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GEOMETRIC AUSLANDER CRITERION FOR FLATNESS

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Abstract. Our aim is to understand the algebraic notion of flatness in explicit geometric terms. Let $\varphi: X \rightarrow Y$ be a morphism of complex-analytic spaces, where Y is smooth. We prove that nonflatness of φ is equivalent to a severe discontinuity of the fibres—the existence of a *vertical component* (a local irreducible component at a point of the source whose image is nowhere-dense in Y)—after passage to the n -fold fibred power of φ , where $n = \dim Y$. Our main theorem is a more general criterion for flatness over Y of a coherent sheaf of modules \mathcal{F} on X . In the case that φ is a morphism of complex algebraic varieties, the result implies that the stalk \mathcal{F}_ξ of \mathcal{F} at a point $\xi \in X$ is flat over $R := \mathcal{O}_{Y, \varphi(\xi)}$ if and only if its n -fold tensor power is a torsion-free R -module (conjecture of Vasconcelos in the case of \mathbb{C} -algebras).

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1. Introduction. Flatness is a subtle algebraic notion that expresses continuity of the fibres of a mapping, and therefore the idea of a family of varieties parametrized by a given variety Y . But flatness has remained geometrically elusive because “it depends on infinitesimal data which are frequently invisible at the level of topology” (Kollár [16]). This article is a contribution towards attempts to understand flatness in geometric terms. The following is a special case of our main result (see Theorem 1.9 and Corollary 1.10).

THEOREM 1.1. *Let $\varphi: X \rightarrow Y$ denote a morphism of complex-analytic spaces, where Y is smooth, and let $\xi \in X$. Let $\varphi^{\{n\}}: X^{\{n\}} \rightarrow Y$ denote the induced morphism from the n -fold fibred power of X over Y , where $n = \dim Y$, and let $\xi^{\{n\}} \in X^{\{n\}}$ denote the diagonal point corresponding to ξ . Then φ is not flat at ξ if and only if $\varphi^{\{n\}}$ has a vertical component at $\xi^{\{n\}}$; i.e., a local irreducible component (perhaps embedded) of $X^{\{n\}}$ at $\xi^{\{n\}}$ whose image is nowhere-dense in Y .*

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Theorem 1.9 and Corollary 1.10 provide a more general criterion for \mathcal{O}_Y -flatness of a coherent sheaf of \mathcal{O}_X -modules. The theorem is an extension of a result of Galligo and Kwieciński [11].

Our work is of origin in M. Auslander’s criterion for freeness of a finitely generated module over a regular local ring:

THEOREM 1.2. [5, Thm. 3.2] *Let R be a regular local ring of dimension $n > 0$, and let F be a finite R -module. Then F is R -free if and only if the n -fold tensor power $F^{\otimes_R^n}$ is a torsion-free R -module.*

(Theorem 1.2 was proved in the case that R is unramified by Auslander, and extended to arbitrary regular local rings by Lichtenbaum [19].)

Flat is the appropriate analogue of *free* for modules that are not necessarily finitely generated—“flatness ... [embodies] that part of freeness which can be expressed in terms of linear equations” (Mumford [21]). Flat is equivalent to free for finite modules over a local ring. Vasconcelos [22] and Kwieciński [18] were the first to consider extending Auslander’s criterion from finite modules to finite algebras over a regular local ring.

We use our main theorem to prove the following result—a generalization of Vasconcelos’s conjecture [22, Conj. 6.2], [23, Conj. 2.6.1] in the case of \mathbb{C} -algebras (see Section 1.2).

THEOREM 1.3. *Let R be a regular \mathbb{C} -algebra of finite type. Let A denote an R -algebra essentially of finite type, and let F denote a finitely generated A -module. Then F is R -flat if and only if the n -fold tensor power $F^{\otimes_R^n}$ is a torsion-free R -module, where $n = \dim R$.*

An R -algebra *essentially of finite type* means a localization of an R -algebra of finite type.

Remark 1.4. By Theorem 1.3 and the prime avoidance lemma [8, Lemma 3.3], in order to verify that F is not R -flat, it is enough to find an associated prime of $F^{\otimes_R^n}$ in $A^{\otimes_R^n}$ which contains a nonzero element $r \in R$. Thus Theorem 1.3 together with Gröbner-basis algorithms for primary decomposition (see [23] or [13]) provides a tool for checking flatness by effective computation.

Frisch’s generic flatness theorem [10, Prop. VI,14] plays an important part in the proof of our main theorem, in the proof of *verticality* of torsion modules [11, Prop. 4.5] (see Proposition 3.1(4) below)—a property which is immediate in the case of finitely generated modules over an integral domain.

The assertion of Theorem 1.3 for any field of characteristic zero follows from Theorem 1.3 as stated, using the Tarski-Lefschetz Principle (see [3, Thm. 2.1]). L. Avramov and S. Iyengar have more recently proved Theorem 1.3 for an arbitrary field [6]. In a subsequent paper [4], we prove another geometric flatness criterion using techniques which differ from but share a common approach with

those here—successive reduction in fibre dimension via Weierstrass preparation, to eventually reduce the problem to a question about flatness of finitely generated modules that is well-understood. The inductive criterion is a convenient tool to prove generic flatness results like Frisch’s theorem.

1.1. Vertical components in fibred powers. Kwieciński [18] proved that, if R is a finitely generated \mathbb{C} -algebra which is a normal domain and A is a finitely generated R -algebra, then A is R -flat if and only if *all* tensor powers $A^{\otimes_R k}$ are R -torsion-free. He used techniques of complex-analytic geometry, introducing the idea of a *vertical component* of an analytic mapping as a geometric analogue of torsion in commutative algebra. Following [1], we distinguish algebraic and geometric versions of Kwieciński’s idea: Let $\varphi_\xi : X_\xi \rightarrow Y_\eta$ denote a morphism of germs of complex-analytic spaces.

Definition 1.5. Let W_ξ denote an irreducible component of X_ξ (isolated or embedded). We say that W_ξ is an *algebraic* (respectively, *geometric*) *vertical component* of φ_ξ (or over Y_η) if φ_ξ maps W_ξ to a proper analytic (respectively, nowhere-dense) subgerm of Y_η .

We are allowing ourselves some imprecision of language. The notation $\varphi_\xi : X_\xi \rightarrow Y_\eta$ is meant to suggest the germ at a point $\xi \in X$ of a morphism of complex-analytic spaces $\varphi : X \rightarrow Y$, $\eta = \varphi(\xi)$. In particular, we will write $\mathcal{O}_{X,\xi}$ for the local ring of X_ξ . Consider a representative $\varphi : X \rightarrow Y$ of φ_ξ . The “if” clause in Definition 1.5 means more precisely that, for a sufficiently small representative W of W_ξ in X , the germ $\varphi(W)_\eta$ lies in a proper analytic subgerm of Y_η (respectively, $\varphi(W)$ is nowhere-dense in Y).

By the prime avoidance lemma, $\varphi_\xi : X_\xi \rightarrow Y_\eta$ has an algebraic (respectively, geometric) vertical component if and only if there exists a nonzero element $m \in \mathcal{O}_{X,\xi}$ such that the zero-set germ $\mathcal{V}(\text{Ann}_{\mathcal{O}_{X,\xi}}(m))$ of the annihilator of m in $\mathcal{O}_{X,\xi}$ is mapped to a proper analytic (respectively, nowhere-dense) subgerm of Y_η .

We can extend the notion of vertical component to a finitely generated $\mathcal{O}_{X,\xi}$ -module F :

Definition 1.6. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_s$ be the associated primes of F in $\mathcal{O}_{X,\xi}$ and, for $j = 1, \dots, s$, let $(Z_j)_\xi$ be the germ of a complex analytic subspace of X , defined by $\mathcal{O}_{Z_j,\xi} := \mathcal{O}_{X,\xi}/\mathfrak{p}_j$. We say that F has a *geometric vertical component* over Y_η (or over $\mathcal{O}_{Y,\eta}$) if, for some j , φ_ξ maps $(Z_j)_\xi$ to a nowhere-dense subgerm of Y_η ; equivalently, there exists a nonzero $m \in F$ such that $\mathcal{V}(\text{Ann}_{\mathcal{O}_{X,\xi}}(m))$ is mapped to a nowhere-dense subgerm of Y_η . We will call such m a *geometric vertical element* (or simply a *vertical element*) of F over Y_η (or over $\mathcal{O}_{Y,\eta}$).

Note that an analogously defined “algebraic vertical element” of F over Y_η is simply a (nonzero) zero-divisor of F over $\mathcal{O}_{Y,\eta}$, so there is no need to define algebraic vertical elements. A *vertical element* will always mean geometric vertical.

Remark 1.7. In the special case that $F = \mathcal{O}_{X,\xi}$, X_ξ has no geometric (respectively, algebraic) vertical components over Y_η if and only if $\mathcal{O}_{X,\xi}$ (as an $\mathcal{O}_{X,\xi}$ -module) has no vertical elements (respectively, no zero-divisors) over $\mathcal{O}_{Y,\eta}$.

Now let R denote a regular local analytic \mathbb{C} -algebra of dimension n . Then R is isomorphic to the ring $\mathbb{C}\{y\} = \mathbb{C}\{y_1, \dots, y_n\}$ of convergent power series in n variables. A *local analytic R -algebra* A means a ring of the form $R\{x\}/I = \mathbb{C}\{y, x\}/I$, where I is an ideal in $\mathbb{C}\{y, x\} = \mathbb{C}\{y_1, \dots, y_n, x_1, \dots, x_m\}$, with the canonical homomorphism $R \rightarrow A$. Let F denote an R -module. We say that F is an *almost finitely generated R -module* (following [11]) if F is a finitely generated A -module, for some local analytic R -algebra A . In this case, there is a morphism of germs of analytic spaces $\varphi_\xi : X_\xi \rightarrow Y_\eta$ such that $R \cong \mathcal{O}_{Y,\eta}$, $A \cong \mathcal{O}_{X,\xi}$, $R \rightarrow A$ is the induced homomorphism $\varphi_\xi^* : \mathcal{O}_{Y,\eta} \rightarrow \mathcal{O}_{X,\xi}$, and F is a finitely generated $\mathcal{O}_{X,\xi}$ -module. We say that a nonzero element $m \in F$ is *vertical over R* if m is vertical over $\mathcal{O}_{Y,\eta}$ in the sense of Definition 1.6.

Remark 1.8. It is easy to see that the notion of vertical element is well-defined; i.e., independent of a choice of local R -algebra A such that F is a finitely generated A -module. In particular, given an almost finitely generated R -module F , we can assume without loss of generality that F is finitely generated over the regular ring $A = R\{x\} \cong \mathbb{C}\{y, x\}$, where $x = (x_1, \dots, x_m)$, for some $m \geq 0$.

1.2. Main results. Our main theorem is the following flatness criterion.

THEOREM 1.9. *Let R be a regular local analytic \mathbb{C} -algebra and let F denote an almost finitely generated R -module. Let $n = \dim R$. Then F is R -flat if and only if the n -fold analytic tensor power $F^{\otimes_R^n}$ has no vertical elements over R .*

(See Section 2 for the notion of analytic tensor power.) Theorem 1.9 can be restated as follows.

COROLLARY 1.10. *Let $\varphi : X \rightarrow Y$ denote a morphism of complex-analytic spaces, where Y is smooth, and let \mathcal{F} denote a coherent sheaf of \mathcal{O}_X -modules. Let $\xi \in X$ and $\eta = \varphi(\xi)$. Then \mathcal{F}_ξ is $\mathcal{O}_{Y,\eta}$ -flat if and only if the n -fold analytic tensor power $\mathcal{F}_\xi^{\otimes_{\mathcal{O}_{Y,\eta}}^n}$ has no vertical elements over $\mathcal{O}_{Y,\eta}$, where $n = \dim_\eta Y$.*

Theorem 1.1 is equivalent to Corollary 1.10 in the special case that $\mathcal{F} = \mathcal{O}_X$, according to Remark 1.7 and the canonical isomorphism $\mathcal{O}_{X,\xi} \tilde{\otimes}_{\mathcal{O}_{Y,\eta}} \cdots \tilde{\otimes}_{\mathcal{O}_{Y,\eta}} \mathcal{O}_{X,\xi} \cong \mathcal{O}_{X^{(n)}, \xi^{(n)}}$.

Theorem 1.1 in the special case that X_ξ is equidimensional is the theorem of Galligo and Kwieciński [11]. The assumption that X_ξ is equidimensional guarantees that $A = \mathcal{O}_{X,\xi}$ is a finite *torsion-free* module over some R -flat algebra S (where $R = \mathcal{O}_{Y,\eta}$), and Auslander's techniques can be extended to this case.

We are happy to acknowledge the influence of [11] on our paper. We are also grateful to L. Avramov and S. Iyengar for pointing out an error in an earlier version of the article (arXiv:0901.2744v2).

We prove Theorem 1.9 in Section 5 below. We reduce to the case that $A = R\{x_1, \dots, x_m\}$ using Remark 1.8, and then argue by induction on m . The case $m = 0$ follows directly from Auslander’s theorem. We divide the inductive step into three cases, according as F is torsion-free over A , F is a torsion A -module, or neither. The proof of the first case is independent of the inductive hypothesis, and again uses the argument of [5] (in the same way it is used in [11]). The second case follows from the inductive assumption using the Weierstrass Preparation Theorem.

The most difficult new situation in the inductive step is the case that F is neither torsion-free nor a torsion A -module. Our proof involves the geometry of the support of F over A . If F is A -torsion-free, then the support of F coincides with that of A . In general, an analysis of the support of F allows us either to reduce the fibre dimension m over R , or otherwise to use Proposition 4.2 on the variation of fibre dimension to produce zero-divisors over R already for topological reasons:

We show (in Section 4) how nonconstancy of the dimension of the fibres of a morphism $\varphi : X \rightarrow Y$ leads to *isolated algebraic* vertical components in fibred powers of φ .

By way of comparison with Corollary 1.10, we note that a lack of *isolated algebraic* vertical components in the n -fold fibred power (where n is the target dimension) characterizes *openness* of an analytic mapping with irreducible target (see [1], [2]). Proposition 4.2 is a simpler version of the latter result.

Frisch’s generic flatness theorem asserts that if \mathcal{F} is a coherent sheaf of \mathcal{O}_X -modules over a morphism $\varphi : X \rightarrow Y$ of complex-analytic spaces, then $\{\xi \in X : \mathcal{F}_\xi \text{ is not } \mathcal{O}_{Y, \varphi(\xi)}\text{-flat}\}$ has image nowhere-dense in Y . Frisch’s theorem is responsible for the criterion of Theorem 1.9 in terms of *geometric* vertical elements. The existence of a vertical element in $F^{\tilde{\otimes}_R^n}$ guarantees the existence of a zero-divisor of $F^{\tilde{\otimes}_R^k}$ over R , for *some* $k \geq n$ (by [1] and Theorem 1.9). The following question seems to be open.

QUESTION 1.11. *With the assumptions of Theorem 1.9, if F is not R -flat, does $F^{\tilde{\otimes}_R^n}$ have a zero-divisor over R ?*

If $\varphi_\xi : X_\xi \rightarrow Y_\eta$ and F have an underlying algebraic structure as in Theorem 1.3, then the notions of geometric and algebraic vertical components coincide, so that Theorem 1.3 follows from Theorem 1.9:

Proof of Theorem 1.3. If F is R -flat, then $F^{\otimes_R^k}$ is R -flat and therefore R -torsion-free, for all k .

On the other hand, suppose that F is not R -flat. Since A is a localization of a quotient of a polynomial R -algebra $B = R[x_1, \dots, x_m]$ and F is a finitely generated A -module, then F is also finite over $S^{-1}B$, for some multiplicative subset S of B .

Therefore, $F \cong S^{-1}M$, for some finitely generated B -module M . Flatness and torsion-freeness are both local properties; i.e., F is R -flat (respectively, R -torsion-free) if and only if $F_{\mathfrak{b}}$ is R -flat (respectively, R -torsion-free), for every maximal ideal \mathfrak{b} of $S^{-1}B$. Since F is not R -flat, there is a prime ideal \mathfrak{p} in B such that $\mathfrak{p} \cap S = \emptyset$ and $M_{\mathfrak{p}}$ is not R -flat, and it suffices to prove that $M_{\mathfrak{p}}^{\otimes_R^n}$ is not R -torsion-free.

Now, the nonflatness of $M_{\mathfrak{p}}$ over R is equivalent to that of $M_{\mathfrak{n}}$, for every maximal ideal \mathfrak{n} in B containing \mathfrak{p} (indeed, for every such \mathfrak{n} , we have $(M_{\mathfrak{n}})_{\mathfrak{p}} \cong M_{\mathfrak{p}}$, and a localization of an R -flat B -module is R -flat).

Consider a maximal ideal \mathfrak{n} in B containing \mathfrak{p} . We will show that $M_{\mathfrak{n}}^{\otimes_R^n}$ has a zero-divisor in R . Let $\varphi : X \rightarrow Y$ be the morphism of complex-analytic spaces associated to the morphism $\text{Spec } B \rightarrow \text{Spec } R$ and let \mathcal{F} be the coherent sheaf of \mathcal{O}_X -modules associated to M . Let $\xi \in X$ be the point corresponding to the maximal ideal \mathfrak{n} of $\text{Spec } B$. It follows from faithful flatness of completion that \mathcal{F}_{ξ} is not $\mathcal{O}_{Y,\eta}$ -flat, where $\eta = \varphi(\xi)$. By Theorem 1.9, $\mathcal{F}_{\xi}^{\otimes_{\mathcal{O}_{Y,\eta}}^n}$ has a vertical element over $\mathcal{O}_{Y,\eta}$. Since $\varphi^{\{n\}}$ is the holomorphic map induced by the ring homomorphism $R \rightarrow B^{\otimes_R^n}$, it follows from Chevalley’s Theorem that $\mathcal{F}_{\xi}^{\otimes_{\mathcal{O}_{Y,\eta}}^n}$ has a zero-divisor in $\mathcal{O}_{Y,\eta}$. Hence $M_{\mathfrak{n}}^{\otimes_R^n}$ has a zero-divisor in R , as required.

Finally, let $\mathfrak{q}_1, \dots, \mathfrak{q}_s$ be the primes in R whose union is the set of zero-divisors of $M^{\otimes_R^n}$. It follows from the preceding paragraph that if \mathfrak{n} is a maximal ideal containing \mathfrak{p} , then $\mathfrak{n} \cap R \supset \mathfrak{q}_j$, for some $1 \leq j \leq s$; hence $\mathfrak{n} \cap R \supset \mathfrak{q}_1 \dots \mathfrak{q}_s$. Since B is a Jacobson ring (see [8, Thm. 4.19]), \mathfrak{p} is the intersection of all maximal ideals containing \mathfrak{p} , and consequently $\mathfrak{p} \cap R \supset \mathfrak{q}_1 \dots \mathfrak{q}_s$. Then $\mathfrak{p} \cap R \supset \mathfrak{q}_j$, for some j , because $\mathfrak{p} \cap R$ is prime. Therefore the zero-divisors from \mathfrak{q}_j do not vanish after localizing in \mathfrak{p} , and hence $M_{\mathfrak{p}}^{\otimes_R^n}$ has a zero-divisor in R . □

2. Analytic tensor product and fibred product. We briefly recall the concepts of analytic tensor product and fibred product of analytic spaces, which are used throughout the paper.

The analytic tensor product is defined in the category of finitely generated modules over local analytic \mathbb{C} -algebras (i.e., rings of the form $\mathbb{C}\{z_1, \dots, z_n\}/I$ for some ideal I) by the usual universal mapping property for tensor product (cf. [12]): Let $\varphi_i : R \rightarrow A_i$ ($i = 1, 2$) be homomorphisms of local analytic \mathbb{C} -algebras. Then there is a unique (up to isomorphism) local analytic \mathbb{C} -algebra $A_1 \tilde{\otimes}_R A_2$, together with homomorphisms $\theta_i : A_i \rightarrow A_1 \tilde{\otimes}_R A_2$ ($i = 1, 2$), such that (1) $\theta_1 \circ \varphi_1 = \theta_2 \circ \varphi_2$, and (2) for every pair of homomorphisms of local analytic \mathbb{C} -algebras $\psi_1 : A_1 \rightarrow B$, $\psi_2 : A_2 \rightarrow B$ satisfying $\psi_1 \circ \varphi_1 = \psi_2 \circ \varphi_2$, there is a unique homomorphism of local analytic \mathbb{C} -algebras $\psi : A_1 \tilde{\otimes}_R A_2 \rightarrow B$ making the associated diagram commute. The algebra $A_1 \tilde{\otimes}_R A_2$ is called the *analytic tensor product* of A_1 and A_2 over R .

For finite modules M_1 and M_2 over local analytic R -algebras A_1 and A_2 , respectively, there is a unique (up to isomorphism) finite $A_1 \tilde{\otimes}_R A_2$ -module $M_1 \tilde{\otimes}_R M_2$, together with an R -bilinear mapping $\rho : M_1 \times M_2 \rightarrow M_1 \tilde{\otimes}_R M_2$, such that for every R -bilinear $\kappa : M_1 \times M_2 \rightarrow N$, where N is a finite $A_1 \tilde{\otimes}_R A_2$ -module, there is a unique homomorphism of $A_1 \tilde{\otimes}_R A_2$ -modules $\lambda : M_1 \tilde{\otimes}_R M_2 \rightarrow N$ satisfying $\kappa = \lambda \circ \rho$. The module $M_1 \tilde{\otimes}_R M_2$ is called the *analytic tensor product* of M_1 and M_2 over R .

It is sometimes convenient to express the analytic tensor product of modules over a local analytic \mathbb{C} -algebra in terms of ordinary tensor product of certain naturally associated modules: given homomorphisms of local analytic \mathbb{C} -algebras $\varphi : R \rightarrow A_i$, and finitely generated A_i -modules M_i ($i = 1, 2$), the modules $M_1 \tilde{\otimes}_R A_2$ and $A_1 \tilde{\otimes}_R M_2$ are finitely generated over $A_1 \tilde{\otimes}_R A_2$, and there is a canonical isomorphism

$$M_1 \tilde{\otimes}_R M_2 \cong (M_1 \tilde{\otimes}_R A_2) \otimes_{A_1 \tilde{\otimes}_R A_2} (A_1 \tilde{\otimes}_R M_2).$$

In particular, if $A_1 = R\{x\}/I_1$ and $A_2 = R\{t\}/I_2$, where $x = (x_1, \dots, x_l)$, $t = (t_1, \dots, t_m)$ are systems of variables and $I_1 \subset R\{x\}$, $I_2 \subset R\{t\}$ are ideals, then

$$\begin{aligned} A_1 \tilde{\otimes}_R A_2 &\cong (A_1 \tilde{\otimes}_R R\{t\}) \otimes_{R\{x\} \tilde{\otimes}_R R\{t\}} (R\{x\} \tilde{\otimes}_R A_2) \\ &\cong (R\{x, t\}/I_1 R\{x, t\}) \otimes_{R\{x, t\}} (R\{x, t\}/I_2 R\{x, t\}) \\ &\cong R\{x, t\}/(I_1 R\{x, t\} + I_2 R\{x, t\}). \end{aligned}$$

The fibred product of analytic spaces is defined by a dual universal mapping property (see [9]): let $\varphi_i : X_i \rightarrow Y$ ($i = 1, 2$) denote holomorphic mappings of complex analytic spaces. Then there exists a unique (up to isomorphism) complex analytic space $X_1 \times_Y X_2$, together with holomorphic maps $\pi_i : X_1 \times_Y X_2 \rightarrow X_i$ ($i = 1, 2$), such that (1) $\varphi_1 \circ \pi_1 = \varphi_2 \circ \pi_2$, and (2) for every pair of holomorphic maps $\psi_1 : X \rightarrow X_1$, $\psi_2 : X \rightarrow X_2$ satisfying $\varphi_1 \circ \psi_1 = \varphi_2 \circ \psi_2$, there is a unique holomorphic map $\psi : X \rightarrow X_1 \times_Y X_2$ making the associated diagram commute. The space $X_1 \times_Y X_2$ is called the *fibred product* of X_1 and X_2 over Y (more precisely, *over* φ_1 and φ_2). There is a canonical holomorphic mapping $\varphi_1 \times_Y \varphi_2 : X_1 \times_Y X_2 \rightarrow Y$, given by $\varphi_1 \times_Y \varphi_2 = \varphi_i \circ \pi_i$ (where $i = 1$ or 2).

Given a holomorphic map $\varphi : X \rightarrow Y$ of complex analytic spaces, with $\varphi(\xi) = \eta$, let $\varphi_\xi : X_\xi \rightarrow Y_\eta$ denote the germ of φ at ξ . We denote by $\varphi^{\{d\}} : X^{\{d\}} \rightarrow Y$ the canonical map from the d -fold fibred power of X over Y to Y , and by $\varphi_{\xi^{\{d\}}}^{\{d\}} : X_{\xi^{\{d\}}}^{\{d\}} \rightarrow Y_\eta$ its germ at the point $\xi^{\{d\}} := (\xi, \dots, \xi) \in X^{\{d\}}$.

Suppose that $\varphi_1 : X_1 \rightarrow Y$ and $\varphi_2 : X_2 \rightarrow Y$ are holomorphic mappings of analytic spaces, with $\varphi_1(\xi_1) = \varphi_2(\xi_2) = \eta$. Then the local rings \mathcal{O}_{X_i, ξ_i} ($i = 1, 2$) are $\mathcal{O}_{Y, \eta}$ -modules and, by the uniqueness of fibred product and of analytic tensor product, the local ring $\mathcal{O}_{Z, (\xi_1, \xi_2)}$ of the fibred product $Z = X_1 \times_Y X_2$ at (ξ_1, ξ_2) is canonically isomorphic to $\mathcal{O}_{X_1, \xi_1} \tilde{\otimes}_{\mathcal{O}_{Y, \eta}} \mathcal{O}_{X_2, \xi_2}$. Therefore, given a holomorphic

germ $\varphi_\xi : X_\xi \rightarrow Y_\eta$, we will identify the d -fold analytic tensor power $\mathcal{O}_{X,\xi}^{\tilde{\otimes}^d \mathcal{O}_{Y,\eta}} = \mathcal{O}_{X,\xi} \tilde{\otimes} \mathcal{O}_{Y,\eta} \cdots \tilde{\otimes} \mathcal{O}_{Y,\eta} \mathcal{O}_{X,\xi}$ d times with the local ring of the d -fold fibred power $\mathcal{O}_{X^{(d)},\xi^{(d)}}$, for $d \geq 1$.

3. Homological properties of almost finitely generated modules. We first recall some homological properties of almost finitely generated modules, established by Galligo and Kwieciński [11], that generalize the corresponding properties of finite modules used by Auslander [5]. We then generalize a lemma of Auslander [5, Lemma 3.1] to almost finitely generated modules (Lemma 3.3 below).

Our proof of the main theorem 1.9 in the case that F is torsion-free over A (Case (1) in Section 5) follows the argument of [5]. Auslander’s main tools are two addition formulas: the Auslander-Buchsbaum formula (see [17, Ch. VII, Prop. 1.12]) and additivity of projective dimension [5, Cor. 1.3]. We replace these by tools adapted to almost finitely generated modules: an Auslander-Buchsbaum type formula for flat dimension (Proposition 3.1(2)) and additivity of flat dimension (Lemma 3.3).

Let $R = \mathbb{C}\{y_1, \dots, y_n\}$ denote a regular local analytic \mathbb{C} -algebra of dimension n . Let $\tilde{\otimes}_R$ denote the analytic tensor product over R , and let $\widetilde{\text{Tor}}^R$ be the corresponding derived functor.

Let F denote an almost finitely generated R -module. We define the *flat dimension* $\text{fd}_R(F)$ of F over R as the minimal length of a flat resolution of F (i.e., a resolution by R -flat modules). It is easy to see that

$$(3.1) \quad \text{fd}_R(F) = \max \{ i \in \mathbb{N} : \widetilde{\text{Tor}}_i^R(F, N) \neq 0 \text{ for some } N \}.$$

Indeed, if M is an almost finitely generated R -module, then

$$(3.2) \quad M \text{ is } R\text{-flat} \iff \widetilde{\text{Tor}}_1^R(M, R/\mathfrak{m}_R) = 0,$$

where \mathfrak{m}_R is the maximal ideal of R (cf. [15, Prop. 6.2]). Let (A, \mathfrak{m}_A) be a regular local R -algebra such that F is a finite A -module. Then (3.1) follows from (3.2) applied to the kernels of a minimal A -free (hence R -flat) resolution

$$\mathcal{F}_* : \cdots \xrightarrow{\alpha_{i+1}} F_{i+1} \xrightarrow{\alpha_i} F_i \xrightarrow{\alpha_{i-1}} \cdots \xrightarrow{\alpha_1} F_1 \xrightarrow{\alpha_0} F_0 \longrightarrow F$$

of F . (\mathcal{F}_* *minimal* means that $\alpha_i(F_{i+1}) \subset \mathfrak{m}_A F_i$, for all $i \in \mathbb{N}$).

The *depth* $\text{depth}_R(F)$ of F as an R -module is defined as the length of a maximal F -sequence in R (i.e., a sequence $a_1, \dots, a_s \in \mathfrak{m}_R$ such that a_j is not a zero-divisor in $F/(a_1, \dots, a_{j-1})F$, for $j = 1, \dots, s$). Since all the maximal F -sequences in R have the same length, depth is well defined: As observed in [11, Lemma 2.4], the classical proof of Northcott-Rees for finitely generated modules (see, e.g., [17,

Section VI, Prop. 3.1]), carries over to the case of almost finitely generated modules.

PROPOSITION 3.1. *Let M and N be almost finitely generated R -modules. Then the following properties hold.*

(1) Rigidity of $\widetilde{\text{Tor}}^R$ [11, Prop. 2.2(4)]. *If $\widetilde{\text{Tor}}_{i_0}^R(M, N) = 0$ for some $i_0 \in \mathbb{N}$, then $\widetilde{\text{Tor}}_i^R(M, N) = 0$ for all $i \geq i_0$.*

(2) Auslander-Buchsbaum-type formula [11, Thm. 2.7].

$$\text{fd}_R(M) + \text{depth}_R(M) = n.$$

(3) Additivity of flat dimension [11, Prop. 2.10]. *If $\widetilde{\text{Tor}}_i^R(M, N) = 0$ for all $i \geq 1$, then*

$$\text{fd}_R(M) + \text{fd}_R(N) = \text{fd}_R(M \widetilde{\otimes}_R N).$$

(4) Verticality of $\widetilde{\text{Tor}}^R$ (cf. [11, Prop. 4.5]). *For all $i \geq 1$, $\widetilde{\text{Tor}}_i^R(M, N)$ is an almost finitely generated R -module, and every element of $\widetilde{\text{Tor}}_i^R(M, N)$ is vertical over R (recall Definition 1.6).*

Remark 3.2. The analytic $\widetilde{\text{Tor}}_i^R$ need not be torsion R -modules, except in the case that M and N are finitely generated over R . (In this case, $\widetilde{\text{Tor}} = \text{Tor}$.) It seems to be unknown whether the $\widetilde{\text{Tor}}_i^R$ necessarily contain R -zero-divisors (cf. Question 1.12).

LEMMA 3.3. *Let $A = R\{x\}$ denote a regular local analytic R -algebra, $x = (x_1, \dots, x_m)$. Let F be a finitely generated A -torsion-free module, and let N be a module which is finitely generated over $B = A^{\widetilde{\otimes}_R j}$, for some $j \geq 1$. Suppose that $F \widetilde{\otimes}_R N$ has no vertical elements over R . Then:*

(1) N has no vertical elements over R ;

(2) $\widetilde{\text{Tor}}_i^R(F, N) = 0$, for all $i \geq 1$;

(3) $\text{fd}_R(F) + \text{fd}_R(N) = \text{fd}_R(F \widetilde{\otimes}_R N)$.

Proof. To prove (1), consider $N' = \{n \in N : n \text{ is vertical over } R\}$. It is easy to see that N' is a B -submodule of N . Indeed, if $n, n_1, n_2 \in N'$ and $b \in B$, then

$$\text{Ann}_B(n_1 + n_2) \supset \text{Ann}_B(n_1) \cdot \text{Ann}_B(n_2) \quad \text{and} \quad \text{Ann}_B(bn) \supset \text{Ann}_B(n);$$

hence the zero set germ $\mathcal{V}(\text{Ann}_B(n_1 + n_2))$ is mapped into the union of the (nowhere-dense) images of $\mathcal{V}(\text{Ann}_B(n_1))$ and $\mathcal{V}(\text{Ann}_B(n_2))$, and $\mathcal{V}(\text{Ann}_B(bn))$ is mapped into the image of $\mathcal{V}(\text{Ann}_B(n))$. Therefore, we get an exact sequence of B -modules,

$$0 \longrightarrow N' \longrightarrow N \longrightarrow N'' = N/N' \longrightarrow 0.$$

Tensoring with F induces a long exact sequence of $A \tilde{\otimes}_R B \cong A^{\tilde{\otimes}_R^{j+1}}$ -modules,

$$(3.3) \quad \begin{aligned} \cdots &\longrightarrow \widetilde{\mathrm{Tor}}_{i+1}^R(F, N') \longrightarrow \widetilde{\mathrm{Tor}}_{i+1}^R(F, N) \longrightarrow \widetilde{\mathrm{Tor}}_{i+1}^R(F, N'') \\ &\longrightarrow \widetilde{\mathrm{Tor}}_i^R(F, N') \longrightarrow \cdots \longrightarrow \widetilde{\mathrm{Tor}}_1^R(F, N'') \\ &\longrightarrow F \tilde{\otimes}_R N' \longrightarrow F \tilde{\otimes}_R N \longrightarrow F \tilde{\otimes}_R N'' \longrightarrow 0. \end{aligned}$$

Since every element of N' is vertical over R , the same is true for $F \tilde{\otimes}_R N'$ (indeed, $\mathrm{Ann}_{A \tilde{\otimes}_R B}(f \tilde{\otimes}_R n) \supset 1 \tilde{\otimes}_R \mathrm{Ann}_B(n)$ for all $f \in F$, $n \in N'$). But $F \tilde{\otimes}_R N$ has no vertical elements, by assumption, so that $F \tilde{\otimes}_R N' \rightarrow F \tilde{\otimes}_R N$ is the zero map; hence $F \tilde{\otimes}_R N \cong F \tilde{\otimes}_R N''$. In particular, $F \tilde{\otimes}_R N''$ has no vertical elements over R .

Since F is A -torsion-free, there is an injection of F into a finite free A -module L (obtained by composing the natural map $F \rightarrow (F^*)^*$, which is injective in this case, with the dual of a presentation of F^*). The exact sequence of A -modules $0 \rightarrow F \rightarrow L \rightarrow L/F \rightarrow 0$ induces a long exact sequence of $A \tilde{\otimes}_R B$ -modules,

$$\begin{aligned} \cdots &\longrightarrow \widetilde{\mathrm{Tor}}_{i+1}^R(L, N'') \longrightarrow \widetilde{\mathrm{Tor}}_{i+1}^R(L/F, N'') \longrightarrow \widetilde{\mathrm{Tor}}_i^R(F, N'') \\ &\longrightarrow \widetilde{\mathrm{Tor}}_i^R(L, N'') \longrightarrow \cdots \longrightarrow \widetilde{\mathrm{Tor}}_1^R(L, N'') \longrightarrow \widetilde{\mathrm{Tor}}_1^R(L/F, N'') \\ &\longrightarrow F \tilde{\otimes}_R N'' \longrightarrow L \tilde{\otimes}_R N'' \longrightarrow L/F \tilde{\otimes}_R N'' \longrightarrow 0. \end{aligned}$$

Since L is a free A -module and therefore R -flat, $\widetilde{\mathrm{Tor}}_i^R(L, N'') = 0$ for all $i \geq 1$, and we obtain isomorphisms

$$(3.4) \quad \widetilde{\mathrm{Tor}}_{i+1}^R(L/F, N'') \cong \widetilde{\mathrm{Tor}}_i^R(F, N''), \quad i \geq 1,$$

as well as injectivity of $\widetilde{\mathrm{Tor}}_1^R(L/F, N'') \rightarrow F \tilde{\otimes}_R N''$. But $F \tilde{\otimes}_R N''$ has no vertical elements, while every element of $\widetilde{\mathrm{Tor}}_1^R(L/F, N'')$ is vertical over R (by Prop. 3.1(4)); hence $\widetilde{\mathrm{Tor}}_1^R(L/F, N'') \rightarrow F \tilde{\otimes}_R N''$ is the zero map. Therefore, $\cdots \xrightarrow{0} \widetilde{\mathrm{Tor}}_1^R(L/F, N'') \xrightarrow{0} \cdots$ is exact; hence $\widetilde{\mathrm{Tor}}_1^R(L/F, N'') = 0$. By rigidity of $\widetilde{\mathrm{Tor}}^R$ (Prop. 3.1(1)), $\widetilde{\mathrm{Tor}}_{i+1}^R(L/F, N'') = 0$ for all $i \geq 1$, so by (3.4),

$$(3.5) \quad \widetilde{\mathrm{Tor}}_i^R(F, N'') = 0, \quad i \geq 1.$$

In particular, $\widetilde{\mathrm{Tor}}_1^R(F, N'') = 0$, hence, $\cdots \xrightarrow{0} F \tilde{\otimes}_R N' \xrightarrow{0} \cdots$ is exact, by (3.3), so that $F \tilde{\otimes}_R N' = 0$. However, $F \tilde{\otimes}_R N' \cong (F \tilde{\otimes}_R B) \otimes_{A \tilde{\otimes}_R B} (A \tilde{\otimes}_R N')$ is an (ordinary) tensor product of finitely generated modules over a regular local ring $A \tilde{\otimes}_R B$, so it is zero only if one of the factors is zero. We conclude that $A \tilde{\otimes}_R N' = 0$, and therefore $N' = 0$, by R -flatness of A . This proves assertion (1).

Now, $\widetilde{\mathrm{Tor}}_i^R(F, N') = 0$ for all $i \geq 0$; hence $\widetilde{\mathrm{Tor}}_i^R(F, N) \cong \widetilde{\mathrm{Tor}}_i^R(F, N'')$ for all $i \geq 1$, by (3.3). Therefore, $\widetilde{\mathrm{Tor}}_i^R(F, N) = 0$ for all $i \geq 1$, by (3.5), proving (2).

Assertion (3) follows from Proposition 3.1(3) and (2). \square

4. Vertical components and variation of fibre dimension. In this section, we describe a relationship between the filtration of the target of an analytic mapping $\varphi : X \rightarrow Y$ by fibre dimension and the isolated irreducible components of the n -fold fibred power $X^{\{n\}}$, where $n = \dim Y$.

Let $\varphi_\xi : X_\xi \rightarrow Y_\eta$ be a morphism of germs of analytic spaces, where Y_η is irreducible and of dimension n . Let Y be an irreducible representative of Y_η , and let X be a representative of X_ξ , such that the components of X are precisely the representatives in X of the components of X_ξ , and $\varphi(X) \subset Y$, where φ represents the germ φ_ξ . Let $\text{fbd}_x \varphi$ denote the fibre dimension $\dim_x \varphi^{-1}(\varphi(x))$ of φ at a point $x \in X$.

We will use the following notation in this section: $l := \min\{\text{fbd}_x \varphi : x \in X\}$, $k := \max\{\text{fbd}_x \varphi : x \in X\}$, and $A_j := \{x \in X : \text{fbd}_x \varphi \geq j\}$, $l \leq j \leq k$. Then $X = A_l \supset A_{l+1} \supset \dots \supset A_k$ and, by upper-semicontinuity of fibre dimension (see Cartan-Remmert theorem [20, Section V.3.3, Thm. 5]), the A_j are analytic in X . Define $B_j := f(A_j) = \{y \in Y : \dim \varphi^{-1}(y) \geq j\}$, $l \leq j \leq k$. Upper-semicontinuity of $\text{fbd}_x \varphi$ (as a function of x) implies that the germs $(A_j)_\xi$ and $(B_j)_\eta$ are independent of the choices of representatives made above.

Note that, except for B_k (cf. proof of Proposition 4.2 below), the B_j may not even be semianalytic in general. This fact is responsible for a complicated relationship between the algebraic vertical and geometric vertical components in the fibred powers of X over Y (see [2] for a detailed discussion), but will not affect our considerations here, which rely only on the properties of B_k .

PROPOSITION 4.1. [1, Prop. 2.1] *Under the assumptions above, let $\bigcup_{i \in I} W_i$ denote the decomposition of $(X^{\{n\}})_{\text{red}}$ into finitely many isolated irreducible components through $\xi^{\{n\}}$. Then:*

- (1) *For each $j = l, \dots, k$, there is an index subset $I_j \subset I$ such that*

$$B_j = \bigcup_{i \in I_j} \varphi^{\{n\}}(W_i).$$

- (2) *Let $y \in B_j$ and let $s = \dim \varphi^{-1}(y)$ ($s \geq j$). If Z is an isolated irreducible component of the fibre $(\varphi^{\{n\}})^{-1}(y)$, of dimension ns , and W is an irreducible component of $X^{\{n\}}$ containing Z , then $\varphi^{\{n\}}(W) \subset B_j$.*

Proof. For (2), fix $j \geq l + 1$. (The statement is trivial for $j = l$, since $B_l = \varphi(X)$.) Suppose that there exists $x = (x_1, \dots, x_n) \in W$ such that $\varphi(x_1) \in Y \setminus B_j$ (and hence $\varphi(x_i) \in Y \setminus B_j$, $i \leq n$). Then $\text{fbd}_{x_i} \varphi \leq j - 1$, $i = 1, \dots, n$; hence $\text{fbd}_x \varphi^{\{n\}} \leq n(j - 1) = nj - n$. In particular, the generic fibre dimension of $\varphi^{\{n\}}|_W$ is at most $nj - n$. Since $\text{rank}(\varphi^{\{n\}}|_W) \leq \dim Y = n$, then $\dim W \leq (nj - n) + n = nj$ (see, e.g., [20, Section V.3]).

Now we have $W \supset Z$, $\dim W \leq nj$, $\dim Z = ns \geq nj$, and both W and Z are irreducible analytic sets in $X^{\{n\}}$. This is possible only if $W = Z$; hence $\varphi^{\{n\}}(W) = \varphi^{\{n\}}(Z) = \{y\} \subset B_j$; a contradiction. Therefore $\varphi^{\{n\}}(W) \subset B_j$, completing the proof of (2).

Part (1) follows immediately, since if $y \in B_j$ and Z is an irreducible component of $(\varphi^{\{n\}})^{-1}(y)$ of the highest dimension, then there exists an isolated irreducible component W of $X^{\{n\}}$ that contains Z . □

The following is a simplified variant of an openness criterion of [1, Thm. 2.2], proved here under somewhat weaker assumptions.

PROPOSITION 4.2. *Let $\varphi_\xi : X_\xi \rightarrow Y_\eta$ be a morphism of germs of analytic spaces. Suppose that Y_η is irreducible, $\dim Y_\eta = n$, $\dim X_\xi = m$, and the maximal fibre dimension of φ_ξ is not generic on some m -dimensional irreducible component of X_ξ . Then the n -fold fibred power $\varphi_{\xi^{\{n\}}}^{\{n\}} : X_{\xi^{\{n\}}}^{\{n\}} \rightarrow Y_\eta$ contains an isolated algebraic vertical component.*

Proof. As above, let $\varphi : X \rightarrow Y$ be a representative of φ_ξ , where Y is irreducible and of dimension n . Let $k := \max\{\text{fbd}_x \varphi : x \in X\}$, $A_k := \{x \in X : \text{fbd}_x \varphi = k\}$, and $B_k := \varphi(A_k) = \{y \in Y : \dim \varphi^{-1}(y) = k\}$. Then the fibre dimension of φ is constant on the analytic set A_k . By the Remmert Rank theorem (see [20, Section V.6, Thm. 1]), B_k is locally analytic in Y , of dimension $\dim A_k - k \leq \dim X - k$. Since $\eta \in B_k$, after shrinking Y if necessary, we can assume that B_k is an analytic subset of Y . Therefore, by Proposition 4.1, it is enough to show that the analytic germ $(B_k)_\eta$ is a proper subgerm of Y_η . Let U be an isolated irreducible component of X , of dimension $m = \dim X$, and such that k is not the generic fibre dimension of $\varphi|_U$. It follows that

$$\dim Y \geq \dim U - \text{generic fbd } \varphi|_U \geq m - k + 1.$$

Then $\dim B_k \leq m - k < \dim Y$; hence $\dim(B_k)_\eta < \dim Y = \dim Y_\eta$, so that $(B_k)_\eta \subsetneq Y_\eta$. □

5. Proof of the main theorem. Let F be an almost finitely generated module over $R := \mathbb{C}\{y_1, \dots, y_n\}$. By Remark 1.8, there exists $m \geq 0$ such that F is finitely generated as a module over $A = R\{x\}$, where $x = (x_1, \dots, x_m)$. Let X and Y be connected open neighborhoods of the origins in \mathbb{C}^{m+n} and \mathbb{C}^n (respectively), and let $\varphi : X \rightarrow Y$ be the canonical coordinate projection. Let \mathcal{F} be a coherent sheaf of \mathcal{O}_X -modules whose stalk at the origin in X equals F . We can identify R with $\mathcal{O}_{Y,0}$ and A with $\mathcal{O}_{X,0}$. Then F is R -flat if and only if \mathcal{F}_0 is $\mathcal{O}_{Y,0}$ -flat.

The “only if” direction of Theorem 1.9 is easy to establish (see Section 5.1). Our proof of the more difficult “if” direction will be divided into three cases according to the following plan. We consider a short exact sequence $0 \rightarrow K \rightarrow F \rightarrow N \rightarrow 0$, where K is the A -torsion submodule of F and N is A -torsion-free. Then

N can be treated by Auslander’s techniques (as extended in Section 3); see Case (1) of the proof below. The A -torsion module K is supported over a subgerm of X_0 of strictly smaller fibre dimension over Y_0 , so it can be treated by induction (Case (2)). For the general case (3), we want to show that the analytic tensor powers of either N or K embed into the corresponding powers of F , and hence so do their R -vertical elements.

The latter would be automatic if F were the *direct sum* of K and N . This is not true, in general. We can, however, choose N to be a submodule of F of the form $g \cdot F$, for a suitable choice of $g \in A$. It follows that the analytic tensor powers of K embed into those of F , unless gF is not R -flat. In the latter case, in turn, we show that the analytic tensor powers of gF embed into those of F .

The following lemma will be used in Case (3) of the proof of Theorem 1.9 below. It will allow us to conclude that the analytic tensor powers of gF embed into the corresponding powers of F , provided gF is A -torsion-free.

LEMMA 5.1. *Let $R = \mathbb{C}\{y_1, \dots, y_n\}$, and let A and B be regular local analytic R -algebras. Suppose that M and N are finite A - and B -modules (respectively). Let $g \in A$, $h \in B$, and $m \in gM \tilde{\otimes}_R hN$ all be nonzero elements. If $m = 0$ as an element of $M \tilde{\otimes}_R N$, then $(g \tilde{\otimes}_R h) \cdot m = 0$ in $gM \tilde{\otimes}_R hN$. In other words, if $g \tilde{\otimes}_R h$ is not a zero-divisor of $gM \tilde{\otimes}_R hN$, then the canonical homomorphism $gM \tilde{\otimes}_R hN \rightarrow M \tilde{\otimes}_R N$ is an embedding.*

Proof. Using the identification

$$gM \tilde{\otimes}_R hN \cong (gM \tilde{\otimes}_R B) \otimes_{A \tilde{\otimes}_R B} (A \tilde{\otimes}_R hN),$$

we can write $m = \sum_{i=1}^k m_i \otimes n_i$, where the $m_i \in gM \tilde{\otimes}_R B$, and n_1, \dots, n_k generate $A \tilde{\otimes}_R hN$. The latter can be extended to a sequence $n_1, \dots, n_k, n_{k+1}, \dots, n_t$ generating $A \tilde{\otimes}_R N$. Setting $m_{k+1} = \dots = m_t = 0$, we get $m = \sum_{i=1}^t m_i \otimes n_i \in (M \tilde{\otimes}_R B) \otimes_{A \tilde{\otimes}_R B} (A \tilde{\otimes}_R N)$. By [8, Lemma 6.4], $m = 0$ in $M \tilde{\otimes}_R N$ if and only if there are $m'_1, \dots, m'_s \in M \tilde{\otimes}_R B$ and $a_{ij} \in A \tilde{\otimes}_R B$, such that

$$(5.1) \quad \sum_{j=1}^s a_{ij} m'_j = m_i \text{ in } M \tilde{\otimes}_R B, \quad \text{for all } i;$$

$$(5.2) \quad \sum_{i=1}^t a_{ij} n_i = 0 \text{ in } A \tilde{\otimes}_R N, \quad \text{for all } j.$$

Multiplying the equations (5.1) by $g \tilde{\otimes}_R 1$, we get

$$(5.3) \quad \sum_{j=1}^s a_{ij} (g \tilde{\otimes}_R 1) m'_j = (g \tilde{\otimes}_R 1) m_i \text{ in } gM \tilde{\otimes}_R B, \quad \text{for all } i;$$

hence $(g \tilde{\otimes}_R 1) m = 0$ in $gM \tilde{\otimes}_R hN$, by (5.2) and (5.3).

Now write $(g\tilde{\otimes}_R 1)m = \sum_{i=1}^l m_i \otimes n_i$, where the $n_i \in A\tilde{\otimes}_R N$, and m_1, \dots, m_l generate $gM\tilde{\otimes}_R B$. Then $(g\tilde{\otimes}_R 1)m = 0$ in $(gM\tilde{\otimes}_R B) \otimes_{A\tilde{\otimes}_R B} (A\tilde{\otimes}_R N)$ if and only if there are $n'_1, \dots, n'_p \in A\tilde{\otimes}_R N$ and $b_{ij} \in A\tilde{\otimes}_R B$, such that

$$(5.4) \quad \sum_{j=1}^p b_{ij}n'_j = n_i \text{ in } A\tilde{\otimes}_R N, \quad \text{for all } i;$$

$$(5.5) \quad \sum_{i=1}^l b_{ij}m_i = 0 \text{ in } gM\tilde{\otimes}_R B, \quad \text{for all } j.$$

Multiplying the equations (5.4) by $1\tilde{\otimes}_R h$, we get

$$(5.6) \quad \sum_{j=1}^p b_{ij}(1\tilde{\otimes}_R h)n'_j = (1\tilde{\otimes}_R h)n_i \text{ in } A\tilde{\otimes}_R hN, \quad \text{for all } i;$$

hence $(g\tilde{\otimes}_R h)m = 0$ in $gM\tilde{\otimes}_R hN$, by (5.5) and (5.6). Thus $g\tilde{\otimes}_R h$ is a zero-divisor of $gM\tilde{\otimes}_R hN$, as required. \square

5.1. Proof of Theorem 1.9. Let F be an almost finitely generated module over $R := \mathbb{C}\{y_1, \dots, y_n\}$. By Remark 1.8, there exists $m \geq 0$ such that F is finitely generated as a module over $A := R\{x\} = R\{x_1, \dots, x_m\}$. Let X and Y be connected open neighborhoods of the origins in \mathbb{C}^{m+n} and \mathbb{C}^n (respectively), and let $\varphi : X \rightarrow Y$ be the canonical coordinate projection. Let \mathcal{F} be a coherent sheaf of \mathcal{O}_X -modules whose stalk at the origin in X equals F , and let \mathcal{G} be a coherent $\mathcal{O}_{X^{\{n\}}}$ -module whose stalk at the origin $0^{\{n\}}$ in $X^{\{n\}}$ equals $F^{\hat{\otimes}_R^n}$. We can identify R with $\mathcal{O}_{Y,0}$ and A with $\mathcal{O}_{X,0}$. Then F is R -flat if and only if \mathcal{F}_0 is $\mathcal{O}_{Y,0}$ -flat.

We first prove the “only if” direction of Theorem 1.9, by contradiction. Assume that F is R -flat. Since flatness is an open condition, by Douady’s theorem [7], we can assume that \mathcal{F} and \mathcal{G} are \mathcal{O}_Y -flat. Suppose that $F^{\hat{\otimes}_R^n}$ has a vertical element over $\mathcal{O}_{Y,0}$. In other words, (after shrinking X and Y if necessary) there exist a nonzero section $\tilde{m} \in \mathcal{G}$ and an analytic subset $Z \subset X^{\{n\}}$, such that $Z_0 = \mathcal{V}(\text{Ann}_{\mathcal{O}_{X^{\{n\}},0^{\{n\}}}}(\tilde{m}_0))$ and the image $\varphi^{\{n\}}(Z)$ has empty interior in Y . Let $\tilde{\varphi}$ denote the restriction $\varphi^{\{n\}}|_Z : Z \rightarrow Y$. Consider $\xi \in Z$ such that the fibre dimension of $\tilde{\varphi}$ at ξ is minimal. Then the fibre dimension $\text{fbd}_x \tilde{\varphi}$ is constant on some open neighborhood U of ξ in Z . By the Remmert Rank theorem, $\tilde{\varphi}(U)$ is locally analytic in Y near $\eta = \tilde{\varphi}(\xi)$. Since $\tilde{\varphi}(Z)$ has empty interior in Y , it follows that there is a holomorphic function g in a neighborhood of η in Y , such that $(\tilde{\varphi}(U))_\eta \subset \mathcal{V}(g_\eta)$. Therefore, $\tilde{\varphi}_\xi^*(g_\eta) \cdot \tilde{m}_\xi = 0$ in \mathcal{G}_ξ ; i.e., \mathcal{G}_ξ has a (nonzero) zero-divisor in $\mathcal{O}_{Y,\eta}$, contradicting flatness.

We will now prove the more difficult “if” direction of the theorem, by induction on m . If $m = 0$, then F is finitely generated over R , and the result follows from Auslander’s theorem 1.2 (because flatness of finitely generated modules over a

local ring is equivalent to freeness, the analytic tensor product equals the ordinary tensor product for finite modules, and vertical elements in finite modules are just zero-divisors).

The inductive step will be divided into three cases:

- (1) F is torsion-free over A ;
- (2) F is a torsion A -module;
- (3) F is neither A -torsion-free nor a torsion A -module.

Case 1. We prove this case independently of the inductive hypothesis. We essentially repeat the argument of Galligo and Kwieciński [11], which itself is an adaptation of Auslander [5] to the almost finitely generated context.

Suppose that $F^{\tilde{\otimes}_R^n}$ has no vertical elements over R . Then it follows from Lemma 3.3(1) that $F^{\tilde{\otimes}_R^i}$ has no vertical elements, for $i = 1, \dots, n$. By Lemma 3.3(3),

$$\text{fd}_R(F^{\tilde{\otimes}_R^n}) = \text{fd}_R(F) + \text{fd}_R(F^{\tilde{\otimes}_R^{n-1}}) = \dots = n \cdot \text{fd}_R(F).$$

On the other hand, since $F^{\tilde{\otimes}_R^n}$ has no vertical elements over R , it has no zero-divisors over R , so that $\text{depth}_R(F^{\tilde{\otimes}_R^n}) \geq 1$. It follows from Proposition 3.1(2) that $\text{fd}_R(F^{\tilde{\otimes}_R^n}) < n$. Hence $n \cdot \text{fd}_R(F) < n$. This is possible only if $\text{fd}_R(F) = 0$; i.e., F is R -flat.

Case 2. Suppose that F is not R -flat and a torsion A -module. We will show that then $F^{\tilde{\otimes}_R^n}$ contains vertical elements over R . Let $I = \text{Ann}_A(F)$. Since every element of F is annihilated by some nonzero element of A , and F is finitely generated over A , then I is a nonzero ideal in A . Put $B = A/I$; then F is finitely generated over B . Let $I(0)$ denote the *evaluation* of I at $y = 0$ (i.e., $I(0)$ is the ideal generated by I in $A(0) := A \tilde{\otimes}_R R/\mathfrak{m}_R \cong \mathbb{C}\{x_1, \dots, x_m\}$).

First suppose that $I(0) \neq (0)$. Then there exists $g \in I$ such that $g(0) := g(0, x) \neq 0$, and F is a finite $A/(g)A$ -module. It follows that (after an appropriate linear change in the x -coordinates) g is regular in x_m and hence, by the Weierstrass Preparation Theorem, that F is finite over $R\{x_1, \dots, x_{m-1}\}$. Therefore, $F^{\tilde{\otimes}_R^n}$ has a vertical element over R , by the inductive hypothesis.

On the other hand, suppose that $I(0) = (0)$; i.e., $I \subset \mathfrak{m}_R A$. Then $B \tilde{\otimes}_R R/\mathfrak{m}_R = (A/I) \tilde{\otimes}_R R/\mathfrak{m}_R$ equals $\mathbb{C}\{x_1, \dots, x_m\}$. Let Z be a closed analytic subspace of X such that $\mathcal{O}_{Z,0} \cong B$, and let $\tilde{\varphi} := \varphi|_Z$. It follows that the fibre $\tilde{\varphi}^{-1}(0)$ equals \mathbb{C}^m . Of course, m is not the generic fibre dimension of $\tilde{\varphi}$ on any irreducible component of Z , because otherwise all its fibres would equal \mathbb{C}^m , so we would have $B = A$ and $I = (0)$, contrary to the choice of I . Therefore, by Proposition 4.2, there is an isolated algebraic vertical component in the n -fold fibred power of $\tilde{\varphi}_0$; i.e., $B^{\tilde{\otimes}_R^n}$ has a zero-divisor in R . But $F^{\tilde{\otimes}_R^n}$ is a faithful $B^{\tilde{\otimes}_R^n}$ -module, so itself it has a zero-divisor (hence a vertical element) over R .

Case 3. Suppose that F is not R -flat, F has zero-divisors in A , but $\text{Ann}_A(F) = (0)$. Let

$$K := \{f \in F : af = 0 \text{ for some nonzero } a \in A\};$$

i.e., K is the A -torsion submodule of F . Since K is a submodule of a finitely generated module over a Noetherian ring, K is finitely generated; say $K = \sum_{i=1}^s A \cdot f_i$. Take $a_i \in A \setminus \{0\}$ such that $a_i f_i = 0$, and put $g = a_1 \cdots a_s$. Then the sequence of A -modules

$$(5.7) \quad 0 \longrightarrow K \longrightarrow F \xrightarrow{g} gF \longrightarrow 0$$

is exact, and gF is a torsion-free A -module.

First suppose that gF is R -flat. Then by applying $\tilde{\otimes}_R K$ and $F \tilde{\otimes}_R$ to (5.7), we get short exact sequences

$$\begin{aligned} 0 &\longrightarrow K \tilde{\otimes}_R K \longrightarrow F \tilde{\otimes}_R K \longrightarrow gF \tilde{\otimes}_R K \longrightarrow 0, \\ 0 &\longrightarrow F \tilde{\otimes}_R K \longrightarrow F \tilde{\otimes}_R F \longrightarrow F \tilde{\otimes}_R gF \longrightarrow 0. \end{aligned}$$

So we have injections

$$K \tilde{\otimes}_R K \hookrightarrow F \tilde{\otimes}_R K \hookrightarrow F \tilde{\otimes}_R F,$$

and by induction, an injection $K \tilde{\otimes}_R^i \hookrightarrow F \tilde{\otimes}_R^i$, for all $i \geq 1$. In particular, $K \tilde{\otimes}_R^n$ is a submodule of $F \tilde{\otimes}_R^n$. Since gF is R -flat and F is not R -flat, it follows that K is not R -flat. Therefore, by Case (2), $K \tilde{\otimes}_R^n$ (and hence $F \tilde{\otimes}_R^n$) has a vertical element over R .

Now suppose that gF is not R -flat. Then $(gF) \tilde{\otimes}_R^n$ has a vertical element over R , by Case (1). We will show that $(gF) \tilde{\otimes}_R^n$ embeds into $F \tilde{\otimes}_R^n$, and hence so do its vertical elements. By Lemma 5.1, in order for $(gF) \tilde{\otimes}_R^n$ to embed into $F \tilde{\otimes}_R^n$, it suffices to prove that $g \tilde{\otimes}_R^n$ is not a zero-divisor of $(gF) \tilde{\otimes}_R^n$.

To simplify the notation, let B denote the ring $A \tilde{\otimes}_R^n$, and let $h := g \tilde{\otimes}_R^n \in B$. Since $(gF) \tilde{\otimes}_R^n$ is a finite B -module, we can write $(gF) \tilde{\otimes}_R^n = B^q/M$, where $q \geq 1$ and M is a B -submodule of B^q . Given $b \in B$, let $(M : b)$ denote the B -submodule of B^q consisting of those elements $m \in B^q$ for which $b \cdot m \in M$. Since

$$(M : h) \subset (M : h^2) \subset \cdots \subset (M : h^l) \subset \cdots$$

is an increasing sequence of submodules of a Noetherian module B^q , it stabilizes; i.e., there exists $k \geq 1$ such that $(M : h^{k+1}) = (M : h^k)$. In other words, there exists $k \geq 1$ such that h is not a zero-divisor in $h^k \cdot B^q/M$; i.e., $g \tilde{\otimes}_R^n$ is not a zero-divisor in $(g^{k+1}F) \tilde{\otimes}_R^n$.

Observe though that (because gF is A -torsion-free), multiplication by g induces an isomorphism $gF \rightarrow g^2F$ of A -modules, and in general, $gF \cong g^lF$, for $l \geq 1$. We thus have isomorphisms $(gF) \tilde{\otimes}_R^n \cong (g^lF) \tilde{\otimes}_R^n$ of B -modules, for $l \geq 1$.

In particular, for every $l \geq 1$, $g^{\otimes_R^n}$ is a zero-divisor of $(gF)^{\otimes_R^n}$ if and only if it is a zero-divisor of $(g^l F)^{\otimes_R^n}$. Therefore, by Lemma 5.1, we have an embedding $(gF)^{\otimes_R^n} \hookrightarrow F^{\otimes_R^n}$. This completes the proof of Theorem 1.9.

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