Problem Set 2
February 7, 2019.

1. Verify properties (7) and (8) of Remark 4.11.

2. Prove Lemmas 5.1 and 5.4.

3. Let A be a UFD, and let B be an integral domain, integral over A. Prove that, for every $b \in B$, there is a unique monic polynomial $P \in A[w]$ of minimal degree, such that $P(b) = 0$.

4. Let X be an analytic subset of a manifold M, and let $\xi \in X$. We define the *local ring of X at ξ* as

$$\mathcal{O}_{X,\xi} := \mathcal{O}_{M,\xi}/\mathfrak{I}(X_\xi)\,,$$

where $\mathfrak{I}(X_\xi)$ denotes the full ideal of the analytic germ X_ξ. Prove the following:

(a) The ring $\mathcal{O}_{X,\xi}$ is an integral domain if and only if X_ξ is an irreducible germ.

(b) The ring $\mathcal{O}_{X,\xi}$ need not be a UFD, even if X is locally principal and locally irreducible (give an example and justify).

5. Recall that the *Krull dimension* of a Noetherian local ring $(R, \mathfrak{m})$ is the maximal length d of a chain of prime ideals $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_d = \mathfrak{m}$ in R. Let X be an analytic subset of a manifold M, and let $\xi \in X$.

(a) Prove that the zero ideal (0) is prime in $\mathcal{O}_{X,\xi}$ if and only if X_ξ is an irreducible germ.

(b) Suppose that X_ξ is irreducible. Prove that the Krull dimension of $\mathcal{O}_{X,\xi}$ equals $\dim_\xi X$.

6. Recall that a Noetherian local ring $(R, \mathfrak{m})$ is called *regular* when the maximal ideal $\mathfrak{m}$ can be generated by n elements, where n is the Krull dimension of R. Show that, if X_ξ is smooth, then the local ring $\mathcal{O}_{X,\xi}$ is regular.