and the map of 2) is a local weak equivalence) if X is locally connected in the sense that $\tilde{\pi}_0X \cong *$ and there is a uniform bound on cohomological dimension for all sheaves $\tilde{\pi}X|_U$. See [12].

8 Non-abelian cohomology

Suppose that G is a sheaf of groups. A G-torsor is traditionally defined to be a sheaf X with a free G-action such that $X/G \cong *$ in the sheaf category.

The requirement that the action $G \times X \to X$ is free means that the isotropy subgroups of G for the action are trivial in all sections, which is equivalent to requiring that all sheaves of fundamental groups for the Borel construction $EG \times_G X$ are trivial. There is an isomorphism of sheaves

$$\tilde{\pi}_0(EG \times_G X) \cong X/G.$$

Also the simplicial sheaf $EG \times_G X$ is the nerve of a sheaf of groupoids, which is given in each section by the translation category for the action of $G(U)$ on $X(U)$; this means, in particular, that all sheaves of higher homotopy groups for $EG \times_G X$ vanish.

It follows that a G-sheaf X is a G-torsor if and only if the map $EG \times_G X \to *$ is a local weak equivalence.

The category $G - \mathbf{Tors}$ is the category whose objects are all G-torsors and whose maps are all G-equivariant maps between them.

If $f : X \to Y$ is a map of G-torsors, then f is induced as a map of fibres by the comparison of local fibrations

$$\begin{array}{ccc} EG \times_G X & \longrightarrow & EG \times_G Y \\ & \searrow \quad \swarrow & \\ & BG & \end{array}$$

It follows that $f : X \to Y$ is a weak equivalence of constant simplicial sheaves, and is therefore an isomorphism. The category of G-torsors is therefore a groupoid.

Suppose that the picture

$$* \xleftarrow{\simeq} Y \xrightarrow{\alpha} BG$$

is an object of the cocycle category $H(*, BG)$, and form the pullback

$$\begin{array}{ccc} \mathrm{pb}(Y) & \longrightarrow & Y \\ \downarrow & & \downarrow \alpha \\ EG & \xrightarrow[\pi]{} & BG \end{array}$$

where $EG = B(G/*) = EG \times_G G$ and $\pi : EG \to BG$ is the canonical map. Then $\mathrm{pb}(Y)$ inherits a G-action from the G-action on EG, and the map $EG \times_G \mathrm{pb}(Y) \to Y$ is a sectionwise weak equivalence. Also, the square is homotopy cartesian in sections where $Y(U) \neq \emptyset$, so that $Y(U) \simeq G(U)$ in those sections. It follows that the canonical map $\mathrm{pb}(Y) \to \tilde{\pi}_0 \,\mathrm{pb}(Y)$ is a G-equivariant local weak equivalence, and hence that the maps

$$EG \times_G \tilde{\pi}_0 \,\mathrm{pb}(Y) \leftarrow EG \times_G \mathrm{pb}(Y) \to Y \simeq *$$

are natural local weak equivalences. In particular, the G-sheaf $\tilde{\pi}_0 \,\mathrm{pb}(Y)$ is a G-torsor.

We therefore have a functor

$$H(*, BG) \to G - \mathbf{Tors}$$

defined by sending $* \xleftarrow{\simeq} Y \to BG$ to the object $\tilde{\pi}_0 \, \mathrm{pb}(Y)$. The Borel construction defines a functor

$$G - \mathbf{Tors} \to H(*, BG):$$

the G-torsor X is sent to the cocycle

$$* \xleftarrow{\simeq} EG \times_G X \to BG.$$

It is elementary to check that these two functors, together, induce a bijection

$$\pi_0 H(*, BG) \cong \pi_0(G - \mathbf{Tors}).$$

In view of the fact that $\pi_0(G - \mathbf{Tors})$ is isomorphism classes of G-torsors, and we know that

$$\pi_0 H(*, BG) \cong [*, BG],$$

we have proved

Theorem 19. *Suppose that G is a sheaf of groups on a small Grothendieck site $\mathcal{C}$. Then there is a bijection*

$$[*, BG] \cong \{\textit{isomorphism classes of } G\textit{-torsors}\}$$

Remarks:

1) The theorem was first proved, by a different method, in "Universal Hasse-Witt classes" [13].

2) The non-abelian invariant $H^1(\mathcal{C}, G)$ is traditionally defined to be the collection of isomorphism classes of G-torsors. The theorem therefore gives an identification

$$H^1(\mathcal{C}, G) \cong [*, BG]$$

3) Suppose that G is a sheaf of groups. When you take the cocycle $C(U) \to BG$ associated to a G-torsor P by the classical construction, what are you actually doing?

First of all, note that there is a canonical morphism of simplicial sheaves

$$EG \times_G P \to C(P)$$

where $C(P)$ is the Čech resolution for the covering $P \to *$. This simplicial sheaf map is induced by a morphism of sheaves of groupoids

$$\begin{array}{ccc} G \times P & \rightrightarrows & P \\ \downarrow & & \downarrow 1 \\ P \times P & \rightrightarrows & P \end{array}$$

taking values in the trivial groupoid on the sheaf P. The vertical maps are isomorphisms of sheaves, so that the induced simplicial sheaf map $EG \times_G P \to C(P)$ is actually an isomorphism. Any trivialization

$$\begin{array}{ccc} & & P \\ & \overset{\sigma}{\nearrow} & \downarrow \\ U & \longrightarrow & * \end{array}$$

($U \to *$ is a sheaf epimorphism) therefore induces a composite simplicial sheaf map

$$C(U) \xrightarrow{\sigma_*} C(P) \cong EG \times_G P \to BG$$

and this cocycle is the classical cocycle associated to the trivialization.

Example: Suppose that k is a field. Let $\mathcal{C}$ be the étale site $et|_k$ for k, and identify the orthogonal group O_n with a sheaf of groups on this site. The non-abelian cohomology object $H^1_{et}(k, O_n)$ is well known to coincide with the set of isomorphism classes of non-degenerate symmetric bilinear forms over k of rank n. Thus, every such form q determines a morphism $* \to BO_n$ in the simplicial (pre)sheaf homotopy category, and this morphism determines the form q up to isomorphism.

9 Presheaves of groupoids

Write $\operatorname{Pre}\mathbf{Gpd}(\mathcal{C})$ for the category of presheaves of groupoids on a Grothendieck site $\mathcal{C}$.

Say that a map $f : G \to H$ of presheaves of groupoids is a weak equivalence (respectively fibration) if the induced map $f : BG \to BH$ is a local weak equivalence (respectively global fibration) of simplicial presheaves. Cofibrations in this category are defined by a left lifting property with respect to trivial fibrations.

Corresponding definitions may be made for sheaves of groupoids. Write $\operatorname{Shv}\mathbf{Gpd}(\mathcal{C})$ for the category of sheaves of groupoids on the site $\mathcal{C}$.

Theorem 20 (Joyal-Tierney [30], Hollander [9])**.** *1) With these definitions,* $\operatorname{Pre}\mathbf{Gpd}(\mathcal{C})$ *and* $\operatorname{Shv}\mathbf{Gpd}(\mathcal{C})$ *satisfy the axioms for a right proper closed simplicial model category.*

2) The adjoint pair

$$L^2 : \operatorname{Pre}\mathbf{Gpd}(\mathcal{C}) \leftrightarrows \operatorname{Shv}\mathbf{Gpd}(\mathcal{C}) : i$$

determines a Quillen equivalence of the model structures.

Proof. The nerve BG of an ordinary groupoid G is a Kan complex. The fundamental groupoid functor

$$\pi : s\mathbf{Set} \to \mathbf{Gpd}$$

(left adjoint of nerve) takes every anodyne extension $\Lambda^n_k \to \Delta^n$ to a strong deformation retraction of groupoids $\pi(\Lambda^n_k) \to \pi(\Delta^n)$, so these maps push out to weak equivalences. The factorization axiom follows, and the rest of the model axioms for ordinary groupoids follow by standard tricks. The n-simplices of the simplicial function complex $\mathbf{hom}(G,H)$ are morphisms $G \times \pi(\Delta^n) \to H$, or equivalently n-simplices of the function complex $\mathbf{hom}(BG,BH)$. The fundamental groupoid functor preserves finite products, and this is used to verify the simplicial model structure. Right properness is trivial.

For the presheaves of groupoids case, $f : G \to H$ is a weak equivalence if and only if the induced map $p^*\tilde{G} \to p^*\tilde{H}$ is a sectionwise weak equivalence for some Boolean localization $p : \mathrm{Shv}(\mathcal{B}) \to \mathrm{Shv}(\mathcal{C})$, since the nerve functor B commutes with the associated sheaf functor and the inverse image p^* up to isomorphism.

Suppose given a pushout diagram

$$\begin{array}{ccc} \tilde{\pi}A & \longrightarrow & G \\ {\scriptstyle i_*}\downarrow & & \downarrow{\scriptstyle i'} \\ \tilde{\pi}B & \longrightarrow & H \end{array} \tag{5}$$

of sheaves of groupoids, where $i : A \to B$ is a cofibration and a local weak equivalence. The proof of the factorization axioms, and hence the proof of the Theorem, follows if we can show that the map i' is a local weak equivalence.

The fundamental groupoid sheaf construction $A \mapsto \tilde{\pi}A$ commutes with p^*, so it suffices to prove the result for sheaves of groupoids on a complete Boolean algebra $\mathcal{B}$.

Consider the diagram

$$\begin{array}{ccc} A & \xrightarrow{\eta} & B\tilde{\pi}A \\ {\scriptstyle i}\downarrow & & \downarrow \\ B & \xrightarrow[\eta]{} & B\tilde{\pi}B \end{array}$$

in the category of simplicial sheaves on $\mathcal{B}$, and note that the objects $B\tilde{\pi}A$ and $B\tilde{\pi}B$ are presheaves of Kan complexes as well as simplicial sheaves. There is a factorization of this diagram

$$\begin{array}{ccccc} A & \xrightarrow{j} & A_f & \xrightarrow{q} & B\tilde{\pi}A \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i'} & & \downarrow \\ B & \xrightarrow[j]{} & B_f & \xrightarrow[q]{} & B\tilde{\pi}B \end{array}$$

in the simplicial presheaf category such that A_f and B_f are projective fibrant and i' is a cofibration and a local weak equivalence. Applying the associated sheaf functor gives a diagram of the same shape, in which i' is a sectionwise weak equivalence as well as a cofibration. The adjoint of this diagram, namely

$$\begin{array}{ccccc} \tilde{\pi} A & \xrightarrow{j_*} & \tilde{\pi}\tilde{A}_f & \longrightarrow & \tilde{\pi} A \\ {\scriptstyle i_*}\downarrow & & \downarrow{\scriptstyle i'_*} & & {\scriptstyle i_*}\downarrow \\ \tilde{\pi} B & \xrightarrow[j_*]{} & \tilde{\pi}\tilde{B}_f & \longrightarrow & \tilde{\pi} B \end{array}$$

shows that i_* is a retract of a map i'_* which is induced by map $i' : \tilde{A}_f \to \tilde{B}_f$ which is a cofibration and a weak equivalence in each section. It therefore suffices to assume that the cofibration i has this form.

The diagram (5) can be constructed by applying the associated sheaf functor to a pushout diagram

$$\begin{array}{ccc} \pi A & \longrightarrow & G \\ {\scriptstyle i_*}\downarrow & & \downarrow{\scriptstyle i''} \\ \pi B & \longrightarrow & H' \end{array}$$

On account of the assumptions on i the induced map i_* is a sectionwise cofibration and sectionwise equivalence of presheaves of groupoids. It follows from the model structure on ordinary groupoids that the map i'' has the same properties. The induced map $i' : G \to H$ on associated sheaves is therefore a local weak equivalence, as desired. □

Remark: One can show that if G is a sheaf of groupoids, then G is a stack if and only if G satisfies descent in the sense that any fibrant model $G \to G_f$ is a sectionwise weak equivalence. We can therefore identify stacks with homotopy types of presheaves of groupoids [17].

10 Torsors and stacks

Suppose now that G is a sheaf of groupoids on some site $\mathcal{C}$.

A *G-diagram* X consists of functors $X(U) : G(U) \to \mathbf{sSet}$ with $x \mapsto X(U)_x$ such that the simplicial set maps

$$\bigsqcup_{x \in \mathrm{Ob}(G(U))} X_x \to \mathrm{Ob}(G(U))$$

define a simplicial presheaf map $X \to \mathrm{Ob}(G)$, and such that all diagrams

$$\begin{array}{ccc} X(U)_x & \xrightarrow{\alpha} & X(U)_y \\ {\scriptstyle \phi^*}\downarrow & & \downarrow{\scriptstyle \phi^*} \\ X(V)_{\phi^* x} & \xrightarrow[\phi^*(\alpha)]{} & X(V)_{\phi^* y} \end{array}$$

commute for all $\alpha : x \to y$ in $G(U)$ and all morphisms $\phi : V \to U$ of $\mathcal{C}$.

Equivalently, a G-diagram X is a simplicial presheaf map $\pi : X \to \mathrm{Ob}(G)$ with an action

$$\begin{array}{ccc} X \times_s \mathrm{Mor}(G) & \xrightarrow{m} & X \\ {\scriptstyle pr}\downarrow & & \downarrow{\scriptstyle \pi} \\ \mathrm{Mor}(G) & \xrightarrow[t=d_0]{} & \mathrm{Ob}(G) \end{array}$$

suitably defined ($s = d_1$).

One can form the homotopy colimit $\underrightarrow{\mathrm{holim}}_G X$ section by section, and there is a canonical simplicial presheaf map $\underrightarrow{\mathrm{holim}}_G X \to BG$.

A *G-torsor* is a G-diagram X such that the canonical map $\underrightarrow{\mathrm{holim}}_G X \to *$ is a weak equivalence. A map $X \to Y$ of G-torsors is just a natural transformation, or equivalently it's a simplicial presheaf map

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow \quad \swarrow & \\ & \mathrm{Ob}(G) & \end{array}$$

fibred over $\mathrm{Ob}(G)$ which respects the actions. Write $G - \mathbf{tors}$ for the category of G-torsors.

If X is a G-torsor then the diagram

$$\begin{array}{ccc} X & \longrightarrow & \underrightarrow{\mathrm{holim}}_G X \\ {\scriptstyle \pi}\downarrow & & \downarrow \\ \mathrm{Ob}(G) & \longrightarrow & BG \end{array}$$

is homotopy cartesian by Quillen's Theorem B, and so every map $X \to Y$ of G-torsors is a weak equivalence fibred over $\mathrm{Ob}(G)$. Note that if X and Y are sheaves (concentrated in degree 0) as well as G-torsors, then this same argument implies that the map $f : X \to Y$ is an isomorphism.

Quillen's Theorem B (in the form that I like, at least) says that if $X : I \to s\mathbf{Set}$ is a small diagram of simplicial sets such that all morphisms $i \to j$ of I induce weak equivlences $X(i) \to X(j)$, then the diagram

$$\begin{array}{ccc} X & \longrightarrow & \underrightarrow{\mathrm{holim}}_I X \\ \downarrow & & \downarrow \\ \mathrm{Ob}(I) & \longrightarrow & BI \end{array}$$

is homotopy cartesian. Here, $X = \bigsqcup_i X(i)$ and the map $X \to \mathrm{Ob}(I)$ is the collection of maps $X(i) \to *$.

Suppose that $* \xleftarrow{\simeq} Y \to BG$ is a cocycle, and form the G-diagram $\mathrm{pb}(Y)$ in sections by the pullbacks

$$\begin{array}{ccc} \mathrm{pb}(Y)(U)_x & \longrightarrow & Y(U) \\ \downarrow & & \downarrow \\ BG(U)/x & \longrightarrow & BG(U) \end{array}$$

Then by formal nonsense there is a weak equivalence

$$\underrightarrow{\mathrm{holim}}_G\, \mathrm{pb}(Y) \xrightarrow{\simeq} Y \simeq *$$

and so $\mathrm{pb}(Y)$ is a G-torsor.

If X is a G-torsor then $* \xleftarrow{\simeq} \underrightarrow{\mathrm{holim}}_G X \to BG$ is a cocycle, and there is a weak equivalence G-diagrams $X \to \mathrm{pb}(\underrightarrow{\mathrm{holim}}_G X)$, again by Quillen's Theorem B.

Theorem 21. *These constructions and equivalences are natural, giving bijections*

$$\pi_0(G - \mathbf{tors}) \cong \pi_0 H(*, BG) \cong [*, BG].$$

A *discrete G-torsor* is a G-diagram Z taking values in sheaves (identified with simplicial presheaves concentrated in degree 0) such that

$$\underrightarrow{\mathrm{holim}}_G Z = B(E_G Z) \to *$$

is a weak equivalence.

Suppose that X is a G-torsor as above. Then Quillen's Theorem B implies that the canonical map $X \to \tilde{\pi}_0 X$ is a weak equivalence of G-diagrams, where $\tilde{\pi}_0 X$ is the sheaf of path components of X. It follows that $\tilde{\pi}_0 X$ is a discrete G-torsor, and every G-torsor X is naturally equivalent to a discrete G-torsor.

Write $G - \mathbf{tors}_d$ for the category of discrete G-torsors. Then this category is a groupoid, and there are natural bijections

$$\pi_0(G - \mathbf{tors}_d) \cong \pi_0(G - \mathbf{tors}) \cong [*, BG].$$

I should write $G - \mathbf{tors}_d(*)$ for $G - \mathbf{tors}_d$, since it's really global sections of a presheaf of groupoids $G - \mathbf{tors}_d$. In effect, all G-diagrams X restrict to $G|_U$-diagrams $X|_U$ on $\mathcal{C}/U$ for all $U \in \mathcal{C}$.

There is a map $j : G \to G - \mathbf{tors}_d$ of presheaves of groupoids. One way to describe it is that we send $x \in G(*)$ to the cocycle

$$* \xleftarrow{\simeq} B(G/x) \to BG$$

The path component sheaf of the pullback

$$B(G/y) \times_{BG} B(G/x)$$

is the sheaf $G(y, x)$ and the action of $\alpha : z \to y$ is the map $G(z, x) \to G(y, x)$ defined by precomposition with α^{-1}. In other words, $j(x) \cong G(\ , x)$.

Theorem 22. *1) The map $j : G \to G - \mathbf{tors}_d$ is a weak equivalence.*

2) The presheaf of groupoids $G - \mathbf{tors}_d$ satisfies descent, and is therefore sectionwise weakly equivalent to its associated stack.

Remark: In other words, $G - \mathbf{tors}_d$ is an explicit model for the associated stack of G.

Theorem 22 is a a consequence of the following three lemmas. This result is the subject of [21].

Lemma 23. *The map $j : BG \to B(G - \mathbf{tors}_d)$ is a local weak equivalence.*

Proof. The map j is fully faithful in all sections, so it suffices to show that the map

$$\pi_0 BG \to \pi_0 B(G - \mathbf{tors}_d)$$

is a local epimorphism.

If a discrete G-torsor X has a section $x \in X(U)_y$ then there is a G-equivariant map $\hom(\ ,y) \to X|_U$ which classfies x, in which case there is an isomorphism $\hom(\ ,y) \cong X|_U$.

If X is a discrete G-torsor then

$$\pi_0 \underrightarrow{\mathrm{holim}}_G X = \varinjlim_G X \to *$$

is a local epimorphism, so there is a covering $U \to *$ which lifts to $\varinjlim_G X$ and hence to X. In other words, X has sections locally, so X is locally isomorphic to some objects in the image of j. □

Lemma 24. *If $f : G \to H$ is a weak equivalence of sheaves of groupoids, then*

$$f_* : B(G - \mathbf{tors}_d) \to B(H - \mathbf{tors}_d)$$

is a sectionwise weak equivalence.

The map f_* in the statement of Lemma 24 is defined by left Kan extension. In sections, for a G-torsor X and for each $x \in H$, there are pullback diagrams

$$\begin{array}{ccc}
\underrightarrow{\mathrm{holim}}_{f/x} X|_{f/x} & \longrightarrow & \underrightarrow{\mathrm{holim}}_G X \\
\downarrow & & \downarrow \\
B(f/x) & \longrightarrow & BG \\
\downarrow & & \downarrow \\
B(H/x) & \longrightarrow & BH
\end{array}$$

and these diagrams are homotopy cartesian by Quillen's Theorem B. The left Kan extension of X is defined by sending x to $\pi_0(\underrightarrow{\mathrm{holim}}_{f/x} X|_{f/x})$. At the same time, the map $\underrightarrow{\text{holim}}_G X \to *$ is a local weak equivalence since X is a G-torsor, and so the maps

$$\underrightarrow{\text{holim}}_{f/x} X|_{f/x} \to \pi_0(\underrightarrow{\text{holim}}_{f/x} X|_{f/x})$$

define a local weak equivalence. But finally, the map

$$\underrightarrow{\text{holim}}_{x \in H}(\underrightarrow{\text{holim}}_{f/x} X|_{f/x}) \to \underrightarrow{\text{holim}}_G X$$

is a weak equivalence, so that the left Kan extension of X represents a sheaf which is a discrete G-torsor.

Proof of Lemma 24. The map f_* is a local weak equivalence, and therefore induces isomorphisms

$$[*, BG] \cong [*, BH].$$

This takes care of π_0 (in global sections).

Every torsor X is a sheaf which is locally isomorphic to torsors $j(x_U)$. The maps j and f are fully faithful, so f induces an isomorphism $\text{Aut}(X) \to \text{Aut}(f_* X)$. □

Lemma 25. *Suppose that H is globally fibrant. Then $j : H \to H - \mathbf{tors}_d$ is a sectionwise equivalence.*

Proof. The map $j : H \to H - \mathbf{tors}_d$ is sectionwise fully faithful. It suffices to show that

$$\pi_0 H(*) \to \pi_0 H - \mathbf{tors}_d(*)$$

is a surjection. But H is globally fibrant, so the lifting exists up to homotopy in all diagrams

$$\begin{array}{ccc} X & \longrightarrow & BH \\ {\scriptstyle\simeq}\downarrow & \nearrow & \\ * & & \end{array}$$

□

Remark: The description of torsors given here fits is part of a very general story [23]. Suppose that I is a presheaf of categories enriched in simplicial sets. An I-diagram is an enriched functor $X \to \text{Ob}(I)$ taking values in simplicial presheaves, defined as above. An I-torsor is an I-diagram which is a diagram of equivalences in the sense that all action diagrams

$$\begin{array}{ccc} X \times_s \text{Mor}(I) & \xrightarrow{m} & X \\ \downarrow & & \downarrow \\ \text{Mor}(I) & \xrightarrow[t]{} & \text{Ob}(I) \end{array}$$

are homotopy cartesian, and such that the map $\underrightarrow{\text{holim}}_I X \to *$ is a weak equivalence. There is a corresponding category of $I - \mathbf{tors}$ of I-torsors (diagrams and natural transformations), and there is a homotopy classification result

$$\pi_0(I - \mathbf{tors}) \cong [*, BI].$$

There is a motivic version of this definition and homotopy classification of I-torsors.

11 Simplicial groupoids

Write $s\mathbf{Gpd}$ for the category of groupoids enriched in simplicial sets. Following a common abuse, an object of this category will be called a simplicial groupoid here. Not all simplicial object in the category of groupoids are simplicial groupoids in this sense, because we are requiring that the objects of a simplicial groupoid form a discrete simplicial set.

Examples:

1) Suppose that G is an ordinary groupoid. There is a corresponding (discrete) simplicial groupoid whose objects and morphisms are both discrete simplicial sets.

2) Suppose that H is a 2-groupoid, aka. a groupoid enriched in groupoids. Then there is a simplicial groupoid whose objects are the objects of H and whose morphisms are the classifying simplicial sets $BH(x,y)$ of the morphism groupoids $H(x,y)$.

3) Suppose that G is a group. Then there is a one-object 2-groupoid $\text{Aut}(G)$ whose 1-cells are the automorphisms of G and whose 2-cells are the homotopies (ie. conjugations) of automorphisms.

Every simplicial groupoid H has an associated groupoid $\pi_0 H$ with the same objects as H, and with

$$(\pi_0 H)(x,y) = \pi_0(H(x,y)).$$

There is a simplicial groupoid morphism $H \to \pi_0 H$ which is initial among maps from H to discrete simplicial groupoids. The group $\pi_0 \text{Aut}(G)$ is the group $\text{Out}(G)$ of outer automorphisms of G. The corresponding map $\text{Aut}(G) \to \text{Out}(G)$ is special: in sheaf land its homotopy fibres classify gerbes locally equivalent to G with specific choices of bands (which are torsors for the group $\text{Out}(G)$) — see [25].

4) Suppose that $p : G \to H$ is a group epimorphism. There is a 2-groupoid $\tilde{p}$ with the same 0-cells and 1-cells as G, and with a unique 2-cell for each pair of elements x, y of G such that $p(x) = p(y)$. The canonical map $\tilde{p} \to H$ turns out to be a weak equivalence, and if K is the kernel of p there is a 2-groupoid morphism $\tilde{p} \to \text{Aut}(K)$ which is defined by conjugation. The resulting picture

$$H \xleftarrow{\simeq} \tilde{p} \to \text{Aut}(K)$$

is a cocycle in 2-groupoids which is canonically associated to the extension of K by H.

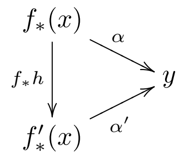
5) The 2-groupoid $\mathrm{Aut}(G)$ of automorphisms for a group G has a big brother, namely the groupoid **Grp** of groups, isomorphisms between groups and homotopies of isomorphisms.

Suppose that H is a groupoid. Then there is a 2-groupoid $\tilde{H}$ whose 0-cells and 1-cells are those of H, and which has a unique 2-cell $f \to g$ for any pair of morphisms $f, g : x \to y$ in H. Then H determines a 2-groupoid morphism $F(H) : \tilde{H} \to \mathbf{Grp}$, which takes $x \in H$ to the group $H(x, x)$, takes a morphism (isomorphism) $f : x \to y$ of H to the isomorphism $H(x, x) \to H(y, y)$ defined by conjugation by f, ie. $\alpha \mapsto f\alpha f^{-1}$. Finally, $F(H)$ takes a 2-cell $f \to g$ to conjugation by gf^{-1}. There is a map of 2-groupoids $H \to \tilde{H}$ defined by identifying H with the vertices of $\tilde{H}$, and this map induces an isomorphism $\pi_0 H \cong \pi_0(\tilde{H})$. The object $\tilde{H}$ has no higher homotopy groups. Thus, we get a cocycle

$$\pi_0 H \xleftarrow{\simeq} \tilde{H} \xrightarrow{F(H)} \mathbf{Grp}.$$

If H is connected then $\pi_0 H = *$, and this canonical cocycle construction defines a function from path components of the category of weak equivalences of connected groupoids to $\pi_0 H(*, \mathbf{Grp})$, meaning path components of 2-cocycles.

Going backwards involves a generalization of the Grothendieck construction. starting with a 2-functor $K : A \to \mathbf{cat}$ defined on a 2-groupoid A, one can form a category $E_A K$ with objects (a, x) with a a 0-cell of A and $x \in K(a)$. A morphism $(a, x) \to (b, y)$ of $E_A K$ is an equivalence class of pairs $(f, \alpha) : (a, x) \to (b, y)$ where $f : a \to b$ is a 1-cell of A and $\alpha : f_*(x) \to y$ is a morphism of $K(b)$. The pairs $(f, \alpha), (f', \beta)$ are equivalent if there is a 2-cell $h : \alpha \to \alpha'$ of A such that the diagram



commutes.

Fiddling with this a little bit shows that these two operations set up a bijection between weak equivalence classes of connected groupoids and the class $\pi_0 H(*, \mathbf{Grp})$ of path components of 2-cocyles in **Grp**. These constructions can be made local, and set up a "homotopy" classification of gerbes. Gerbes are locally connected stacks. Again, see [25].

There is a functor

$$\overline{W} : s\mathbf{Gpd} \to s\mathbf{Set}$$

which generalizes the nerve construction on groupoids as well as the Eilenberg-Mac Lane $\overline{W}$-construction for simplicial groups, and is sometimes called the universal cocycle construction.

This functor is a bit tricky to describe, and it's easier to define and keep track of it for simplicial objects in groupoids. Such a thing G consists of groupoids G_n, with simplicial structure morphisms $G_n \to G_m$. Then G has an associated Grothendieck construction $E_{\mathbf{\Delta}} G$ which is a category with objects consisting of pairs $(\mathbf{n}, x)$ with $\mathbf{n}$ an ordinal number and $x \in \mathrm{Ob}(G)_n$. A morphism

$$(\theta, f) : (\mathbf{m}, y) \to (\mathbf{n}, x)$$

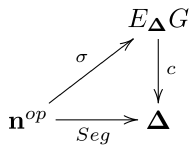
in $E_{\mathbf{\Delta}} G$ consists of an ordinal number map $\theta : \mathbf{m} \to \mathbf{n}$ and a morphism $f : y \to \theta^* x$ of G_m. There is a functor

$$Seg : \mathbf{n}^{op} \to \mathbf{\Delta}$$

which is defined by taking $i \in \mathbf{n}$ to the segment

$$[i, n] = \{j \mid i \leq j \leq n\} \subset \mathbf{n}.$$

Finally, an n-simplex σ of $\overline{W} G_n$ is a lifting



of the segment functor, where c is the canonical functor. In other words σ consists of objects $x_i \in G([i, n])$, $0 \leq i \leq n$ together with morphisms

$$x_j \to (i, j)^* x_i \quad \text{in } G([j, n]),$$

for each morphism $i \leq j$ of $\mathbf{n}$. Note that if $i \leq j$, then there is an inclusion of segments

$$(i, j) : [j, n] \subset [i, n].$$

This map (i, j) is an iterated composite of coface maps d^0.

If $\theta : \mathbf{m} \to \mathbf{n}$ is an ordinal morphism, then θ restricts to ordinal number morphisms

$$\theta_i : [i, m] \to [\theta(i), n]$$

which respect segment inclusions and $\sigma \in \overline{W} G_n$ is as above, then $\theta^*(\sigma)$ consists of the objects $\theta_i^* x_{\theta(i)}$ of $G([i, m])$ and the morphisms

$$\theta_j^* x_{\theta(j)} \to \theta_j^* (\theta(i), \theta(j))^* x_{\theta(i)} = (i, j)^* \theta_i^* x_{\theta(i)}.$$

Suppose that $\sigma : \mathbf{n} \to G_n$ is an n-simplex of the diagonal dBG of the bisimplicial set BG, and that it consists of objects y_i and morphisms $\alpha : y_i \to y_j$ for $i \leq j$. Define $\gamma(x) \in \overline{W} G_n$ by setting

$$\gamma(x)_i = (0, i)^* x_i.$$

The maps

$$(0,j)^*x_j \xrightarrow{(0,j)^*(\alpha^{-1})} (0,j)^*x_i = (i,j)^*(0,i)^*x_i$$

defined for all $i \leq j$ complete the structure of an n-simplex $\gamma(\sigma)$ of $\overline{W}G$. One shows that the maps γ define a natural simplicial set map

$$\gamma : dBG \to \overline{W}G.$$

Then we have

Lemma 26. *The map $\gamma : dBG \to \overline{W}G$ is a weak equivalence for groupoids G enriched in simplicial sets.*

Both constructions preserve homotopies, and each simplicial groupoid G is homotopy equivalent to a disjoint union of simplicial groups. For simplicial groups the satement is essentially classical, because γ induces an equivariant comparison of princicpal G-bundles with contractible total spaces.

Lemma 27. *The functor $G \mapsto \overline{W}G$ takes values in Kan complexes.*

This is true if G is a simplicial group, and all lifting problems for $\overline{W}G$ can be pushed into lifting problems for $\overline{W}G(x,x)$ for some simplicial group of automorphisms of G is connected, via a contracting homotopy.

Note that an n-simplex σ of $\overline{W}G$ can be completely specified by the images of the morphisms $i \leq i+1$ in $\mathbf{n}$. Thus σ can be thought of as a string of arrows

$$x_0 \xrightarrow{g_1} x_1 \xrightarrow{g_2} \dots \xrightarrow{g_n} x_n$$

with g_i a morphism of the groupoid G_{i-1}.

These same observations lead to the definition of the left adjoint

$$G : s\mathbf{Set} \to s\mathbf{Gpd}$$

of the functor $\overline{W}$, which is sometimes called the loop groupoid functor. Specifically $G(X)_n$ is the free groupoid on the graph

$$x : x_1 \to x_0$$

with $x \in X_{n+1}$, subject to the relation $s_0x = 1_{x_0}$. Here, x_i is the image of the vertex $i \in \mathbf{n}$ under the simplicial set map $x : \Delta^n \to X$. The objects of this simplicial groupoid are the vertices of X. $G(X)$ is sometimes called the loop groupoid of X.

Say that a weak equivalence of $s\mathbf{Gpd}$ is a map $f : G \to H$ such that $\overline{W}(f)$ is a weak equivalence of simplicial sets. In view of Lemma 26, we could equally well require that the induced map $dBG \to dBH$ is a weak equivalence of simplicial sets.

Observe that if G is a simplicial groupoid, then all simplicial structure functors $G_n \to G_m$ induce bijections $\pi_0G_n \cong \pi_0G_m$. Write $\pi_0G = \pi_0G_0$, and observe that there are natural bijections

$$\pi_0G \cong \pi_0 dBG \cong \pi_0\overline{W}G.$$

Proposition 28. *Suppose that $f : G \to H$ is a map of simplicial groupoids. Then f is a weak equivalence if and only if*

1) *the map $\pi_0 G \to \pi_0 H$ is a bijection, and*

2) *the simplicial set map $G(x,y) \to H(f(x), f(y))$ is a weak equivalence for all pairs x, y of objects of G.*

Proof. Suppose that G is a simplicial groupoid, and let x be an object of G. Then the assignment $y \mapsto H(x,y)$ defines an enriched functor on G taking values in simplicial sets. This functor has homotopy colimit $B(x/G)$, where x/G is the simplicial object in groupoids defined by the slice groupoids x/G_n. It follows from the simplicially enriched version of Quillen's Theorem B (originally due to Moerdijk — see Lemma 2 of [23]) that the diagram

$$\begin{array}{ccc} G(x,y) & \longrightarrow & Bx/G \\ \downarrow & & \downarrow{\scriptstyle \pi} \\ * & \xrightarrow[y]{} & BG \end{array} \tag{6}$$

is homotopy cartesian for each pair of objects x, y of G. Of course the space dBx/G is contractible, so that the simplicial set $G(x,y)$ is a model for the loop space ΩdBG at the vertex y when $G(x,y)$ is non-empty.

It follows that if the simplicial set map $dBG \to dBH$ induced by a morphism $f : G \to H$ of simplicial groupoids having discrete object sets is a weak equivalence then the function $\pi_0 G \to \pi_0 H$ is a bijection and the simplicial set maps $G(x,y) \to H(fx, fy)$ are weak equivalences for all objects x, y.

For the converse, if $f : G \to H$ satisfies conditions 1) and 2), then a comparison of homotopy cartesian diagrams of the form (6) implies that f induces a weak equivalence $G_x \to H_{f(x)}$ for all $x \in \mathrm{Ob}(G)$. Here, G_x denotes the path component of an object x in G. The object dBG_x is the path component of the vertex x in dBG. It follows that the map $f_* : dBG \to dBH$ induces weak equivalences $dBG_x \to dBH_{f(X)}$ in all path components, and induces a bijection $\pi_0 dBG \to \pi_0 dBH$. The map $dBG \to dBH$ is therefore a weak equivalence. □

The criteria for a weak equivalence appearing in Proposition 28 amount to the Dwyer-Kan definition of weak equivalence of simplicial groupoids [8, p.297]. Dwyer and Kan also say that a map $p : G \to H$ of simplicial groupoids is a fibration if the morphism of groupoids $p : G_0 \to H_0$ has the path lifting property, and if all maps $p : G(x,y) \to H(f(x), f(y))$ are Kan fibrations.

Theorem 29 (Dwyer-Kan)**.**

1) *With these definitions, the category $s\mathbf{Gpd}$ of simplicial groupoids satisfies the axioms for a right proper closed model category.*

2) *The functor $\overline{W}$ preserves fibrations. A simplicial set map $K \to \overline{W}X$ is a weak equivalence if and only if its adjoint $GK \to X$ is a weak equivalence of $s\mathbf{Gpd}$.*

See [8, pp.300–305] for proofs.

Say that the fibrations of this last result are Dwyer-Kan fibrations. I say that a map $p : G \to H$ of simplicial groupoids is a fibration if the induced map $p_* : \overline{W}G \to \overline{W}H$ is a fibration of simplicial sets. Then we have the following:

Theorem 30. *1) With these definitions, s**Gpd** has the structure of a right proper closed model category.*

2) The functors G and $\overline{W}$ determine a Quillen equivalence

$$G : s\mathbf{Set} \leftrightarrows s\mathbf{Gpd} : \overline{W}.$$

Proof. A map $p : H \to K$ is a fibration (respectively trivial cofibration) for this theory if and only if it has the right lifting property with respect to all maps $GA \to GB$ induced by a generating set of trivial cofibrations (respectively cofibrations) $A \to B$ of simplicial sets. It follows in particular that every map $f : G \to H$ has a factorization

$$\begin{array}{ccc} G & \xrightarrow{i} & X \\ & \searrow^{f} & \downarrow q \\ & & H \end{array}$$

where i is a cofibration and q is a trivial fibration.

The map $f : G \to H$ also has a factorization

$$\begin{array}{ccc} G & \xrightarrow{j'} & Y \\ & \searrow^{f} & \downarrow p' \\ & & H \end{array}$$

such that p' is a Dwyer-Kan fibration and j' is a Dwyer-Kan cofibration and a weak equivalence. The map p' is a fibration by the second statement of the Dwyer-Kan Theorem, while j' has a factorization $j' = \pi \cdot j$ where j is a cofibration and π is a trivial fibration. But then j is also a weak equivalence and the composite $p' \cdot \pi$ is a fibration, and the factorization axiom is proved. □

Write $\mathrm{Pre}(s\mathbf{Gpd})(\mathcal{C})$ for the category of presheaves on $\mathcal{C}$ taking values in groupoids enriched in simplicial sets. The functors $\overline{W}$ and G together determine an adjoint pair of functors

$$G : s\,\mathrm{Pre}(\mathcal{C}) \leftrightarrows \mathrm{Pre}(s\mathbf{Gpd})(\mathcal{C}) : \overline{W},$$

and I say that a map $f : G \to H$ of presheaves of groupoids enriched in simplicial sets is a weak equivalence (respectively fibration) if and only if the induced map $f_* : \overline{W}G \to \overline{W}H$ is a local weak equivalence (respectively global fibration) of simplicial presheaves.

Here is the local version of Theorem 30

Theorem 31. *1) With these definitions, the category* $\mathrm{Pre}(s\mathbf{Gpd})(\mathcal{C})$ *has the structure of a right proper closed model category.*

2) A simplicial presheaf map $K \to \overline{W}X$ *is a weak equivalence if and only if its adjoint* $GK \to X$ *is a weak equivalence of simplicial groupoids. In particular, the functors* G *and* $\overline{W}$ *preserve weak equivalences and determine a Quillen equivalence*

$$G : s\,\mathrm{Pre}(\mathcal{C}) \leftrightarrows \mathrm{Pre}(s\mathbf{Gpd})(\mathcal{C}) : \overline{W}.$$

Much of 2) is trivial. The statement about adjoints of weak equivalences reflecting weak equivalences is a consequence of the Dwyer-Kan theorem.

The verification of the model structure axioms in 1) is the same as that for the corresponding statement for presheaves of groupoids. One uses a Boolean localization argument to show that in a pushout diagram

$$\begin{array}{ccc} G(A) & \longrightarrow & H \\ {\scriptstyle i_*}\downarrow & & \downarrow{\scriptstyle i'} \\ G(B) & \longrightarrow & H' \end{array}$$

for which i_* is induced by a trivial cofibration $i : A \to B$ of simplicial presheaves the map i' is a weak equivalence of presheaves of simplicial groupoids. In particular, it's enough to prove this property for pushout diagrams of sheaves on a complete Boolean algebra. But $\overline{W}$ takes values in presheaves of Kan complexes, so we can use the same trick as before to assume that the simplicial sheaves A and B are both locally fibrant. But then $i : A \to B$ is a sectionwise weak equivalence, so the result follows from the Dwyer-Kan model structure.

Say that a presheaf of simplicial groupoids H is a *(higher) stack* if it satisfies descent in the sense that some fibrant model $j : G \to H$ (hence any) is a sectionwise weak equivalence. The definition means, just as it did for stacks in groupoids, that you can identify a stack with a homotopy type in presheaves of simplicial groupoids.

There is also a notion of 2-stack which arises in an analogous way from a model structure on presheaves of 2-groupoids, which we shall now describe. These are the objects which are involved in the homotopy theoretic classifications of the various flavours of gerbes that one finds in [25].

Suppose that G is a 2-groupoid, with morphism groupoids $G(x, y)$ for all pairs of objects x, y of G. Then there is a simplicial groupoid BG having the same objects as G and with morphism sets $BG(x, y)$. This construction is natural in 2-groupoids G, and so there is a functor

$$B : 2 - \mathbf{Gpd} \to s\mathbf{Gpd}.$$

The functor B has a left adjoint

$$\pi : s\mathbf{Gpd} \to 2 - \mathbf{Gpd}.$$

In particular, if H is a simplicial groupoid with object simplicial sets $H(x, y)$, then $\pi(H)$ is the 2-groupoid with the same objects as H and with morphism groupoids $\pi H(x, y)$ defined by applying the fundamental groupoid functor π to the simplicial sets $H(x, y)$..

Say that a morphism $f : G \to H$ of 2-groupoids is a weak equivalence (respectively fibration) if the induced map $f_* : BG \to BH$ of simplicial groupoids is a weak equivalence (respectively fibration).

Observe that the canonical morphism

$$\epsilon : \pi BG \to G$$

is a natural isomorphism of 2-groupoids.

Lemma 32. *The functor* $\pi : s\mathbf{Gpd} \to 2 - \mathbf{Gpd}$ *preserves weak equivalences.*

Proof. Suppose that $f : G \to H$ is a weak equivalence of simplicial groupoids The canonical map $\eta : H \to B\pi H$ is an isomorphism in groupoids in simplicial degree 0, and therefore induces a natural bijection $\pi_0 H \cong \pi_0 B\pi H$. It follows that the induced map

$$\pi_0 B\pi G \to \pi_0 B\pi H$$

is a bijection.

All simplicial set maps $G(x, y) \to H(f(x), f(y))$ are weak equivalences, so that the induced maps

$$B\pi G(x, y) \to B\pi H(f(x), f(y))$$

are weak equivalences. □

Write $\text{Pre}(2 - \mathbf{Gpd})(\mathcal{C})$ for the category of presheaves of 2-groupoids on a site $\mathcal{C}$. The functors B and π relating 2-groupoids and simplicial groupoids determine an adjoint pair of functors

$$\pi : \text{Pre}(s\mathbf{Gpd})(\mathcal{C}) \leftrightarrows \text{Pre}(2 - \mathbf{Gpd})(\mathcal{C}) : B$$

in an obvious way. I say that a map $f : G \to H$ is a weak equivalence (respectively fibration) of presheaves of 2-groupoids if the induced map $BG \to BH$ is a weak equivalence (respectively fibration) of presheaves of simplicial groupoids.

Lemma 33. *Suppose that* $f : G \to H$ *is a weak equivalence of presheaves of simplicial groupoids. Then the induced map* $f_* : \pi G \to \pi H$ *is a weak equivalence of presheaves of* 2*-groupoids.*

Proof. It suffices to prove the claim for a map $f : G \to H$ of sheaves of simplicial groupoids on a complete Boolean algebra $\mathcal{B}$. But then the induced map $\overline{W}G \to \overline{W}H$ is a local weak equivalence of locally fibrant simplicial sheaves, so that it is a sectionwise weak equivalence. In particular, all maps $G(U) \to H(U)$ in sections are weak equivalences of ordinary simplicial groupoids, so that the maps $\pi G(U) \to \pi H(U)$ are weak equivalences of 2-groupoids by Lemma 32. This means that the map $B\pi G \to B\pi H$ is a sectionwise weak equivalence of presheaves of simplicial groupoids. □

Lemma 34. *Suppose that $i : A \to B$ is a trivial cofibration of presheaves of simplicial groupoids, and form the pushout diagram*

$$\begin{array}{ccc} \pi A & \xrightarrow{\alpha} & G \\ {\scriptstyle i_*}\downarrow & & \downarrow{\scriptstyle i'} \\ \pi B & \longrightarrow & H \end{array} \tag{7}$$

in presheaves of 2-groupoids. Then the induced map $i' : G \to H$ is a weak equivalence of presheaves of 2-groupoids.

Proof. Let $\alpha_* : A \to BG$ be the adjoint of α, and form the pushout diagram

$$\begin{array}{ccc} A & \xrightarrow{\alpha_*} & BG \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i''} \\ B & \longrightarrow & X \end{array}$$

The pushout diagram (7) is obtained from the one just given, up to isomorphism, by applying the functor π. It therefore suffices to show that the map $i''_* : \pi BG \to \pi X$ is a weak equivalence. But i'' is a trivial cofibration of presheaves of simplicial groupoids and π preserves weak equivalences by Lemma 33. □

Proposition 35. *1) With the definitions given above, the category* $\mathrm{Pre}(2-\mathbf{Gpd})(\mathcal{C})$ *of presheaves of 2-groupoids on a small Grothendieck site $\mathcal{C}$ satisfies the axioms for a right proper closed model category.*

2) The functors π and B determine a Quillen adjunction

$$\pi : \mathrm{Pre}(s\mathbf{Gpd})(\mathcal{C}) \leftrightarrows \mathrm{Pre}(2-\mathbf{Gpd})(\mathcal{C}) : B.$$

Proof. The one interesting detail in the proof is the verification that every map $f : G \to H$ of 2-groupoids has a factorization

$$\begin{array}{ccc} G & \xrightarrow{i} & K \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle p} \\ & & H \end{array}$$

such that i is a trivial cofibration and p is a fibration, but this is a consequence of Lemma 34. □

12 Cubical sets

Simplicial sets are contravariant set-valued functors defined on the category of $\mathbf{\Delta}$ of finite sets and order preserving maps, and as such are artifacts of the combinatorics of finite sets. Cubical sets depend on or represent the combinatorics of the power sets of finite ordered sets.

Write

$$\underline{\mathrm{n}} = \{1, 2, \ldots, n\},$$

and write

$$\mathbf{1}^n = \mathbf{1}^{\times n}, \quad \mathbf{1} = \{0, 1\}.$$

$\mathbf{1}^0$ is the category consisting of one object and one morphism. $\mathcal{P}(\underline{\mathrm{n}})$ is the poset of subsets of the set $\underline{\mathrm{n}}$.

Fact: There is an isomorphism of posets

$$\Omega_n : \mathbf{1}^n \xrightarrow{\cong} \mathcal{P}(\underline{\mathrm{n}})$$

$\Omega_n(\epsilon) = \{i \mid \epsilon_i = 1\}$ for $\epsilon = (\epsilon_1, \ldots, \epsilon_n) \in \mathbf{1}^n$.

Every finite totally ordered set A has a unique order-preserving bijection $\underline{n} \to A$, and it is convenient to represent box category morphisms as poset morphisms $\mathcal{P}(A) \to \mathcal{P}(B)$ where A and B are finite ordered sets. There are two distinguished families of poset maps $\mathcal{P}(A) \to \mathcal{P}(B)$:

1) Suppose $A \subset B \subset C$ (finite ordered sets).

$$[A, B] = \{D \subset C | \ A \subset D \subset B\} \subset \mathcal{P}(C)$$

is called an *interval* of subsets. There is a canonical poset isomorphism

$$\mathcal{P}(B - A) \xrightarrow{\cong} [A, B]$$

defined by $E \mapsto A \cup E$. The composite

$$\mathcal{P}(B - A) \xrightarrow{\cong} [A, B] \subset \mathcal{P}(C)$$

is called a *face functor*, and is also denoted by $[A, B]$.

2) Suppose $\emptyset \neq B \subset C$ (finite ordered sets). There is a poset morphism

$$s_B : \mathcal{P}(C) \to \mathcal{P}(B)$$

defined by $E \mapsto E \cap B$. s_B is called a *degeneracy functor*.

The *box category* $\square$ is the subcategory of the category of poset morphisms $\mathbf{1}^m \to \mathbf{1}^n$ which is generated by the face and degeneracy functors.

Suppose that $A \subset B$ and E are subsets of a finite ordered set C. There is a commutative diagram

$$\begin{array}{ccc}
\mathcal{P}(B - A) & \xrightarrow{[A,B]} & \mathcal{P}(C) \\
{\scriptstyle s_{E \cap (B-A)}} \downarrow & & \downarrow {\scriptstyle s_E} \\
\mathcal{P}((B \cap E) - (A \cap E)) & \xrightarrow[{[A \cap E, B \cap E]}]{} & \mathcal{P}(E)
\end{array}$$

which allows one to show that all morphisms of the box category $\square$ are composites

$$\mathcal{P}(C) \xrightarrow{s_E} \mathcal{P}(E) \xrightarrow{[A, A\cup E]} \mathcal{P}(D).$$

These decompositions are unique

Examples:

1) Every $i \in C$ determines two intervals in $\mathcal{P}(C)$, namely $[\{i\}, C]$ and $[\emptyset, C - \{i\}]$. For $i \leq n$ the interval $[\{i\}, \underline{n}]$ uniquely determines a functor

$$d^{(i,1)} : \mathbf{1}^{n-1} \to \mathbf{1}^n,$$

while $[\emptyset, \underline{n} - \{i\}]$ determines a functor

$$d^{(i,0)} : \mathbf{1}^{n-1} \to \mathbf{1}^n.$$

For $\epsilon = 0, 1$ $d^{(i,\epsilon)} : \mathbf{1}^{n-1} \to \mathbf{1}^n$ is defined by

$$d^{(i,\epsilon)}(\gamma_1, \dots, \gamma_{n-1}) = (\gamma_1, \dots, \overset{i}{\epsilon}, \dots, \gamma_{n-1}).$$

2) Every $j \in \underline{n}$ determines a poset map $s_{\underline{n}-\{j\}} : \mathcal{P}(\underline{n}) \to \mathcal{P}(\underline{n} - \{j\})$, or $s^j : \mathbf{1}^n \to \mathbf{1}^{n-1}$. s^j is the projection which drops the j^{th} entry:

$$s^j(\gamma_1, \dots, \gamma_n) = (\gamma_1, \dots, \gamma_{j-1}, \gamma_{j+1}, \dots, \gamma_n)$$

$s^1 : \mathbf{1} \to \mathbf{1}^0$ is the map to the terminal object $\mathbf{1}^0$.

A *cubical set* X is a contravariant set-valued functor $X : \square^{op} \to \mathbf{Set}$. A morphism of cubical sets $f : X \to Y$ is a natural transformation of functors, and we have a category $c\mathbf{Set}$ of cubical sets.

Write $X_n = X(\mathbf{1}^n)$, and call this set the set of *n-cells* of X.

Examples:

1) *standard n-cell* $\square^n = \hom_\square(\ ,\mathbf{1}^n)$ Every $x \in X_n$ is classified by a cubical set map $x : \square^n \to X$. The faces $d_{(i,\epsilon)}(x)$ of x are the composites

$$\square^{n-1} \xrightarrow{d^{(i,\epsilon)}} \square^n \xrightarrow{x} X$$

The degeneracy $s_j(x)$ is represented by

$$\square^{n+1} \xrightarrow{s^j} \square^n \xrightarrow{x} X$$

A cell y is *degenerate* if $y = s_j x$; otherwise it is *non-degenerate.*

2) The *boundary* $\partial\square^n$ is the union of the images of the maps $d^{(i,\epsilon)} : \square^{n-1} \to \square^n$. There is a coequalizer

$$\bigsqcup_{\substack{(\epsilon_1,\epsilon_2) \\ 0\leq i<j\leq n}} \square^{n-2} \rightrightarrows \bigsqcup_{(i,\epsilon)} \square^{n-1} \to \partial\square^n$$

where $\epsilon_i \in \{0,1\}$.

3) $\sqcap^n_{(i,\epsilon)}$ is the subobject of $\partial\square^n$ which is generated by all faces except for $d^{(i,\epsilon)} : \square^{n-1} \to \square^n$. There is a coequalizer diagram

$$\bigsqcup \square^{n-2} \rightrightarrows \bigsqcup_{(j,\gamma)\neq(i,\epsilon)} \square^{n-1} \to \sqcap^n_{(\epsilon,i)}$$

where the first disjoint union is indexed over all pairs (j_1,γ_1), (j_2,γ_2) with $0 \leq j_1 < j_2 \leq n$ and $(j_k,\gamma_k) \neq (i,\epsilon)$, $k = 1,2$.

4) The assignment $\mathbf{1}^n \mapsto B(\mathbf{1}^n)$ defines a simplicial set-valued functor $\square \to \mathbf{S}$. If X is a simplicial set, there is an associated *singular* cubical set $S(X)$, with n-cells

$$S(X)_n = \hom(B(\mathbf{1}^n), X).$$

Note that $B(\mathbf{1}^n) \cong (\Delta^1)^{\times n}$. The singular functor $S : \mathbf{S} \to c\mathbf{Set}$ has a left adjoint $|\ | : c\mathbf{Set} \to \mathbf{S}$, called *triangulation*, which is defined by

$$|Y| = \varinjlim_{\square^n \to Y} B(\mathbf{1}^n).$$

The colimit is indexed by members of the *cell category* $i_\square Y$: the objects of the cell category are the cells $\square^m \to Y$ and the morphisms of $i_\square Y$ are the incidence relations

$$\begin{array}{ccc} \square^r & \longrightarrow & \square^m \\ & \searrow \quad \swarrow & \\ & Y & \end{array}$$

NB: there are similarly defined realization and singular functors

$$|\ | : c\mathbf{Set} \leftrightarrows \mathbf{Top} : S$$

relating cubical sets and topological spaces; realization is left adjoint to the singular functor.

4) Suppose that C is a small category. The *cubical nerve* $B_\square C$ is the cubical set with n-cells

$$B_\square C_n = \hom_{cat}(\mathbf{1}^n, C).$$

The cells of $B_\square C$ are the hypercube diagrams in C.

There is a good notion of skeleta for cubical sets: the n-skeleton $\operatorname{sk}_n X$ of a cubical set X is the subobject of X which is generated by the cells X_k, $0 \leq k \leq n$. Clearly, $\operatorname{sk}_{n-1} X \subset \operatorname{sk}_n X$, and one can show that there is a pushout

$$\begin{array}{ccc} \bigsqcup_{x\in NX_n} \partial\square^n & \longrightarrow & \operatorname{sk}_{n-1} X \\ \downarrow & & \downarrow \\ \bigsqcup_{x\in NX_n} \square^n & \longrightarrow & \operatorname{sk}_n X \end{array}$$

where NX_n denotes the non-degenerate part of X_n. Proving this requires showing that if x, y are degenerate n-cells with the same boundary, then $x = y$ — see Lemma 18 of "Categorical homotopy theory"

A *cofibration* of cubical sets is a monomorphism, a *weak equivalence* of cubical sets is a map $f : X \to Y$ which induces a weak equivalence $|X| \to |Y|$ of triangulations. A fibration of cubical sets is a map which has the RLP wrt to all inclusions $\sqcap^n_{(i,\epsilon)} \subset \square^n$.

Theorem 36 (Cisinski)**.** *1) With these definitions c***Set** *satisfies the axioms for a proper closed model category.*

2) The cubical singular and triangulation functors induce a Quillen equivalence

$$| \ | : \mathrm{Ho}(c\mathbf{Set}) \simeq \mathrm{Ho}(\mathbf{S}) : S.$$

Cisinski's theorem is perhaps the deepest result in abstract homotopy theory. It is mentioned for cultural reasons here, and will not be needed in the sequel. It is proved in Cisinski's thesis [6], and again in [20].

One can use standard techniques to show that there is a model structure on cubical sets for which the cofibrations are monomorphisms and weak equivalences are those maps which induce weak equivalences of triangulations, and that the resulting homotopy category is Quillen equivalent to the standard homotopy category for simplicial sets. Showing that the fibrations are as described is the interesting part.

There's one little problem: categorical products of cubical sets are very badly behaved.

Example: An n-cell $(\sigma, \tau) : \square^n \to \square^1 \times \square^1$ is a pair of n-cells of $\square^1$. $\square^1 \times \square^1$ has two distinct non-degenerate 2-cells, namely the identity on $\mathbf{1}^2$ and the twist automorphism $\tau : \mathbf{1}^2 \to \mathbf{1}^2$. These 2-cells have the common boundary that one expects, namely $\partial \square^2$ (up to a twist), but there is an additional non-degenerate 1-cell $\Delta : \mathbf{1} \to \mathbf{1}^2$ given by the diagonal map. It follows that $|\square^1 \times \square^1|$ has the homotopy type of $S^2 \vee S^1$.

The problem is fixed as follows: define

$$\square^n \otimes \square^m = \square^{n+m},$$

and more generally set

$$X \otimes Y = \varinjlim_{\square^n \to X,\ \square^m \to Y} \square^n \otimes \square^m.$$

Then one can show that there is a natural isomorphism

$$|X \otimes Y| \cong |X| \times |Y|.$$

Remark: I did not say that c**Set** has a simplicial model structure, because it doesn't. It has, instead, a cubical model structure, with function complex object $\mathbf{hom}(X, Y)$ specified by

$$\mathbf{hom}(X, Y)_n = \hom(X \otimes \square^n, Y).$$

Note that if $i : A \to B$ and $j : K \to L$ are cofibrations of cubical sets, then the induced map

$$(B \otimes K) \cup_{(A \otimes K)} (A \otimes L) \to (B \otimes L)$$

is a cofibration (for this you need to know that triangulation reflects monics — Corollary 23 of [25]) which is trivial if either i or j is trivial. It follows that if $p : X \to Y$ is a fibration and if $i : A \to B$ is a cofibration as above, then the map

$$\mathbf{hom}(B, X) \to \mathbf{hom}(A, X) \times_{\mathbf{hom}(A,Y)} \mathbf{hom}(B, Y)$$

is a fibration which is a weak equivalence if either i or p is trivial. This is the cubical analogue of Quillen's axiom **SM7** and so I say that cubical sets has a *cubical model structure*

13 Localization

Suppose that $\mathcal{A}$ is a small category, and say that an *$\mathcal{A}$-set* is a functor $X : \mathcal{A}^{op} \to \mathbf{Set}$, ie. a contravariant set-valued functor on $\mathcal{A}$. The $\mathcal{A}$-sets and natural transformations form a category, called the category of $\mathcal{A}$-sets and denoted by $\mathcal{A} - \mathbf{Set}$.

Examples:

1) $\mathcal{A} = \mathbf{\Delta}$: $\mathbf{\Delta} - \mathbf{Set}$ = simplicial sets.

2) $\mathcal{A} = \square$: $\square - \mathbf{Set}$ = cubical sets.

3) $\mathcal{A} = \mathcal{B} \times \mathcal{C}$ models presheaves of $\mathcal{B}$-sets (provided $\mathcal{B}$ and $\mathcal{C}$ are small).

Given $a \in \mathcal{A}$, the *standard a-cell* Δ^a is the functor represented by a, ie.

$$\Delta^a = \hom_{\mathcal{A}}(\ , a).$$

The *cell category* $i_{\mathcal{A}} X$ for an $\mathcal{A}$-set X has all morphisms $\Delta^a \to X$ as objects and morphisms all commutative diagrams

$$\begin{array}{ccc} \Delta^a & \longrightarrow & \Delta^b \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$

A map $f : X \to Y$ of $\mathcal{A}$-sets is said to be a "simplicial" weak equivalence (∞-equivalence in [6]) if $Bi_{\mathcal{A}} X \to Bi_{\mathcal{A}} Y$ is a weak equivalence of simplicial sets.

Examples:

1) Suppose that X is a simplicial set. The cell category $i_{\mathbf{\Delta}}X = \mathbf{\Delta}/X$ for X is its simplex category. There are canonical weak equivalences

$$\begin{array}{ccc} \underrightarrow{\mathrm{holim}}_{\,\Delta^n \to X}\ \Delta^n & \xrightarrow{\simeq} & X \\ \simeq \downarrow & & \\ Bi_{\mathbf{\Delta}}X & & \end{array}$$

so that $X \to Y$ is a weak equivalence of simplicial sets if and only if $Bi_{\mathbf{\Delta}}X \to Bi_{\mathbf{\Delta}}Y$ is a weak equivalence

2) Suppose that Y is a cubical set. There are natural weak equivalences

$$\begin{array}{ccc} \underrightarrow{\mathrm{holim}}_{\,\square^n \to Y}\ |\square^n| & \xrightarrow{\simeq} & |Y| \\ \simeq \downarrow & & \\ Bi_{\square}Y & & \end{array}$$

The horizontal equivalence is a bit subtle — it's a consequence of the "regularity" property of cubical sets, in Cisinski's language, which asserts that the map

$$\underrightarrow{\mathrm{holim}}_{\,\square^n \to X}\square^n \to X$$

is a weak equivalence of cubical sets for an internal description of homotopy colimit in cubical sets; you also need to know that realization preserves homotopy colimits. The regularity property itself results from the skeletal decomposition for cubical sets, and the fact that $X \mapsto Bi_{\mathcal{A}}X$ takes pushout squares to homotopy cocartesian diagrams (which, in itself, was a surprise). Once again, it follows that a map $f : X \to Y$ of cubical sets is a weak equivalence if and only if the map $Bi_{\square}X \to Bi_{\square}Y$ is a weak equivalence of cubical sets.

The pairing

$$(\square^n, \square^m) \mapsto \square^n \otimes \square^m = \square^{n+m}$$

defines a monoidal structure $\otimes : \square \times \square \to \square$ on the box category.

An *interval theory* on the category of $\mathcal{A}$-sets is a coherent action

$$\otimes : \mathcal{A} - \mathbf{Set} \times \square \to \mathcal{A} - \mathbf{Set}$$

of $\square$ on the category of $\mathcal{A}$-sets, written as

$$(X, \mathbf{1}^n) \mapsto X \otimes \square^n,$$

subject to the following conditions:

DH1: The functor $X \mapsto X \otimes \square^1$ preserves filtered colimits and monomorphisms.

DH2: Given a monomorphism $i : X \to Y$ and a $d^{(i,\epsilon)} : \square^{n-1} \to \square^n$ the following is a pullback

$$\begin{array}{ccc} X \otimes \square^{n-1} & \xrightarrow{i\otimes 1} & Y \otimes \square^{n-1} \\ {\scriptstyle 1\otimes d^{(i,\epsilon)}}\downarrow & & \downarrow{\scriptstyle 1\otimes d^{(i,\epsilon)}} \\ X \otimes \square^{n} & \xrightarrow[i\otimes 1]{} & Y \otimes \square^{n} \end{array}$$

DH3: The following is a pullback for $1 \leq i \leq n$:

$$\begin{array}{ccc} \emptyset & \longrightarrow & X \otimes \square^{n-1} \\ \downarrow & & \downarrow{\scriptstyle d^{(i,0)}} \\ X \otimes \square^{n-1} & \xrightarrow[d^{(i,1)}]{} & X \otimes \square^{n} \end{array}$$

Examples:

1) If I is any $\mathcal{A}$-set with a monomorphism $(d_0, d_1) : * \sqcup * \to I$ (ie. d_0 and d_1 are disjoint "rational points" of the interval I) then $(X, \mathbf{1}^n) \mapsto X \times I^{\times n}$ defines an interval theory

$$I : \mathcal{A} - \mathbf{Set} \times \square \to \mathcal{A} - \mathbf{Set}$$

2) The pairing $(X, Y) \mapsto X \otimes Y$ defines a monoidal structure on the category of cubical sets, and this structure induces a coherent action

$$\otimes : c\mathbf{Set} \times \square \to c\mathbf{Set}$$

of the box category on the category of cubical sets.

In the presence of an interval theory, we always have a cubical function complex construction. Explicitly, if $X \in \mathcal{A} - \mathbf{Set}$ and K is a cubical set, define

$$X \otimes K = \varinjlim_{\square^n \to K} X \otimes \square^n.$$

For $X, Y \in \mathcal{A} - \mathbf{Set}$, define a cubical set $\mathbf{hom}_\square(X, Y)$ by

$$\mathbf{hom}_\square(X, Y)_n = \hom(X \otimes \square^n, Y).$$

There is a natural bijection

$$\hom(X \otimes K, Y) \cong \hom(K, \mathbf{hom}_\square(X, Y)).$$

Some basics:

1) The inclusion $\partial\square^n \subset \square^n$ induces a monomorphism

$$X \otimes \partial\square^n \to X \otimes \square^n.$$

It follows that any cubical set inclusion $K \to L$ induces a monic $X \otimes K \to X \otimes L$.

2) If $X \to Y$ is a monomorphism of $\mathcal{A}$-sets and $K \to L$ is a monomorphism of cubical sets, then the map

$$(Y \otimes K) \cup_{(X \otimes K)} (X \otimes L) \to Y \otimes L$$

is a monomorphism. The map $X \otimes L \to Y \otimes L$ is a monomorphism for all L.

3) In cubical sets, the map

$$(\sqcap^n_{(i,\epsilon)} \otimes \square^k) \cup (\square^n \otimes \partial\square^k) \subset \square^n \otimes \square^k$$

is isomorphic to the inclusion $\sqcap^{n+k}_{(i,\epsilon)} \subset \square^{+k}$. Similarly, the map

$$(\partial\square^n \otimes \square^k) \cup (\square^n \otimes \partial\square^k) \subset \square^n \otimes \square^k$$

is isomorphic to $\partial\square^{n+k} \subset \square^{n+k}$.

Assumption: Suppose that S is a fixed set of monomorphisms of $\mathcal{A}$-sets

The class of *anodyne cofibrations* is the saturation of the set of inclusions

$$(Y \otimes \square^n) \cup (\Delta^a \otimes \sqcap^n_{(i,\epsilon)}) \subset \Delta^a \otimes \square^n \tag{8}$$

arising from the set of all inclusions of subobjects $Y \subset \Delta^a$, together with the set of inclusions

$$(A \otimes \square^n) \cup (B \otimes \partial\square^n) \subset B \otimes \square^n \tag{9}$$

induced by the maps $A \to B$ of the set S. Write $\Lambda(S)$ for the set of all maps appearing in (8) and (9).

Lemma 37. *1) Any inclusion $C \to D$ of $\mathcal{A}$-sets induces an anodyne cofibration*

$$(C \otimes \square^n) \cup (D \otimes \sqcap^n_{(i,\epsilon)}) \subset D \otimes \square^n.$$

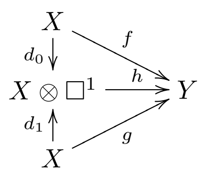
2) If $C \to D$ is an anodyne cofibration, then so is

$$(C \otimes \square^1) \cup (D \otimes \partial\square^1) \subset D \otimes \square^1.$$

Proof. It's enough to prove this for $C \to D$ of the form (8) or (9) above, but this is just fun with the identifications of cubical set morphisms given in 4) above. $\square$

Say that a map $p : X \to Y$ of $\mathcal{A}$-sets is *injective* if it has the right lifting property with respect to all anodyne cofibrations. An $\mathcal{A}$-set X is injective if $X \to *$ is injective.

A *naive homotopy* between maps $f, g : X \to Y$ is a map $h : X \otimes \square^1 \to Y$ which makes the obvious diagram commute:



Lemma 38. *Naive homotopy of maps $X \to Z$ is an equivalence relation if Z is injective.*

Proof. Suppose that $f_0 \simeq f_1$ and $f_1 \simeq f_2$ via homotopies $h_1, h_2 : X \otimes \square^1 \to Z$, respectively. Then h_1, h_2 and the constant homotopy c at f_2 together determine a map

$$H : X \otimes \sqcap^2_{(2,0)} \to Z$$

which can be represented by the picture

$$\begin{array}{ccc} f_1 & \xrightarrow{h_2} & f_2 \\ {\scriptstyle h_1}\uparrow & & \uparrow{\scriptstyle c} \\ f_0 & \dashrightarrow & f_2 \end{array}$$

The map H extends to a map $H' : X \otimes \square^2 \to Z$ since the cofibration $X \otimes \sqcap^2_{(2,0)} \to X \otimes \square^2$ is anodyne. Restriction to the $(2,0)$ face gives a homotopy $f_0 \simeq f_2$. Symmetry has a similar proof, and reflexivity is trivial. □

Write $\pi(X, Y)$ for the set of naive homotopy classes of maps from X to Y, meaning collapse $\hom(X, Y)$ by the equivalence relation generated by naive homotopy.

A map $f : X \to Y$ is said to be a *weak equivalence* if it induces a bijection

$$\pi(Y, Z) \xrightarrow{\cong} \pi(X, Z)$$

for all injective Z. A *cofibration* is a monomorphism. A *fibration* is a map which has the right lifting property with respect to all trivial cofibrations.

We'll see later on that all injective objects are fibrant. Every fibrant object is obviously injective, so that the classes of fibrant and injective objects coincide.

Lemma 39. *Every X has an injective model $j : X \to \mathcal{L}X$, meaning that j is anodyne and $\mathcal{L}X$ is injective.*

The proof of this Lemma is a standard transfinite small object argument.

Lemma 40. *All anodyne cofibrations are weak equivalences.*

Proof. Suppose that $i : A \to B$ is anodyne and Z is injective. The lifting exists in any diagram

$$\begin{array}{ccc} A & \longrightarrow & Z \\ {\scriptstyle i}\downarrow & \nearrow & \\ B & & \end{array}$$

so that $\pi(B, Z) \to \pi(A, Z)$ is surjective.

Suppose that $f, g : B \to Z$ become homotopic on restriction to A, via a homotopy $h : A \otimes \square^1 \to Z$. Then the lifting exists in the diagram

$$\begin{array}{ccc} (B \otimes \partial\square^1) \cup (A \otimes \square^1) & \xrightarrow{((f,g),h)} & Z \\ \downarrow & \nearrow & \\ B \otimes \square^1 & & \end{array}$$

and so $f \simeq g$. Thus $\pi(B, Z) \to \pi(A, Z)$ is a monomorphism. $\square$

Lemma 41. *Suppose that X and Y are injective objects. Then $f : X \to Y$ is a weak equivalence if and only if it is a naive homotopy equivalence.*

Proof. $f_* : \pi(Y, X) \to \pi(Y, Y)$ is a bijection, so there is a unique naive homotopy class $g : Y \to X$ such that $fg \simeq 1$. $f_* : \pi(X, X) \to \pi(X, Y)$ is a bijection and $fgf \simeq f$. Thus $gf \simeq 1$. $\square$

Corollary 42. *$f : X \to Y$ is a weak equivalence if and only if $\mathcal{L}X \to \mathcal{L}Y$ is a naive homotopy equivalence.*

Theorem 43 (Cisinski, "Swiss army knife theorem")**.** *With the definitions given above, the category of $\mathcal{A}$-sets has the structure of cubical model category.*

The model structure given by this theorem is called the $(S, \otimes)$*-model structure* on the category $\mathcal{A}$-sets, reflecting the fact that it depends only on the interval theory $\otimes$ and the generating set of cofibrations S. If $\otimes$ is specified by an interval I, one calls this the (S, I)*-model structure.*

Theorem 44. *Suppose that the interval theory $\otimes$ on $\mathcal{A}$-sets is defined by an interval I in the sense that*

$$Z \otimes \square^n = Z \times I^{\times n}$$

Suppose that all cofibrations in the set S pull back to weak equivalences along all fibrations $p : X \to Y$ with Y fibrant. Then the corresponding model structure on $\mathcal{A}$-sets is proper.

The proof is the "localization script". It is an abstraction of the standard argument which produces localizations of the model structure on simplicial sets (or simplicial presheaves) by formally inverting some cofibration. The catch/kicker is that, at this level of generality, you are not localizing an underlying model structure. See [20].

Here are some consequences:

Example 1: $\mathcal{A} = \mathcal{C} \times \mathbf{\Delta}$: $\mathcal{A}$-sets are simplicial presheaves on $\mathcal{C}$, S = generating set of local trivial cofibrations for the standard (injective) model structure on $s\,\mathrm{Pre}(\mathcal{C})$, $I = \Delta^1$.

The (S, Δ^1)-model structure given by Theorem 43 is the standard model structure for $s\,\mathrm{Pre}(\mathcal{C})$: every injective object is globally fibrant and the map

$X \to \mathcal{L}X$ is a local weak equivalence, so $f : X \to Y$ is a local weak equivalence iff f is a weak equivalence for the (S, Δ^1)-structure.

NB: The case $\mathcal{C} = *$ gives the standard model structure for simplicial sets. In that case S is the set of all inclusions $\Lambda^n_k \subset \Delta^n$, $n \geq 0$.

Example 2: S can be empty: the interval theory $X \times (\Delta^1)^{\times n}$ alone gives a model structure for simplicial sets which is a priori weaker (has fewer weak equivalences) than the standard model structure. Say that the case $S = \emptyset$ is a *primitive model structure.*

Example 3: Back to $\mathcal{A} = \mathcal{C} \times \mathbf{\Delta}$: suppose that $f : A \to B$ is a monomorphism (or a set of monomorphisms) of simplicial presheaves on $\mathcal{C}$.

Take the set S of generating cofibrations from Example 1, and add the set of all cofibrations

$$(Y \times B) \cup (L_U \Delta^n \times A) \to L_U \Delta^n \times B$$

induced by all subobjects $Y \subset L_U \Delta^n$. Denote the enlarged set of cofibrations by S_f. Let $I = \Delta^1$, as before. The resulting (S_f, Δ^1)-model structure on $s\operatorname{Pre}(\mathcal{C})$ is the f-local model structure on $s\operatorname{Pre}(\mathcal{C})$ [7]. The f-local model structure is proper if f is a map $* \to J$ for some simplicial presheaf J.

Example 4: Suppose that $\mathcal{C} = (Sm|_S)_{Nis}$ where S is a scheme of finite dimension, and let $f : * \to \mathbb{A}^1$ be the rational point 0 (or any other). The f-local structure of the previous example, in this case, is the *motivic model structure* on $s\operatorname{Pre}(Sm|_S)_{Nis}$. This model structure is proper.

Different construction: Use the interval theory $\mathbb{A}^1$ given by the presheaf $\mathbb{A}^1$ and the rational points $0, 1 : * \to \mathbb{A}^1$. Let S be the generating set of trivial cofibrations for the standard model structure on $s\operatorname{Pre}(Sm|_S)_{Nis}$. Then the $(\mathbb{A}^1, S)-$model structure on $s\operatorname{Pre}(Sm|_S)_{Nis}$ is the motivic model structure.

Example 5: Recall that a map $f : X \to Y$ of simplicial presheaves on $\mathcal{C}$ is a homology sheaf isomorphism if $\tilde{H}_*(X) \to \tilde{H}_*(Y)$ is an isomorphism of sheaves. Suppose that $\alpha > |\operatorname{Mor}(\mathcal{C})|$.

Exercise: Suppose that the cofibration $i : X \to Y$ is a homology sheaf isomorphism, and that A is an α-bounded subobject of Y. Then there is an α-bounded subobject $B \subset Y$ with $A \subset B$ such that $B \cap X \to B$ is a homology sheaf isomorphism.

Let S be the set of α-bounded cofibrations which are homology sheaf isomorphisms, and let $I = \Delta^1$. Then the (S, I)-model structure on $s\operatorname{Pre}(\mathcal{C})$ is integral homology localization structure, and the fibrant models $X \to \mathcal{L}X = L_{\mathbb{Z}}(X)$ are homology (sheaf) localizations.

a) If $\mathcal{C}$ has no topology, this construction specializes to sectionwise integral homology localization on $\mathcal{C}^{op}$-diagrams.

b) If $\mathcal{C} = *$, this construction specializes further to Bousfield's integral homology localization theory for simplicial sets. This construction generalizes to a localization construction for any homology theory, sheaf theoretic or not.

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