ON ARC-ANALYTIC FUNCTIONS AND ARC-SYMMETRIC SETS

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In this note, we will mostly deal with semialgebraic geometry, that is, the study of real solutions of systems of polynomial equations and inequalities. A semialgebraic set $E$ in $\mathbb{R}^n$ is a finite union of sets of the form

$$\{x \in \mathbb{R}^n : f(x) = 0, g_1(x) > 0, \ldots, g_s(x) > 0\},$$

where $s \in \mathbb{N}$ and $f, g_1, \ldots, g_s$ are polynomials in real variables $x = (x_1, \ldots, x_n)$. A function $f : E \to \mathbb{R}$ is called semialgebraic if its graph $\Gamma_f$ is a semialgebraic subset of $\mathbb{R}^n \times \mathbb{R}$. Given an open semialgebraic $U \subset \mathbb{R}^n$, a real analytic semialgebraic function $f : U \to \mathbb{R}$ is called Nash.

Our main object of interest here are the so called arc-analytic functions. A function $f : S \to \mathbb{R}$ on a set $S \subset \mathbb{R}^n$ is said to be arc-analytic when $f \circ \gamma$ is analytic for every real analytic arc $\gamma : (-\varepsilon, \varepsilon) \to S$.

Arc-analytic functions, although relatively unknown among non-specialists, play an important role in modern real algebraic and analytic geometry (see, e.g., [10] and the references therein). Indeed, Bierstone and Milman [3] proved that arc-analytic semialgebraic functions on a Nash manifold are precisely those that can be made Nash after composition with a finite sequence of blowings-up with smooth algebraic nowhere dense centres. In fact, this criterion is often the quickest way to determine arc-analyticity of a given function. Many classical examples in calculus are arc-analytic but not analytic.

**Example 1.** (a) The function $f : \mathbb{R}^2 \to \mathbb{R}$ defined as $f(x, y) = x^3/(x^2 + y^2)$ for $(x, y) \neq (0, 0)$ and $f(0,0) = 0$ is arc-analytic but not differentiable at the origin. Observe that $f$ is made Nash after composition with a single blowing-up of the origin; for instance, $f(x, xy) = x/(1 + y^2)$. Note also that the graph $\Gamma_f$ of $f$ is not real analytic. In fact, the smallest real analytic subset of $\mathbb{R}^3$ containing $\Gamma_f$ is the Cartan umbrella $\{(x, y, z) \in \mathbb{R}^3 : z(x^2 + y^2) = x^3\}$ (cf. [9, Ex. 1.2(1)]).

(b) The function $g : \mathbb{R}^2 \to \mathbb{R}$ defined as $g(x, y) = \sqrt{x^4 + y^4}$ is arc-analytic but not $C^2$. The graph $\Gamma_g$ of $g$ is not real analytic. Indeed, the Zariski closure $\{(x, y, z) \in \mathbb{R}^3 : z^2 = x^4 + y^4\}$ of $\Gamma_g$ has two $C^1$ sheets $z = \pm \sqrt{x^4 + y^4}$, but it is irreducible at the origin as a real analytic set (cf. [3, Ex. 1.2(3)]).

In general, the behaviour of arc-analytic functions may be surprising, if not pathological. For example, in [4] the authors construct an arc-analytic function $f : \mathbb{R}^2 \to \mathbb{R}$ which is not even continuous. However, in the semialgebraic setting, arc-analytic functions form a very nice family.

Arc-analytic functions were first considered by Kurdyka [9] on arc-symmetric semialgebraic sets. A set $E$ in $\mathbb{R}^n$ is called arc-symmetric when, for every analytic arc $\gamma : (-1, 1) \to \mathbb{R}^n$ with $\gamma((-1, 0)) \subset E$, one has $\gamma((-1,1)) \subset E$. By a fundamental theorem [9, Thm. 1.4], the arc-symmetric semialgebraic sets are precisely
the closed sets of a certain noetherian topology on \( \mathbb{R}^n \). (A topology is called noetherian when every descending sequence of its closed sets is stationary.) Following [9], we will call it the \( \mathcal{A} \mathcal{R} \) topology, and the arc-symmetric semialgebraic sets will henceforth be called \( \mathcal{A} \mathcal{R} \) closed sets.

Given an \( \mathcal{A} \mathcal{R} \)-closed set \( X \) in \( \mathbb{R}^n \), we will denote by \( \mathcal{A}_a(X) \) the ring of arc-analytic semialgebraic functions on \( X \). By [9, Prop. 5.1], the zero locus of every \( f \in \mathcal{A}_a(X) \) is \( \mathcal{A} \mathcal{R} \)-closed. Interestingly, despite noetherianity of the \( \mathcal{A} \mathcal{R} \) topology, the ring \( \mathcal{A}_a(\mathbb{R}^n) \) is not noetherian (see [9, Ex. 6.11]).

The usefulness of \( \mathcal{A} \mathcal{R} \) topology comes from the fact that it contains and is strictly finer than the Zariski topology on \( \mathbb{R}^n \). Moreover, it follows from the semialgebraic Curve Selection Lemma that \( \mathcal{A} \mathcal{R} \)-closed sets are closed in the Euclidean topology in \( \mathbb{R}^n \).

Noetherianity of the \( \mathcal{A} \mathcal{R} \) topology allows one to make sense of the notions of irreducibility and components of a semialgebraic set much like in the algebraic case: An \( \mathcal{A} \mathcal{R} \)-closed set \( X \) is called \( \mathcal{A} \mathcal{R} \)-irreducible if it cannot be written as a union of two proper \( \mathcal{A} \mathcal{R} \)-closed subsets. Every \( \mathcal{A} \mathcal{R} \)-closed set admits a unique decomposition \( X = X_1 \cup \cdots \cup X_r \) into \( \mathcal{A} \mathcal{R} \)-irreducible sets satisfying \( X_i \not\subseteq \bigcup_{j \neq i} X_j \) for each \( i = 1, \ldots, r \), the sets \( X_1, \ldots, X_r \) are called the \( \mathcal{A} \mathcal{R} \)-components of \( X \). The decomposition into \( \mathcal{A} \mathcal{R} \)-components is finer than that into algebraic or Nash components and encodes more algebro-differential information (see [11]). In particular, by a beautiful characterisation of Kurdyka, there is a one-to-one correspondence between the \( \mathcal{A} \mathcal{R} \)-components of \( X \) of maximal dimension and the connected components of a desingularization of the Zariski closure of \( X \).

Desingularization arguments play a very important role in the study of arc-symmetry and arc-analyticity. Together with H. Seyedinejad [1], we used them recently to prove that every \( \mathcal{A} \mathcal{R} \)-closed set \( X \) in \( \mathbb{R}^n \) is precisely the zero locus of a certain arc-analytic function \( f \in \mathcal{A}_a(\mathbb{R}^n) \). It thus follows that the \( \mathcal{A} \mathcal{R} \) topology coincides with the one defined by the vanishing of semialgebraic arc-analytic functions, which is not at all apparent from the intrinsic definition above.

Extending the techniques of [1], most recently we also proved in [2] an arc-analytic analogue of Efroymson’s extension theorem [5]: Every arc-analytic semialgebraic function \( f : X \to \mathbb{R} \) on an \( \mathcal{A} \mathcal{R} \)-closed set \( X \subset \mathbb{R}^n \) is, in fact, a restriction of an arc-analytic function \( F \in \mathcal{A}_a(\mathbb{R}^n) \). Moreover, the function \( F \) may be chosen real analytic outside the Zariski closure of \( X \). This result is particularly interesting in the context of the so-called continuous rational functions, which form one of the most active research areas in contemporary real algebraic geometry (see, e.g., [7] and the references therein). A continuous function \( f \) is called continuous rational if it is generically of the form \( \frac{p}{q} \), with \( p \) and \( q \) polynomial. Continuous rational functions on an \( \mathcal{A} \mathcal{R} \)-closed set \( X \) form a subring of \( \mathcal{A}_a(X) \), and the following example of Kollar-Nowak [8] shows that not every continuous rational function on an \( \mathcal{A} \mathcal{R} \)-closed set admits an extension to the ambient space as a continuous rational function. Nonetheless, by [2], it does admit an extension as an arc-analytic one.

**Example 2.** The function \( f(x, y, z) = \sqrt[3]{1 + z^2} \) is continuous rational on the real algebraic surface \( S = \{(x, y, z) \in \mathbb{R}^3 : x^3 = (1 + z^2)y^3\} \), since \( f|_S \) coincides with \( \frac{x}{y}|_S \), but it has no continuous rational extension to \( \mathbb{R}^3 \) (see [8, Ex. 2]). Note that \( f \) is Nash, and hence arc-analytic, on \( \mathbb{R}^3 \).
References


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