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Topics in Complex Analytic Geometry

by

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Lecture Notes
PART II

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Contents

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1 Analytic tensor product and fibre product of analytic spaces

In this section we sketch the proof of existence of the fibre product in the category of complex analytic spaces, and relate it to the analytic tensor product of local analytic $\mathbb{C}$-algebras (see [Fi] for details).

1.1 Analytic tensor product

Definition 1.1. Given homomorphisms of local analytic $\mathbb{C}$-algebras $\varphi_i : R \to A_i$ $(i = 1, 2)$, there is a unique (up to isomorphism) local analytic $\mathbb{C}$-algebra, denoted $A_1\tilde{\otimes}_R A_2$, together with homomorphisms $\theta_i : A_i \to A_1\tilde{\otimes}_R A_2$ $(i = 1, 2)$, such that $\theta_1 \circ \varphi_1 = \theta_2 \circ \varphi_2$ and for every pair of homomorphisms of local analytic $\mathbb{C}$-algebras $\psi_1 : A_1 \to B$, $\psi_2 : A_2 \to B$ satisfying $\psi_1 \circ \varphi_1 = \psi_2 \circ \varphi_2$, there is a unique homomorphism of local analytic $\mathbb{C}$-algebras $\psi : A_1\tilde{\otimes}_R A_2 \to B$ making the whole diagram commute. The algebra $A_1\tilde{\otimes}_R A_2$ is called the *analytic tensor product* of algebras A_1 and A_2 *over* R.

For finitely generated A_i-modules M_i $(i = 1, 2)$, the $M_1\tilde{\otimes}_R A_2$ and $A_1\tilde{\otimes}_R M_2$ are finitely generated $A_1\tilde{\otimes}_R A_2$-modules, and we may define

$$M_1\tilde{\otimes}_R M_2 \ = \ (M_1\tilde{\otimes}_R A_2) \otimes_{A_1\tilde{\otimes}_R A_2} (A_1\tilde{\otimes}_R M_2)\,.$$

Remark 1.2. The analytic tensor product enjoys the following properties:

(i) For any two local analytic $\mathbb{C}$-algebras A_1 and A_2 over a local analytic $\mathbb{C}$-algebra R, the product $A_1\tilde{\otimes}_R A_2$ exists and is unique up to isomorphism (of R-analytic algebras) (see [GR], or verify the universal mapping property is satisfied with formulas below).

(ii) If $A_1 \cong R\{x\}/I_1$ and $A_2 \cong R\{z\}/I_2$, then

$$A_1\tilde{\otimes}_R A_2 \cong \frac{R\{x, z\}}{I_1\tilde{\otimes}_R 1 + 1\tilde{\otimes}_R I_2}\,,$$

where $I_1\tilde{\otimes}_R 1+1\tilde{\otimes}_R I_2$ denotes the ideal in $R\{x, z\}$ generated by the generators of (the extensions of) I_1 and I_2.

This essentially follows from the (easy to verify) isomorphisms

$$\mathbb{C}\{x\}\tilde{\otimes}_{\mathbb{C}}\mathbb{C}\{z\} \cong \mathbb{C}\{x, z\} \qquad \text{and} \qquad \mathbb{C}\{x\}/I_1\tilde{\otimes}_{\mathbb{C}}\mathbb{C}\{z\}/I_2 \cong \frac{\mathbb{C}\{x, z\}}{I_1\tilde{\otimes}_{\mathbb{C}} 1 + 1\tilde{\otimes}_{\mathbb{C}} I_2}\,.$$

(iii) If $M_1 \cong A_1^q/N_1$ and $M_2 \cong A_2^p/N_2$ for some finite A_1-submodule N_1 of A_1^q and a finite A_2-submodule N_2 of A_2^p, then

$$\begin{aligned} M_1\tilde{\otimes}_R M_2 \ &\cong \ (\frac{A_1^q}{N_1}\tilde{\otimes}_R A_2) \otimes_{A_1\tilde{\otimes}_R A_2} (A_1\tilde{\otimes}_R \frac{A_2^p}{N_2}) \\ &\cong \frac{(A_1\tilde{\otimes}_R A_2)^q}{N_1\tilde{\otimes}_R 1} \otimes_{A_1\tilde{\otimes}_R A_2} \frac{(A_1\tilde{\otimes}_R A_2)^p}{1\tilde{\otimes}_R N_2} \ \cong \ \frac{(A_1\tilde{\otimes}_R A_2)^{pq}}{N_1\tilde{\otimes}_R 1 + 1\tilde{\otimes}_R N_2}\,. \end{aligned}$$

(iv) If M_1 and M_2 are finitely generated R-modules, then $M_1\tilde{\otimes}_R M_2 \ \cong \ M_1 \otimes_R M_2$.

1.2 Fibre product of analytic spaces

Definition 1.3. Given holomorphic mappings of complex analytic spaces $\varphi_1 : X_1 \to Y$ and $\varphi_2 : X_2 \to Y$, a complex analytic space $X_1 \times_Y X_2$ together with holomorphic maps $\pi_i : X_1 \times_Y X_2 \to X_i$ $(i = 1, 2)$ such that $\varphi_1 \circ \pi_1 = \varphi_2 \circ \pi_2$, is called a *fibre product* of X_1 and X_2 over Y (or, more precisely, over φ_1 and φ_2), when it satisfies the following universal mapping property: For every complex analytic space X with mappings $\psi_i : X \to X_i$ $(i = 1, 2)$ satisfying $\varphi_1 \circ \psi_1 = \varphi_2 \circ \psi_2$, there exists unique $\psi : X \to X_1 \times_Y X_2$ making the whole diagram commute.
If $Y = (\{\eta\}, \mathbb{C})$ is a simple point, the product of X_1 and X_2 over Y is called the *direct product* of X_1 and X_2, and denoted $X_1 \times X_2$.

Existence of the global fibre product will follow from the existence of fibre product of local models via the following glueing of local data:

Proposition 1.4. *Let $\{X_i\}_{i\in I}$ be a family of complex analytic spaces. Assume that, for every $i \in I$, there is a family $\{X_{ij}\}_{j\in I}$ of open subspaces $X_{ij} \subset X_i$, with biholomorphic maps $\varphi_{ij} : X_{ij} \to X_{ji}$ for every pair $i, j \in I$, satisfying:*

(a) $X_{ii} = X_i$, $\varphi_{ii} = \mathrm{id}_{X_i}$ for all $i \in I$.

(b) For all $i, j, k \in I$, $\varphi_{ij}(X_{ij} \cap X_{ik}) \subset X_{ji} \cap X_{jk}$ and
$$\varphi_{ik}|_{(X_{ij}\cap X_{ik})} = [\varphi_{jk}|_{(X_{ji}\cap X_{jk})}] \circ [\varphi_{ij}|_{(X_{ij}\cap X_{ik})}] \,.$$

Let $M = \bigsqcup_{i\in I} |X_i|$, and call $p, q \in M$ equivalent *($p \sim q$) if there exist $i, j \in I$ such that $p \in X_i$, $q \in X_j$, and $q = \varphi_{ij}(p)$. Assume that $M/\sim$ is Hausdorff. Then there exists a complex analytic space X with an open covering $\{U_i\}_{i\in I}$ and a family of biholomorphic $\varphi_i : U_i \to X_i$ such that, for all $i, j \in I$, the following diagram commutes*

$$\begin{array}{ccc} X_i \supset X_{ij} & \xrightarrow{\varphi_{ij}} & X_{ji} \subset X_j \\ {\scriptstyle\varphi_i}\big\downarrow & & \big\downarrow{\scriptstyle\varphi_j} \\ U_i \supset U_i \cap U_j & \xrightarrow{\mathrm{id}} & U_j \cap U_i \subset U_j \,. \end{array}$$

We will now sketch the proof of existence of fibre product of local models:

Step 1. $\mathbb{C}^{m+n}$ together with the canonical projections $\pi_1 : \mathbb{C}^{m+n} \to \mathbb{C}^m$ and $\pi_2 : \mathbb{C}^{m+n} \to \mathbb{C}^n$ is a direct product of $\mathbb{C}^m$ and $\mathbb{C}^n$.

Indeed, for analytic $\psi_1 : X \to \mathbb{C}^m$ and $\psi_2 : X \to \mathbb{C}^n$, we may identify ψ_1 with $(\psi_1^* z_1, \dots, \psi_1^* z_m) = (f_1, \dots, f_m) \in (\mathcal{O}_X(X))^m$, and similarly ψ_2 with $(\psi_2^* w_1, \dots, \psi_2^* w_n) = (g_1, \dots, g_n) \in (\mathcal{O}_X(X))^n$. Then (ψ_1, ψ_2) identifies with the analytic mapping $(f_1, \dots, f_m, g_1, \dots, g_n) : X \to \mathbb{C}^{m+n}$.

Step 2. Let $\varphi : X \to Y$ be an analytic map, and let Y' be an open (resp. closed) subspace of Y. Then $X \times_Y Y'$ exists and is an open (resp. closed) subspace of X, denoted $\varphi^{-1}(Y')$, called the (*analytic*) *inverse image* of Y'. In particular, if $Y' = (\{\eta\}, \mathbb{C})$ is a simple point, then $\varphi^{-1}(Y') = \varphi^{-1}(\eta)$ is called the fibre over η.

Idea of a proof: If Y' is open in Y, there is nothing to show. If Y' is closed, defined by a coherent ideal sheaf J, say, then $\varphi^{-1}(Y')$ is defined by a coherent ideal sheaf $I \lhd \mathcal{O}_X$ given as follows: if $\eta \in Y$ and J_η is generated by $a_1, \dots, a_k \in \mathcal{O}_{Y,\eta}$, then for every $\xi \in X$ with $|\varphi|(\xi) = \eta$, the stalk I_ξ is generated by $\varphi_\xi^*(a_1), \dots, \varphi_\xi^*(a_k) \in \mathcal{O}_{X,\xi}$.
By construction of I, φ restricts to $\varphi' : \varphi^{-1}(Y') \to Y'$, and one checks that the universal mapping property holds for this $\varphi^{-1}(Y')$.

Step 3. Let complex analytic spaces X, Y with open (resp. closed) subspaces X', Y' be given. If the product $X \times Y$ exists, then $X' \times Y'$ exists and is an open (resp. closed) subspace of $X \times Y$.

Indeed, by *Step 2*, if Z_1, Z_2 are open (resp. closed, defined by coherent I_1 and I_2) subspaces of Z, then $Z_1 \cap Z_2$ (defined in Z by the coherent ideal $I_1 + I_2$) is the fibre product of $Z_1 \overset{\iota_1}{\hookrightarrow} Z$ and $Z_2 \overset{\iota_2}{\hookrightarrow} Z$. Therefore we have the following commutative diagram:

$$\begin{tikzcd}
X' \times Y' \arrow[r] \arrow[d] & X'' = \pi_1^{-1}(X') \arrow[r, "\pi_1'"] \arrow[d, "\iota'"] & X' \arrow[d, "\iota"] \\
Y'' = \pi_2^{-1}(Y') \arrow[r, "\kappa'"] \arrow[d, "\pi_2'"'] & X \times Y \arrow[r, "\pi_1"] \arrow[d, "\pi_2"] & X \arrow[d] \\
Y' \arrow[r, "\kappa"] & Y \arrow[r] & \{0\}\ .
\end{tikzcd}$$

Corollary 1.5. *If $X \hookrightarrow V \subset \mathbb{C}^m$, $Y \hookrightarrow W \subset \mathbb{C}^n$ are closed complex subspaces, then $X \times Y$ exists and is a closed complex subspace of $V \times W$.*

Step 4. For any complex analytic space X, there is a closed subspace $D_X \hookrightarrow X \times X$, called the *diagonal*, such that $X \cong D_X$.

Step 5. Given $\varphi_i : X_i \to Y$ $(i = 1, 2)$, a fibre product $X_1 \times_Y X_2$ exists and is a closed subspace of $X_1 \times X_2$.

Indeed, by *Step 4*, there is an embedding $Y \cong D_Y \hookrightarrow Y \times Y$; by *Step 3*, there are $X_1 \times X_2$ and $Y \times Y$; and by *Step 2*, we get the following commutative diagram:

$$\begin{tikzcd}
X_1 \times_Y X_2 \arrow[r] \arrow[d] & X_1 \times X_2 \arrow[d, "\varphi_1 \times \varphi_2"] \\
D_Y \arrow[r] & Y \times Y\ .
\end{tikzcd}$$

1.3 Fibre product vs. analytic tensor product

Given holomorphic $\varphi_1 : X_1 \to Y$ and $\varphi_2 : X_2 \to Y$, let $\xi_1 \in X_1$ and $\xi_2 \in X_2$ be such that $|\varphi_1|(\xi_1) = |\varphi_2|(\xi_2) = \eta \in Y$. Then, by the duality [I, Prop.9.29], the fibre product square

$$\begin{tikzcd}
X_1 \times_Y X_2 \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] & X_1 \arrow[d, "\varphi_1"] \\
X_2 \arrow[r, "\varphi_2"] & Y
\end{tikzcd}$$

induces a commutative diagram of homomorphisms of local analytic $\mathcal{O}_{Y,\eta}$-algebras

$$\begin{tikzcd}
\mathcal{O}_{Y,\eta} \arrow[r, "(\varphi_1)^*_{\xi_1}"] \arrow[d, "(\varphi_2)^*_{\xi_2}"'] & \mathcal{O}_{X_1,\xi_1} \arrow[d] \\
\mathcal{O}_{X_2,\xi_2} \arrow[r] & \mathcal{O}_{X_1 \times_Y X_2, (\xi_1, \xi_2)}\ .
\end{tikzcd}$$

The uniqueness of the fibre product and of the analytic tensor product then implies the following canonical isomorphism

$$\mathcal{O}_{X_1 \times_Y X_2, (\xi_1,\xi_2)} \ \cong\ \mathcal{O}_{X_1,\xi_1} \tilde{\otimes}_{\mathcal{O}_{Y,\eta}} \mathcal{O}_{X_2,\xi_2}\,. \tag{†}$$

2 Rank and fibre dimension of analytic mappings

In the remainder of these notes we will often make use of the so-called *local installation* of a holomorphic mapping of analytic spaces, that allows one to replace (locally) a given map with a restriction of a canonical projection from a Cartesian product. The details of the following construction are left as an **Exercise**.

Assume a holomorphic mapping $\varphi : X \to Y$ is given, where $X \hookrightarrow V \subset \mathbb{C}^m$ and $Y \hookrightarrow W \subset \mathbb{C}^n$ are local models, defined by coherent ideals I_X and I_Y resprectively. Define Γ_φ as the analytic subspace of $V \times W$ given by the coherent ideal

$$J := I_X + (z_1 - \varphi_1(x), \dots, z_n - \varphi_n(x))\,,$$

where (after shrinking V and W if necessary) $\varphi = (\varphi_1, \dots, \varphi_n)$ in local coordinates at the origins in $\mathbb{C}^m$ and $\mathbb{C}^n$.

We claim that there exists a natural biholomorphism $\gamma : X \overset{\cong}{\to} \Gamma_\varphi$ such that $\varphi = \pi_2 \circ \gamma$, where $\pi_2 : V \times W \to W$ is a canonical projection.

For that, define $\gamma = (|\gamma|, \gamma^*) : X \to \Gamma_\varphi$ as

$$|\gamma| : |X| \ni x \;\mapsto\; (x, \varphi_1(x), \dots, \varphi_n(x)) \in |\Gamma_\varphi|\,,$$

and

$$\gamma^* : \mathcal{O}_{\Gamma_\varphi} \ni f(x,z) \;\mapsto\; f(x, \varphi(x)) \in |\varphi|_* \mathcal{O}_X\,.$$

One now easily finds a two-sided inverse of γ and verifies the composite equality.

2.1 Remmert Rank Theorem

For a holomorphic map $\varphi : X \to Y$ and a point $\xi \in X$, we define the *fibre dimension* of φ *at* ξ as the dimension of the germ at ξ of the fibre of φ through ξ, that is

$$\mathrm{fbd}_\xi \varphi := \dim_\xi [\varphi^{-1}(\varphi(\xi))]\,.$$

Theorem 2.1 (Remmert Rank Theorem). *Let $\varphi : X \to Y$ be a holomorphic mapping of analytic spaces, with constant fibre dimension; i.e., such that there exists $r \in \mathbb{N}$ with*

$$\mathrm{fbd}_x \varphi = r \quad \text{for all } x \in X\,.$$

Then every $\xi \in X$ has arbitrarily small open neighbourhoods in X with images locally analytic in Y, of dimension $\dim_\xi X - r$.

Proof. The problem being local, we may assume that X and Y are local models in open subsets V and W of $\mathbb{C}^m$ and $\mathbb{C}^n$ respectively. Then, by passing to the graph Γ_φ of φ, it suffices to show that:

If A is a locally analytic subset of $V \times W$, $0 \in A$, with $1-1$ proper projection κ to V, and such that

$$\mathrm{fbd}_a \pi = r \quad \text{for all } a \in A\,,$$

where π is the restriction to A of the canonical projection $\mathbb{C}^m \times \mathbb{C}^n \to \mathbb{C}^n$, then a sufficiently small open neighbourhood of 0 in A is mapped by π onto a locally analytic subset of W (through $0 \in \mathbb{C}^n$), of dimension $\dim_0 A - r$.

For this, consider the fibre $\pi^{-1}(0) \subset A$, which by assumption is an analytic subset of A, of pure dimension r. Then so is $\kappa(\pi^{-1}(0))$, and by [I, Thm.6.1], we can find $\epsilon > 0$ and an $(m-r)$-dimensional linear subspace T in $\mathbb{C}^m = T^\perp \times T$ such that

$$\kappa(\pi^{-1}(0)) \cap \Omega \cap (\{0\} \times T) \;=\; \{0\} \quad \text{in } \mathbb{C}^m\,,$$

and $A \cap (\Omega \times \Delta)$ is analytic (that is, closed) in $\Omega \times \Delta$, where $\Omega = \epsilon\Delta^m$ and $\Delta = \epsilon\Delta^n$ are small open polydiscs at the origins of $\mathbb{C}^m$ and $\mathbb{C}^n$ respectively. It follows that

$$A \cap (\Omega \times \Delta) \cap [(\{0\} \times T) \times \{0\}] = \{0\} \quad \text{in } \mathbb{C}^m \times \mathbb{C}^n .$$

Then, for $\delta > 0$ small enough,

$$[(\{0\} \times (\overline{2\delta\Delta^{m-r}} \setminus \delta\Delta^{m-r})) \times \{0\}] \cap A = \varnothing .$$

By compactness of $(\{0\} \times (\overline{2\delta\Delta^{m-r}} \setminus \delta\Delta^{m-r})) \times \{0\}$, we can shrink ϵ so that

$$[(\epsilon\Delta^r \times (\overline{2\delta\Delta^{m-r}} \setminus \delta\Delta^{m-r})) \times \epsilon\Delta^n] \cap A = \varnothing$$

again disjoint, which by [I, Lem.5.1], implies that the projection

$$\lambda : A \cap [(\Lambda \times \Sigma) \times \Delta] \to \Lambda \times \Delta$$

is proper, where $\Omega = \Lambda \times \Sigma$ according to the decomposition of $\mathbb{C}^m = T^\perp \times T$, $\Lambda = \epsilon\Delta^r$, and $\Sigma = 2\delta\Delta^{m-r}$. After shrinking ϵ again, we may assume that $\tilde{A} := A \cap (\Omega \times \Delta)$ is analytic in $\Omega \times \Delta$, of dimension $\dim \tilde{A} = \dim_0 A$. By Remmert Proper Mapping Theorem and [I, Thm.5.17], $\lambda(\tilde{A})$ is analytic in $\Lambda \times \Delta$, of dimension $\dim_0 A$. We now claim that

$$\lambda(\tilde{A}) = \Lambda \times \pi(\tilde{A}) . \tag{‡}$$

Indeed, for $y \in \pi(\tilde{A})$, the fibre $\pi^{-1}(y)$ is analytic in $\Omega \times \Delta$, of pure dimension r, hence $\kappa(\pi^{-1}(y))$ is analytic in Ω, and of pure dimension r. Moreover, the projection $\rho : \kappa(\pi^{-1}(y)) \to \Lambda$ is proper (as ρ is the restriction of λ to the closed set $\lambda^{-1}(\Lambda \times \{y\})$). Then, by [I, Thm.5.17] again, the image $\rho(\kappa(\pi^{-1}(y)))$ is analytic, of dimension r. Since $r = \dim \Lambda$, it follows from the irreducibility of Λ that $\rho(\kappa(\pi^{-1}(y))) = \Lambda$, which proves (‡).

To complete the proof, observe that, by (‡), $\pi(\tilde{A})$ is the preimage of the analytic set $\lambda(\tilde{A})$ under the analytic mapping $\gamma : \Delta \ni y \mapsto (0, y) \in \Lambda \times \Delta$. Thus $\pi(\tilde{A})$ is analytic in Δ, and

$$\dim \pi(\tilde{A}) = \dim \lambda(\tilde{A}) - \dim \Lambda = \dim_0 A - r .$$

□

2.2 Analytically constructible sets

Definition 2.2. A subset E of a manifold M is called (*analytically*) *constructible* when

$$E = \bigcup_{\iota \in I} (V_\iota \setminus W_\iota) ,$$

where $\{V_\iota\}$ is a locally finite family of irreducible analytic sets, and $\{W_\iota\}$ is a locally finite family of analytic sets.

The following properties of constructible sets follow easily from the results of [I, Sec.7].

Proposition 2.3. *(i) If a set $E \subset M$ is constructible, then its closure $\bar{E}$ is analytic in M.*

(ii) In particular, a constructible subset of M is analytic iff it is closed.

(iii) The class of constructible subsets of M is precisely the smallest collection of subsets of M containing analytic sets and closed under the complement and locally finite union.

2.3 Rank and fibre dimension

Suppose $\varphi : X \to Y$ is a holomorphic mapping of analytic subsets of manifolds $X \subset M$ and $Y \subset N$. Then, for $\xi \in \mathrm{reg}X$, we can define the *rank* of φ *at* ξ, denoted $\mathrm{rk}_{\,\xi}\varphi$, as the rank at ξ of $\varphi|_U : U \to N$ for some open neighbourhood U of ξ in X such that $U \subset \mathrm{reg}X$.

Theorem 2.4. *Let $\varphi : X \to Y$ be as above. Then, for every connected component W of $\mathrm{reg}X$ and any $k \in \mathbb{N}$, the set*

$$\{x \in W \,:\, \mathrm{rk}_{\,x}\varphi \leq k\}$$

is analytically constructible in M.

Proof. By [I, Sec.7], there exists an irreducible component X_i of X such that $W = \mathrm{reg}X_i \setminus \mathrm{sng}X$, and hence

$$\{x \in W \,:\, \mathrm{rk}_{\,x}\varphi \leq k\} \;=\; \{x \in \mathrm{reg}X_i \,:\, \mathrm{rk}_{\,x}\varphi|_{X_i} \leq k\} \setminus \mathrm{sng}X\,.$$

Therefore, without loss of generality, we may assume that X is of pure dimension, and it suffices to show that the set

$$E = \{x \in \mathrm{reg}X \,:\, \mathrm{rk}_{\,x}\varphi \leq k\}$$

is constructible in M. The question being local, we may assume that $N = \mathbb{C}^n$, and φ is the restriction to X of a holomorphic mapping $\Phi = (\Phi_1, \dots, \Phi_n) : M \to \mathbb{C}^n$, and it suffices to show that every point $\xi \in E$ has an open neighbourhood U in M such that $E \cap U$ is constructible in U.

Now, $\mathrm{reg}X$ being a submanifold of M, there is an open neighbourhood U of a given point $\xi \in E$ and holomorphic functions $G_1, \dots, G_r \in \mathcal{O}(U)$ such that

$$T_x(\mathrm{reg}X) = \bigcap_{i=1}^{r} \ker d_x G_i \quad \text{for all } x \in U \cap \mathrm{reg}X\,.$$

Put $G = (G_1, \dots, G_r) : U \to \mathbb{C}^r$. Then $\ker d_x\varphi = \ker d_x(\Phi, G)$ for $x \in U \cap \mathrm{reg}X$, and hence

$$E \cap U = \{x \in U \,:\, \mathrm{rk}_{\,x}(\Phi, G) \leq k + \mathrm{codim}_M(\mathrm{reg}X)\} \cap \mathrm{reg}X\,,$$

which is constructible as a locally finite union of analytic sets. □

Theorem 2.5. *Let $\varphi : X \to Y$ be as before. Given $k \in \mathbb{N}$, we have:*

(i) If $\varphi^{-1}(y) \geq k$ for all $y \in \varphi(X)$, and $X \neq \varnothing$, then

$$\dim X \geq k + \dim \varphi(X).$$

(i) If $\varphi^{-1}(y) \leq k$ for all $y \in \varphi(X)$, then

$$\dim X \leq k + \dim \varphi(X).$$

Proof. The proof of (i) can be reduced to the case of Y being an l-dimensional affine space (for some $l \in \mathbb{N}$) and $\varphi(X)$ open in Y. Indeed, there exists a submanifold $\Gamma \subset \varphi(X)$ biholomorphic with an open subset of $\mathbb{C}^l$ for $l = \dim \varphi(X)$, and showing the inequality for $\varphi|_{\varphi^{-1}(\Gamma)} : \varphi^{-1}(\Gamma) \to \Gamma$ will imply the general case, as $(\varphi|_{\varphi^{-1}(\Gamma)})^{-1}(y) = \varphi^{-1}(y)$ for $y \in \Gamma$.

We will proceed by induction on $l = \dim Y$. The case $l = 0$ being trivial, suppose that $l > 0$ and the statement holds for all affine targets of dimension at most $l-1$. Let $\chi : Y \to \mathbb{C}$ be a non-constant affine mapping. Of all the irreducible components X_i of X, there are at most countably many with constant restrictions $(\chi \circ \varphi)|_{X_i} \equiv c_i \in \mathbb{C}$. On the other hand, as $\varphi(X)$ is open nonempty in Y, the image $(\chi \circ \varphi)(X)$ is open nonempty in $\mathbb{C}$. Therefore, there exists $c \in \mathbb{C}$ such that $c \neq c_i$ for all c_i as above, and hence $(\chi \circ \varphi)^{-1}(c) \cap X_i$ is a proper analytic subset of X_i for every irreducible component X_i of X. Put $X_0 = (\chi \circ \varphi)^{-1}(c)$. It follows that $\dim X_0 < \dim X$. Put $Y_0 = \chi^{-1}(c)$. Then Y_0 is affine, of dimension $l-1$, and by the inductive hypothesis applied to $\varphi|_{\varphi^{-1}(Y_0)} : \varphi^{-1}(Y_0) = X_0 \to Y_0$, we have $\dim X_0 \geq k + \dim \varphi(X_0)$. Hence

$$\dim X > \dim X_0 \geq k + \dim \varphi(X_0) = k + \dim Y_0 = k + \dim Y - 1 = k + \dim \varphi(X) - 1\,,$$

so that

$$\dim X \geq k + \dim \varphi(X)\,.$$

For the proof of (ii), we may replace φ with its local installation $\pi : \Gamma_\varphi \to N$. Indeed, we have $\pi(\Gamma_\varphi) = \varphi(X)$, and $\pi^{-1}(y) = (\varphi^{-1}(y), y)$, hence $\dim \pi^{-1}(y) \leq k$ for all $y \in \varphi(X)$; it thus suffices to show that $\dim \Gamma_\varphi \leq k + \dim \pi(\Gamma_\varphi)$.

Let then $\Lambda \subset \Gamma_\varphi$ be a submanifold of $M \times N$, of dimension $\dim \Gamma_\varphi$, and let $z_0 \in \Lambda$ be such that the rank of $\pi|_\Lambda : \Lambda \to N$ is maximal at z_0. Then, in an open neighbourhood Λ_0 of z_0 in Λ, $\mathrm{rk}_z(\pi|_{\Lambda_0}) = \mathrm{rk}_{z_0}(\pi|_\Lambda)$. Hence, by the Rank Theorem (after shrinking Λ_0 if necessary), $\pi(\Lambda_0)$ is a submanifold of N, of dimension $\mathrm{rk}\,(\pi|_{\Lambda_0})$, and nonempty fibres of $\pi|_{\Lambda_0}$ are submanifolds of M, of dimension $\dim \Lambda_0 - \mathrm{rk}\,(\pi|_{\Lambda_0})$, which is $\leq k$. Thus,

$$\dim \Gamma_\varphi = \dim \Lambda = \dim \Lambda_0 \leq k + \mathrm{rk}\,(\pi|_{\Lambda_0}) = k + \dim \pi(\Lambda_0) = k + \dim \pi(\Gamma_\varphi)\,.$$

□

Corollary 2.6. *If $\varphi : X \to Y$ has 0-dimensional fibres, then $\dim \varphi(X) = \dim X$.*

We now return to the study of the fibre dimension $\mathrm{fbd}_x\varphi$, and its relationship with the rank of φ.

Theorem 2.7. *Given analytic $\varphi : X \to Y$ as above ($X \subset M$, $Y \subset N$), the function*

$$X \ni x \mapsto \mathrm{fbd}_x\varphi \in \mathbb{N}$$

is upper semi-continuous; i.e., every point $\xi \in X$ admits an open neighbourhood U in X such that $\mathrm{fbd}_x\varphi \leq \mathrm{fbd}_\xi\varphi$ for all $x \in U$.

Proof. The problem being local, we may assume that $M = \mathbb{C}^m$ and $N = \mathbb{C}^n$. Fix $k \in \mathbb{N}$, and assume without loss of generality that $\xi = 0 \in M$. The condition $\mathrm{fbd}_\xi\varphi \leq k$ is then (by [I, Thm.6.1]) equivalent to saying that there exists a linear subspace $L \subset M$ of dimension $m - k$, such that the origin is an isolated point of $\varphi^{-1}(\varphi(\xi)) \cap L$. The latter, in turn, is equivalent to the existence of an $(m-k)$-dimensional linear subspace $L \subset M$ such that $(\xi, \varphi(\xi))$ is an isolated point of the intersection

$$(\varphi^{-1}(\varphi(\xi)) \cap L) \times \{\varphi(\xi)\} \;=\; \Gamma_\varphi \cap (L \times \{\varphi(\xi)\})\,.$$

Now, as in the proof of the Remmert Rank Theorem, we get that in a small open neighbourhood of the point $(\xi, \varphi(\xi))$, the projection from Γ_φ in the direction of L is proper. Hence, there exists an open neighbourhood U of ξ in X such that, for every $x \in U$,

$$\Gamma_\varphi \cap ((x + L) \times \{\varphi(x)\}) \;=\; (\varphi^{-1}(\varphi(x)) \cap (x + L)) \times \{\varphi(x)\}$$

contains $(x, \varphi(x))$ as an isolated point. Therefore, $\mathrm{fbd}_x\varphi \leq k$ for all $x \in U$, by [I, Thm.6.1] again. □

Corollary 2.8. *Given $\varphi : X \to Y$ as above, we have*

$$\mathrm{fbd}_x\varphi \geq \dim_x X - \dim \varphi(X)$$

for every $x \in X$.

Proof. For a proof by contradiction, suppose there exists $\xi \in X$ for which $\mathrm{fbd}_\xi\varphi < \dim_\xi X - \dim\varphi(X)$, and put $p := \dim_\xi X - \dim\varphi(X)$. By the upper semi-continuity of the fibre dimension, there is an open neighbourhood U of ξ in X such that $\mathrm{fbd}_x\varphi \leq \mathrm{fbd}_\xi\varphi$ for all $x \in U$, and $\dim U = \dim_\xi X$. Then, for all $y \in \varphi(U)$, we have $\dim(\varphi|_U)^{-1}(y) \leq p-1$, and hence, by Theorem 2.5(ii),

$$\dim_\xi X = \dim U \leq (p-1) + \dim\varphi(U) < p + \dim\varphi(U)\,,$$

which contradicts the definition of p. □

Definition 2.9. Let, as before, $\varphi : X \to Y$ be a holomorphic mapping of analytic subsets of manifolds $X \subset M$ and $Y \subset N$. The *rank* of φ is defined as

$$\mathrm{rk}\,\varphi \;=\; \max\{\mathrm{rk}_x\varphi \,:\, x \in \mathrm{reg}X\} \qquad (\text{if } X = \varnothing,\ \text{put } \mathrm{rk}\,\varphi = -1)\,.$$

Given $\xi \in X$, the *generic rank* of φ *at* ξ, denoted $r^1_\xi(\varphi)$, is defined as

$$r^1_\xi(\varphi) \;=\; \min\{\mathrm{rk}\,(\varphi|_U) \,:\, U \text{ open neighbourhood of } \xi \text{ in } X\}\,.$$

Proposition 2.10. *Let $\varphi : X \to Y$ be as above, and suppose that $X \subset M$ is irreducible. Then*

(i) $\mathrm{rk}\,(\varphi|_U) = \mathrm{rk}\,\varphi$ *for every open nonempty $U \subset X$.*

(ii) $\mathrm{fbd}_x\varphi \geq \dim X - \dim\varphi(X)$ *for every $x \in X$.*

(iii) $\dim(\varphi^{-1}(y)) \geq \dim X - \dim\varphi(X))$ *for every $y \in \varphi(X)$.*

Proof. The property (i) follows from the fact that $\mathrm{reg}X$ is now a connected manifold, open and dense in X, and its locus of maximal rank, $\Omega = \{x \in \mathrm{reg}X \,:\, \mathrm{rk}_x\varphi = \mathrm{rk}\,\varphi\}$ is open and dense in $\mathrm{reg}X$. (iii) follows from (ii) immediately. For the proof of (ii), observe that, by the Rank Theorem ([I, Thm.1.10]), the equality $\mathrm{fbd}_x\varphi = \dim X - \mathrm{rk}\,\varphi$ holds for all $x \in \Omega$. Hence, by Theorem 2.7, $\mathrm{fbd}_x\varphi \geq \dim X - \mathrm{rk}\,\varphi$ for all $x \in X$. □

Remark 2.11. The number $\rho_\xi\varphi := \dim_\xi X - \mathrm{fbd}_\xi\varphi$, from the Remmert Rank Theorem, is called the *Remmert rank* of φ at ξ. It need not be, in general, equal to the rank $\mathrm{rk}_\xi\varphi$. For instance, for $\varphi : \mathbb{C} \ni z \mapsto z^2 \in \mathbb{C}$, we have $\mathrm{rk}_0\varphi = 0$ and $\rho_0\varphi = 1$. In light of Prop. 2.10 though, for irreducible X, the two ranks are equal on a dense open subset of X.

Proposition 2.12 (Whitney Lemma)**.** *Let $\varphi : X \to Y$ be a holomorphic mapping of analytic subsets of manifolds $X \subset M$ and $Y \subset N$. Then*

$$\mathrm{rk}\,(\varphi|_{\mathrm{sng}X}) \leq \mathrm{rk}\,\varphi\,.$$

Proof. Put $r = \mathrm{rk}\,(\varphi|_{\mathrm{sng}X})$. We may assume that $\mathrm{sng}X \neq \varnothing$, and then there is an open subset $\Delta \neq \varnothing$ of $\mathrm{reg}(\mathrm{sng}X)$ such that $\mathrm{rk}_x(\varphi|_{\mathrm{sng}X}) = r$ for all $x \in \Delta$. Put $k = \dim\Delta$. By the Rank Theorem, after shrinking Δ if necessary, we may assume that $\Gamma = \varphi(\Delta)$ is an r-dimensional submanifold of N, and the non-empty fibres of $\varphi|_\Delta : \Delta \to \Gamma$ are $(k-r)$-dimensional submanifolds of M, hence $\mathrm{fbd}_x(\varphi|_\Delta) = k - r$ for all $x \in \Delta$.

Suppose now that $\mathrm{rk}\,\varphi < r$; i.e., that $\mathrm{rk}_x\varphi < r$ for every $x \in \mathrm{reg}X$. By the Baire Category Theorem and Rank Theorem, it follows that $\dim\varphi(\mathrm{reg}X) < r$, and thus $\Gamma \not\subset \varphi(\mathrm{reg}X)$. Thus, there exists $\xi \in \Delta$ such that $\varphi(\xi) \notin \varphi(\mathrm{reg}X)$, and hence $\varphi^{-1}(\varphi(\xi)) \subset \mathrm{sng}X$. In particular,

$$[\varphi^{-1}(\varphi(\xi))]_\xi \;=\; [(\varphi|_{\mathrm{sng}X})^{-1}((\varphi|_{\mathrm{sng}X})(\xi))]_\xi\,,$$

hence $\mathrm{fbd}_\xi\varphi = k - r$. By the upper semi-continuity of the fibre dimension, $\mathrm{fbd}_x\varphi \le k - r$ for all x in some open neighbourhood U of ξ in X. Put $d = \dim_\xi X$. By the density of $\mathrm{reg}X$ in X, we may find a point $x_0 \in U$ such that $x_0 \in \mathrm{reg}X$, $\dim_{x_0} X = d$, and $\mathrm{rk}_x\varphi = s$ for all x in an open neighbourhood U_0 of x_0 in $\mathrm{reg}X$, where $s = \mathrm{rk}_{x_0}\varphi$. By our hypothesis, $s < r$. By the Rank Theorem again, we get that

$$d = \dim U_0 = \dim\varphi(U_0) + \mathrm{fbd}_{x_0}(\varphi|_{U_0}) = \mathrm{rk}_{x_0}\varphi + \mathrm{fbd}_{x_0}(\varphi|_{U_0}) \le s + (k - r) < r + (k - r) = k\,.$$

Hence $\dim_\xi X = d < k = \dim\Delta = \dim_\xi\Delta$; a contradiction. □

Theorem 2.13. *Let $\varphi : X \to Y$ be as above. Then $\mathrm{rk}\,\varphi = \dim\varphi(X)$.*

Proof. By the Rank Theorem, $\varphi(X)$ contains submanifolds of dimension $\mathrm{rk}\,\varphi$ (namely, the images of open pieces of $\mathrm{reg}X$, where the rank is maximal), hence $\mathrm{rk}\,\varphi \le \dim\varphi(X)$. For the opposite inequality, put $k = \dim X$, and proceed by induction on k.

If $k = -1$, then $X = \varnothing$, so $\mathrm{rk}\,\varphi = -1$ and we have equality. Suppose then that $k \ge 0$ and the inequality holds for domains of dimensions at most $k - 1$. Since $\dim X \ge 0$, then $\dim(\mathrm{sng}X) \le k - 1$, and by the inductive hypothesis $\dim\varphi(\mathrm{sng}X) \le \mathrm{rk}\,(\varphi|_{\mathrm{sng}X})$, which by the Whitney Lemma implies $\dim\varphi(\mathrm{sng}X) \le \mathrm{rk}\,\varphi$. But Baire Category Theorem with Rank Theorem imply that also $\dim\varphi(\mathrm{reg}X) \le \mathrm{rk}\,\varphi$, so $\dim\varphi(X) = \max\{\dim\varphi(\mathrm{sng}X), \dim\varphi(\mathrm{reg}X)\} \le \mathrm{rk}\,\varphi$. □

Let now $\varphi : X \to Y$ be a holomorphic mapping of analytic subsets of manifolds $X \subset M$ and $Y \subset N$, where X is irreducible. Then $\mathrm{reg}X$ is a connected manifold, and consequently (Proposition 2.10(i)) $\mathrm{rk}\,(\varphi|_U) = \mathrm{rk}\,\varphi$ for every nonempty open $U \subset X$.

Corollary 2.14. *Under the above assumptions, the image $\varphi(X)$ is of pure dimension.*

Proof. Let $y \in \varphi(X)$. For arbitrary open neighbourhood V of y in N, we have $\varphi(X) \cap V = \varphi|_{\varphi^{-1}(V)}(\varphi^{-1}(V))$, hence $\dim(\varphi(X)\cap V) = \mathrm{rk}\,(\varphi|_{\varphi^{-1}(V)})$, by Theorem 2.13, and thus $\dim(\varphi(X)\cap V) = \mathrm{rk}\,\varphi$ (by openness of $\varphi^{-1}(V)$). Therefore, $\dim_y\varphi(X)$ is constant along $\varphi(X)$. □

Definition 2.15. Under the irreducibility assumption on X, we define the *generic fibre dimension* of φ, denoted $\lambda(\varphi)$, as $\lambda(\varphi) = -1$ if $X = \varnothing$, and otherwise

$$\lambda(\varphi) \;=\; \min\{\mathrm{fbd}_x\varphi \,:\, x \in X\}\,.$$

Observe that, by the upper semi-continuity of the fibre dimension of φ, we have

$$\lambda(\varphi) \;=\; \min\{\mathrm{fbd}_x\varphi \,:\, x \in \Omega\} \quad \text{for any dense subset } \Omega \subset X\,.$$

We will now show that the name "generic" is well justified, namely, the fibre dimension of φ is equal to $\lambda(\varphi)$ on a dense Zariski-open subset of X. Consider the set

$$C(\varphi) \;=\; \{x \in \mathrm{reg}X \,:\, \mathrm{rk}_x\varphi < \mathrm{rk}\,\varphi\} \cup \mathrm{sng}X\,.$$

By connectedness of $\mathrm{reg}X$, $C(\varphi)$ is nowhere dense in X. Also, $C(\varphi)$ is analytic in M. Indeed, $C(\varphi)$ is closed in X (as its complement is open in X), and hence in M, and is analytically constructible in M, by Theorem 2.4; therefore $C(\varphi)$ analytic in M, by Proposition 2.3(ii).

Proposition 2.16 (Dimension Formula)**.** *We have*

$$\mathrm{fbd}_x\varphi \;=\; \lambda(\varphi) \qquad \text{for all } x \in X \setminus C(\varphi)\,,$$

and hence

$$\dim X \;=\; \lambda(\varphi) + \mathrm{rk}\,\varphi \;=\; \lambda(\varphi) + \dim\varphi(X)\,.$$

Proof. By the definition of $C(\varphi)$, the Rank Theorem implies that $\mathrm{fbd}_x\varphi$ is constant on $X \setminus C(\varphi)$. The upper semi-continuity of fibre dimension together with the density of $X \setminus C(\varphi)$ in X then imply that this constant equals $\lambda(\varphi)$, which proves the first equality. Therefore, by the Rank Theorem again, we have $\dim X = \lambda(\varphi) + \mathrm{rk}_x\varphi$ for every $x \in X \setminus C(\varphi)$. Hence $\dim X = \lambda(\varphi) + \mathrm{rk}\,\varphi$, by openness in X of the non-empty $X \setminus C(\varphi)$. The last equality of the proposition follows from Theorem 2.13. $\square$

Finally, without the irreducibility assumption on X, we prove the following:

Theorem 2.17 (Cartan–Remmert Theorem)**.** *Let $\varphi : X \to Y$ be a holomorphic mapping of analytic subsets of manifolds $X \subset M$ and $Y \subset N$. For every $k \in \mathbb{N}$, the set*

$$A_k(\varphi) = \{x \in X : \mathrm{fbd}_x\varphi \geq k\}$$

is analytic in M.

Proof. We will proceed by induction on $l = \dim X$. If $l = 0$, there is nothing to show. Let then $l = \dim X \geq 1$, and assume the theorem holds for domains of dimensions less than l. For a holomorphic $\psi : Z \to W$, denote $A_k(\psi) = \{z \in Z : \mathrm{fbd}_z\psi \geq k\}$.

We may assume that X is irreducible, because if $X = \bigcup_\iota X_\iota$ is the decomposition into irreducible components, then for every $x \in X$,

$$[\varphi^{-1}(\varphi(x))]_x = \bigcup_\iota [(\varphi|_{X_\iota})^{-1}((\varphi|_{X_\iota})(x))]_x\,,$$

and only finitely many summands on the right hand side are non-empty, hence $A_k(\varphi) = \bigcup_\iota A_k(\varphi|_{X_\iota})$.

Now, for irreducible X, we may consider $C(\varphi)$ and $\lambda(\varphi)$ as above. For $k \leq \lambda(\varphi)$, we have $A_k(\varphi) = X$. Let's assume then that $k > \lambda(\varphi)$. Since $\dim C(\varphi) < \dim X$, then by the inductive hypothesis $A_k(\varphi|_{C(\varphi)})$ is analytic. It thus suffices to show that $A_k(\varphi) = A_k(\varphi|_{C(\varphi)})$.

Of course, $\mathrm{fbd}_x\varphi|_{C(\varphi)} \geq k \Rightarrow \mathrm{fbd}_x\varphi \geq k$, so $A_k(\varphi) \supset A_k(\varphi|_{C(\varphi)})$. Let then $\xi \in A_k(\varphi)$; i.e., $\mathrm{fbd}_\xi\varphi \geq k$. As $k > \lambda(\varphi)$, it follows from the Dimension Formula (Proposition 2.16 above) that $\xi \in C(\varphi)$. Consider the fibre $Z := \varphi^{-1}(\varphi(\xi))$. By the Rank Theorem and Proposition 2.16, $Z \setminus C(\varphi)$ is a submanifold of $\mathrm{reg}X$ of dimension $\lambda(\varphi)$, which is less than k by assumption. On the other hand, $\dim Z \geq k$, since $\dim_\xi Z \geq k$. But

$$Z_\xi = [\varphi^{-1}(\varphi(\xi))]_\xi = [Z \setminus C(\varphi)]_\xi \cup [(\varphi|_{C(\varphi)})^{-1}((\varphi|_{C(\varphi)})(\xi))]_\xi\,,$$

which implies that the dimension of the second summand, $[(\varphi|_{C(\varphi)})^{-1}((\varphi|_{C(\varphi)})(\xi))]_\xi$, must be at least k; i.e., $\xi \in A_k(\varphi|_{C(\varphi)})$. $\square$

2.4 Remmert Open Mapping Theorem

We will now prove a fundamental openness criterion for holomorphic mappings of analytic spaces, which asserts that openness is equivalent to the (geometric) continuity in the family of fibres of a mapping. As we will see in the later sections, this property allows to think of openness as a geometric equivalent of flatness.

Theorem 2.18 (Remmert Open Mapping Theorem)**.** *Let $\varphi : X \to Y$ be a holomorphic mapping of analytic spaces, where X is of pure dimension m, and Y is locally irreducible of dimension n. Then φ is an open mapping if and only if one of the following equivalent conditions holds:*

(i) $\mathrm{fbd}_x\varphi = m - n$ *for all* $x \in X$

(ii) $\dim \varphi^{-1}(y) = m - n$ *for all* $y \in \varphi(X)$.

Proof. We first show the equivalence of conditions (i) and (ii). The implication (i) $\Rightarrow$ (ii) is trivial. Suppose then that (ii) holds. Then, of course, $\mathrm{fbd}_x \leq m - n$ for all $x \in X$. For the proof of the opposite inequality, fix a point $\xi \in X$, and let $X = \bigcup_\iota X_\iota$ be the decomposition of X into isolated irreducible components. Then

$$[\varphi^{-1}(\varphi(\xi))]_\xi = \bigcup_\iota [(\varphi|_{X_\iota})^{-1}((\varphi|_{X_\iota})(\xi))]_\xi ,$$

and only finitely many germs on the right hand side are non-empty, hence $\mathrm{fbd}_\xi\varphi = \max_\iota \mathrm{fbd}_\xi(\varphi|_{X_\iota})$. For each ι, the Dimension Formula implies

$$\mathrm{fbd}_\xi(\varphi|_{X_\iota}) \geq \lambda(\varphi|_{X_\iota}) = \dim X_\iota - \dim \varphi(X_\iota) \geq m - n .$$

The latter inequality follows from the puredimensionality of X ($\dim X_\iota = \dim X$), and the fact that $\dim \varphi(X_\iota) \leq \dim Y = n$. Therefore, $\mathrm{fbd}_\xi\varphi \geq m - n$.

Next we show that (i) implies openness of φ. Notice that this is the only place where we need the local irreducibility of Y (in other words, the other implications hold true without this assumption).

By the Remmert Rank Theorem 2.1 and puredimensionality of X, we have

every $x \in X$ *has an open nbhd* Δ *in* X *with* $\varphi(\Delta)$ *locally analytic in* Y, *of dimension* n. (*)

Let $\xi \in X$ and let U be an arbitrary open neighbourhood of ξ in X. We want to show that $\varphi(U)$ contains an open neighbourhood of $\varphi(\xi)$ in Y. For this, let Δ be an open neighbourhood of ξ, as in $(*)$, and let Ω be an irreducible open neighbourhood of $\varphi(\xi)$ in Y, small enough so that $\varphi(\Delta)\cap\Omega$ is analytic in Ω. Let next $\Delta_0 \subset \Delta$ be as in $(*)$ and such that $\varphi(\Delta_0) \subset \Omega$. Then $\varphi(\Delta_0) \subset \varphi(\Delta)\cap\Omega \subset Y$, and both $\varphi(\Delta_0)$ and Y are n-dimensional. Hence $\dim(\varphi(\Delta) \cap \Omega) = n$. By irreducibility of Ω, $\varphi(\Delta) \cap \Omega = \Omega$, and hence $\varphi(U)$ contains an open neighbourhood $\varphi(\Delta) \cap \Omega$ of $\varphi(\xi)$ in Y.

Finally, we will prove that openness of φ implies (ii), with help of the following lemma.

Lemma 2.19. *Let* Δ *and* Ω *be open polydiscs centered at the origins in* $M = \mathbb{C}^d$ *and* $N = \mathbb{C}^n$ *respectively, and let* A *be analytic in* $\Delta \times \Omega$. *If the canonical projection* $\pi : A \to \Omega$ *is open, and* $A \supset \Delta \times \{0\}$, *then* $A = \Delta \times \Omega$.

Proof. Assume first that $n = 1$. Suppose $A \subsetneq \Delta \times \Omega$. Then $\dim A \leq \dim(\Delta \times \Omega) - 1 = d$. Then the constructible set $A \setminus (\Delta \times \{0\})$ is also of dimension at most d, and hence its closure (which is analytic, by Proposition 2.3) satisfies

$$\dim \overline{A \setminus (\Delta \times \{0\})} \leq d .$$

The set $\overline{A \setminus (\Delta \times \{0\})} \cap (\Delta \times \{0\})$, being nowhere dense in $\overline{A \setminus (\Delta \times \{0\})}$, is thus of dimension at most $d - 1$. In particular, $\Delta \times \{0\} \not\subset \overline{A \setminus (\Delta \times \{0\})}$ (as $\dim(\Delta \times \{0\}) = d$), and consequently $(\Delta \times \{0\}) \setminus \overline{A \setminus (\Delta \times \{0\})}$ is open non-empty in A. Therefore, there exists $\xi \in X$ and its open neighbourhood U in X such that $U \subset (\Delta \times \{0\}) \setminus \overline{A \setminus (\Delta \times \{0\})}$. Then $\pi(U) = \{0\}$, which contradics the openness of π.

For arbitrary $n \geq 1$, to show that $A \supset \Delta \times \Omega$, it suffices to show that, for any complex line $L \subset N$, $A \cap [\Delta \times (\Omega \cap L)] = \Delta \times (\Omega \cap L)$. Let then L be any such a line. The projection $\tilde{\pi} : A \cap [\Delta \times (\Omega \cap L)] \to \Delta \times (\Omega \cap L)$ is open, as a restriction of π to $\pi^{-1}(\Omega \cap L)$. Also, $\Delta \times \{0\} \subset A \cap [\Delta \times (\Omega \cap L)]$, since $0 \in L$. Therefore, by the first part of the proof, $\Delta \times (\Omega \cap L) \subset A \cap [\Delta \times (\Omega \cap L)]$. $\square$

Now, we claim that it suffices to prove the required implication for the graph Γ_φ of φ. Indeed, the problem being local, we may assume that $X \subset \mathbb{C}^d$ and $Y \subset \mathbb{C}^p$ are local models. Then φ is open iff the projection from Γ_φ to Y is open; and the corresponding fibres are biholomorphic, hence of the same (even local) dimension.

Next, we may assume that Y is open in $\mathbb{C}^p$ (and hence, after shrinking and translation, an open polydisc Ω at the origin in $\mathbb{C}^p$). Indeed, as a locally irreducible n-dimensional analytic set in an open polydisc $W \subset \mathbb{C}^p$, Y has a proper coordinate projection π' onto some $W \cap \mathbb{C}^n$ (after a linear change of coordinates, at worst). As, for every $y \in Y$, $\dim_{\pi'(y)} \pi'(Y) = \dim_y Y = n = \dim_{\pi'(y)}(W \cap \mathbb{C}^n)$, the projection is open. Therefore, φ is open iff $\pi' \circ \varphi$ is open, and fibres of φ and $\pi' \circ \varphi$ are of the same local dimension.

Let then Δ and Ω be open polydiscs centered at the origins of $\mathbb{C}^d$ and $\mathbb{C}^n$ respectively, and let A be an analytic set through $(0,0)$ in $\Delta \times \Omega$, of pure dimension m, such that the canonical projection $\pi : A \to \Omega$ is open, and the canonical projection $\kappa : A \to \Delta$ is $1-1$ and proper. Put $r = \dim_0 \pi^{-1}(0)$. After shrinking Δ and Ω if necessary, we may assume that $\pi^{-1}(0)$ is of pure dimension r. Then so is $\kappa(\pi^{-1}(0))$, and by [I, Thm.6.1], we can find $\epsilon > 0$ and a $(d-r)$-dimensional linear subspace T in $\mathbb{C}^d = T^\perp \times T$ such that 0 is an isolated point of

$$\kappa(\pi^{-1}(0)) \cap (\{0\} \times T) \quad \text{in } \mathbb{C}^d .$$

Then, as in the proof of the Remmert Rank Theorem 2.1, we can shrink Δ so that $\Delta = \Lambda \times \Sigma$, with Λ and Σ open polydiscs at the origins in $\mathbb{C}^{d-r}$ and $\mathbb{C}^r$ respectively, such that the canonical projection

$$\lambda : A \cap [(\Lambda \times \Sigma) \times \Omega] \to \Sigma \times \Omega$$

is proper. Put $\tilde{A} = A \cap [(\Lambda \times \Sigma) \times \Omega]$. Then $\lambda(\tilde{A})$ is analytic in $\Sigma \times \Omega$, of dimension $\dim \lambda(\tilde{A}) = \dim A = m$.

Let next $\rho : \lambda(\tilde{A}) \to \Omega$ be the restriction of the canonical projection from $\Sigma \times \Omega$. Then ρ is open, as $\pi = \rho \circ \lambda$. Moreover, $\rho^{-1}(0) = \Sigma$, because $\kappa(\pi^{-1}(0))$ is of pure dimension r with proper projection to the r-dimensional Σ. Therefore, $\lambda(\tilde{A}) \supset \Sigma \times \{0\}$. By the above Lemma, it follows that $\lambda(\tilde{A}) = \Sigma \times \Omega$. Hence

$$m \,=\, \dim \tilde{A} \,=\, \dim \lambda(\tilde{A}) \,=\, \dim(\Sigma \times \Omega) \,=\, r + n .$$

Also, every fibre $\pi^{-1}(y)$ over $y \in \Omega$ is of dimension r, because $\rho^{-1}(y) = \Sigma$. Thus, for every $y \in \Omega$, we have

$$\dim \pi^{-1}(y) \,=\, r \,=\, m - n ,$$

as required. $\square$

3 Vertical components and effective openness criterion

In this section, we introduce the so-called vertical components, that are a natural geometric analogue of torsion elements in algebraic geometry. We will first use them to establish an effective (that is, computable) openness criterion, and in later sections, to the study of flatness.

In the remainder of these notes, we shall concentrate on the *local* geometry of analytic mappings. We will tacitly use the results of [I, Sec.9], particularly the duality of the germs of complex analytic spaces and local analytic $\mathbb{C}$-algebras. Recall that the local ring of a germ X_ξ of a complex analytic space can be thought of as the quotient $\mathbb{C}\{x\}/I$ for some ideal I in the ring of convergent power series in variables $x = (x_1, \dots, x_m)$ with complex coefficients. Then the irreducible components of X_ξ are precisely the zero set germs $\mathcal{V}(\mathfrak{p})$, where $\mathfrak{p} \in \mathrm{Ass}_{\mathbb{C}\{x\}} I$ are the associated primes of I (the *isolated* components corresponding to the minimal among those primes, and the *embedded* components being the zero set germs of the non-minimal associated primes).

3.1 Vertical components

Definition 3.1. Let $\varphi_\xi : X_\xi \to Y_\eta$ be a germ of a holomorphic mapping of analytic spaces, and assume that Y_η is irreducible. An irreducible (isolated or embedded) component W_ξ of X_ξ is called *algebraic vertical* when there exists a nonzero element $a \in \mathcal{O}_{Y,\eta}$ such that (the pullback by φ_ξ^* of) a belongs to the associated prime $\mathfrak{p}$ in $\mathcal{O}_{X,\xi}$ corresponding to W_ξ. Equivalently, W_ξ is algebraic vertical when an arbitrarily small representative W of W_ξ is mapped into a proper analytic subset of a neighbourhood of η in Y. We say that W_ξ is *geometric vertical* when an arbitrarily small representative W of W_ξ is mapped into a nowhere dense subset of a neighbourhood of η in Y.

The concept of vertical component comes up naturally as an equivalent of torsion in algebraic geometry: Let $\varphi : X \to Y$ be a polynomial map of algebraic varieties with Y irreducible. Then the coordinate ring of the source, $A(X)$ has nonzero torsion as a module over the coordinate ring $A(Y)$ of the target if and only if there exists a nonzero element $a \in A(Y)$ such that its pullback $\varphi^* a$ is a zerodivisor in $A(X)$. Since the set of zerodivisors equals the union of the associated primes, it follows that $A(X)$ has nonzero torsion over $A(Y)$ if and only if there exists an irreducible (isolated or embedded) component of X whose image under φ is contained in a proper algebraic subset of Y (or, equivalently, is nowhere dense in Y). There are two natural ways of generalizing this property of irreducible components to the analytic case. For a morphism $\varphi_\xi : X_\xi \to Y_\eta$ of germs of analytic spaces (with Y_η irreducible), one can either consider the components of the source that are mapped into nowhere dense subgerms of the target (the *geometric vertical* components), or the components that are mapped into proper analytic subgerms of the target (the *algebraic vertical* components).

Note that, in principle, the existence of the algebraic vertical components is a weaker condition than the presence of the geometric vertical ones. Indeed, any algebraic vertical component (over an irreducible target) is geometric vertical, since a proper analytic subset of a locally irreducible analytic set has empty interior. The converse is not true though, as can be seen in the following example of Osgood (cf. [GR]):

$$\varphi : \mathbb{C}^2 \ni (x, y) \mapsto (x, xy, xy\mathrm{e}^y) \in \mathbb{C}^3.$$

Here the image of an arbitrarily small neighbourhood of the origin is nowhere dense in $\mathbb{C}^3$, but its Zariski closure has dimension 3 and therefore the image is not contained in a proper locally analytic subset of the target.

Remark 3.2. On the other hand, the algebraic approach has an advantage that all the statements about algebraic vertical components (as opposed to geometric vertical) can be restated in terms of torsion freeness of the local rings. Namely,

$\varphi_\xi : X_\xi \to Y_\eta$ has no (isolated or embedded) algebraic vertical components if and only if the local ring $\mathcal{O}_{X,\xi}$ is a torsionfree $\mathcal{O}_{Y,\eta}$-module.

In view of Remark 3.2, an interesting question is under what conditions are the two approaches equivalent. In the next section we show that one can expect a positive answer to this question under some minor constraints on the source space. In Section 3.3 below, we show that the geometric and the algebraic approaches are not equivalent in general; that is, there are examples of bad behaviour of analytic mappings that can be detected by means of geometric vertical components but not by the algebraic vertical ones.

3.2 Effective openness criterion

We begin with a technical observation carrying the kernel of all the subsequent results of this section. throughout Section 3, we shall keep the following assumptions: Let $\varphi_\xi : X_\xi \to Y_\eta$ be a germ of a holomorphic mapping of complex analytic spaces, where Y_η is irreducible of dimension n. Let Y be an irreducible representative of Y_η, and let X be a representative of X_ξ, such that $\varphi(X) \subset Y$ and $X_{\mathrm{red}} = X_1 \cup \dots \cup X_s$ consists of finitely many irreducible components through ξ that are precisely the representatives in X of the isolated irreducible components of the germ X_ξ. (At this moment, we are not interested in the nilpotent structure of X_ξ. In fact, if one takes into account the embedded components as well, one obtains criteria for flatness instead.)

Let $l = \min\{\mathrm{fbd}_x\varphi : x \in X\}$ and let $k = \max\{\mathrm{fbd}_x\varphi : x \in X\}$. For $l \leq j \leq k$, define $A_j = \{x \in X : \mathrm{fbd}_x\varphi \geq j\}$. Then, for each j, A_j is analytic in X (by the Cartan–Remmert Theorem) and $X = A_l \supset A_{l+1} \supset \dots \supset A_k$. Define $B_j = \varphi(A_j) = \{y \in Y : \dim\varphi^{-1}(y) \geq j\}$, for $l \leq j \leq k$. Note that, except for B_k (cf. proof of Proposition 3.3 below), the B_j may not even be semianalytic in general. Nonetheless, there is an interesting connection between the filtration of the target by fibre dimension $Y \supset B_l \supset B_{l+1} \supset \dots \supset B_k$ and the isolated irreducible components of the n-fold fibre power $X^{\{n\}}_{\xi^{\{n\}}}$ that we describe below.

Proposition 3.3 ([A2, Prop.2.1]). *Under the above assumptions, let $(X^{\{n\}})_{\mathrm{red}} = \bigcup_{i\in I} W_i$ be the decomposition into finitely many isolated irreducible components through $\xi^{\{n\}} = (\xi,\dots,\xi) \in X^n$. Then*

(a) For each $j = l,\dots,k$, there exists an index subset $I_j \subset I$ such that

$$B_j = \bigcup_{i\in I_j} \varphi^{\{n\}}(W_i)\,.$$

(b) If $y \in B_j$ with $\dim\varphi^{-1}(y) = s$ $(s \geq j)$, Z is an isolated irreducible component of the fibre $(\varphi^{\{n\}})^{-1}(y)$ of dimension ns, and W is an irreducible component of $X^{\{n\}}$ containing Z, then $\varphi^{\{n\}}(W) \subset B_j$.

Proof. For the part (b), fix $j \geq l+1$ (the statement is trivial for $j = l$, as $B_l = \varphi(X)$). Let $y \in B_j$, and let $s = \dim\varphi^{-1}(y)$. Then $s \geq j$. Let Z be an irreducible component of the fibre $(\varphi^{\{n\}})^{-1}(y)$ of dimension ns, and let W be an irreducible component of $X^{\{n\}}$ containing Z. We will show that $\varphi^{\{n\}}(W) \subset B_j$.

Suppose to the contrary that $W \cap (X^{\{n\}} \setminus (\varphi^{\{n\}})^{-1}(B_j)) \neq \varnothing$, that is, suppose that there exists $x = (x_1,\dots,x_n) \in W$ such that $\varphi(x_1) \in Y \setminus B_j$ (and hence $\varphi(x_i) \in Y \setminus B_j$ for $i \leq n$). Then $\mathrm{fbd}_{x_i}\varphi \leq j-1$, $i = 1,\dots,n$, and hence $\mathrm{fbd}_x\varphi^{\{n\}} \leq n(j-1) = nj-n$. In particular, the generic fibre dimension of $\varphi^{\{n\}}|W$ is not greater than $nj-n$. Since $\mathrm{rk}\,(\varphi^{\{n\}}|W) \leq \dim Y = n$, then $\dim W \leq (nj-n)+n = nj$, by Proposition 2.16.

Now we have: $W \supset Z$, $\dim W \leq nj$, $\dim Z = ns \geq nj$, and both W and Z irreducible analytic sets in $X^{\{n\}}$. This is only possible when $W = Z$, and hence $\varphi^{\{n\}}(W) = \varphi^{\{n\}}(Z) = \{y\} \subset B_j$; a contradiction. Therefore $\varphi^{\{n\}}(W) \subset B_j$, which completes the proof of part (b).

Part (a) follows immediately, since for any $y \in B_j$ and any irreducible component Z of $(\varphi^{\{n\}})^{-1}(y)$ of the highest dimension, there exists an isolated irreducible component W of $X^{\{n\}}$ that contains Z. □

Remark 3.4. In the context of the above proposition, it should be noted that the fibre product of (germs of) reduced analytic spaces over a reduced analytic space need not be reduced itself. Consider for example $X \subset \mathbb{C}^4$ which is a union of two copies of $\mathbb{C}^2$ that intersect precisely at the origin, say

$$X_1 = \{(y_1, y_2, t_1, t_2) \in \mathbb{C}^4 \, : \, t_1 = t_2 = 0\} \quad \text{and} \quad X_2 = \{(y_1, y_2, t_1, t_2) \in \mathbb{C}^4 \, : \, t_1 - y_1 = t_2 - y_2 = 0\},$$

and let $\varphi : X \to \mathbb{C}^2$ be the projection onto the y variables. Then the fibre power $X^{\{2\}}$ has an embedded component, namely the origin in $\mathbb{C}^4$ (see, e.g., [Har, III, Ex.9.3(b)]).

Theorem 3.5 ([A2, Thm.2.2]). *Let $\varphi_\xi : X_\xi \to Y_\eta$ be a germ of a holomorphic mapping of complex analytic spaces, with X_ξ puredimensional and Y_η irreducible of dimension n. Then the following conditions are equivalent:*

(i) *φ_ξ is a germ of an open mapping*

(ii) *$X^{\{n\}}_{\xi^{\{n\}}}$ has no isolated geometric vertical components*

(iii) *$X^{\{n\}}_{\xi^{\{n\}}}$ has no isolated algebraic vertical components.*

Proof. First, note that if $\varphi_\xi : X_\xi \to Y_\eta$ is open, then $\varphi^{\{n\}}_{\xi^{\{n\}}} : X^{\{n\}}_{\xi^{\{n\}}} \to Y_\eta$ is open for every $i \geq 1$, and hence every isolated irreducible component of $X^{\{n\}}_{\xi^{\{n\}}}$ is mapped onto Y_η. This proves (i) ⇒ (ii). The implication (ii) ⇒ (iii) is trivial, as every algebraic vertical component is geometric vertical over an irreducible target.

For the proof of (iii) ⇒ (i), choose representatives φ, X and Y of φ_ξ, X_ξ and Y_η as in Proposition 3.3, put $m = \dim X$, and suppose that φ is not open. We will show that then $X^{\{n\}}_{\xi^{\{n\}}}$ contains an isolated algebraic vertical component over Y_η. Consider the sets A_k and $B_k = \varphi(A_k)$ as in Proposition 3.3. As the fibre dimension $\mathrm{fbd}_x\varphi$ is constant on A_k (and equal to k), the Remmert Rank Theorem implies that B_k is locally analytic in Y, of dimension $m - k$. After shrinking Y and X if necessary, we may assume that B_k is closed in Y, and hence analytic. By Proposition 3.3(a), there is an isolated irreducible component W of $X^{\{n\}}$ (through $\xi^{\{n\}}$) such that $\varphi^{\{n\}}(W) \subset B_k$. We claim that $W_{\xi^{\{n\}}}$ is algebraic vertical. For that it suffices to show that $(B_k)_\eta$ is a proper subgerm of Y_η.

Without loss of generality, we may assume that k is not the generic fibre dimension along every irreducible component of X. Indeed, if k is the generic (that is, minimal) fibre dimension of $\varphi|_{X_\iota}$ for every irreducible component X_ι of X, then the fibre dimension of φ is constantly k on X. Then, by Remmert Open Mapping Theorem, φ is open (contrary to our hypothesis), else $m - k < n$. But then *every* component of X, and hence of any fibre power of X, is algebraic vertical and there is nothing to show.

Now, let X_ι be an irreducible component of X, for which $\lambda(\varphi|_{X_\iota}) \leq k - 1$. By the Dimension Formula (Prop. 2.16) and puredimensionality of X, we have

$$m \,=\, \dim X_\iota \,=\, \lambda(\varphi|_{X_\iota}) + \dim \varphi(X_\iota) \,\leq\, (k-1) + \dim Y \,=\, n + k - 1\,,$$

hence $\dim Y = n \leq m - k + 1$. Consequently,

$$\dim Y_\eta = \dim Y > \dim B_k \geq \dim(B_k)_\eta\,,$$

and thus $(B_k)_\eta$ is a proper analytic subgerm of Y_η. $\square$

In light of Remark 3.2, and the fact that isolated irreducible components correspond to the minimal associated primes, the equivalence (i) $\Leftrightarrow$ (iii) in the above theorem could be restated as follows:

Corollary 3.6. *Under the assumptions of Theorem 3.5, $\varphi_\xi : X_\xi \to Y_\eta$ is open if and only if the reduced local ring $\left(\mathcal{O}_{X^{\{n\}},\xi^{\{n\}}}\right)_{\mathrm{red}}$ is a torsionfree $\mathcal{O}_{Y,\eta}$-module.*

Remark 3.7. The true effectiveness of the above openness criterion can be seen in the algebraic setting. When X and Y are algebraic sets (or even schemes of finite type over a field) and $\varphi : X \to Y$ is a polynomial mapping, one can actually *calculate* openness of φ with a computer algebra software. We will not discuss this topic here, as it belongs to algebraic (rather than analytic) geometry. For details and state-of-the-art results, we refer the reader to [ABM3].

Example 3.8. The following example shows that the puredimensionality constraint on X in Theorem 3.5 is unavoidable. If X is not puredimensional, it may happen that the exceptional fibres of one component are generic for a component of higher dimension, and therefore they do not give rise to an isolated algebraic vertical component in any fibre power.

Let $X = X_1 \cup X_2$, where $X_1 = \{(x,y,s,t,z) \in \mathbb{C}^5 : s = t = z = 0\}$ and $X_2 = \{(x,y,s,t,z) \in \mathbb{C}^5 : x = 0\}$. Define $\varphi : X \to Y = \mathbb{C}^3$ as

$$\varphi(x,y,s,t,z) = (x+s, xy+t, xye^y + z).$$

Observe that (the germ at the origin of) $\varphi|_{X_1}$ is an Osgood mapping and hence it is not open. Therefore φ is not open, as openness of φ implies openness of its restrictions to irreducible components. But the exceptional fibre $\{x = s = t = z = 0\}$ of $\varphi|_{X_1}$ is in no sense exceptional for $\varphi|_{X_2}$. One can easily verify that in any fibre power of X over Y, any isolated (!) irreducible component is either *purely* geometric vertical (with the image equal to that of $\varphi|_{X_1}$) or maps onto (the germ at the origin of) Y and therefore is not vertical in neither sense.

Remark 3.9. By Theorem 3.5, the existence of isolated geometric vertical components in $X^{\{n\}}_{\xi^{\{n\}}}$ is equivalent to the existence of isolated algebraic vertical components, provided X_ξ is puredimensional. Nevertheless, it is not true that every isolated geometric vertical component in $X^{\{n\}}_{\xi^{\{n\}}}$ is algebraic vertical. As we show in Proposition 3.12 and Example 3.13 in the next section, this is not even true in the case of a smooth domain.

3.3 Fibre powers and Gabrielov regularity

In this section, we show that fibre product doesn't behave well with respect to the so-called Gabrielov regularity. We begin with defining two natural *ranks* of a holomorphic germ. Let, as before, $\varphi_\xi : X_\xi \to Y_\eta$ denote a germ of a holomorphic mapping of complex analytic spaces, and assume that both X_ξ and Y_η are irreducible. Consider the pull-back homomorphism $\varphi^*_\xi : \mathcal{O}_{Y,\eta} \to \mathcal{O}_{X,\xi}$ induced by φ, and its natural extension $\widehat{\varphi}_\xi^* : \widehat{\mathcal{O}}_{Y,\eta} \to \widehat{\mathcal{O}}_{X,\xi}$ to completions in the $\mathfrak{m}$-adic topologies of $\mathcal{O}_{Y,\eta}$ and $\mathcal{O}_{X,\xi}$ respectively. We define

$$r_\xi^2(\varphi) \ := \ \dim \frac{\widehat{\mathcal{O}}_{Y,\eta}}{\ker \widehat{\varphi}_\xi^*}\ , \qquad r_\xi^3(\varphi) \ := \ \dim \frac{\mathcal{O}_{X,\xi}}{\ker \varphi_\xi^*}$$

(where dim denotes the Krull dimension). Recall that the generic rank $r_\xi^1(\varphi)$ of φ at ξ is defined as the minimum over ranks of restrictions of φ to open neighbourhoods of ξ in X. Now, when X is assumed irreducible, Proposition 2.10 together with Theorem 2.13 imply that $r_\xi^1(\varphi)$ equals to the dimension $\dim \varphi(U)$, where U is an arbitrarily small open neighbourhood of ξ in X. Since $r_\xi^2(\varphi)$ (resp. $r_\xi^3(\varphi)$) is the dimension of the locus of common zeros of all the formal (resp. convergent) relations among the components of φ (in local coordinates at ξ and η), or in other words, of common zeros of all the formal (resp. convergent) power series that vanish after the substitution of φ, it follows that

$$r_\xi^1(\varphi) \ \le \ r_\xi^2(\varphi) \ \le \ r_\xi^3(\varphi)\,.$$

In [Ga], Gabrielov proved a very deep theorem stating that

$$r_\xi^1(\varphi) = r_\xi^2(\varphi) \implies r_\xi^2(\varphi) = r_\xi^3(\varphi)\,,$$

which can be rephrased as follows:

if there are enough formal relations among the components of φ, then the ideal of formal relations is generated by the convergent relations.

Definition 3.10. We say that a holomorphic map $\varphi : X \to Y$ is *Gabrielov regular at a point $\xi \in X$*, when $r_\xi^1(\varphi) = r_\xi^3(\varphi)$. The map φ is *Gabrielov regular* when it is Gabrielov regular at every point of X. For a holomorphic germ $\varphi_\xi : X_\xi \to Y_\eta$, we will say that φ_ξ is Gabrielov regular, when it is a germ at ξ of a holomorphic φ which is Gabrielov regular at ξ. Then, for irreducible X_ξ, φ_ξ is Gabrielov regular iff an arbitrarily small open neighbourhood U of ξ in X satisfies the equality of dimensions

$$\dim \varphi(U) = \dim \overline{\varphi(U)}\,,$$

where $\overline{Z}$ denotes the analytic Zariski closure of Z in Y. In general, when $X_\xi = \bigcup_\iota (X_\iota)_\xi$ is a decomposition into isolated irreducible components, then we say that φ_ξ is Gabrielov regular when $\varphi_\xi|_{(X_\iota)_\xi}$ is regular for every ι.

Remark 3.11. Gabrielov regularity makes perfect sense for real-analytic mappings as well, and it plays an important role in subanalytic geometry, where it is responsible for certain “tameness” properties of subanalytic sets (see, e.g., [ABM1] for details).

In regard with the local geometry of complex analytic mappings, one can think of irregularity in the sense of Gabrielov as the fundamental distinction between transcendental and algebraic mappings. Indeed, by a theorem of Chevalley, images of algebraic varieties under polynomial mappings are algebraic constructible, and hence taking their Zariski closure does not increase the dimension. The relationship between vertical components and Gabrielov regularity is that:

a geometric vertical component W of $\varphi : X \to Y$ is algebraic vertical if and only if $\varphi|_W$ is Gabrielov regular.

The following result is an immediate corollary of Proposition 3.3. Let $\varphi_\xi : X_\xi \to Y_\eta$ be a germ of a holomorphic mapping of analytic spaces, with X_ξ of pure dimension, and Y_η irreducible of dimension n. Let φ, X, and Y be representatives of φ_ξ, X_ξ, and Y_η respectively, as at the beginning of Section 3.2. Put $S = \{y \in Y \ : \ \dim \varphi^{-1}(y) > l\}$, where as before $l = \min\{\mathrm{fbd}_x\varphi : x \in X\}$. Then $\dim S < \dim Y$, by pure-dimensionality of X (**Exercise**).

Proposition 3.12 ([A2, Prop.3.1]). *Suppose that* $\dim_\eta \overline{S} = n$*, where* $\overline{S}$ *denotes the analytic Zariski closure of* S *in* Y*. Then* $X^{\{n\}}_{\xi^{\{n\}}}$ *contains an isolated* purely geometric *vertical component; i.e., a geometric vertical component which is not algebraic vertical.*

Proof. By Proposition 3.3, there exist irreducible components $W_1, \dots, W_p$ of $X^{\{n\}}$ such that $S = \bigcup_{i=1}^p \varphi^{\{n\}}(W_i)$. (Recall that $S = B_{l+1}$ according to the notation from Proposition 3.3). We claim that $\dim_\eta \overline{\varphi^{\{n\}}(W_j)} = n$ for some $j \in \{1, \dots, p\}$. Indeed, if $\dim_\eta \overline{\varphi^{\{n\}}(W_i)} < n$ for all i, then we would have

$$n = \dim_\eta \overline{S} = \dim_\eta \overline{\bigcup_{i=1}^p \varphi^{\{n\}}(W_i)} = \max\{\dim_\eta \overline{\varphi^{\{n\}}(W_i)} \colon i = 1, \dots, p\} < n\,;$$

a contradiction. Then the component W_j is not algebraic vertical, and it is geometric vertical, since $\dim_\eta \varphi^{\{n\}}(W_j) \leq \dim_\eta S < n$. □

The above proposition implies that fibre powers of Gabrielov regular mappings need not be regular themselves.

Example 3.13. Let $X = Y = \mathbb{C}^4$ and define φ as follows

$$(x,\, y,\, s,\, t) \mapsto (x,\ (x+s)y,\ x^2y^2e^y,\ x^2y^2e^{y(1+xe^y)} + st).$$

It is easy to check that the generic fibre dimension of φ equals 0, hence the generic rank of φ is 4, and so φ is dominant, hence Gabrielov regular.

On the other hand, S contains the image of the set $\mathbb{C}^2 \times \{0\} \times \mathbb{C}$, consisting of points (z_1, z_2, z_3, z_4) of the form $(x,\, xy,\, x^2y^2e^y,\, x^2y^2e^{y(1+xe^y)})$, whose Zariski closure equals $\mathbb{C}^4$ (indeed, this set is the image of a composite of Osgood type mappings, $\mathbb{C}^2 \to \mathbb{C}^3$ followed by $\mathbb{C}^3 \to \mathbb{C}^4$). Hence also $\overline{S} = \mathbb{C}^4$. Thus, by Proposition 3.12, $X^{\{4\}}$ contains an isolated purely geometric vertical component over Y, which means that $\varphi^{\{4\}}$ is not Gabrielov regular.

In light of the above example, a natural question is under what conditions the fibre powers of a Gabrielov regular mapping remain regular. The answer turns out to be the regularity of the filtration by fibre dimension (recall Proposition 3.3). More precisely:

Proposition 3.14 ([A4, Prop.4.2]). *Let* $\varphi_\xi : X_\xi \to Y_\eta$ *be a morphism of germs of analytic spaces, with* Y_η *irreducible of dimension* n*. The following conditions are equivalent:*

(i) $\varphi^{\{i\}}_{\xi^{\{i\}}} : X^{\{i\}}_{\xi^{\{i\}}} \to Y_\eta$ *is Gabrielov regular for all* $i \geq 1$*;*

(ii) the restrictions $\varphi|_{A_j}$ *are Gabrielov regular* $(j = l, \dots, k)$*.*

Proof. Suppose first that $\varphi|_{A_j}$ $(j = l, \dots, k)$ are regular. Fix a positive integer i and let W be an isolated irreducible component of $X^{\{i\}}$. Since the components of $X^{\{i\}}$ are precisely the representatives of those of $X^{\{i\}}_{\xi^{\{i\}}}$, it suffices to show that $\varphi^{\{i\}}|W$ is regular.

Let q be the greatest integer for which the generic fibre germ $F_{(\xi_1,\dots,\xi_i)} = (F_1)_{\xi_1} \times \dots \times (F_i)_{\xi_i}$ of $\varphi^{\{i\}}|W$ contains a component $(F_m)_{\xi_m}$ of dimension q. Then $\varphi^{\{i\}}(W) \subset B_q = \varphi(A_q)$. Indeed, by the upper semi-continuity of fibre dimension, the fibre dimensions $\mathrm{fbd}_{x_p}\varphi$ can only drop for $x = (x_1, \dots, x_i)$ in an open neighbourhood of ξ. On the other hand, the sum $\mathrm{fbd}_{x_1}\varphi + \dots + \mathrm{fbd}_{x_i}\varphi$ is already minimal, so all its summands must remain constant. When, in turn, a sequence $(\xi^\nu)_{\nu\in\mathbb{N}}$, where $\xi^\nu = (\xi^\nu_1, \dots, \xi^\nu_i)$, converges to a point in $C(\varphi^{\{i\}}|_W)$ (cf. the Dimension Formula), the sequence of its m'th components $(\xi^\nu_m)_{\nu\in\mathbb{N}}$ converges to a point in A_q (as A_q is closed). The property of being a fibre of dimension

q is a Zariski open condition on A_q (by Cartan–Remmert). Hence W is induced by an irreducible component V of A_q with the generic fibre dimension of $\varphi|_V$ equal to q, in the sense that there exists a component V of A_q such that $\varphi^{\{i\}}(W) = \varphi(V)$. By assumption, $\dim_\eta \overline{\varphi(V)} = \dim_\eta \varphi(V)$ (where closure is in the Zariski topology in Y), hence also $\dim_\eta \overline{\varphi^{\{i\}}(W)} = \dim_\eta \varphi^{\{i\}}(W)$; i.e., $\varphi^{\{i\}}|W$ is Gabrielov regular.

Suppose now that there exists $j \in \{l, \dots, k\}$ for which $\varphi|_{A_j}$ is *not* regular. We shall show that then regularity of $\varphi^{\{n\}}_{\xi^{\{n\}}} : X^{\{n\}}_{\xi^{\{n\}}} \to Y_\eta$ fails, where $n = \dim Y$.

Fix $j \in \{l, \dots, k\}$ such that $\varphi|_{A_j}$ is not regular. Pick $y \in B_j$ with $\dim \varphi^{-1}(y) = j$, and let Z be an isolated irreducible component of the fibre $(\varphi^{\{n\}})^{-1}(y)$ of dimension nj. Let W be an isolated irreducible component of $X^{\{n\}}$ containing Z. Then $\varphi^{\{n\}}(W) \subset B_j$, by Proposition 3.3(b). Moreover, $\varphi^{\{n\}}|_W$ has no fibres of dimension less than or equal to $n(j-1)$. Indeed, otherwise the generic fibre dimension of $\varphi^{\{n\}}|W$ would be at most $n(j-1)$, so that $\dim W \le n(j-1) + n = nj = \dim Z$, and hence $W = Z$, a contradiction (see the proof of Proposition 3.3). Thus, the generic fibre germ $F_{(\xi_1,\dots,\xi_i)} = (F_1)_{\xi_1} \times \cdots \times (F_n)_{\xi_i}$ of $\varphi^{\{n\}}|W$ contains a component F_m of dimension j.

Now, there is an isolated irreducible component V of A_j such that $\dim_\eta \overline{\varphi(V)} > \dim_\eta \varphi(V)$ and the generic fibre dimension of $\varphi|V$ is j. Our $y \in B_j$ can then be chosen from $\varphi(V)$, and Z a component of $\big((\varphi|_V)^{-1}(y)\big)^n$. Since being a j-dimensional fibre is an open condition on A_j, then (as in the first part of the proof) we find that $\varphi^{\{n\}}(W) = \varphi(V)$, so that $\dim_\eta \overline{\varphi^{\{n\}}(W)} > \dim_\eta \varphi^{\{n\}}(W)$. Thus $\varphi^{\{n\}}_{\xi^{\{n\}}} : X^{\{n\}}_{\xi^{\{n\}}} \to Y_\eta$ is not regular. $\square$

4 Flatness in complex analytic geometry

In this section, we continue our study of local regularity of holomorphic mappings with the discussion of flatness. Until quite recently we had a rather vague geometric understanding of this very algebraic concept, which on the other hand proves very useful in geometry. As David Mumford adequatlity put it, "the concept of flatness is a riddle that comes out of algebra, but which technically is the answer to many prayers". Among the many prayers, just to give an idea of the power of flatness, is the fact that families of varieties parametrized as fibres of a flat morphism share the same Hilbert polynomial (see, e.g., [Ei] or [Mu]). The purpose of this section is to present recent (and very recent) *geometric* criteria for flatness in the complex analytic context.

4.1 Algebraic study of flatness

We recall here the definition, basic properties, and some classical algebraic criteria for flatness. For details see, e.g., [Ei].

Let A be a commutative ring with identity, and let M be a unitary A-module. We say that M is *flat* over A (or A-*flat*) when, for every exact sequence of A-modules

$$N' \to N \to N'' \, ,$$

the sequence

$$N' \otimes_A M \to N \otimes_A M \to N'' \otimes_A M$$

is exact.

Let N be an A-module. Consider a *free resolution* of N; i.e., an exact sequence

$$\mathcal{F}_* : \quad \dots \xrightarrow{\alpha_{i+1}} F_{i+1} \xrightarrow{\alpha_i} F_i \xrightarrow{\alpha_{i-1}} \dots \xrightarrow{\alpha_1} F_1 \xrightarrow{\alpha_0} F_0 \to N \, ,$$

where the F_i are free A-modules. Tensor with M over A:

$$\dots \xrightarrow{\alpha_{i+1}\otimes 1} F_{i+1} \otimes_A M \xrightarrow{\alpha_i \otimes 1} F_i \otimes_A M \xrightarrow{\alpha_{i-1}\otimes 1} \dots \xrightarrow{\alpha_1 \otimes 1} F_1 \otimes_A M \xrightarrow{\alpha_0 \otimes 1} F_0 \otimes_A M \to N \otimes_A M \, .$$

Definition 4.1. One defines the Tor A-modules as the cohomology of the above complex:

$$\mathrm{Tor}_0^A(N, M) := \mathrm{coker}\,(\alpha_0 \otimes 1)\, , \quad \mathrm{Tor}_i^A(N, M) := \frac{\ker(\alpha_i \otimes 1)}{\mathrm{im}\,(\alpha_{i+1} \otimes 1)}\, , \quad i \geq 1 \, .$$

Up to isomorphism, the Tor modules are independent of the resolution of N. It is also easy to see that

$$\mathrm{Tor}_i^A(M, N) = \mathrm{Tor}_i^A(N, M) \quad \text{and} \quad \mathrm{Tor}_0^A(N, M) = N \otimes_A M \, .$$

A homomorphism of A-modules $N' \xrightarrow{\varphi} N$ induces homomorphisms $\varphi_i : \mathrm{Tor}_i^A(N', M) \to \mathrm{Tor}_i^A(N, M)$; φ_0 identifies with $\varphi \otimes 1 : N' \otimes_A M \to N \otimes_A M$. A short exact sequence of A-modules

$$0 \to N' \xrightarrow{\varphi} N \xrightarrow{\psi} N'' \to 0$$

induces a long exact sequence of Tori

$$\dots \xrightarrow{\varphi_{i+1}} \mathrm{Tor}_{i+1}^A(N, M) \xrightarrow{\psi_{i+1}} \mathrm{Tor}_{i+1}^A(N'', M) \xrightarrow{\partial} \mathrm{Tor}_i^A(N', M) \xrightarrow{\varphi_i} \mathrm{Tor}_i^A(N, M) \xrightarrow{\psi_i} \dots$$
$$\dots \xrightarrow{\varphi_1} \mathrm{Tor}_1^A(N, M) \xrightarrow{\psi_1} \mathrm{Tor}_1^A(N'', M) \xrightarrow{\partial} N' \otimes_A M \xrightarrow{\varphi_0} N \otimes_A M \xrightarrow{\psi_0} N'' \otimes_A M \to 0 \, .$$

One easily verifies the following proposition (**Exercise**).

Proposition 4.2. *The following conditions are equivalent:*

(i) *M is A-flat.*

(ii) *For every exact sequence of A-modules $0 \to N' \to N \to N'' \to 0$, the sequence*

$$0 \to N' \otimes_A M \to N \otimes_A M \to N'' \otimes_A M \to 0$$

is exact.

(iii) *For every injective homomorphism of A-modules $N' \to N$, the homomorphism $N' \otimes_A M \to N \otimes_A M$ is injective.*

(iv) *For every A-module N and all $i \geq 1$, $\mathrm{Tor}_i^A(N, M) = 0$.*

(v) *For every A-module N, $\mathrm{Tor}_1^A(N, M) = 0$.*

Now, let $(A, \mathfrak{m}_A), (B, \mathfrak{m}_B)$ be noetherian local rings, let $\theta : A \to B$ be a local homomorphism (i.e., $\theta(\mathfrak{m}_A) \subset \mathfrak{m}_B$), and let M be a finitely generated B-module. Then M can be viewed as an A-module via the homomorphism θ. Then we have the following fundamental local flatness criterion (see, e.g., [Ei]).

Proposition 4.3. *M is A-flat if and only if* $\mathrm{Tor}_1^A(A/\mathfrak{m}_A, M) = 0$.

Suppose now that $R = \mathbb{C}\{y\}/J$ is a local analytic $\mathbb{C}$-algebra (with maximal ideal $\mathfrak{m}_R$), $A = R\{x\}/I$ is a local analytic R-algebra (where $y = (y_1, \dots, y_n)$, $x = (x_1, \dots, x_m)$), and let M be a finite A-module (cf. Section 1). One may then consider the left derived functor $\widetilde{\mathrm{Tor}}^R$ of the analytic tensor product $\tilde{\otimes}$ over R, which enjoys all the above described properties of the ordinary Tor when applied to the so-called *almost finitely generated* R-modules (i.e., modules finite over a local analytic R-algebra). In general, given two R-modules, M and N, finitely generated over some local analytic R-algebras, the ordinary tensor product $M \otimes_R N$ only embeds into the analytic one $M \tilde{\otimes}_R N$ (**Exercise** [Hint: Show first that the ordinary tensor product $A \otimes_R B$ of two local R-analytic algebras embeds into $A \tilde{\otimes}_R B$ by mapping $f \otimes g \mapsto (f \cdot g)|_{X_1 \times_Y X_2}$, where X_1, X_2 and Y are complex analytic spaces with mappings $\varphi_i : X_i \to Y$ $(i = 1, 2)$, whose germs correspond to the algebras A_1, A_2, and R respectively; then extend to modules via finite presentations]). Nonetheless, tensoring with finite R-modules is equivalent with analytic-tensoring with such modules (see Remark 1.2), hence, in particular, the functors "$\otimes_R (R/\mathfrak{m}_R)$" and "$\tilde{\otimes}_R (R/\mathfrak{m}_R)$" are identical. Consequently we obtain (cf. [Hi, Prop.6.2]):

Proposition 4.4. *Let R be a local analytic $\mathbb{C}$-algebra, and let M be a module finitely generated over some local analytic R-algebra. Then M is R-flat if and only if*

$$\widetilde{\mathrm{Tor}}_1^R(R/\mathfrak{m}_R, M) = 0\,.$$

Let again $(R, \mathfrak{m}_R)$ be a local analytic $\mathbb{C}$-algebra, let A be a local analytic R-algebra, and let M be a finitely generated A-module. Then $M \cong A^p/N$ for some $p \geq 1$ and some A-submodule N of A^p.

Corollary 4.5. *Suppose that A is R-flat. Then M is R-flat if and only if*

$$N \cap \mathfrak{m}_R A^p = \mathfrak{m}_R N\,.$$

Proof. By Proposition 4.4, M is R-flat iff $\widetilde{\mathrm{Tor}}_1^R(R/\mathfrak{m}_R, M) = \widetilde{\mathrm{Tor}}_1^R(M, \mathbb{C}) = 0$. The result is thus a consequence of the following commutative diagram, where all sequences are exact:

$$
\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & & \\
 & & \mathfrak{m}_R N \arrow[d] & \mathfrak{m}_R A^p \arrow[d] & & \\
 & 0 \arrow[r] & N \arrow[r] \arrow[d] & A^p \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & 0 \\
 {} \arrow[r] & \widetilde{\mathrm{Tor}}_1^R(M,\mathbb{C}) \arrow[r] & N\tilde{\otimes}_R\mathbb{C} \arrow[r] & A^p\tilde{\otimes}_R\mathbb{C} \arrow[r] & M\tilde{\otimes}_R\mathbb{C} \arrow[r] & 0\,.
\end{tikzcd}
$$

□

4.2 Hironaka division and flatness

Definition 4.6. Let $\varphi : X \to Y$ be a holomorphic mapping of complex analytic spaces. We say that the map φ is *flat at* $\xi \in X$, when the local ring $\mathcal{O}_{X,\xi}$ is a flat $\mathcal{O}_{Y,\varphi(\xi)}$-module via the pull-back homomorphism $\varphi^*_\xi : \mathcal{O}_{Y,\varphi(\xi)} \to \mathcal{O}_{X,\xi}$. We say that φ is a *flat mapping*, or that X is flat over Y, when this is so at every point $\xi \in X$. More generally, for a coherent sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules, we say that $\mathcal{F}$ is *flat over* Y, when $\mathcal{F}_\xi$ is a flat $\mathcal{O}_{Y,\varphi(\xi)}$-module for every $\xi \in X$.

The following result of Hironaka (cf. [Hi]) gives a criterion for flatness of a holomorphic function at a point in terms of the division algorithm (recall [I, Sec.3]). Let $\varphi : X \to Y$ be a holomorphic mapping of analytic spaces, with $\varphi(\xi) = \eta$. We will consider the flatness of φ at ξ. For that, we can replace φ by its local installation near ξ, that is, we may assume that $X \subset Y \times \mathbb{C}^m$ and φ is (the restriction of) a coordinate projection to Y. Then the structure of $\mathcal{O}_{Y,\eta}$-module of $\mathcal{O}_{X,\xi}$ is given by $\mathcal{O}_{X,\xi} \cong \mathcal{O}_{Y,\eta}\{x\}/I$ for some ideal I in $\mathcal{O}_{Y,\eta}\{x\}$, where $x = (x_1, \dots, x_m)$. Let $y = (y_1, \dots, y_n)$ denote the local coordinate system of Y at η. Let $\Delta \subset \mathbb{N}^m$ be the complement of the diagram of initial exponents $\mathfrak{N}(I(0))$, where evaluation is at $y = 0$ (which corresponds to tensoring with $\tilde{\otimes}_{\mathcal{O}_{Y,\eta}}\mathbb{C}$). Denote by $\mathcal{O}_{Y,\eta}\{x\}^\Delta$ the $\mathcal{O}_{Y,\eta}$-module

$$\{F \in \mathcal{O}_{Y,\eta}\{x\} \,:\, \mathrm{supp}\,(F) \subset \Delta\}$$

of series supported outside of the diagram $\mathfrak{N}(I(0))$, and let

$$\kappa : \mathcal{O}_{Y,\eta}\{x\}^\Delta \to \mathcal{O}_{X,\xi}$$

be the restriction to $\mathcal{O}_{Y,\eta}\{x\}^\Delta$ of the canonical epimorphism $\mathcal{O}_{Y,\eta}\{x\} \to \mathcal{O}_{Y,\eta}\{x\}/I$. Then, by Hironaka Division ([I, Thm.3.3]), κ is surjective, as every series $F \in \mathcal{O}_{Y,\eta}\{x\}$ is congruent modulo I to its remainder in Hironaka division, which is supported outside of $\mathfrak{N}(I(0))$.

Theorem 4.7 (Hironaka)**.** *The map φ is flat at ξ if and only if κ is injective. That is, φ is flat at ξ if and only if every series $F \in I$ whose support is contained in Δ is identically zero.*

Proof. Let $\mathfrak{m}$ denote the maximal ideal of $\mathcal{O}_{Y,\eta}$. By Corollary 4.5, flatness of φ at ξ is equivalent to the inclusion

$$I \cap \mathfrak{m}\cdot\mathcal{O}_{Y,\eta}\{x\} \subset \mathfrak{m}\cdot I\,. \tag{*}$$

Let $G_1, \dots, G_t$ be such that the initial exponents $\exp(G_i(0))$ are the vertices of the diagram $\mathfrak{N}(I(0))$, and let Δ_i be the partition of $\mathbb{N}^m \setminus \Delta$ as in [I, Section 3].

Assume first that $\ker\kappa = (0)$. Let $F \in I \cap \mathfrak{m}\cdot\mathcal{O}_{Y,\eta}\{x\}$. Then F is generated over $\mathcal{O}_{Y,\eta}\{x\}$ by the $G_1, \dots, G_t$:

$$F = \sum_i Q_i G_i\,,$$

where $\exp(G_i(0)) + \operatorname{supp} Q_i \subset \Delta_i$, $i = 1, \ldots, t$. Indeed, in general $F = \sum_i Q_i G_i + R$, with $\operatorname{supp} R \subset \Delta$. Thus, if $F \in I$, we get $R = F - \sum_i Q_i G_i \in I$, which by injectivity of κ implies $R = 0$.

Since $F(0) = 0$, then each $Q_i(0) = 0$ by uniqueness in the division algorithm; i.e., $Q_i \in \mathfrak{m}\cdot\mathcal{O}_{Y,\eta}\{x\}$ for $i = 1, \ldots, t$, so that $F \in \mathfrak{m}\cdot I$, which proves (*).

Suppose now that (*) is satisfied. Observe that $\ker\kappa$ is an $\mathcal{O}_{Y,\eta}$-submodule of I consisting of series which equal their own remainder after division by $G_1, \ldots, G_t$. For a proof by contradiction, suppose that $\ker\kappa \neq (0)$. Then we may define r to be the greatest integer such that $\ker\kappa \subset \mathfrak{m}^r\cdot\mathcal{O}_{Y,\eta}\{x\}$. Choose $F \in \ker\kappa$ with $F \notin \mathfrak{m}^{r+1}\cdot\mathcal{O}_{Y,\eta}\{x\}$. Then $F(0) = 0$, since $F(0) \in I(0)$ and $F(0)$ equals its remainder after division by the $G_1(0), \ldots, G_t(0)$. Therefore $F \in I \cap \mathfrak{m}\cdot\mathcal{O}_{Y,\eta}\{x\}$, and hence, by (*), $F = \sum_{k=1}^{s} a_k \cdot F_k$, where $a_k \in \mathfrak{m}$, $F_k \in I$ $(i = 1, \ldots, s)$. Divide each F_k by $G_1, \ldots, G_t$:

$$F_k = \sum_{i=1}^{t} Q_{ki} G_i + R_k \,,$$

where $\exp(G_i(0)) + \operatorname{supp} Q_{ki} \subset \Delta_i$, $\operatorname{supp} R \subset \Delta$. Then $R_k = F_k - \sum_{i=1}^{t} Q_{ki} G_i \in \ker\kappa$ (as $F_k - \sum_i Q_{ki} G_i \in I$ and $\operatorname{supp} R_k \subset \Delta$), and

$$F = \sum_{i=1}^{t} (\sum_{k=1}^{s} a_k Q_{ki}) G_i + \sum_{k=1}^{s} a_k R_r \,.$$

By uniqueness in the division algorithm, since $F \in \ker\kappa$, it follows that $F = \sum_k a_k R_r$. But $R_k \in \ker\kappa \subset \mathfrak{m}^r\cdot\mathcal{O}_{Y,\eta}\{x\}$ and $a_k \in \mathfrak{m}$ for all k, hence $F = \sum_k a_k R_r \in \mathfrak{m}^{r+1}\cdot\mathcal{O}_{Y,\eta}\{x\}$; a contradiction. $\square$

4.3 Vertical components and flatness

Recall Theorem 3.5 characterizing openness in terms of the lack of isolated vertical components in fibre powers. Below we show that taking into account the embedded components as well implies a criterion for flatness. Theorems 3.5 and 4.8 therefore allow us to thing of openness as a geometric analogue of flatness.

Theorem 4.8 (cf. [Kw, Thm.1.1])**.** *Let $\varphi_\xi : X_\xi \to Y_\eta$ be a germ of a holomorphic mapping of complex analytic spaces, with Y_η irreducible. Then the following are equivalent:*

(i) *φ_ξ is flat.*

(ii) *For any $i \geq 1$, the i-fold fibre power $\varphi^{\{i\}}_{\xi^{\{i\}}} : X^{\{i\}}_{\xi^{\{i\}}} \to Y_\eta$ has no (isolated or embedded) algebraic vertical components.*

Proof. Note that if φ_ξ is flat, then so is $\varphi^{\{i\}}_{\xi^{\{i\}}}$ for every $i \geq 1$, since flatness is preserved by any base change (see [Hi, Prop.6.8]) and the composition of flat maps is flat. Therefore the implication (i)⇒(ii) is an immediate consequence of the definition of flatness in terms of relations (see, e.g., [BM, Prop.7.3]). This in fact is the only place where the irreducibility assumption is needed.

For the proof of (ii)⇒(i), assume that φ_ξ is not flat. We will show that then there exist $i \geq 1$, a nonzero $a \in \mathcal{O}_{Y,\eta}$ and a nonzero $b \in \mathcal{O}_{X^{\{i\}},\xi^{\{i\}}}$ such that $ab = 0$ in $\mathcal{O}_{X^{\{i\}},\xi^{\{i\}}}$. In other words, by Remark 3.2, there is an algebraic vertical component (possibly embedded) in $X^{\{i\}}_{\xi^{\{i\}}}$ over Y_η.

Since φ_ξ is non-flat, then Hironaka's Theorem 4.7 implies that $\ker\kappa \neq \{0\}$. Pick any nontrivial $F = \sum_{\beta\in\Delta} a_\beta x^\beta$ from $\ker\kappa$. Let $a_{\beta_1}, \ldots, a_{\beta_i}$ be distinct nonzero coefficients of F which generate the ideal in $\mathcal{O}_Y$ generated by all the coefficients of the series F (exist, for some i, by noetherianity). Then $F = a_{\beta_1}\tilde{F}_1 + \cdots + a_{\beta_i}\tilde{F}_i$ for some $\tilde{F}_j(x) \in \mathcal{O}_{Y,\eta}[[x]]$. Hence, by faithfull flatness of the completion $\widehat{\mathcal{O}_{Y,\eta}\{x\}} = \mathcal{O}_{Y,\eta}[[x]]$ over $\mathcal{O}_{Y,\eta}\{x\}$ (see, e.g., [Bou]), we have

$$F = a_{\beta_1}F_1 + \cdots + a_{\beta_i}F_i\,,$$

with $F_j(x) \in \mathcal{O}_{Y,\eta}\{x\}$. Moreover, by comparing the supports, we may require that

$$F_j(x) = x^{\beta_j} + \sum_{\beta\in\Delta\setminus\{\beta_1,\dots,\beta_i\}} f_\beta x^\beta\,, \qquad j = 1,\dots,i\,.$$

Define $a_j = a_{\beta_j}$ and $h_j = \kappa(F_j)$ for $j = 1,\dots,i$. It follows that $a_1h_1 + \cdots + a_ih_i = 0$, but $h_1(0),\dots,h_i(0)$ are linearly independent (over $\mathbb{C}$) in $\mathcal{O}_{X,\xi}/\mathfrak{m}\mathcal{O}_{X,\xi} = \mathbb{C}\{x\}/I(0)$, where $\mathfrak{m}$ is the maximal ideal in $\mathcal{O}_{Y,\eta}$.

Next consider the following commutative diagram of canonical maps of $\mathcal{O}_{Y,\eta}$-modules, where both tensor products are taken i times, and the vertical arrows are given by tensoring with $\otimes_{\mathcal{O}_{Y,\eta}}\mathcal{O}_{Y,\eta}/\mathfrak{m}$.

$$\begin{array}{ccccc}
\bigwedge^i_{\mathcal{O}_{Y,\eta}}\mathcal{O}_{X,\xi} & \xrightarrow{\rho} & \mathcal{O}_{X,\xi}\otimes_{\mathcal{O}_{Y,\eta}}\cdots\otimes_{\mathcal{O}_{Y,\eta}}\mathcal{O}_{X,\xi} & \xrightarrow{\lambda} & \mathcal{O}_{X^{\{i\}},\xi^{\{i\}}} \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle\mu} \\
\bigwedge^i_{\mathbb{C}}\mathcal{O}_{X,\xi}/\mathfrak{m}\mathcal{O}_{X,\xi} & \xrightarrow{\bar\rho} & \mathcal{O}_{X,\xi}/\mathfrak{m}\mathcal{O}_{X,\xi}\otimes_{\mathbb{C}}\cdots\otimes_{\mathbb{C}}\mathcal{O}_{X,\xi}/\mathfrak{m}\mathcal{O}_{X,\xi} & \xrightarrow{\bar\lambda} & \mathcal{O}_{(\varphi^{-1}(\eta))^i,\xi^{\{i\}}}
\end{array}$$

Put $a = a_1$ and $b = \lambda\circ\rho(h_1\wedge\cdots\wedge h_i)$. Since a_1h_1 is an $\mathcal{O}_{Y,\eta}$-linear combination of the $h_2,\dots,h_i$, then $ab = \lambda\circ\rho((a_1h_1)\wedge h_2\cdots\wedge h_i) = 0$. Finally $b \neq 0$, because $\mu(b) = \bar\lambda\circ\bar\rho(h_1(0)\wedge\cdots\wedge h_i(0))$ is nonzero, as $h_1(0),\dots,h_i(0)$ are linearly independent and $\bar\rho,\bar\lambda$ are injective. □

4.4 Flatness vs. openness

The comparison of Theorems 4.8 and 3.5 (the criteria for flatness and openness in terms of vertical components in fibre powers) shows that openness can be viewed as a geometric analogue of flatness. Flatness intuitively means that fibres of a morphism change in a continuous way. If one disregards the embedded structure of the fibres, that is, if one only considers their geometry, then continuity in the family of fibres means openness of the morphism. This roughly explains why for the study of openness it is enough to consider the isolated irreducible components in fibre powers, whilst both the isolated and embedded components must be considered in the study of flatness.

In fact, every flat morphism is open. This result was first obtained by A. Douady. Combining Theorems 4.8 and 3.5 provides an alternative, simple proof (see Proposition 4.10 below).

Our proof of Proposition 4.10 is based on the observation that if $\varphi_\xi : X_\xi \to Y_\eta$ is flat then every restriction of φ_ξ to an irreducible component of X_ξ is flat as well (Lemma 4.9 below). The converse is not true though: flatness of all the restrictions to irreducible components does not imply flatness of the morphism, as can be observed in the example of Remark 3.4.

It is worth pointing out that for openness the opposite is true: Clearly, openness of all the restrictions to isolated irreducible components implies openness of the morphism, but openness of a morphism does not imply openness of all its restrictions to isolated irreducible components. Consider, for instance, the mapping given by

$$\varphi : X \ni (x,y,s,t) \mapsto (x+s, xy+t) \in Y = \mathbb{C}^2,$$

where $X = X_1 \cup X_2$, $X_1 = \{(x,y,s,t) \in \mathbb{C}^4 : s = t = 0\}$, and $X_2 = \{(x,y,s,t) \in \mathbb{C}^4 : x = 0\}$. Then φ is open, but $\varphi|_{X_1}$ is not. (In particular, the above is a simple example of an open and non-flat mapping, by Proposition 4.10 and Lemma 4.9 below.)

Lemma 4.9. *If $\varphi_\xi : X_\xi \to Y_\eta$ is a flat morphism of germs of analytic spaces with Y_η irreducible, then every restriction $\varphi_\xi|_{W_\xi} : W_\xi \to Y_\eta$ to an (isolated or embeddded) irreducible component W_ξ of X_ξ is also flat.*

Proof. Let $\varphi_\xi : X_\xi \to Y_\eta$ be a flat morphism of germs of analytic spaces with Y_η irreducible, and let W_ξ be an (isolated or embedded) irreducible component of X_ξ. Without loss of generality, we can assume that X_ξ is a subgerm of $\mathbb{C}^m_0$ for some $m \geq 1$. We can then embed X_ξ into $Y_\eta \times \mathbb{C}^m_0$ via the graph of φ. Therefore the local ring $\mathcal{O}_{X,\xi}$ can be thought of as a quotient of the local ring of $Y_\eta \times \mathbb{C}^m_0$; i.e., $\mathcal{O}_{X,\xi} = \mathcal{O}_{Y,\eta}\{x\}/I$ for some ideal I in $\mathcal{O}_{Y,\eta}\{x\}$, where $x = (x_1, \dots, x_m)$ is a system of m complex variables.

Let $\mathfrak{p}$ be the associated prime of I in $\mathcal{O}_{Y,\eta}\{x\}$ corresponding to W_ξ (i.e., such that $\mathcal{V}(\mathfrak{p}) = W_\xi$). Now, $\varphi_\xi|_{W_\xi} : W_\xi \to Y_\eta$ is flat if and only if $\mathcal{O}_{Y,\eta}\{x\}/\mathfrak{p}$ is a flat $\mathcal{O}_{Y,\eta}$-module. By Proposition 4.4, the latter is equivalent to

$$\widetilde{\mathrm{Tor}}_1^{\mathcal{O}_{Y,\eta}}(\mathcal{O}_{Y,\eta}\{x\}/\mathfrak{p}, \mathcal{O}_{Y,\eta}/\mathfrak{m}) = 0,$$

where, as before, $\mathfrak{m}$ is the maximal ideal in $\mathcal{O}_{Y,\eta}$.

The short exact sequence of $\mathcal{O}_{Y,\eta}$-modules

$$0 \to \mathfrak{p}/I \to \mathcal{O}_{Y,\eta}\{x\}/I \to \mathcal{O}_{Y,\eta}\{x\}/\mathfrak{p} \to 0$$

induces a long exact sequence of $\widetilde{\mathrm{Tor}}$ of which a part is

$$\begin{aligned}\widetilde{\mathrm{Tor}}_1^{\mathcal{O}_{Y,\eta}}(\mathcal{O}_{X,\xi}, \mathcal{O}_{Y,\eta}/\mathfrak{m}) \to \widetilde{\mathrm{Tor}}_1^{\mathcal{O}_{Y,\eta}}(\mathcal{O}_{Y,\eta}\{x\}/\mathfrak{p}, \mathcal{O}_{Y,\eta}/\mathfrak{m}) &\xrightarrow{\psi} \\ (\mathfrak{p}/I)\tilde{\otimes}_{\mathcal{O}_{Y,\eta}}(\mathcal{O}_{Y,\eta}/\mathfrak{m}) \xrightarrow{\chi} \mathcal{O}_{X,\xi}\tilde{\otimes}_{\mathcal{O}_{Y,\eta}}(\mathcal{O}_{Y,\eta}/\mathfrak{m}) &\to (\mathcal{O}_{Y,\eta}\{x\}/\mathfrak{p})\tilde{\otimes}_{\mathcal{O}_{Y,\eta}}(\mathcal{O}_{Y,\eta}/\mathfrak{m}) \to 0\,.\end{aligned}$$

Since $\mathcal{O}_{X,\xi}$ is $\mathcal{O}_{Y,\eta}$-flat, then $\widetilde{\mathrm{Tor}}_1^{\mathcal{O}_{Y,\eta}}(\mathcal{O}_{X,\xi}, \mathcal{O}_{Y,\eta}/\mathfrak{m}) = 0$, and hence ψ is injective. Therefore, $\widetilde{\mathrm{Tor}}_1^{\mathcal{O}_{Y,\eta}}(\mathcal{O}_{Y,\eta}\{x\}/\mathfrak{p}, \mathcal{O}_{Y,\eta}/\mathfrak{m}) = 0$ if and only if χ is injective.

Observe that χ is *not* injective if and only if there exists a nonzero series $F \in \mathfrak{p} \setminus I$ that factors (in $\mathcal{O}_{Y,\eta}\{x\}$, but not in $\mathfrak{p}$!) as $F = GH$ for some $G \in \mathfrak{m}$ and $H \in \mathcal{O}_{Y,\eta}\{x\} \setminus \mathfrak{p}$. Then $G \in \mathfrak{p}$, since $\mathfrak{p}$ is prime and $H \notin \mathfrak{p}$. Thus, the maximal ideal $\mathfrak{m}$ of $\mathcal{O}_{Y,\eta}$ contains a nonzero element of an associated prime of I, and hence $\mathcal{O}_{Y,\eta}$ contains a zerodivisor in $\mathcal{O}_{X,\xi}$. It follows from the characterization of flatness in terms of relations that $\mathcal{O}_{Y,\eta}$ has a zerodivizor itself, which contradicts the assumption that Y_η be irreducible. We thus showed that χ must be injective, which completes the proof. $\square$

Proposition 4.10. *Let $\varphi : X \to Y$ be a holomorphic map of analytic spaces. If φ is flat, then it is open.*

Proof. The problem being local, we can replace X and Y by germs of analytic spaces X_ξ and Y_η, where $\varphi(\xi) = \eta$. We will proceed in three steps:

Step 1. The result follows immediately from Theorems 3.5 and 4.8 in the case Y_η is irreducible and X_ξ is of pure dimension.

Step 2. Assume now that Y_η is irreducible and $\varphi_\xi : X_\xi \to Y_\eta$ is a flat morphism of germs of analytic spaces. Since openness of φ follows from openness of every restriction $\varphi|_{W_\xi}$ to an isolated irreducible component W_ξ of X_ξ, by Lemma 4.9 the problem reduces to Step 1.

Step 3. In the general case we may assume that Y is reduced. Let $\nu\colon \widetilde{Y} \to Y$ denote the normalization map. Then $\widetilde{Y}$ is irreducible and there is a commutative square

$$\begin{array}{ccc} X \times_Y \widetilde{Y} & \longrightarrow & X \\ {\scriptstyle \widetilde{\varphi}}\big\downarrow & & \big\downarrow{\scriptstyle \varphi} \\ \widetilde{Y} & \xrightarrow{\ \nu\ } & Y\,. \end{array}$$

Now, $\widetilde{\varphi}$ is flat, because flatness is preserved by any base change, and hence open, by Step 2. Since Y has the quotient topology with respect to ν, this implies that φ is open. $\square$

5 Auslander-type effective flatness criterion

We finish these notes with a proof of an effective, that is computable, flatness criterion that generalizes a well-known Auslander's freeness criterion (below) to modules not finitely generated over the base ring. This is a recent work [ABM2], joint with E. Bierstone and P.D. Milman.

The main inspiration for characterizing flatness of a module in terms of torsion freeness of tensor powers of this module comes from a seminal paper [Au] of Auslander. As a consequence of rigidity of the Tor modules over regular local rings, Auslander obtains the following freeness criterion for finitely generated modules.

Theorem 5.1 (Auslander, [Au, Thm.3.2])**.** *Let R be a regular local ring of dimension $n > 0$. A finite R-module F is R-free if and only if the n-fold tensor power $F^{\otimes^n_R}$ is a torsionfree R-module.*

In fact, the original result of Auslander was formulated for modules over unramified local rings $(R, \mathfrak{m}_R)$, that is, such that the characteristic of R equals that of its residue class field $R/\mathfrak{m}_R$. It was later extended by Lichtenbaum [Li] to arbitrary regular local rings.

From the geometric point of view, it is more natural not to restrict oneself to finite R-modules, but rather to consider modules F finite over some local R-algebra of finite type. Recall that we call such modules *almost finitely generated* over R. The corresponding geometric picture is a morphism $\varphi : X \to Y$ of complex algebraic varieties, with Y nonsingular, where $F = \mathcal{F}_\xi$ is a stalk of a coherent sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules. We shall prove the following geometric generalization of Auslander's criterion for the almost finitely generated modules over regular local $\mathbb{C}$-algebras.

Theorem 5.2. *Let R be a regular $\mathbb{C}$-algebra of finite type. Let A denote an R-algebra essentially of finite type, and let F denote a finitely generated A-module. Then F is R-flat if and only if the n-fold tensor power $F^{\otimes^n_R}$ is a torsion-free R-module, where $n = \dim R$.*

An R-algebra *essentially of finite type* means a localization of an R-algebra of finite type. (In particular, the above theorem includes the case when A is just a finitely generated R-algebra.)

One may interpret this criterion as follows. Let $\varphi : X \to Y$ be a morphism of complex algebraic varieties, with Y smooth of dimension $n > 0$. Let $\xi \in X$, $\eta = \varphi(\xi)$, and let $\mathcal{F}$ be a coherent sheaf of $\mathcal{O}_X$-modules. Then the flatness of the stalk $\mathcal{F}_\xi$ over the local ring $\mathcal{O}_{Y,\eta}$ is equivalent to the lack of zerodivisors (over $\mathcal{O}_{Y,\eta}$) in the n-fold tensor power of $\mathcal{F}_\xi$ over $\mathcal{O}_{Y,\eta}$.

As freeness of a module F finitely generated over a local ring R is equivalent to flatness of F, Auslander's criterion over $\mathbb{C}$ is indeed a special case of Theorem 5.2. We obtain Theorem 5.2 as a consequence of a stronger, analytic result (Theorem 5.10 below).

Remark 5.3. From the computational point of view, a general method of determining R-flatness of an almost finitely generated R-module F is to find an element $r \in R$, such that (i) r is not a zerodivisor in both R and F, and (ii) the localization R_r is a regular local ring (see [V2]). Condition (i) allows for the reduction of dimension by passage from R to $R/(r)$. To benefit from (ii), one needs to handle the case of the regular ring directly. Paired with the algorithms for primary decomposition (see [V2] again), Theorem 5.2 provides an effective criterion for flatness in this case. Indeed, to verify R-flatness of a finite A-module F, it suffices to check if among the associated primes of $F^{\otimes^n_R}$ in $A^{\otimes^n_R}$ there is one that contains a nonzero $r \in R$. (Notice that the power n is independent of the module F, which is crucial for effectiveness.) On the other hand, the power $n = \dim R$ is best possible, already in the finite module case (see [Au]).

We will now generalize the notion of a vertical component to *vertical elements* in almost finitely generated modules:

Definition 5.4. Let $I := \mathrm{Ann}_{\mathcal{O}_{X,\xi}}(F)$ and let Z_ξ be the germ of a complex analytic subspace of X, defined by $\mathcal{O}_{Z,\xi} := \mathcal{O}_{X,\xi}/I$. We say that F has an *algebraic vertical* (resp. *geometric vertical*) component over Y_η when Z_ξ has an algebraic vertical (resp. geometric vertical) component over Y_η. Equivalently, there exists a nonzero $m \in F$ such that the zero set germ $\mathcal{V}(\mathrm{Ann}_{\mathcal{O}_{X,\xi}}(m))$ is mapped into a proper analytic (resp. nowhere-dense) subgerm of Y_η. A nonzero element $m \in F$ with the property that $\mathcal{V}(\mathrm{Ann}_{\mathcal{O}_{X,\xi}}(m))$ is mapped into a nowhere dense subgerm of Y_η, will be called (*geometric*) *vertical* over Y_η (or *vertical* over $\mathcal{O}_{Y,\eta}$).

Notice that there is no need to define algebraic vertical elements, since a nonzero element $m \in F$ with the property that $\mathcal{V}(\mathrm{Ann}_{\mathcal{O}_{X,\xi}}(m))$ is mapped into a proper analytic subgerm of Y_η, is simply a zerodivisor of F over $\mathcal{O}_{Y,\eta}$. Indeed, $\mathcal{V}(\mathrm{Ann}_{\mathcal{O}_{X,\xi}}(m))$ is mapped into the zero set germ of some $r \in \mathcal{O}_{Y,\eta}$ iff $\varphi_\xi^*(r)\cdot m = 0$ in F. Therefore, a *vertical* element shall always mean "vertical in the geometric sense".

Remark 5.5. In particular, when $\mathcal{O}_{X,\xi}$ itself is viewed as an $\mathcal{O}_{X,\xi}$-module, then $\mathcal{O}_{X,\xi}$ has no geometric vertical components over $\mathcal{O}_{Y,\eta}$ iff $\mathcal{O}_{X,\xi}$ has no vertical elements (as an $\mathcal{O}_{X,\xi}$-module) over $\mathcal{O}_{Y,\eta}$. Similarly, $\mathcal{O}_{X,\xi}$ has no algebraic vertical components over $\mathcal{O}_{Y,\eta}$ iff $\mathcal{O}_{X,\xi}$ has no zerodivisors (as an $\mathcal{O}_{X,\xi}$-module) over $\mathcal{O}_{Y,\eta}$.

Let now $R = \mathbb{C}\{y_1, \dots, y_n\}$ be the ring of convergent power series in n complex variables, and let F be an R-module. We call F an *almost finitely generated* R-module if there exists a local analytic R-algebra A (i.e., a ring of the form $R\{x\}/I = \mathbb{C}\{y_1, \dots, y_n, x_1, \dots, x_m\}/I$ for some ideal I, with the canonical homomorphism $\theta : R \to A$) such that F is a finitely generated A-module. Given such $\theta : R \to A$, there are analytic spaces X, Y, and a morphism of germs $\varphi_\xi : X_\xi \to Y_\eta$ such that $\mathcal{O}_{Y,\eta} \cong R$, $\mathcal{O}_{X,\xi} \cong A$ and $\varphi_\xi^* = \theta$. In particular, F can be thought of as a finite $\mathcal{O}_{X,\xi}$-module. We will then say that a nonzero element $m \in F$ is vertical over R if m is vertical over $\mathcal{O}_{Y,\eta}$ in the sense of Definition 5.4.

Remark 5.6. It is easy to verify that this notion of a vertical element is well-defined. That is, if F is simultaneously finitely generated over local analytic R-algebras A and B, and a nonzero $m \in F$ is vertical over R when F is considered as a module over A, then it is also vertical when F is considered as a module over B (see [GK, Prop.3.6]).
In particular, given an almost finitely generated R-module F, one can without loss of generality assume that F is finitely generated over a regular algebra $A = R\{x\} \cong \mathbb{C}\{y, x\}$, where $x = (x_1, \dots, x_m)$ is a system of m complex variables, $m \geq 0$.

5.1 Puredimensional case

The idea of generalizing Auslander's criterion from finite modules to finite algebras over a regular local ring appeared first in Vasconcelos' [V1], who proved ([V1, Prop.6.1]) a very special case of such a generalization:

If R is a regular local ring of dimension 2, then a finite R-algebra A is R-flat if and only if $A \otimes_R A$ is a torsionfree R-module.

Vasconcelos also conjectured ([V1, Conj.6.2] and [V2, 2.6]) that, if the dimension of a regular local ring R is $n > 2$, then a finite R-algebra A is R-flat if and only if the algebra $A^{\otimes_R^n}$ is torsionfree over R. Galligo and Kwieciński [GK] proved a version of Vasconcelos' conjecture in a complex equidimensional case:

Theorem 5.7 ([GK, Thm.1.1])**.** *If R is a regular local $\mathbb{C}$-algebra of dimension $n > 0$, and A is a finite R-algebra equidimensional over $\mathbb{C}$, then A is R-flat if and only if the n-fold tensor power $A^{\otimes_R^n}$ is torsionfree over R.*

The above result follows, of course, from Theorem 5.2, which we will prove later. At this point, we list a few facts concerning almost finitely generated modules, established by Galligo and Kwieciński in [GK], that generalize the corresponding properties of finite modules used in Auslander's [Au]. Let, as before, $R = \mathbb{C}\{y_1, \dots, y_n\}$ be a regular local analytic $\mathbb{C}$-algebra of dimension n, let $\tilde{\otimes}_R$ denote the analytic tensor product over R, and let $\widetilde{\mathrm{Tor}}^R$ be the corresponding derived functor.

Given an almost finitely generated R-module F, we define its *flat dimension* over R, denoted $\mathrm{fd}_R(F)$, to be the minimal length of a flat resolution of F (i.e., a resolution by R-flat modules). It is not difficult to see that

$$\mathrm{fd}_R(F) = \max\{i \in \mathbb{N} \ : \ \widetilde{\mathrm{Tor}}_i^R(F, N) \neq 0 \ \text{ for some almost finitely generated } N\}\,. \tag{1}$$

Indeed, for every almost finitely generated R-module M, we have

$$M \text{ is } R\text{−flat} \ \text{ iff } \ \widetilde{\mathrm{Tor}}_1^R(M, R/\mathfrak{m}_R) = 0\,, \tag{2}$$

where $\mathfrak{m}_R$ is the maximal ideal of R (see Proposition 4.4). Let $(A, \mathfrak{m}_A)$ be a regular local R-algebra such that F is a finite A-module. Then (1) follows from (2) applied to the kernels of a minimal A-free (and hence R-flat) resolution

$$\mathcal{F}_* : \quad \dots \overset{\alpha_{i+1}}{\to} F_{i+1} \overset{\alpha_i}{\to} F_i \overset{\alpha_{i-1}}{\to} \dots \overset{\alpha_1}{\to} F_1 \overset{\alpha_0}{\to} F_0 \to F$$

of F ($\mathcal{F}_*$ is *minimal* when $\alpha_i(F_{i+1}) \subset \mathfrak{m}_A F_i$ for all $i \in \mathbb{N}$).

We will also consider the *depth* of F as an R-module, defined as the length of a maximal F-sequence in R (i.e., a sequence $a_1, \dots, a_s \in \mathfrak{m}_R$ such that a_j is not a zerodivisor in $F/(a_1, \dots, a_{j-1})F$ for $j = 1, \dots, s$). Since all the maximal F-sequences in R have the same length, the depth is well defined: as observed in [GK, Lemma2.4], the classical proof of Northcott-Rees for finitely generated modules (see, e.g., [Ku, VI, Prop.3.1]), carries over to the case of almost finitely generated modules.

Remark 5.8. Let M and N be almost finitely generated R-modules. Then the following properties hold:

(i) *Rigidity of* $\widetilde{\mathrm{Tor}}^R$ *([GK, Prop.2.2(4)]).*

If $\widetilde{\mathrm{Tor}}_{i_0}^R(M, N) = 0$ for some $i_0 \in \mathbb{N}$, then $\widetilde{\mathrm{Tor}}_i^R(M, N) = 0$ for all $i \geq i_0$.

(ii) *Auslander–Buchsbaum-type formula ([GK, Thm.2.7]).*

$$\mathrm{fd}_R(M) + \mathrm{depth}_R(M) = n\,.$$

(iii) *Additivity of flat dimension ([GK, Prop.2.10]).*

If $\widetilde{\mathrm{Tor}}_i^R(M, N) = 0$ for all $i \geq 1$, then

$$\mathrm{fd}_R(M) + \mathrm{fd}_R(N) = \mathrm{fd}_R(M \tilde{\otimes}_R N)\,.$$

(iv) *Verticality of* $\widetilde{\mathrm{Tor}}^R$ *(cf. [GK, Prop.4.5]).*

For all $i \geq 1$, the module $\widetilde{\mathrm{Tor}}_i^R(M, N)$ is almost finitely generated, and every element of $\widetilde{\mathrm{Tor}}_i^R(M, N)$ is vertical over R (in the sense of Definition 5.4).

The property (iv) above is a consequence of Frisch's theorem stating that, given a holomorphic map $\varphi : X \to Y$ of analytic spaces and a coherent sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules, the image of the non-flatness locus

$$\{x \in X : \mathcal{F}_x \text{ is not } \mathcal{O}_{Y,\varphi(x)} - \text{flat}\}$$

is nowhere dense in Y ([Fri, Prop.VI.14]).

Finally, we generalize Auslander's [Au, Lem.3.1] to almost finitely generated modules.

Lemma 5.9. *Let $x = (x_1, \dots, x_m)$ and let $A = R\{x\}$ be a regular local analytic R-algebra. Let F be a finitely generated A-torsion-free module, and let N be a module finitely generated over $B = A^{\tilde{\otimes}^j_R}$ for some $j \geq 1$, such that $F \tilde{\otimes}_R N$ has no vertical elements over R. Then:*

(a) N has no vertical elements over R

(b) $\widetilde{\mathrm{Tor}}^R_i(F, N) = 0$ for all $i \geq 1$

(c) $\mathrm{fd}_R(F) + \mathrm{fd}_R(N) = \mathrm{fd}_R(F \tilde{\otimes}_R N)$.

Proof. Assertion (c) will follow from Remark 5.8(iii) once we have (b). For the proof of (a), consider $N' = \{n \in N : n \text{ is vertical over } R\}$. It is easy to see that N' is a B-submodule of N. Indeed, for any $n, n_1, n_2 \in N'$ and $b \in B$, we have

$$\mathrm{Ann}_B(n_1 + n_2) \supset \mathrm{Ann}_B(n_1) \cdot \mathrm{Ann}_B(n_2) \quad \text{and} \quad \mathrm{Ann}_B(bn) \supset \mathrm{Ann}_B(n)\,,$$

hence the zero set germ $\mathcal{V}(\mathrm{Ann}_B(n_1 + n_2))$ is mapped into the union of (nowhere dense) images of $\mathcal{V}(\mathrm{Ann}_B(n_1))$ and $\mathcal{V}(\mathrm{Ann}_B(n_2))$, and $\mathcal{V}(\mathrm{Ann}_B(bn))$ is mapped into the image of $\mathcal{V}(\mathrm{Ann}_B(n))$. Therefore, we get an exact sequence of B-modules

$$0 \to N' \to N \to N'' = N/N' \to 0\,.$$

Tensoring with F induces a long exact sequence of $A \tilde{\otimes}_R B \cong A^{\tilde{\otimes}^{j+1}_R}$-modules

$$\begin{aligned} \dots \to \widetilde{\mathrm{Tor}}^R_{i+1}(F, N') \to \widetilde{\mathrm{Tor}}^R_{i+1}(F, N) \to \widetilde{\mathrm{Tor}}^R_{i+1}(F, N'') \to \widetilde{\mathrm{Tor}}^R_i(F, N') \to \dots \\ \dots \to \widetilde{\mathrm{Tor}}^R_1(F, N'') \to F \tilde{\otimes}_R N' \to F \tilde{\otimes}_R N \to F \tilde{\otimes}_R N'' \to 0\,. \end{aligned} \quad (3)$$

Since every element of N' is vertical over R, the same is true for $F \tilde{\otimes}_R N'$ (indeed, $\mathrm{Ann}_{A \tilde{\otimes}_R B}(f \tilde{\otimes}_R n) \supset 1 \tilde{\otimes}_R \mathrm{Ann}_B(n)$ for all $f \in F, n \in N'$). But $F \tilde{\otimes}_R N$ has no vertical elements, by assumption, so $F \tilde{\otimes}_R N' \to F \tilde{\otimes}_R N$ is the zero map, and hence $F \tilde{\otimes}_R N \cong F \tilde{\otimes}_R N''$. In particular, $F \tilde{\otimes}_R N''$ has no vertical elements over R.

Now, F being A-torsion-free, there is an injection of F into a finite free A-module L (obtained by composing the natural map $F \to (F^*)^*$, which is injective in this case, with the dual of a presentation of F^*). Hence an exact sequence of A-modules $0 \to F \to L \to L/F \to 0$, which, in turn, induces a long exact sequence of $A \tilde{\otimes}_R B$-modules

$$\begin{aligned} \dots \to \widetilde{\mathrm{Tor}}^R_{i+1}(L, N'') \to \widetilde{\mathrm{Tor}}^R_{i+1}(L/F, N'') \to \widetilde{\mathrm{Tor}}^R_i(F, N'') \to \widetilde{\mathrm{Tor}}^R_i(L, N'') \to \dots \\ \dots \to \widetilde{\mathrm{Tor}}^R_1(L, N'') \to \widetilde{\mathrm{Tor}}^R_1(L/F, N'') \to F \tilde{\otimes}_R N'' \to L \tilde{\otimes}_R N'' \to L/F \tilde{\otimes}_R N'' \to 0\,. \end{aligned}$$

Since L is R-flat (as a free A-module), then $\widetilde{\mathrm{Tor}}^R_i(L, N'') = 0$ for all $i \geq 1$, and we obtain isomorphisms

$$\widetilde{\mathrm{Tor}}^R_{i+1}(L/F, N'') \cong \widetilde{\mathrm{Tor}}^R_i(F, N'') \quad \text{for all } i \geq 1\,, \quad (4)$$

as well as injectivity of the map $\widetilde{\mathrm{Tor}}_1^R(L/F, N'') \to F\tilde{\otimes}_R N''$. But $F\tilde{\otimes}_R N''$ has no vertical elements, whilst every element of $\widetilde{\mathrm{Tor}}_1^R(L/F, N'')$ is vertical over R (by Remark 5.8(iv)), hence $\widetilde{\mathrm{Tor}}_1^R(L/F, N'') \to F\tilde{\otimes}_R N''$ must be the zero map. Therefore we have $\ldots \xrightarrow{0} \widetilde{\mathrm{Tor}}_1^R(L/F, N'') \xrightarrow{0} \ldots$ exact, and hence $\widetilde{\mathrm{Tor}}_1^R(L/F, N'') = 0$. By rigidity of $\widetilde{\mathrm{Tor}}^R$ (Remark 5.8(i)), $\widetilde{\mathrm{Tor}}_{i+1}^R(L/F, N'') = 0$ for all $i \geq 1$, so by (4),

$$\widetilde{\mathrm{Tor}}_i^R(F, N'') = 0 \quad \text{for all } i \geq 1 \,. \tag{5}$$

In particular, $\widetilde{\mathrm{Tor}}_1^R(F, N'') = 0$, hence, by (3), $\ldots \xrightarrow{0} F\tilde{\otimes}_R N' \xrightarrow{0} \ldots$ is exact, so that $F\tilde{\otimes}_R N' = 0$. However, $F\tilde{\otimes}_R N' \cong (F\tilde{\otimes}_R B)\otimes_{A\tilde{\otimes}_R B}(A\tilde{\otimes}_R N')$ being an (ordinary) tensor product of finitely generated modules over a regular local ring $A\tilde{\otimes}_R B$, it can be zero only if one of the factors is so, which proves that $A\tilde{\otimes}_R N' = 0$, and thus $N' = 0$, by R-flatness of A. This shows (a).

Now, $\widetilde{\mathrm{Tor}}_i^R(F, N') = 0$ for all $i \geq 0$, and hence, by (3), $\widetilde{\mathrm{Tor}}_i^R(F, N) \cong \widetilde{\mathrm{Tor}}_i^R(F, N'')$ for all $i \geq 1$. Thus, by (5), $\widetilde{\mathrm{Tor}}_i^R(F, N) = 0$ for all $i \geq 1$, which proves (b). □

5.2 General case

In the general case, the following criterion holds:

Theorem 5.10. *Let $R = \mathbb{C}\{y_1, \ldots, y_n\}$, and let F be an almost finitely generated R-module. Then F is R-flat if and only if the n-fold analytic tensor power $F^{\tilde{\otimes}_R^n}$ has no vertical elements over R.*

The proof being somewhat involved, we first show that the algebraic version (Theorem 5.2) follows easily, by standard faithfull flatness arguments:

Proof of Theorem 5.2.

If F is R-flat, then $F^{\otimes_R^k}$ is R-flat and therefore R-torsion-free, for all k.

On the other hand, suppose that F is not R-flat. Since A is a localization of a quotient of a polynomial R-algebra $B = R[x_1, \ldots, x_m]$ and F is a finitely generated A-module, then F is also finite over $S^{-1}B$, for some multiplicative subset S of B. Therefore, $F \cong S^{-1}M$, for some finitely generated B-module M. Flatness and torsion-freeness are both local properties; i.e., F is R-flat (respectively, R-torsion-free) if and only if $F_{\mathfrak{b}}$ is R-flat (respectively, R-torsion-free), for every maximal ideal $\mathfrak{b}$ of $S^{-1}B$. Since F is not R-flat, there is a prime ideal $\mathfrak{p}$ in B such that $\mathfrak{p} \cap S = \varnothing$ and $M_{\mathfrak{p}}$ is not R-flat, and it suffices to prove that $M_{\mathfrak{p}}^{\otimes_R^n}$ is not R-torsion-free.

Now, the nonflatness of $M_{\mathfrak{p}}$ over R is equivalent to that of $M_{\mathfrak{n}}$, for every maximal ideal $\mathfrak{n}$ in B containing $\mathfrak{p}$ (indeed, for every such $\mathfrak{n}$, we have $(M_{\mathfrak{n}})_{\mathfrak{p}} \cong M_{\mathfrak{p}}$, and a localization of an R-flat B-module is R-flat).

Consider a maximal ideal $\mathfrak{n}$ in B containing $\mathfrak{p}$. We will show that $M_{\mathfrak{n}}^{\otimes_R^n}$ has a zero-divisor in R. Let $\varphi : X \to Y$ be the morphism of complex-analytic spaces associated to the morphism $\operatorname{Spec} B \to \operatorname{Spec} R$ and let $\mathcal{F}$ be the coherent sheaf of $\mathcal{O}_X$-modules associated to M. Let $\xi \in X$ be the point corresponding to the maximal ideal $\mathfrak{n}$ of $\operatorname{Spec} B$. It follows from faithful flatness of completion that $\mathcal{F}_\xi$ is not $\mathcal{O}_{Y,\eta}$-flat, where $\eta = \varphi(\xi)$. By Theorem 5.10, $\mathcal{F}_\xi^{\tilde{\otimes}^n_{\mathcal{O}_{Y,\eta}}}$ has a vertical element over $\mathcal{O}_{Y,\eta}$. Since $\varphi^{\{n\}}$ is the holomorphic map induced by the ring homomorphism $R \to B^{\otimes_R^n}$, it follows from Chevalley's Theorem (cf. [A3]) that $\mathcal{F}_\xi^{\tilde{\otimes}^n_{\mathcal{O}_{Y,\eta}}}$ has a zero-divisor in $\mathcal{O}_{Y,\eta}$. Hence $M_{\mathfrak{n}}^{\otimes_R^n}$ has a zero-divisor in R, as required.

Finally, let $\mathfrak{q}_1, \ldots, \mathfrak{q}_s$ be the primes in R whose union is the set of zero-divisors of $M^{\otimes_R^n}$. It follows from the preceding paragraph that if $\mathfrak{n}$ is a maximal ideal containing $\mathfrak{p}$, then $\mathfrak{n} \cap R \supset \mathfrak{q}_j$, for some $1 \leq j \leq s$; hence $\mathfrak{n} \cap R \supset \mathfrak{q}_1 \ldots \mathfrak{q}_s$. Since B is a Jacobson ring (see [Ei, Thm. 4.19]), $\mathfrak{p}$ is the intersection of all maximal ideals containing $\mathfrak{p}$, and consequently $\mathfrak{p} \cap R \supset \mathfrak{q}_1 \dots \mathfrak{q}_s$. Then $\mathfrak{p} \cap R \supset \mathfrak{q}_j$, for some j, because $\mathfrak{p} \cap R$ is prime. Therefore the zero-divisors from $\mathfrak{q}_j$ do not vanish after localizing in $\mathfrak{p}$, and hence $M_{\mathfrak{p}}^{\otimes_R^n}$ has a zero-divisor in R. □

5.3 Proof of Theorem 5.10

We will need the following variant of openness criterion (Thm. 3.5).

Proposition 5.11 ([ABM2, Prop. 4.2]). *Let $\varphi_\xi : X_\xi \to Y_\eta$ be a morphism of germs of analytic spaces. Suppose that Y_η is irreducible, $\dim Y_\eta = n$, $\dim X_\xi = m$, and the maximal fibre dimension of φ_ξ is not generic on some m-dimensional irreducible component of X_ξ. Then the n-fold fibred power $\varphi_{\xi^{\{n\}}}^{\{n\}} : X_{\xi^{\{n\}}}^{\{n\}} \to Y_\eta$ contains an isolated algebraic vertical component.*

Proof. As in Section 3.2, let $\varphi : X \to Y$ be a representative of φ_ξ, where Y is irreducible and of dimension n. Let $k := \max\{\mathrm{fbd}_x\varphi : x \in X\}$, $A_k := \{x \in X : \mathrm{fbd}_x\varphi = k\}$, and $B_k := \varphi(A_k) = \{y \in Y : \dim \varphi^{-1}(y) = k\}$. Then the fibre dimension of φ is constant on the analytic set A_k. By the Remmert Rank Theorem, B_k is locally analytic in Y, of dimension $\dim A_k - k \le \dim X - k$. Since $\eta \in B_k$, after shrinking Y if necessary, we can assume that B_k is an analytic subset of Y. Therefore, by Proposition 3.3, it is enough to show that the analytic germ $(B_k)_\eta$ is a proper subgerm of Y_η. Let U be an isolated irreducible component of X, of dimension $m = \dim X$, and such that k is not the generic fibre dimension of $\varphi|_U$. It follows that

$$\dim Y \ \ge\ \dim U - \text{ generic } \mathrm{fbd}\varphi|_U \ \ge\ m - k + 1\,.$$

Then $\dim B_k \le m - k < \dim Y$; hence $\dim(B_k)_\eta < \dim Y = \dim Y_\eta$, so that $(B_k)_\eta \subsetneq Y_\eta$. □

Let F be an almost finitely generated module over $R := \mathbb{C}\{y_1, \dots, y_n\}$. By Remark 5.6, there exists $m \ge 0$ such that F is finitely generated as a module over $A = R\{x\}$, where $x = (x_1, \dots, x_m)$. Let X and Y be connected open neighbourhoods of the origins in $\mathbb{C}^{m+n}$ and $\mathbb{C}^n$ (respectively), and let $\varphi : X \to Y$ be the canonical coordinate projection. Let $\mathcal{F}$ be a coherent sheaf of $\mathcal{O}_X$-modules whose stalk at the origin in X equals F. We can identify R with $\mathcal{O}_{Y,0}$ and A with $\mathcal{O}_{X,0}$. Then F is R-flat if and only if $\mathcal{F}_0$ is $\mathcal{O}_{Y,0}$-flat.

The "only if" direction of Theorem 5.10 is easy to establish (see below). Our proof of the more difficult "if" direction will be divided into three cases according to the following plan. We consider a short exact sequence $0 \to K \to F \to N \to 0$, where K is the A-torsion submodule of F and N is A-torsion-free. Then N can be treated by Auslander's techniques (cf. Remark 5.8); see Case (1) of the proof below. The A-torsion module K is supported over a subgerm of X_0 of strictly smaller fibre dimension over Y_0, so it can be treated by induction (Case (2)). For the general case (3), we want to show that the analytic tensor powers of either N or K *embed* into the corresponding powers of F, and hence so do their R-vertical elements.

The latter would be automatic if F were the *direct* sum of K and N. This is not true, in general. We can, however, choose N to be a submodule of F of the form $g \cdot F$, for a suitable choice of $g \in A$. It follows that the analytic tensor powers of K embed into those of F, unless gF is not R-flat. In the latter case, in turn, we show that the analytic tensor powers of gF embed into those of F.

The following lemma will be used in Case (3) of the proof of Theorem 5.10 below. It will allow us to conclude that the analytic tensor powers of gF embed into the corresponding powers of F, provided gF is A-torsion-free.

Lemma 5.12. *Let $R = \mathbb{C}\{y_1, \dots, y_n\}$, and let A and B be regular local analytic R-algebras. Suppose that M and N are finite A- and B-modules (respectively). Let $g \in A$, $h \in B$, and $m \in gM \tilde{\otimes}_R hN$ all* *be nonzero elements. If $m = 0$ as an element of $M\tilde{\otimes}_R N$, then $(g\tilde{\otimes}_R h)\cdot m = 0$ in $gM\tilde{\otimes}_R hN$. In other words, if $g\tilde{\otimes}_R h$ is not a zero-divisor of $gM\tilde{\otimes}_R hN$, then the canonical homomorphism $gM\tilde{\otimes}_R hN \to M\tilde{\otimes}_R N$ is an embedding.*

Proof. Using the identification

$$gM\tilde{\otimes}_R hN \cong (gM\tilde{\otimes}_R B) \otimes_{A\tilde{\otimes}_R B} (A\tilde{\otimes}_R hN)\,,$$

we can write $m = \sum_{i=1}^k m_i \otimes n_i$, where the $m_i \in gM\tilde{\otimes}_R B$, and $n_1, \dots, n_k$ generate $A\tilde{\otimes}_R hN$. The latter can be extended to a sequence $n_1, \dots, n_k, n_{k+1}, \dots, n_t$ generating $A\tilde{\otimes}_R N$. Setting $m_{k+1} = \dots = m_t = 0$, we get $m = \sum_{i=1}^t m_i \otimes n_i \in (M\tilde{\otimes}_R B) \otimes_{A\tilde{\otimes}_R B} (A\tilde{\otimes}_R N)$. By [Ei, Lemma 6.4], $m = 0$ in $M\tilde{\otimes}_R N$ if and only if there are $m'_1, \dots, m'_s \in M\tilde{\otimes}_R B$ and $a_{ij} \in A\tilde{\otimes}_R B$, such that

$$\sum_{j=1}^s a_{ij} m'_j = m_i \ \text{ in } \ M\tilde{\otimes}_R B, \quad \text{for all } i\,; \tag{1}$$

$$\sum_{i=1}^t a_{ij} n_i = 0 \ \text{ in } \ A\tilde{\otimes}_R N, \quad \text{for all } j\,. \tag{2}$$

Multiplying the equations (1) by $g\tilde{\otimes}_R 1$, we get

$$\sum_{j=1}^s a_{ij}(g\tilde{\otimes}_R 1) m'_j = (g\tilde{\otimes}_R 1) m_i \ \text{ in } \ gM\tilde{\otimes}_R B, \quad \text{for all } i\,; \tag{3}$$

hence $(g\tilde{\otimes}_R 1)m = 0$ in $gM\tilde{\otimes}_R N$, by (2) and (3).

Now write $(g\tilde{\otimes}_R 1)m = \sum_{i=1}^l m_i \otimes n_i$, where the $n_i \in A\tilde{\otimes}_R N$, and $m_1, \dots, m_l$ generate $gM\tilde{\otimes}_R B$. Then $(g\tilde{\otimes}_R 1)m = 0$ in $(gM\tilde{\otimes}_R B) \otimes_{A\tilde{\otimes}_R B} (A\tilde{\otimes}_R N)$ if and only if there are $n'_1, \dots, n'_p \in A\tilde{\otimes}_R N$ and $b_{ij} \in A\tilde{\otimes}_R B$, such that

$$\sum_{j=1}^p b_{ij} n'_j = n_i \ \text{ in } \ A\tilde{\otimes}_R N, \quad \text{for all } i\,; \tag{4}$$

$$\sum_{i=1}^l b_{ij} m_i = 0 \ \text{ in } \ gM\tilde{\otimes}_R B, \quad \text{for all } j\,. \tag{5}$$

Multiplying the equations (4) by $1\tilde{\otimes}_R h$, we get

$$\sum_{j=1}^p b_{ij}(1\tilde{\otimes}_R h) n'_j = (1\tilde{\otimes}_R h) n_i \ \text{ in } \ A\tilde{\otimes}_R hN, \quad \text{for all } i\,; \tag{6}$$

hence $(g\tilde{\otimes}_R h)m = 0$ in $gM\tilde{\otimes}_R hN$, by (5) and (6). Thus $g\tilde{\otimes}_R h$ is a zero-divisor of $gM\tilde{\otimes}_R hN$, as required. □

Proof of Theorem 5.10.

Let F be an almost finitely generated module over $R := \mathbb{C}\{y_1, \dots, y_n\}$. By Remark 5.6, there exists $m \geq 0$ such that F is finitely generated as a module over $A := R\{x\} = R\{x_1, \dots, x_m\}$. Let X and Y be connected open neighbourhoods of the origins in $\mathbb{C}^{m+n}$ and $\mathbb{C}^n$ (respectively), and let $\varphi : X \to Y$ be the canonical coordinate projection. Let $\mathcal{F}$ be a coherent sheaf of $\mathcal{O}_X$-modules whose stalk at the origin in X equals F, and let $\mathcal{G}$ be a coherent $\mathcal{O}_{X^{\{n\}}}$-module whose stalk at the origin

$0^{\{n\}}$ in $X^{\{n\}}$ equals $F^{\tilde{\otimes}^n_R}$. We can identify R with $\mathcal{O}_{Y,0}$ and A with $\mathcal{O}_{X,0}$. Then F is R-flat if and only if $\mathcal{F}_0$ is $\mathcal{O}_{Y,0}$-flat.

We first prove the "only if" direction of Theorem 5.10, by contradiction. Assume that F is R-flat. Since flatness is an open condition, by Douady's theorem [Dou], we can assume that $\mathcal{F}$ and $\mathcal{G}$ are $\mathcal{O}_Y$-flat. Suppose that $F^{\tilde{\otimes}^n_R}$ has a vertical element over $\mathcal{O}_{Y,0}$. In other words, (after shrinking X and Y if necessary) there exist a nonzero section $\tilde{m} \in \mathcal{G}$ and an analytic subset $Z \subset X^{\{n\}}$, such that $Z_0 = \mathcal{V}(\mathrm{Ann}_{\mathcal{O}_{X^{\{n\}},0^{\{n\}}}}(\tilde{m}_0))$ and the image $\varphi^{\{n\}}(Z)$ has empty interior in Y. Let $\tilde{\varphi}$ denote the restriction $\varphi^{\{n\}}|_Z : Z \to Y$. Consider $\xi \in Z$ such that the fibre dimension of $\tilde{\varphi}$ at ξ is minimal. Then the fibre dimension $\mathrm{fbd}_x\tilde{\varphi}$ is constant on some open neighbourhood U of ξ in Z. By the Remmert Rank Theorem, $\tilde{\varphi}(U)$ is locally analytic in Y near $\eta = \tilde{\varphi}(\xi)$. Since $\tilde{\varphi}(Z)$ has empty interior in Y, it follows that there is a holomorphic function g in a neighbourhood of η in Y, such that $(\tilde{\varphi}(U))_\eta \subset \mathcal{V}(g_\eta)$. Therefore, $\tilde{\varphi}^*_\xi(g_\eta)\cdot\tilde{m}_\xi = 0$ in $\mathcal{G}_\xi$; i.e., $\mathcal{G}_\xi$ has a (nonzero) zero-divisor in $\mathcal{O}_{Y,\eta}$, contradicting flatness.

We will now prove the more difficult "if" direction of the theorem, by induction on m. If $m = 0$, then F is finitely generated over R, and the result follows from Auslander's theorem 5.1 (because flatness of finitely generated modules over a local ring is equivalent to freeness, the analytic tensor product equals the ordinary tensor product for finite modules, and vertical elements in finite modules are just zero-divisors).

The inductive step will be divided into three cases:
1. F is torsion-free over A;
2. F is a torsion A-module;
3. F is neither A-torsion-free nor a torsion A-module.

Case (1)

We prove this case independently of the inductive hypothesis. We essentially repeat the argument of Galligo and Kwieciński [GK], which itself is an adaptation of Auslander [Au] to the almost finitely generated context.

Suppose that $F^{\tilde{\otimes}^n_R}$ has no vertical elements over R. Then it follows from Lemma 5.9(a) that $F^{\tilde{\otimes}^i_R}$ has no vertical elements, for $i = 1, \ldots, n$. By Lemma 5.9(c),

$$\mathrm{fd}_R(F^{\tilde{\otimes}^n_R}) = \mathrm{fd}_R(F) + \mathrm{fd}_R(F^{\tilde{\otimes}^{n-1}_R}) = \cdots = n \cdot \mathrm{fd}_R(F)\,.$$

On the other hand, since $F^{\tilde{\otimes}^n_R}$ has no vertical elements over R, it has no zero-divisors over R, so that $\mathrm{depth}_R(F^{\tilde{\otimes}^n_R}) \geq 1$. It follows from Remark 5.8(ii) that $\mathrm{fd}_R(F^{\tilde{\otimes}^n_R}) < n$. Hence $n \cdot \mathrm{fd}_R(F) < n$. This is possible only if $\mathrm{fd}_R(F) = 0$; i.e., F is R-flat.

Case (2)

Suppose that F is not R-flat and a torsion A-module. We will show that then $F^{\tilde{\otimes}^n_R}$ contains vertical elements over R. Let $I = \mathrm{Ann}_A(F)$. Since every element of F is annihilated by some nonzero element of A, and F is finitely generated over A, then I is a nonzero ideal in A. Put $B = A/I$; then F is finitely generated over B. Let $I(0)$ denote the *evaluation* of I *at* $y = 0$ (i.e., $I(0)$ is the ideal generated by I in $A(0) := A\tilde{\otimes}_R R/\mathfrak{m}_R \cong \mathbb{C}\{x_1, \ldots, x_m\}$).

First suppose that $I(0) \neq (0)$. Then there exists $g \in I$ such that $g(0) := g(0, x) \neq 0$, and F is a finite $A/(g)A$-module. It follows that (after an appropriate linear change in the x-coordinates) g is regular in x_m and hence, by the Weierstrass Preparation Theorem, that F is finite over $R\{x_1, \ldots, x_{m-1}\}$. Therefore, $F^{\tilde{\otimes}^n_R}$ has a vertical element over R, by the inductive hypothesis.

On the other hand, suppose that $I(0) = (0)$; i.e., $I \subset \mathfrak{m}_R A$. Then $B\tilde{\otimes}_R R/\mathfrak{m}_R = (A/I)\tilde{\otimes}_R R/\mathfrak{m}_R$ equals $\mathbb{C}\{x_1, \ldots, x_m\}$. Let Z be a closed analytic subspace of X such that $\mathcal{O}_{Z,0} \cong B$, and let $\tilde{\varphi} := \varphi|_Z$. It follows that the fibre $\tilde{\varphi}^{-1}(0)$ equals $\mathbb{C}^m$. Of course, m is not the generic fibre dimension of $\tilde{\varphi}$ on any irreducible component of Z, because otherwise all its fibres would equal $\mathbb{C}^m$, so we would have $B = A$ and $I = (0)$, contrary to the choice of I. Therefore, by Proposition 5.11, there is an isolated algebraic vertical component in the n-fold fibred power of $\tilde{\varphi}_0$; i.e., $B^{\tilde{\otimes}_R^n}$ has a zero-divisor in R. But $F^{\tilde{\otimes}_R^n}$ is a faithful $B^{\tilde{\otimes}_R^n}$-module, so itself it has a zero-divisor (hence a vertical element) over R.

Case (3)

Suppose that F is not R-flat, F has zero-divisors in A, but $\mathrm{Ann}_A(F) = (0)$. Let

$$K := \{f \in F : af = 0 \text{ for some nonzero } a \in A\};$$

i.e., K is the A-torsion submodule of F. Since K is a submodule of a finitely generated module over a noetherian ring, K is finitely generated; say $K = \sum_{i=1}^{s} A \cdot f_i$. Take $a_i \in A \setminus \{0\}$ such that $a_i f_i = 0$, and put $g = a_1 \cdots a_s$. Then the sequence of A-modules

$$0 \to K \to F \stackrel{\cdot g}{\to} gF \to 0 \tag{7}$$

is exact, and gF is a torsion-free A-module.

First suppose that gF is R-flat. Then by applying $\tilde{\otimes}_R K$ and $F\tilde{\otimes}_R$ to (7), we get short exact sequences

$$0 \to K\tilde{\otimes}_R K \to F\tilde{\otimes}_R K \to gF\tilde{\otimes}_R K \to 0,$$
$$0 \to F\tilde{\otimes}_R K \to F\tilde{\otimes}_R F \to F\tilde{\otimes}_R\, gF \to 0\,.$$

So we have injections

$$K\tilde{\otimes}_R K \hookrightarrow F\tilde{\otimes}_R K \hookrightarrow F\tilde{\otimes}_R F\,,$$

and by induction, an injection $K^{\tilde{\otimes}_R^i} \hookrightarrow F^{\tilde{\otimes}_R^i}$, for all $i \geq 1$. In particular, $K^{\tilde{\otimes}_R^n}$ is a submodule of $F^{\tilde{\otimes}_R^n}$. Since gF is R-flat and F is not R-flat, it follows that K is not R-flat. Therefore, by Case (2), $K^{\tilde{\otimes}_R^n}$ (and hence $F^{\tilde{\otimes}_R^n}$) has a vertical element over R.

Now suppose that gF is not R-flat. Then $(gF)^{\tilde{\otimes}_R^n}$ has a vertical element over R, by Case (1). We will show that $(gF)^{\tilde{\otimes}_R^n}$ embeds into $F^{\tilde{\otimes}_R^n}$, and hence so do its vertical elements. By Lemma 5.12, in order for $(gF)^{\tilde{\otimes}_R^n}$ to embed into $F^{\tilde{\otimes}_R^n}$, it suffices to prove that $g^{\tilde{\otimes}_R^n}$ is not a zero-divisor of $(gF)^{\tilde{\otimes}_R^n}$.

To simplify the notation, let B denote the ring $A^{\tilde{\otimes}_R^n}$, and let $h := g^{\tilde{\otimes}_R^n} \in B$. Since $(gF)^{\tilde{\otimes}_R^n}$ is a finite B-module, we can write $(gF)^{\tilde{\otimes}_R^n} = B^q/M$, where $q \geq 1$ and M is a B-submodule of B^q. Given $b \in B$, let $(M : b)$ denote the B-submodule of B^q consisting of those elements $m \in B^q$ for which $b{\cdot}m \in M$. Since

$$(M : h) \ \subset \ (M : h^2) \ \subset \cdots \subset \ (M : h^l) \ \subset \dots$$

is an increasing sequence of submodules of a noetherian module B^q, it stabilizes; i.e., there exists $k \geq 1$ such that $(M : h^{k+1}) = (M : h^k)$. In other words, there exists $k \geq 1$ such that h is not a zero-divisor in $h^k \cdot B^q/M$; i.e., $g^{\tilde{\otimes}_R^n}$ is not a zero-divisor in $(g^{k+1}F)^{\tilde{\otimes}_R^n}$.

Observe though that (because gF is A-torsion-free), multiplication by g induces an isomorphism $gF \to g^2F$ of A-modules, and in general, $gF \cong g^lF$, for $l \geq 1$. We thus have isomorphisms $(gF)^{\tilde{\otimes}_R^n} \cong (g^lF)^{\tilde{\otimes}_R^n}$ of B-modules, for $l \geq 1$. In particular, for every $l \geq 1$, $g^{\tilde{\otimes}_R^n}$ is a zero-divisor of $(gF)^{\tilde{\otimes}_R^n}$ if and only if it is a zero-divisor of $(g^lF)^{\tilde{\otimes}_R^n}$. Therefore, by Lemma 5.12, we have an embedding $(gF)^{\tilde{\otimes}_R^n} \hookrightarrow F^{\tilde{\otimes}_R^n}$. This completes the proof of Theorem 5.10. $\square$