Problem Set 3
due: March 15, 2019.

1. Let $f : \mathbb{R} \to \mathbb{R}$ be a function and let $g : \mathbb{R} \to \mathbb{R}$ be a Borel measurable function.

(a) Suppose that $f = g$ a.e. (with respect to Lebesgue measure). Prove that f is Lebesgue measurable.

(b) Suppose now that f is Lebesgue measurable. Prove that there exists a Borel measurable function $h : \mathbb{R} \to \mathbb{R}$ such that $f = h$ a.e. (with respect to Lebesgue measure).

2. Verify that the integral of a non-negative simple function is well defined; i.e., independent of its representation as a sum of characteristic functions (Exercise 6.1).

3. Given a metric space (X, d) and a function $f : X \to \mathbb{R}$, we define functions $m_f, M_f : X \to \mathbb{R}$ by the formulas

$$m_f(x) = \sup\{\inf(f(U)) : U \text{ an open neighbourhood of } x\}\,,$$

and

$$M_f(x) = \inf\{\sup(f(U)) : U \text{ an open neighbourhood of } x\}\,.$$

(a) Prove that, for every $x \in X$, $m_f(x) \leq f(x) \leq M_f(x)$.

(b) Prove that the functions m_f and M_f are Borel measurable.

(c) Prove that f is continuous at x_0 iff $m_f(x_0) = M_f(x_0)$.

4. Exercises 6.2 and 6.3.

5. Exercise 6.4.

6. Exercise 6.5.

7. Exercise 6.6.

8. Exercise 6.7.

9. Exercises 7.3 and 7.4.