Practice Term Test 2

0. Problems from Problem Sets 5 and 6.

1. State the Monotone Convergence Theorem, Dominated Convergence Theorem and Fatou's Lemma.

2. State the definitions of product σ-algebra and product measure.

3. Prove or give a counterexample:
If $f_n : \mathbb{R} \to \mathbb{R}$ are Lebesgue measurable and pointwise convergent to a function f, then $\int f = \lim_{n\to\infty} \int f_n$.

4. Prove or give a counterexample:
If $f_n : \mathbb{R} \to [0,\infty)$ are Lebesgue integrable and $\lim_{n\to\infty} \int f_n = \int f$ for some function $f : \mathbb{R} \to \mathbb{R}$, then $f_n \longrightarrow f$ a.e..

5. Prove or give a counterexample:
If $(X, \mathcal{M}, \mu)$ is a measure space with $\mu(X) < \infty$, and $(f_n)_{n=1}^{\infty}$ is a sequence of bounded real-valued measurable functions that converge uniformly to a function f, then $\displaystyle\lim_{n\to\infty} \int_X f_n \, d\mu = \int_X f \, d\mu$.

6. Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space, and let $f : X \to \mathbb{R}$ be integrable. Prove that for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\int_A |f(x)| \, d\mu(x) < \varepsilon$$

whenever $A \in \mathcal{M}$ satisfies $\mu(A) < \delta$.

7. Suppose $f : \mathbb{R} \to \mathbb{R}$ is integrable, m denotes the Lebesgue measure on $\mathbb{R}$, $a \in \mathbb{R}$, and we define $F : \mathbb{R} \to \mathbb{R}$ as

$$F(x) = \int_{I_{a,x}} f(x) \, dm(x),$$

where $I_{a,x}$ denotes the closed interval with endpoints a and x. Show that F is continuous.

8. Suppose $(X, \mathcal{M}, \mu)$ is a measure space, f and each f_n is integrable and non-negative, $f_n \longrightarrow f$ a.e., and $\int f_n \longrightarrow \int f$ as $n \to \infty$. Prove that, for each $A \in \mathcal{M}$,

$$\lim_{n\to\infty} \int_A f_n \, d\mu = \int_A f \, d\mu \,.$$

9. Prove that the limit exists and find its value:

$$\lim_{n\to\infty} \int_0^1 \frac{1 + nx^2}{(1 + x^2)^n} \ln(2 + \cos(x/n)) \, dx \,.$$

10. Let $g : \mathbb{R} \to \mathbb{R}$ be Lebesgue integrable and let $f : \mathbb{R} \to \mathbb{R}$ be bounded, Lebesgue measurable, and continuous at 1. Prove that the limit exists and find its value:

$$\lim_{n\to\infty} \int_{-n}^{n} f\Big(1 + \frac{x}{n^2}\Big) \, g(x) \, dx \,.$$

11. Suppose μ is a finite measure on a measurable space $(X, \mathcal{M})$. Prove that a measurable function $f : X \to [0, \infty)$ is integrable if and only if $\displaystyle\sum_{n=1}^{\infty} \mu(\{x \in X : f(x) \geq n\}) < \infty$.

12. Suppose $f : [0,1]^2 \to \mathbb{R}$ is integrable with respect to the 2-dimensional Lebesgue measure m on $[0,1]^2$, and $\displaystyle\int_{[0,a]\times[0,b]} f \, dm = 0$ for all $a, b \in [0,1]$. Prove that $f = 0$ a.e.

13. Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space, and let $f : X \to \mathbb{R}$ be an $\mathcal{M}$-measurable function. Define the *distribution function* of f by

$$\mu_f(t) := \mu(\{x \in X : |f(x)| \geq t\}), \quad t > 0.$$

Show that $\mu_f : (0, \infty) \to [0, \mu(X)]$ is non-increasing and Borel measurable, and $\displaystyle\int_X |f(x)| d\mu(x) = \int_0^{\infty} \mu_f(t) dt$.