SIAM J. MATH. ANAL.
Vol. 57, No. 3, pp. 2880–2910

SPATIAL TEMPORAL DYNAMICS OF NICHOLSON BLOWFLY EQUATION WITH TWO SHIFTING PARAMETERS*

TAISHAN YI†, YURONG ZHANG‡, AND XINGFU ZOU§

Abstract. In this paper, we obtain new results on the spreading speed and asymptotic propagation properties for a diffusive Nicholson's equation with two shifting parameters. By the method of iterative strategies of travelling wave maps, we give a priori estimates on nontrivial solutions which play a key role in the proof of global asymptotic behavior. These estimates in combination with proper test functions allow us to establish the spatial-temporal propagation dynamics of this extended Nicholson's equation. These results enable us to develop a unified method for exploring the spreading speeds and asymptotic propagation phenomena for a class of nontranslation invariant delay reaction-diffusion equations on $\mathbb{R}^N$.

Key words. Nicholson blowfly equation, asymptotic propagation, spreading speed, shifting environment, travelling wave map

MSC codes. 35K57, 45M05, 92D25

DOI. 10.1137/23M1562846

1. Introduction.

The mature population of a species provided with (i) an age structure characterized by immature and mature and (ii) a spatial structure reflected by spatial variable(s) with homogeneous environment, can be described by a reaction diffusion equation with delay:

$$\frac{\partial u}{\partial t}(t,x) = d\Delta u(t,x) - \mu u(t,x) + e^{-\delta\tau}B(u(t-\tau,x)),\ \ x\in\Omega\subset\mathbb{R}^n. \tag{1.1}$$

Here $u(t,x)$ denotes the mature population at location x and time t; and for the *constant parameters,* d stands for the diffusion rate of the matured individuals, μ and δ are the death rates of the mature and immature populations respectively, the delay $\tau > 0$ is the average maturation time, and $B(u)$ is a birth rate function. In (1.1), we have implicitly assumed that the immature individuals do not diffuse and thus, a new born individual, if survived immature period, remains in the same location when becoming mature. Equations of the form (1.1) have been widely studied for various birth functions with both bounded or unbounded domain Ω. See, e.g., [7, 11, 21, 28, 34, 35, 45, 50] and the references therein. Particularly, when the birth function $B(u)$ is the Ricker function $B(u) = bue^{-u}$, then (1.1) reduces to the well-known and well-studied diffusive Nicholson blowfly equation (see, e.g., [27, 28, 29, 30, 36, 38, 44, 45, 48]):

*Received by the editors March 31, 2023; accepted for publication (in revised form) February 13, 2025; published electronically June 3, 2025.
https://doi.org/10.1137/23M1562846
Funding: Research was supported partially by the National Natural Science Foundation of People's Republic of China (grant 12231008) and by NSERC of Canada (grant RGPIN-2022-04744).
†School of Mathematics (Zhuhai), Sun Yat-Sen University, Zhuhai, Guangdong 519082, People's Republic of China (yitaishan@mail.sysu.edu.cn).
‡School of Mathematical Sciences, Chongqing Normal University, Chongqing 401331, People's Republic of China (cqzhangyurong@126.com).
§Department of Applied Mathematics, University of Western Ontario, London, ON, Canada N6A 5B7 (xzou@uwo.ca).

$$\frac{\partial u}{\partial t}(t,x) = d\Delta u(t,x) - \mu u(t,x) + e^{-\delta\tau} b u(t-\tau,x) e^{-u(t-\tau,x)}, \ \ x \in \Omega \subset \mathbb{R}^n. \tag{1.2}$$

In reality, generically a habitat (living environment) is spatially and temporally inhomogeneous. In the context of (1.2), this means that the parameters μ and b (death rate and maximal birth rate) depend on t and x: $\mu = \mu(t,x)$ and $b = b(t,x)$. Among all possible ways of dependence is the form through $\nu \cdot x - \kappa t$: $\mu(t,x) = \mu(\nu \cdot x - \kappa t)$ and $b(t,x) = b(\nu \cdot x - \kappa t)$, where ν is a unit vector in $\mathbb{R}^n$ and $\kappa \in \mathbb{R}$ is a constant. This form of dependence accounts for the scenario of shifting environment along the direction ν with constant shifting speed κ. In such a case the monotonicity of the profile of $\mu(\cdot)$ and $b(\cdot)$ indicates whether the environment is improving or worsening: if $\mu(\cdot)$ is increasing and $b(\cdot)$ is decreasing, it is improving when $\kappa > 0$ and it is worsening when $\kappa < 0$; while if $\mu(\cdot)$ is decreasing and $b(\cdot)$ is increasing, then it is worsening when $\kappa > 0$ and it is improving when $\kappa < 0$. Here, the concurrence of opposite monotonicity for these two functions is due to the biological meaning of these two parameters.

Assuming a shifting environment with constant shifting speed κ in the direction ν and incorporating the above mentioned type of heterogeneity into (1.2), we can obtain a new model equation:

$$\begin{aligned} \frac{\partial u}{\partial t}(t,x) = {} & d\Delta u(t,x) - \mu(x\cdot\nu - \kappa t)u(t,x) \\ & + e^{-\delta\tau} b(x\cdot\nu - \kappa(t-\tau))u(t-\tau,x)e^{-u(t-\tau,x)}, \qquad x \in \mathbb{R}^N. \end{aligned} \tag{1.3}$$

Here we would particularly like to draw the reader's attention to the fact: at time t and location x, due to the shifting nature, the death rate $\mu(\cdot)$ is evaluated at $\nu \cdot x - \kappa t$ while the birth rate term $b(\cdot)$ is evaluated at $\nu \cdot x - \kappa(t-\tau)$.

We point out that in recent years, spatial temporal dynamics for equations or systems with shifting environment has attracted the attention of many mathematicians and ecologists. Many analytical works have been done on models with shifting environments, mainly focusing on the extinction/persistence and the spreading speeds and patterns. See, e.g., [1, 2, 3, 4, 5, 9, 10, 12, 13, 14, 22, 33, 41, 42, 47, 49] and the relevant references therein. However, to the best of the authors' knowledge, these works are on models that only have one shifting parameter. In contrast, the model equation (1.3) has (i) two shifting parameters with solid ecological background which have opposite monotonicity; (ii) a maturation delay in the unknown, and (iii) a shifting mediated phase delay on the profile of the birth rate function. These *three novel features* make (1.3) a novel, interesting, and yet challenging equation and brings in some challenges in mathematics. In addition, we choose to work on a framework that allows *spaces of high dimension* (in contrast to most existing works on similar topics but confined to one-dimensional (1-D) space), because we believe that in high-dimensional spaces, direction should play a role in one way or another, and it turns out the population can, indeed, spread and expand with different speeds and rates in different directions, as observed in the last section.

It is worth pointing out that, under an abstract setting, Weinberger [32] established a theory of traveling waves and spreading speeds for monotone discrete-time systems with spatial translation invariance. This theory has been further extensively developed by other researchers for more general monotone and nonmonotone semiflows in various discrete and continuous-time evolution systems. See, e.g., [6, 16, 17, 18, 40, 47] and the related references therein. There are two common features in the evolution equations in the aforementioned works: (f1) *spatial translation invariance* and (f2) *spatial homogeneity.* In general, (f1) and (f2) are related in the sense that spatial heterogeneity would lead to the lack of spatial translation invariance. Regarding (f1), Yi and Zhao [42, 43] extended the theory of spreading speeds and traveling waves for the monotone and nonmonotone semiflows *without spatial translation invariance* in 1-D space. For (f2), there have also been some works on spreading speeds and traveling waves for some R-D equations with certain *special types of spatial heterogeneities (also in 1-D space)*—e.g., space periodic habitats in [15, 24, 25, 26] and a jump type habitat in [48]. Shifting habitats are another special type of spatial heterogeneity explored in the works mentioned in the preceding paragraph, but the spatial systems in those works only allow one parameter to be of such a heterogeneity and are in 1-D space. Here in this paper, our model arises from age structured population dynamics and it contains *two shifting parameters* together with a *shifting mediated delay* in addition to diffusion in *higher-dimensional* space. For such an equation, the theories and methods developed in aforementioned works cannot be, at least directly, applicable. Indeed, as far as the spreading speed and traveling waves are concerned, we find that there is a lack of theory and method for spatially heterogeneous (any type) systems *in higher dimensions, not to mention that there is a time delay.*

In this paper, we will employ the iterative strategies of travelling wave maps in [37, 48] to investigate the spreading speed of a class of non-translation-invariant delayed reaction diffusion equations (1.3) in $\mathbb{R}^N$. To this end, we introduce the following auxiliary system:

$$\frac{\partial u}{\partial t}(t,x) = d\Delta u(t,x) - \mu(x_1-\kappa t)u(t,x) + b(x_1-\kappa(t-\tau))g(u(t-\tau,x)), \tag{1.4}$$

and establish some a priori estimates on nontrivial solution of (1.4). Then we make use of these estimates and some test functions to investigate the heterogeneous steady state, spreading speed, and asymptotic propagation properties for (1.3). The main goal is to understand how the spatial-temporal dynamics are determined by some key factors such as the shift speed κ, the two limiting equations (as $\nu\cdot x-\kappa t\to\pm\infty$), and the maturation delay τ. Here, for convenience of presentation, based on the biological background, we assume use, throughout this paper, the following properties for the birth and death rate functions:

(H1) $b(s)$ and $\mu(s)$ are continuous and positive;
(H2) $b(s)$ and $\mu(s)$ are monotone but have *opposite monotonicity*;
(H3) $0<b(\pm\infty)<\infty$ and $0<\mu(\pm\infty)<\infty$.

Also the function $g(\cdot)$ in (1.4) is intended to represent a class of functions that includes the Ricker's birth function in (1.2) and (1.3). Hence, we always assume that $g(\cdot)$ possesses the following properties (as the Ricker function does):

(G) $g:\mathbb{R}_+\to\mathbb{R}_+$ is continuous and *nondecreasing* on $\mathbb{R}_+$ with g being continuously differentiable in some right neighborhood of 0 with $g(0)=0$ and $g(u)\le g'(0)u$ $\forall u\in\mathbb{R}_+$; and there exists $\epsilon\in[0,1)$ such that $g'(0)>1-\epsilon$; moreover, for such an ϵ, there exists $u^*\in(0,\infty)$ such that $g(u^*)=(1-\epsilon)u^*$, $g(u)>(1-\epsilon)u$ for all $u\in(0,u^*)$, and $0<g(u)<(1-\epsilon)u$ for all $u\in(u^*,\infty)$.

Let $\mu_\pm=\mu(\pm\infty)$ and $b_\pm=b(\pm\infty)$. If $\mu(\cdot)$ is increasing and $b(\cdot)$ is decreasing (leading to a decreasing $\frac{e^{-\delta\tau}b(\xi)}{\mu(\xi)}$), then $\frac{e^{-\delta\tau}b_+}{\mu_+}<\frac{e^{-\delta\tau}b_-}{\mu_-}$. It is not difficult to observe that the environment is improving when $\kappa>0$ and it is worsening when $\kappa<0$. There can be the following cases for the two limit equations:

(B-1) $\frac{e^{-\delta\tau}b_+}{\mu_+}<1<\frac{e^{-\delta\tau}b_-}{\mu_-}$.
(B-2) $1<\frac{e^{-\delta\tau}b_+}{\mu_+}<\frac{e^{-\delta\tau}b_-}{\mu_-}$.

If $\mu(\cdot)$ is decreasing and $b(\cdot)$ is increasing, we have $\frac{e^{-\delta\tau}b_-}{\mu_-} < \frac{e^{-\delta\tau}b_+}{\mu_+}$ and the environment is worsening scenario when $\kappa > 0$ and it is improving when $\kappa < 0$. Accordingly, there can be the following cases for the two limit equations:

(W-1) $\frac{e^{-\delta\tau}b_-}{\mu_-} < 1 < \frac{e^{-\delta\tau}b_+}{\mu_+}$.

(W-2) $1 < \frac{e^{-\delta\tau}b_-}{\mu_-} < \frac{e^{-\delta\tau}b_+}{\mu_+}$.

By applying the spatial mirror transformation, the case of increasing $\mu(\cdot)$ and decreasing $b(\cdot)$ with (B-1) can be converted to the case of decreasing $\mu(\cdot)$ and increasing $b(\cdot)$ with (W-1). Therefore, in the rest of this paper, unless otherwise specified (in end of section 4), we exclusively work on the case of decreasing $\mu(\cdot)$ and increasing $b(\cdot)$ with (W-1) for $\kappa \in \mathbb{R}$. The cases (B-2) and (W-2) will be investigated in forthcoming papers.

This paper is organized as follows. In section 2, we present some notations. In section 3, we derive some a priori traveling-like estimates on nontrivial solutions to (1.4), which play a key role in the proof of the upward convergence and global asymptotic behavior for (1.4). Then with these estimates and some test function, in section 4, we establish the spatial-temporal propagation dynamics of (1.3). We find that for the cases (W-1) and (B-1), the minimal wave speed of the better limiting equation plays a crucial and actually decisive role in determining the spatial-temporal dynamics of (1.3). We conclude the paper with section 5, in which we present some discussions on the ecological implications of our main theoretical results in terms of the spreading and expansion of populations; particularly in the two-dimensional (2-D) case, we identify certain range of parameters within which, the spreading speed is actually direction dependent.

2. Main results. We first introduce some notations. Let $\mathbb{R}^N$, $\mathbb{R}^N_+ := \mathbb{R}_+ \times \mathbb{R}^{N-1}$, and let $\mathbb{N}$ be the sets of all N-dimensional real vectors, right-half real vectors, positive integers, respectively. Let $X = C(\mathbb{R}^N, \mathbb{R}) \cap L^\infty(\mathbb{R}^N, \mathbb{R})$ and $C = C([-\tau, 0], X)$. Equipped with the usual supremum norm $||\cdot||_X = ||\cdot||_{L^\infty(\mathbb{R}^N,\mathbb{R})}$ and $||\cdot||_C = ||\cdot||_{L^\infty([-\tau,0]\times\mathbb{R}^N,\mathbb{R})}$, respectively, X and C are Banach spaces. Let $X_+ = \{\phi \in X : \phi(x) \geq 0 \text{ for all } x \in \mathbb{R}^N\}$, $X_+^\circ = \{\phi \in X : \phi(x) > 0 \text{ for all } x \in \mathbb{R}^N\}$, $C_+ = \{\varphi \in C : \varphi(\theta, x) \geq 0 \text{ for all } (\theta, x) \in [-\tau, 0] \times \mathbb{R}^N\}$, and $C_+^\circ = \{\varphi \in C : \varphi(\theta, x) > 0 \text{ for all } (\theta, x) \in [-\tau, 0] \times \mathbb{R}^N\}$. Clearly, X_+, C_+ are closed cones in X, C, respectively. For any ξ, $\eta \in X$ (resp., C), we write $\xi \geq \eta$ if $\xi - \eta \in X_+$ (resp., C_+), $\xi > \eta$ if $\xi \geq \eta$ and $\xi \neq \eta$, $\xi \gg \eta$ if $\xi - \eta \in X_+^\circ$ (resp., C_+°). Moreover, for $\gamma \in X_+$, $X_\gamma = \{\phi \in X_+ : \phi(x) \leq \gamma(x) \text{ for all } x \in \mathbb{R}^N\}$ and $C_\gamma = \{\varphi \in C_+ : \varphi(\theta, x) \leq \gamma(x) \text{ for all } (\theta, x) \in [-\tau, 0] \times \mathbb{R}^N\}$. Sometimes, we also write $BC(\mathcal{X}, \mathcal{Y})$ for $C(\mathcal{X}, \mathcal{Y}) \cap L^\infty(\mathcal{X}, \mathcal{Y})$, where $\mathcal{X}, \mathcal{Y}$ are topological spaces. For any $\zeta \in BC(\mathcal{X}, \mathcal{Y})$, we denote supremum norm of ζ by $||\zeta||_{L^\infty}$.

For readers' convenience, we shall also treat an element $\varphi \in C$ as a function from $[-\tau, 0] \times \mathbb{R}^N$ into $\mathbb{R}$. For any $a \in \mathbb{R}$ or $\phi \in X$, we also use a, ϕ to denote the constant function taking constant value a, ϕ in the corresponding function space, when no confusion arises. So, we sometimes consider $\mathbb{R}, X$ as subsets of X, C, respectively, that is, $\mathbb{R} \subseteq X \subseteq C$.

For an interval $I \subseteq \mathbb{R}$, let $I + [-\tau, 0] = \{t + \theta : t \in I \text{ and } \theta \in [-\tau, 0]\}$. For $u : (I + [-\tau, 0]) \times \mathbb{R}^N \to \mathbb{R}$ and $t \in I$, we write $u_t(\cdot, \cdot)$ for the function defined by $u_t(\theta, x) = u(t + \theta, x)$ for $(\theta, x) \in [-\tau, 0] \times \mathbb{R}^N$.

For readers' convenience, by using the traveling wave transformation, we transform system (1.3) to the following initial value problem (IVP) of delayed reaction-diffusion equations in $\mathbb{R}^N$ with spatial switch:

$$(2.1)\quad \begin{cases} \dfrac{\partial w}{\partial t}(t,x) = d\Delta w(t,x) + \kappa\nu\cdot\nabla w(t,x) - \mu(x\cdot\nu)w(t,x) + e^{-\delta\tau}b(x\cdot\nu+\kappa\tau)\cdot \\ \qquad w(t-\tau, x+\kappa\tau\nu)e^{-w(t-\tau,x+\kappa\tau\nu)}, \ (t,x)\in(0,\infty)\times\mathbb{R}^N, \\ w_0 = \varphi \in C_+. \end{cases}$$

For given $\varphi \in C_+$, by the method of steps, it is easy to see (2.1) has a unique solution which exists for all $t \geq -\tau$. Denote by $u^\varphi(t,x)$ the unique solution of (2.1). Then, we easily see that

$$0 < u^\varphi(t,x) \leq \max\left\{||\varphi||_{L^\infty}, \frac{b_+}{\mu_+ e^{\delta\tau+1}}\right\} \text{ for all } t>\tau, \ \varphi\in C_+\setminus\{0\}.$$

Due to the noncompactness of the spatial domain, it is generally difficult and inconvenient to describe the asymptotic behavior of solutions to (1.3) with respect to the L^∞-norm. To overcome this difficulty, we use the following norms on X and C, defined, respectively, by

$$||\phi|| := \sum_{n=1}^{\infty} 2^{-n}\sup\{|\phi(x)| : ||x|| \leq n\} \quad \text{for all } \phi\in X,$$
$$||\varphi|| := \sum_{n=1}^{\infty} 2^{-n}\sup\{|\varphi(\theta,x)| : \theta\in[-\tau,0], ||x|| \leq n\} \quad \text{for all } \varphi\in C.$$

3. Asymptotic propagation persistence. In this section, we shall adapt the iterative strategies of travelling wave maps in [37, 48] to study asymptotic propagation persistence of (1.3).

3.1. Iterative strategy. Let $e_l = (0,\dots,0,1,0,\dots,0) \in \mathbb{R}^N$, $x = (x_1, x') \in \mathbb{R}\times\mathbb{R}^{N-1}$, and

$$k(t,x;\mu) = \frac{\mu e^{-\mu t}}{(4d\pi t)^{\frac{N}{2}}}\exp\left(-\frac{||x||^2}{4dt}\right) \quad \forall (t,x,\mu)\in(0,\infty)\times\mathbb{R}^N\times\mathbb{R}_+.$$

Given $(c,\xi,\eta,\epsilon,\mu)\in\mathbb{R}_+\times S^{N-1}\times S^{N-1}\times[0,1)\times\mathbb{R}_+$, define $k_{c,\xi,\mu}$, $Q_{c,\xi}$, and c^* by

$$k_{c,\xi,\mu}(x) = \begin{cases} \displaystyle\int_{\mathbb{R}_+} k(s, x+cs\xi;\mu)\mathrm{d}s, & x\neq 0, \\ 1, & x=0, \end{cases}$$
$$Q_{c,\xi}[\phi;\epsilon,\mu,g](x) = \int_{\mathbb{R}^N}\Big[\epsilon k_{c,\xi,\mu}(x-y)\phi(y) + k_{c,\xi,\mu}(x+\tau c\xi - y)g(\phi(y))\Big]\mathrm{d}y,$$
$$v^*(c,\epsilon,\mu,g;\xi,\eta) = \inf_{\rho>0}\log\int_{\mathbb{R}^N}\Big[\epsilon k_{c,\xi,\mu}(x) + g'(0)k_{c,\xi,\mu}(x+\tau c\xi)\Big]e^{\rho x\cdot\eta}\mathrm{d}\, x,$$

and

$$c^*(\epsilon,\mu,g) = \inf\left\{\sigma\in\mathbb{R}_+ : \begin{array}{c} \exists\rho\in\left(0, \dfrac{\sigma+\sqrt{\sigma^2+4d\mu}}{2d}\right) \text{ such that} \\ \mu(\epsilon + g'(0)e^{-\sigma\tau\rho}) \leq \mu + \sigma\rho - d\rho^2 \end{array}\right\},$$

where $x\in\mathbb{R}^N$ and $\phi\in X_+ = BC(\mathbb{R}^N, \mathbb{R}_+)$ and $g(\cdot)$ is the function in the auxiliary equation (1.4) related to (1.3), satisfying (G).

LEMMA 3.1. *Let $\xi \in S^{N-1}$, and let $v^*(c,\epsilon,\mu,g;\xi,\eta), c^*(\epsilon,\mu,g)$ be defined as above. Then the following statements are valid.*

(i) *For any $(\gamma,\eta)\in \mathbb{R}_+ \times S^{N-1}$, we have*

$$\int_{\mathbb{R}^N} k_{c,\xi,\mu}(x+c\gamma\xi)e^{\rho x\cdot\eta}\mathrm{d}x = \begin{cases} \dfrac{\mu e^{-c\gamma\rho\xi\cdot\eta}}{\mu+c\rho\xi\cdot\eta-d\rho^2}, \ \mu+c\rho\xi\cdot\eta-d\rho^2>0, \\ \infty, \ \mu+c\rho\xi\cdot\eta-d\rho^2\le 0. \end{cases}$$

Hence, $v^*(c,\epsilon,\mu,g;\eta,\eta)=v^*(c,\epsilon,\mu,g;\xi,\xi)\ \forall \eta\in S^{N-1}$.

(ii) $v^*(c,\epsilon,\mu,g;\xi,\xi)>0$ *if and only if* $v^*(c,\epsilon,\mu,g;\xi,\eta)>0\ \forall\eta\in S^{N-1}$.

(iii) $c^*(\epsilon,\mu,g)=\sup\{c\in\mathbb{R}_+ : v^*(c,\epsilon,\mu,g;\xi,\xi)>0\}$.

(iv) $v^*(c,\epsilon,\mu,g;\xi,\xi)>0$ *if and only if* $c<c^*(\epsilon,\mu,g)$.

(v) $c^*(0,\mu,g)=c^*(1-\frac{\mu}{\overline{\mu}},\overline{\mu},\frac{\mu}{\overline{\mu}}g)\ \forall\overline{\mu}>\mu>0$.

(vi) $c^*(0,\mu,g)=\lim_{\gamma\to 0+}c^*(0,\mu+\gamma,\frac{\mu-\gamma}{\mu+\gamma}g)$ *if* $g'(0)>1$.

Proof. (i) It follows from the Fubini's theorem that for any $(\rho,\gamma,\xi)\in\mathbb{R}_+^2\times S^{N-1}$, there holds

$$\begin{aligned}
&\int_{\mathbb{R}^N} k_{c,\xi,\mu}(x+c\gamma\xi)e^{\rho x\cdot\eta}\mathrm{d}\ x \\
&= \int_{\mathbb{R}^N}\int_{\mathbb{R}_+} k(s,x+c[s+\gamma]\xi;\mu)\mathrm{d}s e^{\rho x\cdot\eta}\mathrm{d}\ x \\
&= \int_{\mathbb{R}^N}\int_{\mathbb{R}_+} \frac{\mu e^{-\mu s}}{(4d\pi s)^{\frac{N}{2}}}\exp\left(-\frac{||x+c[s+\gamma]\xi||^2}{4ds}\right)\mathrm{d}s e^{\rho x\cdot\eta}\mathrm{d}\ x \\
&= \int_{\mathbb{R}_+} \frac{\mu e^{-\mu s}}{(4d\pi s)^{\frac{N}{2}}}\int_{\mathbb{R}^N}\exp\left(-\frac{||x+c[s+\gamma]\xi||^2}{4ds}+\rho x\cdot\eta\right)\mathrm{d}\ x\mathrm{d}s \\
&= \int_{\mathbb{R}_+} \frac{\mu e^{[d\rho^2-c\rho\xi\cdot\eta-\mu]s-c\gamma\rho\xi\cdot\eta}}{(4d\pi s)^{\frac{N}{2}}}\int_{\mathbb{R}^N}\exp\left(-\frac{||x+(cs+c\gamma)\xi-2ds\rho\eta||^2}{4ds}\right)\mathrm{d}\ x\mathrm{d}s \\
&= \int_{\mathbb{R}_+} \mu e^{[d\rho^2-c\rho\xi\cdot\eta-\mu]s-c\gamma\rho\xi\cdot\eta}\mathrm{d}s \\
&= \begin{cases} \dfrac{\mu}{(\mu+c\rho\xi\cdot\eta-d\rho^2)e^{c\gamma\rho\xi\cdot\eta}}, & \mu+c\rho\xi\cdot\eta-d\rho^2>0, \\ \infty, & \mu+c\rho\xi\cdot\eta-d\rho^2\le 0. \end{cases}
\end{aligned}$$

(ii) By (i) and the definition of $v^*(c,\epsilon,\mu,g;\xi,\eta)$, we may obtain that for any $\eta\in S^{N-1}$,

$$\begin{aligned}
v^*(c,\epsilon,\mu,g;\xi,\eta) &= \inf\left\{\log\frac{\mu(\epsilon+g'(0)e^{-c\tau\rho\xi\cdot\eta})}{\mu+c\rho\xi\cdot\eta-d\rho^2} : \rho>0 \text{ and } \mu+c\rho\xi\cdot\eta-d\rho^2>0\right\} \\
&\ge \inf\left\{\log\frac{\mu(\epsilon+g'(0)e^{-c\tau\rho})}{\mu+c\rho-d\rho^2} : \rho>0 \text{ and } \mu+c\rho-d\rho^2>0\right\} \\
&= v^*(c,\epsilon,\mu,g;\xi,\xi),
\end{aligned}$$

which yields (ii).

(iii) By the proof of (ii), we have

$$v^*(c,\epsilon,\mu,g;\xi,\xi)=\inf\left\{\log\frac{\mu(\epsilon+g'(0)e^{-c\tau\rho})}{\mu+c\rho-d\rho^2} : \rho\in\left(0,\frac{c+\sqrt{c^2+4d\mu}}{2d}\right)\right\}, \tag{3.1}$$

and thus $\mathbb{R}_+ \ni c \mapsto v^*(c,\epsilon,\mu,g;\xi,\xi) \in \mathbb{R}$ is nonincreasing. If

$$\sigma < \sup\{c \in \mathbb{R}_+ : v^*(c,\epsilon,\mu,g;\xi,\xi) > 0\},$$

then $v^*(\iota,\epsilon,\mu,g;\xi,\xi) > 0$ for all $\iota \in [0,\sigma]$. It follows from (3.1) that

$$\mu(\epsilon + g'(0)e^{-\iota\tau\rho}) > \mu + \iota\rho - d\rho^2 \ \forall \rho \in \left(0, \frac{\iota + \sqrt{\iota^2 + 4d\mu}}{2d}\right), \quad \iota \in [0,\sigma].$$

This and the definition of $c^*(\epsilon,\mu,g)$ lead to $c^*(\epsilon,\mu,g) \geq \sigma$ and thus

$$c^*(\epsilon,\mu,g) \geq \sup\{c \in \mathbb{R}_+ : v^*(c,\epsilon,\mu,g;\xi,\xi) > 0\}.$$

If $\sigma > \sup\{c \in \mathbb{R}_+ : v^*(c,\epsilon,\mu,g;\xi,\xi) > 0\}$, then $v^*(\sigma,\epsilon,\mu,g;\xi,\xi) \leq 0$. This, together with

$$\lim_{\rho \to 0+} \frac{\mu(\epsilon + g'(0)e^{-\sigma\tau\rho})}{\mu + \sigma\rho - d\rho^2} = \epsilon + g'(0) > 1$$

and

$$\lim_{\rho \to (\frac{\sigma + \sqrt{\sigma^2 + 4d\mu}}{2d})^-} \frac{\mu(\epsilon + g'(0)e^{-c\tau\rho})}{\mu + c\rho - d\rho^2} = \infty,$$

implies that $\mu(\epsilon + g'(0)e^{-\sigma\tau\rho}) \leq \mu + \sigma\rho - d\rho^2$ for some $\rho \in (0, \frac{\sigma + \sqrt{\sigma^2 + 4d\mu}}{2d})$. As a result, $c^*(\epsilon,\mu,g) \leq \sigma$, and by the arbitrariness of σ, we have

$$c^*(\epsilon,\mu,g) \leq \sup\{c \in \mathbb{R}_+ : v^*(c,\epsilon,\mu,g;\xi,\xi) > 0\}.$$

(iv) If $c < c^*(\epsilon,\mu,g)$, then $v^*(c,\epsilon,\mu,g;\xi,\xi) > 0$ follows from (iii) and the monotonicity of $\mathbb{R}_+ \ni c \mapsto v^*(c,\epsilon,\mu,g;\xi,\xi) \in \mathbb{R}$. Suppose that $v^*(c,\epsilon,\mu,g;\xi,\xi) > 0$. Then

$$\mu(\epsilon + g'(0)e^{-c\tau\rho}) > \mu + c\rho - d\rho^2 \ \forall \rho \in \left[0, \frac{c + \sqrt{c^2 + 4d\mu}}{2d}\right]$$

due to $g'(0) > 1 - \epsilon$ and (3.1). It follows that

$$\mu(\epsilon + g'(0)e^{-(c+\delta_0)\tau\rho}) > \mu + (c+\delta_0)\rho - d\rho^2 \ \forall \rho \in \left[0, \frac{c + \delta_0 + \sqrt{(c+\delta_0)^2 + 4d\mu}}{2d}\right]$$

for some $\delta_0 > 0$. Again, by the monotonicity of $\mathbb{R}_+ \ni c \mapsto v^*(c,\epsilon,\mu,g;\xi,\xi) \in \mathbb{R}$, we easily see that

$$\mu(\epsilon + g'(0)e^{-\sigma\tau\rho}) > \mu + \sigma\rho - d\rho^2 \ \forall \rho \in \left[0, \frac{\sigma + \sqrt{\sigma^2 + 4d\mu}}{2d}\right],$$

and hence $v^*(\sigma,\epsilon,\mu,g;\xi,\xi) > 0 \ \forall \sigma \in [0, c+\delta_0]$. Consequently, (iii) gives $c^*(\epsilon,\mu,g) \geq c + \delta_0 > c$.

(v) In view of the definition of $c^*(\epsilon,\mu,g)$ and the fact that $\mu + \sigma\rho - d\rho^2 \leq 0$ for all $\rho \in [\frac{\sigma + \sqrt{\sigma^2 + 4d\mu}}{2d}, \frac{\sigma + \sqrt{\sigma^2 + 4d\overline{\mu}}}{2d}]$, we may obtain that for any $\overline{\mu} > \mu > 0$, there holds

$$\begin{aligned}
&c^*\left(1 - \frac{\mu}{\overline{\mu}}, \overline{\mu}, \frac{\mu}{\overline{\mu}} g\right) \\
&= \inf\left\{\sigma \in \mathbb{R}_+ : \overline{\mu}\left(1 - \frac{\mu}{\overline{\mu}} + \frac{\mu}{\overline{\mu}} g'(0)e^{-\sigma\tau\rho}\right) \leq \overline{\mu} + \sigma\rho - d\rho^2, \exists \rho \in \left(0, \frac{\sigma + \sqrt{\sigma^2 + 4d\overline{\mu}}}{2d}\right)\right\} \\
&= \inf\left\{\sigma \in \mathbb{R}_+ : \mu g'(0)e^{-\sigma\tau\rho} \leq \mu + \sigma\rho - d\rho^2, \exists \rho \in \left(0, \frac{\sigma + \sqrt{\sigma^2 + 4d\mu}}{2d}\right)\right\} \\
&= c^*(0,\mu,g).
\end{aligned}$$

(vi) By the definition of $c^*(\epsilon,\mu,g)$, we have

$$c^*\left(0,\mu+\gamma,\frac{\mu-\gamma}{\mu+\gamma}g\right)$$
$$=\inf\left\{\sigma\in\mathbb{R}_+:(\mu-\gamma)g'(0)e^{-\sigma\tau\rho}\le\mu+\gamma+\sigma\rho-d\rho^2,\exists\rho\in\left(0,\frac{\sigma+\sqrt{\sigma^2+4d(\mu+\gamma)}}{2d}\right)\right\}$$
$$\le\inf\left\{\sigma\in\mathbb{R}_+:\mu g'(0)e^{-\sigma\tau\rho}\le\mu+\sigma\rho-d\rho^2,\exists\rho\in\left(0,\frac{\sigma+\sqrt{\sigma^2+4d\mu}}{2d}\right)\right\}$$
$$=c^*(0,\mu,g)\quad\forall\gamma>0.$$

So, it suffices to prove that $\liminf_{\gamma\to0+}c^*(0,\mu+\gamma,\frac{\mu-\gamma}{\mu+\gamma}g)\ge c^*(0,\mu,g)$; otherwise, there exists $\sigma_0\in\mathbb{R}_+$ such that $\liminf_{\epsilon\to0+}c^*(0,\mu+\gamma,\frac{\mu-\gamma}{\mu+\gamma}g)<\sigma_0<c^*(0,\mu,g)$. The latter inequality and (iv) imply $v^*(\sigma_0,0,\mu,g;\xi,\xi)>0$. Then, by (3.1), we have $\mu g'(0)e^{-\sigma_0\tau\rho}>\mu+\sigma_0\rho-d\rho^2\ \forall\rho\in[0,\frac{\sigma_0+\sqrt{\sigma_0^2+4d\mu}}{2d}]$. Thus, there exists $\delta_0>0$ such that for any $\gamma\in[0,\delta_0]$ and $\rho\in[0,\frac{\sigma_0+\delta_0+\sqrt{(\sigma_0+\delta_0)^2+4d(\mu+\gamma)}}{2d}]$, there holds

$$(\mu-\gamma)g'(0)e^{-(\sigma_0+\delta_0)\tau\rho}>\mu+\gamma+(\sigma_0+\delta_0)\rho-d\rho^2.$$

Hence, by (3.1), we know that $v^*(\sigma_0+\delta_0,0,\mu+\gamma,\frac{\mu-\gamma}{\mu+\gamma}g;\xi,\xi)>0$ and by (iv), we see that $c^*(0,\mu+\gamma,\frac{\mu-\gamma}{\mu+\gamma}g)\ge\sigma_0+\delta_0\ \forall\gamma\in[0,\delta_0]$. Therefore,

$$\liminf_{\gamma\to0+}c^*\left(0,\mu+\gamma,\frac{\mu-\gamma}{\mu+\gamma}g\right)\ge\sigma_0+\delta_0>\sigma_0,$$

a contradiction. ∎

The following lemma itemizes some properties of these maps $Q^n_{c,\xi}[\cdot;\epsilon,\mu,g]$.

LEMMA 3.2. *Let* $(\epsilon,\mu,c,\xi)\in[0,1)\times\mathbb{R}_+\times\mathbb{R}_+\times S^{N-1}$, $Q:=Q^n_{c,\xi}[\cdot;\epsilon,\mu,g]:X_+\to X_+$, *and* $m(y,\mathrm{d}y)=[\epsilon k_{c,\xi,\mu}(y)+g'(0)k_{c,\xi,\mu}(\tau c\xi+y)]\mathrm{d}y\ \forall y\in\mathbb{R}^N$. *Then the following statements hold:*

(i) *Q is a continuous and compact map on X_+ in sense of $Q|_{X_r}:X_r\to X_+$ is a continuous and compact map with the compact-open topology;*
(ii) *Q is order preserving in the sense that $Q[\phi]\ge Q[\psi]$ for all $\phi,\psi\in X_+$ with $\phi\ge\psi$;*
(iii) *Q is spatial translation invariant in the sense that $Q[\phi](x+y)=Q[\phi(\cdot+y)](x)$ for all $(x,y,\phi)\in\mathbb{R}^N\times\mathbb{R}^N\times X_+$;*
(iv) *$Q[0]=0,Q[u^*]=u^*$, $Q[\alpha]>\alpha\ \forall\alpha\in(0,u^*)$, and $Q[\alpha]<\alpha,\forall\alpha\in(u^*,\infty)$;*
(v) *Q and $m(y,\mathrm{d}y)$ satisfy the inequalities in* [47].

Based on Theorem 3.6 in [47], and Lemmas 3.1 and 3.2, we easily verify the following results.

PROPOSITION 3.3. *If $(c,\xi)\in\mathbb{R}_+\times S^{N-1}$ and $c<c^*(\epsilon,\mu,g)$, then for any $\phi\in BC(\mathbb{R}^N,\mathbb{R}_+)\setminus\{0\}$, we have $\lim_{n\to\infty}||Q^n_{c,\xi}[\phi;\epsilon,\mu,g]-u^*||=0$.*

LEMMA 3.4. *Let $c\in[0,c^*(\epsilon,\mu,g))$, and let $\phi\in C(\mathbb{R}^N,[0,u^*))\setminus\{0\}$ have a compact support. Then, for any $\gamma\in(1,\frac{u^*}{||\phi||_{L^\infty(\mathbb{R}^N)}})$, there is an $n=n(c,\phi,\gamma)\in\mathbb{N}$ such that $Q^n_{\sigma,\xi}[\phi;\epsilon,\mu,g]\ge\gamma\phi$ for all $(\sigma,\xi)\in[0,c]\times S^{N-1}$.*

Proof. Clearly, by Proposition 3.3, for any $(\sigma,\xi)\in[0,c]\times S^{N-1}$, $\gamma\in(1,\frac{u^*}{||\phi||_{L^\infty}})$, there is an $\mathscr{N}:=\mathscr{N}(\sigma,\xi,\phi,\gamma)\in\mathbb{N}$ such that $Q^{\mathscr{N}}_{\sigma,\xi}[\phi;\epsilon,\mu,g]\gg\gamma\phi$. Note that

$$[0,c]\times S^{N-1}\times\mathbb{R}^N\ni(\sigma,\xi,x)\mapsto Q^{\mathscr{N}}_{\sigma,\xi}[\phi;\epsilon,\mu,g](x)\in\mathbb{R}$$

is continuous at $(\sigma,\xi,x)\in[0,c]\times S^{N-1}\times\mathbb{R}^N$ due to the definition of $Q^{\mathscr{N}}_{\sigma,\xi}[\phi;\epsilon,\mu,g](x)$. This, together with the compactness of $supp(\phi)$, yields that there is $\delta=\delta(\sigma,\xi,\phi,\gamma)>0$ such that $Q^{\mathscr{N}}_{\varsigma,\eta}[\phi;\epsilon,\mu,g]\gg\gamma\phi$ for all $\varsigma\in[0,c]\cap(\sigma-\delta,\sigma+\delta)$ and $\eta\in S^{N-1}\cap B_\delta(\xi)$. Clearly, by the compactness of $[0,c]\times S^{N-1}$, there exist (c_1,ξ_1), (c_2,ξ_2), …, $(c_l,\xi_l)\in[0,c]\times S^{N-1}$ such that

$$[0,c]\times S^{N-1}\subseteq\bigcup_{i=1}^{l}(c_i-\delta(c_i,\xi_i\phi,\gamma),c_i+\delta(c_i,\xi_i,\phi,\gamma))\times B_{\delta(c_i,\xi_i,\phi,\gamma)}(\xi_i).$$

Let $n:=n_{c,\phi,\gamma}=\prod_{i=1}^{l}\mathscr{N}(c_i,\xi_i,\phi,\gamma)$. Then, by $\gamma>1$, the choice of n, and the monotonicity of $Q_{\sigma,\xi}$, we have $Q^n_{\sigma,\xi}[\phi;\epsilon,\mu,g]\geq\gamma\phi$ for all $(\sigma,\xi)\in[0,c]\times S^{N-1}$. ◻

For any $(c,\xi)\in\mathbb{R}_+\times S^{N-1}$, and α, $\beta>0$, we define linear operators

$$Q_{c,\xi,\alpha}[\cdot;\epsilon,\mu,g]:C([-\alpha,\alpha]^N,\mathbb{R})\to C([-\alpha,\alpha]^N,\mathbb{R})$$

and

$$Q^\infty_{c,\xi}[\cdot;\epsilon,\mu,g],Q^\infty_{c,\xi,\beta}[\cdot;\epsilon,\mu,g]:BC(\mathbb{R}^N_+,\mathbb{R}):=C(\mathbb{R}^N_+,\mathbb{R})\cap L^\infty(\mathbb{R}^N_+,\mathbb{R})\to BC(\mathbb{R}^N_+,\mathbb{R}),$$

respectively, by

$$Q_{c,\xi,\alpha}[\phi;\epsilon,\mu,g](x)=\int_{[-\alpha,\alpha]^N}\Big[\epsilon k_{c,\xi,\mu}(x-y)\phi(y)+k_{c,\xi,\mu}(x+\tau c\xi-y)g(\phi(y))\Big]\mathrm{d}y,\quad x\in[-\alpha,\alpha]^N,$$

$$Q^\infty_{c,\xi}[\zeta;\epsilon,\mu,g](x)=\int_{\mathbb{R}^N_+}\Big[\epsilon k_{c,\xi,\mu}(x-y)\phi(y)+k_{c,\xi,\mu}(x+\tau c\xi-y)g(\phi(y))\Big]\mathrm{d}y,\ x\in\mathbb{R}^N_+,$$

$$Q^\infty_{c,\xi,\beta}[\zeta;\epsilon,\mu,g](x)=\int_0^\beta\int_{\mathbb{R}^N_+}\Big[\epsilon k(s,x+cs\xi-y;\mu)\zeta(y)+k(s,x+c(s+\tau)\xi-y;\mu)g(\zeta(y))\Big]\mathrm{d}y\mathrm{d}s,\ x\in\mathbb{R}^N_+.$$

Here, $\phi\in C([-\alpha,\alpha]^N,\mathbb{R})$, $\zeta\in C(\mathbb{R}^N_+,\mathbb{R})$, and $y\in\mathbb{R}^N_+$. It is easy to verify that these operators are order preserving.

Lemma 3.5. *If $c\in[0,c^*(\epsilon,\mu,g))$, $k\in\mathbb{N}$, and $\phi_1,\phi_2,\dots,\phi_k\in BC(\mathbb{R}^N,\mathbb{R}_+)\setminus\{0\}$ have compact supports, then there exist $n_{c,k}:=n_{c,\phi_1,\phi_2,\dots,\phi_k}\in\mathbb{N}$ and $\alpha^*_{c,k}:=\alpha^*_{c,\phi_1,\phi_2,\dots,\phi_k}>0$ such that $Q^{n_{c,k}}_{\sigma,\xi,\alpha}[\phi_j;\epsilon,\mu,g](0)\geq\frac{2u^*}{3}$ for all $(\sigma,\xi,j)\in[0,c]\times S^{N-1}\times(\mathbb{N}\cap[1,k])$ and $\alpha\geq\alpha^*_{c,k}$.*

Proof. Take $\beta^*>0$ and $\varphi\in C(\mathbb{R}^N,[0,u^*))\setminus\{0\}$ with compact support such that $||\varphi||_{L^\infty(\mathbb{R}^N)}=\varphi(0)$, $supp\varphi\subseteq[-\beta^*,\beta^*]^N$, and $\varphi\leq Q_{\sigma,\xi,\beta^*}[\phi_j]$ for all $(\sigma,\xi,j)\in[0,c]\times S^{N-1}\times(\mathbb{N}\cap[1,k])$.

By Lemma 3.4 with

$$\gamma=\frac{\max\left\{1,\frac{2u^*}{3\varphi(0)}\right\}+\frac{u^*}{\varphi(0)}}{2},$$

we know that there is $n_1\in\mathbb{N}$ such that $Q^{n_1}_{\sigma,\xi}[\varphi;\epsilon,\mu,g]\geq\gamma\varphi$ for all $(\sigma,\xi)\in[0,c]\times S^{N-1}$. According to $\lim_{\alpha\to\infty}Q^{n_1}_{\sigma,\xi,\alpha}[\varphi;\epsilon,\mu,g](0)=Q^{n_1}_{\sigma,\xi}[\varphi;\epsilon,\mu,g](0)>\frac{2u^*}{3}$ for all $(\sigma,\xi)\in[0,c]\times S^{N-1}$, we know that for any $(\sigma,\xi)\in[0,c]\times S^{N-1}$, there exists $\alpha_{\sigma,\xi}>0$ such that $Q^{n_1}_{\sigma,\xi,\alpha_{\sigma,\xi}}[\varphi;\epsilon,\mu,g](0)>\frac{2u^*}{3}$. Based on the continuity of

$$[0,c]\times S^{N-1}\ni(\sigma,\xi)\mapsto Q^{n_1}_{\sigma,\xi,\alpha}[\varphi;\epsilon,\mu,g](0)\in\mathbb{R}\ \forall\alpha\in(0,\infty),$$

we know that there exist $l\in\mathbb{N}$ and (c_1,ξ_1,δ_1), (c_2,ξ_2,δ_2), $\ldots$, $(c_l,\xi_l,\delta_l)\in[0,c]\times S^{N-1}\times(0,1)$ with $[0,c]\times S^{N-1}\subseteq\bigcup_{i=1}^{l}(c_i-\delta_i,c_i+\delta_i)\times B_{\delta_i}(\xi_i)$ such that

$$Q^{n_1}_{\sigma,\xi,\alpha_{c_i,\xi_i}}[\varphi;\epsilon,\mu,g](0)\geq\frac{2u^*}{3}$$

for all $(\sigma,\xi)\in(c_i-\delta_i,c_i+\delta_i)\times(B_{\delta_i}(\xi_i)\bigcap S^{N-1})$ and $i\in[0,l]\cap\mathbb{N}$. Let $\alpha^*=\max\{\alpha_{c_i,\xi_i}: i\in[0,l]\cap\mathbb{N}\}$. Then $Q^{n_1}_{\sigma,\xi,\alpha}[\varphi;\epsilon,\mu,g](0)\geq\frac{2u^*}{3}$ for all $(\sigma,\xi)\in[0,c]\times S^{N-1}$ and $\alpha\geq\alpha^*$ due to the monotonicity of $Q^{n_1}_{\sigma,\xi,\alpha}[\varphi;\epsilon,\mu,g](0)$ with respect to $\alpha\in[0,\infty)$.

Thus, by the choice of φ and the monotonicity of $Q_{\sigma,\xi,\alpha}[\cdot;\epsilon,\mu,g]$, we have

$$Q^{n_1+1}_{\sigma,\xi,\alpha}[\phi_j;\epsilon,\mu,g](0)\geq Q^{n_1}_{\sigma,\xi,\alpha}[\varphi;\epsilon,\mu,g](0)\geq\frac{2u^*}{3}$$

for all $(\sigma,\xi,j)\in[0,c]\times S^{N-1}\times(\mathbb{N}\cap[1,k])$ and $\alpha\geq\alpha^*_{c,k}:=\max\{\alpha^*,\beta^*\}$. ◻

LEMMA 3.6. *Let $c\in[0,c^*(\epsilon,\mu,g))$, $\alpha\in(0,\infty)$, $\phi\in C([-\alpha,\alpha]^N,\mathbb{R}_+)$, $\psi\in BC(\mathbb{R}^N_+,\mathbb{R}_+)$, and $\eta\in\mathbb{R}^N$ with $\eta\geq\alpha e_1$. If $\psi(x+\eta)\geq\phi(x)$ for all $x\in[-\alpha,\alpha]^N$, then $Q^\infty_{c,\xi}[\psi;\epsilon,\mu,g](x+\eta)\geq Q_{c,\xi,\alpha}[\phi;\epsilon,\mu,g](x)$ and hence $(Q^\infty_{c,\xi})^n[\psi;\epsilon,\mu,g](x+\eta)\geq Q^n_{c,\xi,\alpha}[\phi;\epsilon,\mu,g](x)$ for all $(x,\xi)\in[-\alpha,\alpha]^N\times S^{N-1}$ and $n\in\mathbb{N}$.*

Proof. By the definitions of $Q^\infty_{c,\xi}[\cdot;\epsilon,\mu,g]$ and $Q_{c,\xi,\alpha}[\cdot;\epsilon,\mu,g]$, we can conclude that for any $(x,\xi)\in[-\alpha,\alpha]^N\times S^{N-1}$,

$$\begin{aligned}
&Q^\infty_{c,\xi}[\psi;\epsilon,\mu,g](x+\eta)\\
&=\int_{\mathbb{R}^N_+}\Big[\epsilon k_{c,\xi,\mu}(x+\eta-y)\psi(y)+k_{c,\xi,\mu}(x+\eta+\tau c\xi-y)g(\psi(y))\Big]\mathrm{d}y\\
&=\int_{-\eta+\mathbb{R}^N_+}\Big[\epsilon k_{c,\xi,\mu}(x-y)\psi(y+\eta)+k_{c,\xi,\mu}(x+\tau c\xi-y)g(\psi(y+\eta))\Big]\mathrm{d}y\\
&\geq\int_{[-\alpha,\alpha]^N}\Big[\epsilon k_{c,\xi,\mu}(x-y)\psi(y+\eta)+k_{c,\xi,\mu}(x+\tau c\xi-y)g(\psi(y+\eta))\Big]\mathrm{d}y\\
&\geq\int_{[-\alpha,\alpha]^N}\Big[\epsilon k_{c,\xi,\mu}(x-y)\phi(y)+k_{c,\xi,\mu}(x+c\tau\xi-y)g(\phi(y))\Big]\mathrm{d}y\\
&=Q_{c,\xi,\alpha}[\phi;\epsilon,\mu,g](x).
\end{aligned}$$

This completes the proof. ◻

Let $B_r(x)=\{y\in\mathbb{R}^N|\|y-x\|<r\}$ and $B_r:=B_r(0)$, for simplicity. For any $x\in\mathbb{R}^N$, let us denote $U(x)=\{z\in\mathbb{R}^N|B_9\supseteq z+A(x)\}$, where

$$A(x)=\begin{cases}\overline{B_1(-\frac{3x}{\|x\|})}, & x\in\mathbb{R}^N\backslash\{0\},\\ \overline{B_1}, & x=0.\end{cases}$$

LEMMA 3.7. *If $A(x)$ and $U(x)$ as shown above, then the following statements are true:*

(i) *$x+A(x)\subseteq B_9$ and thus $x\in U(x)$ for all $x\in\overline{B}_{10}$.*

(ii) *$U(x)$ is a nonempty open set for all $x\in\overline{B}_{10}$.*

(iii) *There exist $k_0\in\mathbb{N}$ and $x_1,\ldots,x_{k_0}\in\mathbb{R}^N$ such that $\bigcup_{j=1}^{k_0}U(x_j)\supseteq\overline{B}_{10}$.*

Proof. (i) Fix $x\in\overline{B}_{10}$ and $y\in A(x)$. Clearly, we only consider the case of $x\neq0$. Then $||x||\leq10$ and $||y+\frac{3x}{||x||}||\leq1$. It follows that $||x+y||\leq||y+\frac{3x}{||x||}||+||x-\frac{3x}{||x||}||\leq 1+|||x||-3|<9$, which yields (i).

(ii) follows from (i) and the definition of $U(x)$, while (iii) follows from (i), (ii), and the compactness of $\overline{B}_{10}$. ∎

For any $r \in (0,\infty)$, define $h_r, \zeta_j : \mathbb{R}^N \to \mathbb{R}$ by

$$h_r(x) = \begin{cases} 1, & x \in B_r, \\ 0, & x \in \mathbb{R}^N \setminus B_{r+1}, \\ r+1-||x||, & x \in B_{r+1} \setminus B_r, \end{cases}$$

and

$$\zeta_j(x) = \begin{cases} \max\{1-\|x\|, 0\}, & x_j = 0, \\ \max\left\{1-\left\|x+\dfrac{3x_j}{\|x_j\|}\right\|, 0\right\}, & x_j \neq 0, \end{cases}$$

where $j = 1, 2, \ldots, k_0$ and (k_0, x_j) is defined as in Lemma 3.7.

LEMMA 3.8. *For any $r \in [19,\infty)$ and $z \in \overline{B_{r+1}}$, there is $j := j_{r,z} \in \{1, 2, \ldots, k_0\}$ such that $h_r(y+z) \geq \zeta_j(y)$ for all $y \in \mathbb{R}^N$, where k_0 is defined as in Lemma* 3.7.

Proof. Fix $r \in [19,\infty)$ and $z \in \overline{B_{r+1}}$. By applying Lemma 3.7-(iii) and letting

$$z^* = \begin{cases} 0, & z = 0, \\ \dfrac{10}{||z||} z & z \neq 0, \end{cases}$$

we know that there exists $j \in \{1, 2, \ldots, k_0\}$ such that $z^* \in U(x_j)$. It follows that for any $y \in A(x_j)$, there holds

$$\begin{aligned} ||y+z|| &= ||z - z^* + z^* + y|| \\ &\leq ||z - z^*|| + 9 \\ &\leq \big| ||z|| - 10 \big| + 9 \\ &\leq \max\{19, r\} = r. \end{aligned}$$

This, together with the definitions of h_r, ξ_j, implies $h_r(y+z) \geq 1 \geq \zeta_j(y)$ for all $y \in A(x_j)$ and thus $h_r \geq \zeta_j$. ∎

For any given $T > 1$, define the function $h^T : \mathbb{R}^N \to \mathbb{R}_+$ by

$$h^T(x) = h_T(x - (2T, 0, \ldots, 0)^{tr}) \; \forall x \in \mathbb{R}^N.$$

PROPOSITION 3.9. *If $c \in [0, c^*(\epsilon, \mu, g))$ and $\varepsilon \in (0,1]$, then there exist $n_0 = n_0(c,\varepsilon) \in \mathbb{N}$ and $t_0 = t_0(c,\varepsilon) > 19$ such that $(Q^\infty_{\sigma,\xi,\beta})^{n_0}[\varepsilon h^T; \epsilon, \mu, g] \geq \frac{u^*}{2} h^T$ for all $\beta, T \geq t_0$, and $(\sigma,\xi) \in [0,c] \times S^{N-1}$.*

Proof. By applying Lemma 3.5 to $\phi_j = \zeta_j$ and $k = k_0$, there exist $n^* \in \mathbb{N}$ and $\alpha^* > 1 + \tau c$ such that $Q^{n^*}_{\sigma,\xi,\alpha^*}[g; \varepsilon, \mu, \zeta_j](0) \geq \frac{2u^*}{3}$ for all $(\sigma, \xi, j) \in [0,c] \times S^{N-1} \times (\mathbb{N} \cap [1, k_0])$. It follows from Lemma 3.8 and the definitions of h_T, h^T that for any $T \geq 19 + 2\alpha^*$ and $z \in \overline{B_{1+T}}$, there exists $j_{T,z} \in \mathbb{N} \bigcap [1, k_0]$ such that $h^T(x + z + (2T, 0, \ldots, 0)^{tr}) \geq \zeta_{j_{T,z}}(x)$ for all $x \in \mathbb{R}^N$. This, together with Lemma 3.6, implies that for all $T > 19 + 2\alpha^*$, $z \in \overline{B_{1+T}}$, and $(\sigma, \xi) \in [0,c] \times S^{N-1}$, we have

$$(Q^\infty_{\sigma,\xi})^{n^*}[\varepsilon h^T; \epsilon, \mu, g](z + (2T, 0, \ldots, 0)^{tr}) \geq Q^{n^*}_{\sigma,\xi,\alpha^*}[g; \varepsilon, \mu, \zeta_{j_{T,z}}](0) \geq \frac{2u^*}{3}.$$

To complete the proof, we define a new map $R^{\infty}_{\sigma,\xi,\beta}: BC(\mathbb{R}^N_+,\mathbb{R}) \to BC(\mathbb{R}^N_+,\mathbb{R})$ by

$$\begin{aligned} &R^{\infty}_{\sigma,\xi,\beta}[\zeta;\epsilon,\mu,g](x)\\ &\quad= \int_{\beta}^{\infty}\int_{\mathbb{R}^N_+}\Big[\epsilon k(s,x+\sigma\xi-y)\zeta(y)+k(s,x+\sigma(s+\tau)\xi-y)g(\zeta(y))\Big]\mathrm{d}y\mathrm{d}s \end{aligned}$$

for all $(x,\xi)\in\mathbb{R}^N_+\times S^{N-1}$ and $\zeta\in BC(\mathbb{R}^N_+,\mathbb{R})$.

Based on the definitions of $Q^{\infty}_{\sigma,\xi}, R^{\infty}_{\sigma,\xi,\beta}$ and the conditions of g, we easily check that

$$\begin{aligned} ||Q^{\infty}_{\sigma,\xi}[\zeta;\epsilon,\mu,g]||_{L^\infty} &\le (1+\epsilon)\max\{u^*,||\zeta||_{L^\infty}\},\\ ||R^{\infty}_{\sigma,\xi,\beta}[\zeta;\epsilon,\mu,g]||_{L^\infty} &\le (1+\epsilon)\max\{u^*,||\zeta||_{L^\infty}\}e^{-\mu\beta}, \end{aligned}$$

where $\sigma,\beta\in\mathbb{R}_+$, $\xi\in S^{N-1}$, and $\zeta\in BC(\mathbb{R}^N_+,\mathbb{R})$.

By taking $n_0:=n_0(c,\varepsilon)=n^*\in\mathbb{N}$ and $t_0=t_0(c,\varepsilon)>19+2\alpha^*$ such that

$$4^{n_0}\max\{\varepsilon,u^*\}e^{-\mu t_0}\le\frac{u^*}{6},$$

we may obtain that for any $\beta,T\ge t_0$, $\sigma\in[0,c]$, $\xi\in S^{N-1}$, and $x\in(2T,0,\dots,0)^{tr}+\overline{B_{1+T}}$, there holds

$$\begin{aligned} &(Q^{\infty}_{\sigma,\xi,\beta})^{n_0}[\varepsilon h^T;\epsilon,\mu,g](x)\\ &\quad=[Q^{\infty}_{\sigma,\xi}-R^{\infty}_{\sigma,\xi,\beta}]^{n_0}[\varepsilon h^T;\epsilon,\mu,g](x)\\ &\quad\ge[Q^{\infty}_{\sigma,\xi}]^{n_0}[\varepsilon h^T;\epsilon,\mu,g](x)-\sum_{l=1}^{n_0}C^l_{n_0}(1+\epsilon)^{n_0-1}||R^{\infty}_{\sigma,\xi,\beta}[\max\{\varepsilon,u^*\};\epsilon,\mu,g]||_{L^\infty}\\ &\quad\ge[Q^{\infty}_{\sigma,\xi}]^{n_0}[\varepsilon h^T;\epsilon,\mu,g](x)-2^{n_0}(1+\epsilon)^{n_0}\max\{\varepsilon,u^*\}e^{-\mu\beta}\\ &\quad\ge[Q^{\infty}_{\sigma,\xi}]^{n_0}[\varepsilon h^T;\epsilon,\mu,g](x)-4^{n_0}\max\{\varepsilon,u^*\}e^{-\mu\beta}\\ &\quad\ge\frac{u^*}{2}, \end{aligned}$$

which, combining with the definition of h^T, yields the proof. ∎

3.2. Propagation persistence. The relation between solution mapping and integral operator $Q^{\infty}_{\sigma,\xi,\beta}[\cdot;\epsilon,\mu,g]$ will be given below, which will contribute to the propagation persistence with parameters, as well as upward convergence properties in the next section.

Let $\mu_\pm=\mu(\pm\infty)$, $b_\pm=b(\pm\infty)$, $\overline{\mu}=1+\sup_{s\in\mathbb{R}}\mu(s)$, and let $u^{\varphi}(t,x;\mu(\cdot),b(\cdot),g)$ denote the mild solutions of the following initial value problem for the auxiliary equation (1.4):

$$\text{(3.2)}\qquad \begin{cases} \dfrac{\partial u}{\partial t}(t,x)=d\Delta u(t,x)-\mu(x_1-\kappa t)u(t,x)+b(x_1-\kappa(t-\tau))g(u(t-\tau,x)),\\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (t,x)\in(0,\infty)\times\mathbb{R}^N,\\ u(t,x)=\varphi(t,x),\qquad (t,x)\in[-\tau,0]\times\mathbb{R}^N. \end{cases}$$

Now, let $T(t)$ be the semigroup generated by the linear system,

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x)=d\Delta u(t,x), & t>0,\\ u(0,x)=\phi(x), & x\in\mathbb{R}^N, \end{cases}$$

that is, for $(x,\phi)\in\mathbb{R}\times X$,

$$\begin{cases} T(0)[\phi](x) = \phi(x), \\ T(t)[\phi](x) = \dfrac{1}{(4d\pi t)^{N/2}} \displaystyle\int_{\mathbb{R}^N} \phi(y) \exp\left(-\frac{||x-y||^2}{4dt}\right) \mathrm{d}y, \quad t > 0. \end{cases} \tag{3.3}$$

The following result gives the comparative relationships between the solutions of (3.2) and the operators $Q^{\infty}_{c,\xi,\beta}[\cdot;\epsilon,\mu,g]$.

LEMMA 3.10. *Assume that* γ, α, t_0, $\beta > 0$, $\sigma \geq 0$, $\xi \in S^N_+ := S^N \bigcap \mathbb{R}^N_+$, $\varphi \in C_+$, *and* $\zeta \in BC(\mathbb{R}^N_+, \mathbb{R}_+)$ *such that* $b(s-\sigma\tau) \geq b_+ - \gamma > 0$, $\mu(s) \leq \mu_+ + \gamma < \overline{\mu}$ *for all* $s \geq \alpha$. *Suppose*

$$u^{\varphi}(t, x + [\sigma\xi + \kappa e_1]t + \alpha e_1; \mu(\cdot), b(\cdot), g) \geq \zeta(x) \text{ for all } (t,x) \in [t_0 - \tau, t_0 + \beta] \times \mathbb{R}^N_+.$$

Then

$$u^{\varphi}(t, x + [\sigma\xi + \kappa e_1]t + \alpha e_1; \mu(\cdot), b(\cdot), g) \geq Q^{\infty}_{\overline{c},\overline{\xi},t-t_0}\left[\zeta; \frac{\overline{\mu} - \mu_+ - \gamma}{\overline{\mu}}, \overline{\mu}, \frac{b_+ - \gamma}{\overline{\mu}} g\right](x)$$

for all $(t,x) \in [t_0, t_0 + \beta] \times \mathbb{R}^N_+$. *Here,* $\overline{c} = ||\sigma\xi + \kappa e_1||$ *and*

$$\overline{\xi} = \begin{cases} \dfrac{\sigma\xi + \kappa e_1}{||\sigma\xi + \kappa e_1||}, & ||\sigma\xi + \kappa e_1|| \neq 0, \\ e_1, & ||\sigma\xi + \kappa e_1|| = 0. \end{cases}$$

Proof. Let $u(t,x) = u^{\varphi}(t,x;\mu(\cdot),b(\cdot),g)$ for all $(t,x) \in [-\tau,\infty) \times \mathbb{R}^N$. Then by (3.2), we know that for any $(t,x) \in [t_0, t_0+\beta] \times \mathbb{R}^N_+$, there holds

$$\begin{aligned}
&u(t, x + \sigma t\xi + (\kappa t + \alpha)e_1) \\
&\quad \geq \int_{t_0}^{t} e^{-\overline{\mu}(t-s)} T(t-s)\Big[(\overline{\mu} - \mu([\cdot]_1 - \kappa s))u(s,\cdot) \\
&\qquad + b([\cdot]_1 - \kappa s + \kappa\tau) g(u(s-\tau,\cdot))\Big](x + \sigma t\xi + (\kappa t + \alpha)e_1)\mathrm{d}s \\
&\quad = \int_{t_0}^{t}\int_{\mathbb{R}^N} \frac{1}{\overline{\mu}} k(t-s, x + \sigma t\xi + (\kappa t+\alpha)e_1 - y; \overline{\mu})\Big[(\overline{\mu} - \mu(y_1 - \kappa s))u(s,y)\Big]\mathrm{d}y\mathrm{d}s \\
&\qquad + \int_{t_0}^{t}\int_{\mathbb{R}^N} \frac{1}{\overline{\mu}} k(t-s, x + \sigma t\xi + (\kappa t+\alpha)e_1 - y; \overline{\mu})\Big[b(y_1 - \kappa s + \kappa\tau) g(u(s-\tau,y))\Big]\mathrm{d}y\mathrm{d}s \\
&\quad \geq \int_{t_0}^{t}\int_{\sigma s\xi + (\kappa s + \alpha)e_1 + \mathbb{R}^N_+} \frac{k(t-s, x + \sigma t\xi + (\kappa t+\alpha)e_1 - y; \overline{\mu})}{\overline{\mu}}\Big[(\overline{\mu} - \mu(y_1 - \kappa s))u(s,y)\Big]\mathrm{d}y\mathrm{d}s \\
&\qquad + \int_{t_0}^{t}\int_{\sigma(s-\tau)\xi + (\kappa s - \kappa\tau + \alpha)e_1 + \mathbb{R}^N_+} \frac{1}{\overline{\mu}} k(t-s, x + \sigma t\xi + (\kappa t+\alpha)e_1 - y; \overline{\mu}) \\
&\qquad \times \Big[b(y_1 - \kappa s + \kappa\tau) g(u(s-\tau,y))\Big]\mathrm{d}y\mathrm{d}s \\
&\quad \geq \int_{0}^{t-t_0}\int_{\mathbb{R}^N_+} \frac{1}{\overline{\mu}} k(s, x + \sigma s\xi + \kappa s e_1 - y; \overline{\mu})\Big[(\overline{\mu} - \mu_+ - \gamma)u(t-s, y + [\sigma\xi + \kappa e_1](t-s) \\
&\qquad + \alpha e_1)\Big]\mathrm{d}y\mathrm{d}s + \int_{0}^{t-t_0}\int_{\mathbb{R}^N_+} \frac{1}{\overline{\mu}} k(s, x + \sigma(s+\tau)\xi + \kappa(s+\tau)e_1 - y; \overline{\mu}) \\
&\qquad \times [(b_+ - \gamma) g(u(t-s-\tau, y + [\sigma\xi + \kappa e_1](t-s-\tau) + \alpha e_1))]\mathrm{d}y\mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
&\geq \int_0^{t-t_0}\int_{\mathbb{R}^N_+}\frac{1}{\overline{\mu}}k(s,x+\sigma s\xi+\kappa s e_1-y;\overline{\mu})\big[(\overline{\mu}-\mu_+-\gamma)\zeta(y)\big]\mathrm{d}y\mathrm{d}s\\
&\quad+\int_0^{t-t_0}\int_{\mathbb{R}^N_+}\frac{1}{\overline{\mu}}k(s,x+\sigma(s+\tau)\xi+\kappa(s+\tau)e_1-y;\overline{\mu})\big[(b_+-\gamma)g(\zeta(y))\big]\mathrm{d}y\mathrm{d}s\\
&= Q^{\infty}_{\overline{c},\overline{\xi},t-t_0}\left[\zeta;\frac{\overline{\mu}-\mu_+-\gamma}{\overline{\mu}},\overline{\mu},\frac{b_+-\gamma}{\overline{\mu}}g\right](x). \qquad \square
\end{aligned}
$$

The following result produces comparative relationships between solutions of (3.2) and iterations of $Q^{\infty}_{c,\xi,\beta}[\cdot;\epsilon,\mu,g]$, which will be used to derive a priori estimates on nontrivial solutions to (3.2) under the travelling wave transformation.

Lemma 3.11. *Assume that γ, α, t_0, $\beta>0$, $\sigma\geq 0$, $\xi\in S^N_+$, $\varphi\in C_+$, and $\zeta\in BC(\mathbb{R}^N_+,\mathbb{R}_+)$ such that $b(s-\sigma\tau)\geq b_+-\gamma>0$, $\mu(s)\leq\mu_++\gamma<\overline{\mu}$ for all $s\geq\alpha$. Suppose*

$$u^{\varphi}(t,x+[\sigma\xi+\kappa e_1]t+\alpha e_1;\mu(\cdot),b(\cdot),g)\geq\zeta(x) \text{ for all } (t,x)\in[t_0-\tau,t_0+\beta]\times\mathbb{R}^N_+.$$

Then, for any $I\in\mathbb{N}$ and $i\in[1,I]\cap\mathbb{N}$, we have

$$u^{\varphi}(t,x+[\sigma\xi+\kappa e_1]t+\alpha e_1;\mu(\cdot),b(\cdot),g)\geq(Q^{\infty}_{\overline{c},\overline{\xi},\frac{\beta}{1+I}})^i\left[\zeta;\frac{\overline{\mu}-\mu_+-\gamma}{\overline{\mu}},\overline{\mu},\frac{b_+-\gamma}{\overline{\mu}}g\right](x)$$

for all $(t,x)\in[t_0+\frac{i\beta}{1+I}+(i-1)\tau,t_0+\beta]\times\mathbb{R}^N_+$ with $\frac{i\beta}{1+I}+(i-1)\tau<\beta$. Here, $\overline{c},\overline{\xi}$ is defined as in Lemma 3.10.

Proof. First, note that the case where $i=1$ follows from Lemma 3.10. Now, let

$$\hat{I}=\sup\left\{i\in[1,I]\cap\mathbb{N}:\frac{i\beta}{1+I}+(i-1)\tau<\beta\right\},$$

$u(t,x)=u^{\varphi}(t,x;\mu(\cdot),b(\cdot),g)$, and $Q_t[\psi]=Q^{\infty}_{\overline{c},\overline{\xi},t}[\psi;\frac{\overline{\mu}-\mu_+-\gamma}{\overline{\mu}},\overline{\mu},\frac{b_+-\gamma}{\overline{\mu}}g]$. Define

$$i^*=\sup\left\{i\in[1,\hat{I}]\cap\mathbb{N}:\begin{array}{l}u(t,x+[\sigma\xi+\kappa e_1]t+\alpha e_1)\geq(Q_{\frac{\beta}{1+I}})^j[\zeta](x)\text{ for all}\\ j\in[1,i]\text{ and }(t,x)\in[t_0+\frac{j\beta}{1+I}+(j-1)\tau,t_0+\beta]\times\mathbb{R}^N_+\end{array}\right\}.$$

Then $i^*\geq 1$. It suffices to prove $i^*=\hat{I}$. Otherwise, for any $(t,x)\in[t_0+\frac{(1+i^*)\beta}{1+I}+i^*\tau,t_0+\beta]\times\mathbb{R}^N_+$, it follows from Lemma 3.10 and the choice of i^* that

$$
\begin{aligned}
&u(t,x+[\sigma\xi+\kappa e_1]t+\alpha e_1)\\
&\quad\geq Q_{t-t_0-\frac{i^*\beta}{1+I}-i^*\tau}[(Q_{\frac{\beta}{1+I}})^{i^*}[\zeta]](x)\\
&\quad\geq Q_{\frac{\beta}{1+I}}[(Q_{\frac{\beta}{1+I}})^{i^*}[\zeta]](x)\\
&\quad=(Q_{\frac{\beta}{1+I}})^{i^*+1}[\zeta](x),
\end{aligned}
$$

which yields a contradiction with the choice of i^*. This proves the claim and hence the proof is complete. $\square$

The following result produces asymptotic propagation persistence, that is, some a priori traveling-like estimates on nontrivial solutions to (3.2), which play a key role in the proof of the upward convergence and global asymptotic behavior to (3.2).

PROPOSITION 3.12. *Suppose that g is nondereasing and subhomogeneous with $g'(0) > \frac{\mu_+}{b_+}$. Let $c \in [0, c^*(0, \mu_+, \frac{b_+}{\mu_+} g))$. Then there exist $\alpha_0 > 0$, $\varepsilon_0 > 0$, $T_0 > 19$, and $T^* > T_0$ such that, for all $\alpha \in [\alpha_0, \infty)$, $\varepsilon \in (0, \varepsilon_0]$, $T \in [T_0, \infty)$, $\sigma \in [0, c]$, $\xi \in S^{N-1}$, and solutions $u : [-\tau, \infty) \times \mathbb{R}^N \to \mathbb{R}_+$ of (3.2) satisfying $\sigma\xi \in \kappa e_1 + \mathbb{R}^N_+$ and $u(t, \sigma t\xi + \alpha e_1 + \cdot) \geq \varepsilon h^T$ for all $t \in [-\tau, T^*]$, we have $u(t, \sigma t\xi + \alpha e_1 + \cdot) \geq \varepsilon h^T$ for all $t \in \mathbb{R}_+$ and $u(t, \sigma t\xi + \alpha e_1 + \cdot) \gg \varepsilon h^T$ for all $t \in (T^*, \infty)$.*

Proof. Define

$$G : \mathbb{R}_+ \times (0, \infty) \ni (u, \gamma) \mapsto \frac{(b_+ - \gamma) g'(0)}{\mu_+ + \gamma} \min\{u, \gamma\} \in \mathbb{R}.$$

According to the definition of G and Lemma 3.1-(v), (vi), we know that there exists $\gamma \in (0, \min\{\overline{\mu} - \mu_+, b_+\})$ such that

$$\frac{\partial G(0, \gamma)}{\partial u} > 1, \ \frac{b_+ - \gamma}{\mu_+ + \gamma} g \geq G(\cdot, \gamma), \quad \text{and} \quad c \in \left[0, c^*\left(\frac{\overline{\mu} - \mu_+ - \gamma}{\overline{\mu}}, \overline{\mu}, \frac{\mu_+ + \gamma}{\overline{\mu}} G(\cdot, \gamma)\right)\right).$$

By applying Proposition 3.9, we know that there exist $n_0 \in \mathbb{N}$ and $t_0 > 19$ such that

$$(Q^\infty_{\sigma, \xi, \beta})^{n_0} \left[\varepsilon^* h^T; \frac{\overline{\mu} - \mu_+ - \gamma}{\overline{\mu}}, \overline{\mu}, \frac{\mu_+ + \gamma}{\overline{\mu}} G(\cdot, \gamma)\right] \geq 2\varepsilon^* h^T$$

with $\varepsilon^* = (b_+ - \gamma) g'(0) \gamma / 5(\mu_+ + \gamma)$ for all $\beta, T \geq t_0$, and $(\sigma, \xi) \in [0, c] \times S^{N-1}$.

Take $\alpha_0 > 0$ with $b(s - \sigma\tau) \geq b_+ - \gamma > 0$, $\mu(s) \leq \mu_+ + \gamma$ for all $s \geq \alpha_0$, and let $\varepsilon_0 = \frac{\varepsilon^*}{2}$, $T_0 = t_0$, and $T^* = (1 + n_0) t_0 + n_0^2 \tau$.

Suppose that $\alpha \in [\alpha_0, \infty)$, $\varepsilon \in (0, \varepsilon_0]$, $T \in [T_0, \infty)$, $\sigma \in [0, c]$, $\xi \in S^{N-1}$, and $u : [-\tau, \infty) \times \mathbb{R}^N \to \mathbb{R}_+$ is a bounded solution of (3.2) satisfying $\sigma\xi \in \kappa e_1 + \mathbb{R}^N_+$ and $u(t, \sigma t\xi + \alpha e_1 + \cdot) \geq \varepsilon h^T$ for all $t \in [-\tau, T^*]$. Let $T^{**} = \sup\{t \geq 0 : u(t, \sigma t\xi + \alpha e_1 + \cdot) \geq \varepsilon h^T$ for all $s \in [0, t]\}$. Then $T^{**} \geq T^*$. We claim that $T^{**} = \infty$. Otherwise, $T^{**} < \infty$. It follows from Lemma 3.11 that, for any $t \in [\frac{n_0 T^{**}}{n_0 + 1} + (n_0 - 1)\tau, T^{**}]$, we have

$$\begin{aligned} u(t, \sigma t\xi + \alpha e_1 + \cdot) &\geq (Q^\infty_{\sigma, \xi, \frac{T^*}{n_0+1}})^{n_0} \left[\varepsilon h_T; \frac{\overline{\mu} - \mu_+ - \gamma}{\overline{\mu}}, \overline{\mu}, \frac{\mu_+ + \gamma}{\overline{\mu}} G(\cdot, \gamma)\right] \\ &\geq (Q^\infty_{\sigma, \xi, T_0})^{n_0} \left[\frac{\varepsilon}{\varepsilon^*} \varepsilon^* h_T; \frac{\overline{\mu} - \mu_+ - \gamma}{\overline{\mu}}, \overline{\mu}, \frac{\mu_+ + \gamma}{\overline{\mu}} G(\cdot, \gamma)\right] \\ &\geq \frac{\varepsilon}{\varepsilon^*} (Q^\infty_{\sigma, \xi, T_0})^{n_0} \left[\varepsilon^* h_T; \frac{\overline{\mu} - \mu_+ - \gamma}{\overline{\mu}}, \overline{\mu}, \frac{\mu_+ + \gamma}{\overline{\mu}} G(\cdot, \gamma)\right] \geq 2\varepsilon h^T. \end{aligned}$$

In particular, $u(T^{**}, \sigma T^{**}\xi + \alpha e_1 + \cdot) \gg \varepsilon h^T$, and thus there exists $\delta > 0$ such that $u(t, \sigma t\xi + \alpha e_1 + \cdot) \geq \varepsilon h^T$ for all $t \in [T^{**}, T^{**} + \delta]$, a contradiction with the choice of T^{**}. So, $T^{**} = \infty$. ∎

4. Main results. In this section, we shall obtain our main results about propagation dynamics of (1.3). Without loss of generality, we first consider the case $\nu = e_1$ in subsections 4.1, 4.2, and 4.3. Then in subsection 4.4, we will establish some results for (1.3) with a general direction ν.

With $\nu = e_1$, (1.3) reads

(4.1)
$$\frac{\partial u}{\partial t}(t, x) = d\Delta u(t, x) - \mu(x_1 - \kappa t) u(t, x) + e^{-\delta\tau} b(x_1 - \kappa(t - \tau)) u(t - \tau, x) e^{-u(t-\tau, x)}.$$

For convenience, by the travelling wave transformation $w(t,x) = u(t, x+\kappa e_1 t)$, we introduce the auxiliary equations:

$$\frac{\partial w}{\partial t}(t,x) = d\Delta w(t,x) + \kappa w_{x_1} - \mu(x_1)w(t,x) + e^{-\delta\tau} b(x_1+\kappa\tau)w(t-\tau, x+\kappa\tau e_1)e^{-w(t-\tau,x+\kappa\tau e_1)}, \tag{4.2}$$

as well as the associated auxiliary equations:

$$\frac{\partial w}{\partial t}(t,x) = d\Delta w(t,x) + \kappa w_{x_1} - \mu(x_1)w(t,x) + b(x_1+\kappa\tau)g_{\mathcal{M}}(w(t-\tau, x+\kappa\tau e_1)). \tag{4.3}$$

Here, $g_{\mathcal{M}} : \mathbb{R}_+ \to \mathbb{R}$ is defined by

$$g_{\mathcal{M}}(u) = e^{-\delta\tau} \min\{ue^{-u}, \mathcal{M}e^{-\mathcal{M}}\} \tag{4.4}$$

with $\mathcal{M} > 0$, where $u \in \mathbb{R}_+$. Then, for any $\mathcal{M} > 0$, we know that $g_{\mathcal{M}}$ is nondecreasing and subhomogeneous on $[0,\mathcal{M}]$, $g_{\mathcal{M}}(u) \le e^{-\delta\tau}ue^{-u}$ for all $u \in [0,\mathcal{M}]$, and $g_{\mathcal{M}}(u) = e^{-\delta\tau}ue^{-u}$ for all small nonnegative u.

Note that $c^*(0,\mu_+,\frac{b_+}{\mu_+}g_{\mathcal{M}}) = c^*(0,\mu_+,\frac{b_+}{\mu_+}e^{-\delta\tau}Id_{\mathbb{R}_+})$ for all $\mathcal{M}$ in $(0,\infty)$. As a result, we may always denote $c^*(0,\mu_+,\frac{b_+}{\mu_+}g_{\mathcal{M}})$ by c^*. Clearly,

$$c^* = \inf\left\{\sigma \in \mathbb{R}_+ : b_+e^{-\delta\tau}e^{-\sigma\tau\rho} \le \mu_+ + \sigma\rho - d\rho^2, \exists \rho \in \left(0, \frac{\sigma+\sqrt{\sigma^2+4d\mu_+}}{2d}\right)\right\}. \tag{4.5}$$

We point out that this c^* is nothing but the minimal wave speed and the spreading speed of the limiting equation of (1.3) (including (4.1)) with the two shifting parameters replaced by μ^+ and b^+, respectively, under the assumption W_1. By the definition of c^*, we see that the speed of c^* depends on the delay τ and it is decreasing in τ.

For given $\varphi \in C_+$, by the method of steps, is easy to see (4.2) and (4.3) have mild solutions $w^{\varphi}(t,x), w^{\varphi;g_{\mathcal{M}}}(t,x) \in \mathbb{R}_+$ with the initial value $w_0 = \varphi$ which exists for all $t \ge 0$. Clearly, mild solutions $w^{\varphi}(t,x), w^{\varphi;g_{\mathcal{M}}}(t,x)$ are also classical solutions of (4.2), (4.3), respectively, for all $t > \tau$ (see, e.g., [19, 20, 31, 34]).

Denote by Φ the solution semiflow of (4.2), that is, $\Phi : \mathbb{R}_+ \times C_+ \to C_+$ is defined by $\Phi(t,\varphi) = (w^{\varphi})_t$ for all $(t,\varphi) \in \mathbb{R}_+ \times C_+$. Simarly, we write $\Phi(t,\varphi;g_{\mathcal{M}})$ for $(w^{\varphi;g_{\mathcal{M}}})_t$ if there is a need.

The following result shows that any nontrivial solution of (4.3) is far away from the trivial equilibrium, which is key to proving upward convergence of nontrivial solutions and the existence of force steady states.

PROPOSITION 4.1. *Assume that $\mathcal{M} > 0$ and $\kappa \in \mathbb{R}$, and let c^* and $g_{\mathcal{M}}$ be given, respectively, by (4.5) and (4.4). Then, for any $c \in [0,c^*) \cap (\kappa,\infty)$, $\varphi \in C_+ \setminus \{0\}$, there exist $\varepsilon_{c,\varphi} > 0$ and $\alpha_{c,\varphi} > 0$ such that $w^{\varphi;g_{\mathcal{M}}}(t,x) \ge \varepsilon_{c,\varphi}$ for all $(t,x) \in \Omega^*_{c,\alpha_{c,\varphi}}$, where*

$$\Omega^*_{c,\alpha} = \{(t,x) \in \mathbb{R}_+ \times \mathbb{R}^N : t \ge \alpha, x \ge \alpha e_1, \text{ and } ||x+\kappa e_1 t|| \le ct\}$$

for all $\alpha \in \mathbb{R}_+$. Hence, if $|\kappa| < c$, then $\psi(\cdot,x) \ge \varepsilon_{c,\varphi}$ for all $\psi \in \omega(\varphi;g_{\mathcal{M}}) := \omega(\varphi;\Phi(\cdot,\cdot;g_{\mathcal{M}})), x \in \alpha_{c,\varphi}e_1 + \mathbb{R}^N_+$. Moreover, if $|\kappa| < c$, then for any $\gamma > 0$, there exists $\varepsilon_{\gamma,c,\varphi} > 0$ such that $\psi(\cdot,x) \ge \varepsilon_{\gamma,c,\varphi}$ for all $\psi \in \omega(\varphi;g_{\mathcal{M}}), x \in -\gamma e_1 + \mathbb{R}^N_+$.

Proof. Take $\tilde{c} = \frac{c+c^*}{2}$ and select α_0, ε_0, T_0, and T^* with $\tilde{c}$ as in Proposition 3.12. Without loss of generality, we may assume that $u(t,x) := w^{\varphi;g_{\mathcal{M}}}(t, x - \kappa e_1 t) > 0$ for all $(t,x) \in [-\tau, \infty) \times \mathbb{R}^N$. Define

$$\varepsilon_1 = \inf\{u(t,x) : t \in [-\tau, T^*] \quad \text{and} \quad ||x|| \leq 1 + \alpha_0 + (\tilde{c}+6)T^*\},$$

and let $\varepsilon_{c,\varphi} = \min\{\varepsilon_0, \varepsilon_1\}$. Then $\varepsilon_1 > 0$, $\varepsilon_{c,\varphi} > 0$, and $u(t, \sigma t\xi + \alpha_0 e_1 + \cdot) \geq \varepsilon_{c,\varphi} h^T$ for all $t \in [-\tau, T^*]$, $T \in [T_0, 2T_0]$, $\sigma \in [0, \tilde{c}]$, and $\xi \in S^{N-1}$ with $\sigma\xi \in \kappa e_1 + \mathbb{R}^N_+$. It follows from Proposition 3.12 and the choices of α_0, ε_0, T_0, and T^* that $u(t, \sigma t\xi + \alpha_0 e_1 + \cdot) \geq \varepsilon_{c,\varphi} h^T$ for all $t \in \mathbb{R}_+$, $T \in [T_0, 2T_0]$, $\sigma \in [0, \tilde{c}]$, $\xi \in S^{N-1}$ with $\sigma\xi \in \kappa e_1 + \mathbb{R}^N_+$. In particular, $u(t, \sigma t\xi + \alpha_0 e_1 + T_0 e_1) \geq \varepsilon_{c,\varphi}$ for all $t \in \mathbb{R}_+$, $\sigma \in [0, \tilde{c}]$, $\xi \in S^{N-1}$ with $\sigma\xi \in \kappa e_1 + \mathbb{R}^N_+$. Letting

$$\alpha_{c,\varphi} := \max\left\{T_0 + \alpha_0, 1 + \frac{2(\alpha_0 + T_0)}{c^* - c}\right\},$$

we easily check that

$$(t,x) \in \Omega^*_{c,\alpha_{c,\varphi}} \textbf{ implies } x - \alpha_0 e_1 - T_0 e_1 \in \mathbb{R}^N_+ \quad \text{and} \quad \frac{||x + \kappa e_1 t - \alpha_0 e_1 - T_0 e_1||}{t} \in [0, \tilde{c}].$$

As a result, $w^{\varphi;g_{\mathcal{M}}}(t,x) = u(t, x + \kappa e_1 t) \geq \varepsilon_{c,\varphi}$ for all $(t,x) \in \Omega^*_{c,\alpha_{c,\varphi}}$. This, combined with the definition of $\omega(\varphi; g_{\mathcal{M}})$, implies $\psi(\cdot, x) \geq \varepsilon_{c,\varphi}$ for all $\psi \in \omega(\varphi; g_{\mathcal{M}}), x \in \alpha_{c,\varphi} e_1 + \mathbb{R}^N_+$.

Next, let us fix $\gamma > 0$ and take

$$\varepsilon_{\gamma,c,\varphi} = \frac{\varepsilon_{c,\varphi} e^{-\overline{\mu}}}{\sqrt{\pi}} \int_{\frac{\alpha_{c,\varphi}+c+\gamma}{\sqrt{4d}}}^{\infty} \exp\left(-s^2\right) \mathrm{d}s > 0.$$

According to the definition of the mild solutions of (4.3), we know that for any $(x, \psi) \in \mathbb{R}^N \times \omega(\varphi; g_{\mathcal{M}})$, there holds

$$\begin{aligned} w^{\psi;g_{\mathcal{M}}}(1,x) &\geq e^{-\overline{\mu}} T(1)[\psi(0,\cdot)](x + \kappa e_1) \\ &= \int_{\mathbb{R}^N} \frac{e^{-\overline{\mu}}}{(4d\pi)^{\frac{N}{2}}} \exp\left(-\frac{||x + \kappa e_1 - y||^2}{4d}\right) \psi(0,y) \mathrm{d}y \\ &\geq \int_{\alpha_{c,\varphi} e_1 + \mathbb{R}^N_+} \frac{e^{-\overline{\mu}}}{(4d\pi)^{\frac{N}{2}}} \exp\left(-\frac{||x + \kappa e_1 - y||^2}{4d}\right) \varepsilon_{c,\varphi} \mathrm{d}y \\ &= \varepsilon_{c,\varphi} e^{-\overline{\mu}} \int_{\alpha_{c,\varphi}}^{\infty} \frac{1}{\sqrt{4d\pi}} \exp\left(-\frac{|x_1 + \kappa - y_1|^2}{4d}\right) \mathrm{d}y_1 \\ &= \frac{\varepsilon_{c,\varphi} e^{-\overline{\mu}}}{\sqrt{\pi}} \int_{\frac{\alpha_{c,\varphi} - \kappa - x_1}{\sqrt{4d}}}^{\infty} \exp\left(-s^2\right) \mathrm{d}s. \end{aligned}$$

This, together with the invariance of $\omega(\varphi; g_{\mathcal{M}})$, implies that

$$\psi(\cdot, x) \geq \varepsilon_{\gamma,c,\varphi}$$

for all $\psi \in \omega(\varphi; g_{\mathcal{M}}), x \in -\gamma e_1 + \mathbb{R}^N_+$. The proof is complete. ∎

4.1. Asymptotic extinction. In this subsection, we will study the asymptotic extinction properties of (4.2) by constructing some appropriate test functions. We first confirm the asymptotic extinction in the direction $-e_1$.

THEOREM 4.2. *For any $\varphi \in C_+$, there holds*

$$\lim_{\alpha\to\infty}\left[\sup\{w^{\varphi}(t,x):(t,x)\in\mathbb{R}_+\times\mathbb{R}^N \text{ with } t,-x_1\in[\alpha,\infty)\}\right]=0.$$

Hence, $\lim_{t\to\infty}[\sup_{x_1\le-\varepsilon t} w^{\varphi}(t,x)]=0\ \forall\varepsilon>0$.

Proof. Fix $(\varphi,\epsilon_0)\in C_+\times(0,\frac{\mu_- - b_- e^{-\delta\tau}}{1+e^{-\delta\tau}})$. Then, there exist $\alpha_0,\lambda_0,\gamma_0>0$ such that

$$\begin{cases} \lambda_0-\mu_-+\epsilon_0+e^{-\delta\tau}(b_-+\epsilon_0)e^{-\lambda_0\tau}=0,\\ d(\gamma_0)^2+\kappa\gamma_0-\mu_-+\epsilon_0+e^{-\delta\tau}(b_-+\epsilon_0)e^{\gamma_0\kappa\tau}=0,\\ \mu(\alpha-\alpha_0)-\mu_-+\epsilon_0\ge 0, b(\alpha-\alpha_0+\kappa\tau)-b_--\epsilon_0\le 0\ \forall\alpha\in(-\infty,0].\end{cases}$$

Define $Z:[-\tau,\infty)\times\mathbb{R}^N\to\mathbb{R}$ by

$$Z(t,x)=w^{\varphi}(t,x-\alpha_0 e_1)-M_0e^{-\lambda_0 t}-M_0e^{\gamma_0 x_1}\ \forall(t,x)\in[-\tau,\infty)\times\mathbb{R}^N,$$

where $M_0=\sup_{(t,x)\in[-\tau,\infty)\times\mathbb{R}^N}|w^{\varphi}(t,x)|$. It follows from (4.2) that

$$\begin{cases} \dfrac{\partial Z}{\partial t}(t,x)\le d\Delta Z(t,x)+\kappa Z_{x_1}-(\mu_--\epsilon_0)Z(t,x)\\ \qquad\qquad +e^{-\delta\tau}(b_-+\epsilon_0)Z(t-\tau,x+\kappa\tau e_1),(t,x)\in\mathbb{R}_+\times((-\infty,0]\times\mathbb{R}^{N-1}),\\ Z(t,x)\le 0,(t,x)\in[-\tau,0]\times\mathbb{R}^N\bigcup[-\tau,\infty)\times\mathbb{R}^N_+.\end{cases}$$

By applying the step arguments and the Phragmén–Lindelöf type maximum principle [23], we may obtain that $Z(t,x)\le 0$ for all $(t,x)\in[-\tau,\infty)\times\mathbb{R}^N$. This and the definition of Z directly lead to our results. The proof is now complete. ∎

Next, we show the asymptotic extinction beyond a moving hyperplane in the direction e_1.

THEOREM 4.3. *If $\varepsilon>0$ and $\varphi\in C_+$ with $\varphi(\cdot,x)$ being zero for all sufficiently positive x_1, then*

$$\lim_{t\to\infty}\left[\sup\{w^{\varphi}(t,x):(t,x)\in\mathbb{R}_+\times\mathbb{R}^N \text{ with } x_1\ge t(\varepsilon+c^*-\kappa)\}\right]=0.$$

Proof. In view of the definition of c^*, there exist $\epsilon_0\in(0,\frac{\varepsilon}{2})$ and $\lambda>0$ such that

$$-d\lambda^2+(c+\kappa)\lambda+\mu_+=b_+e^{-\delta\tau}e^{-\lambda(c+\kappa)\tau} \text{ with } c:=c^*-\kappa+\epsilon_0.$$

Let $M=e^{\lambda\tau|c|}\sup_{x\in\mathbb{R}^N}(||\varphi(\cdot,x)||e^{\lambda x_1})$ and define $Z:[-\tau,\infty)\times\mathbb{R}^N\to\mathbb{R}$ by

$$Z(t,x)=w^{\varphi}(t,x)-Me^{-\lambda(x_1-ct)}\ \forall(t,x)\in[-\tau,\infty)\times\mathbb{R}^N.$$

It follows from (4.2) that

$$\begin{cases} \dfrac{\partial Z}{\partial t}(t,x)\le d\Delta Z(t,x)+\kappa Z_{x_1}-\mu_+Z(t,x)\\ \qquad\qquad +e^{-\delta\tau}b_+Z(t-\tau,x+\kappa\tau e_1),(t,x)\in(0,\infty)\times\mathbb{R}^N,\\ Z(t,x)\le 0,(t,x)\in[-\tau,0]\times\mathbb{R}^N.\end{cases}$$

By applying the step arguments and the Phragmén–Lindelöf type maximum principle [23], we may obtain that $Z(t,x)\le 0$ for all $(t,x)\in[-\tau,\infty)\times\mathbb{R}^N$. This and the definition of Z directly lead to our results. The proof is now complete. ∎

Now, we explore asymptotic extinction in other directions by using similar argument to Theorem 4.3.

THEOREM 4.4. *If $\varepsilon > 0$, $\xi \in S^{N-1}$, $\varphi \in C_+$ with $\varphi(\cdot, x)$ being zero for all sufficiently positive $x \cdot \xi$, then*

$$\lim_{t\to\infty} \left[\sup\{w^{\varphi}(t,x) : (t,x) \in \mathbb{R}_+ \times \mathbb{R}^N \text{ with } x\cdot\xi \geq t(\varepsilon + c^* - \kappa\xi_1\}\right] = 0.$$

Proof. In view of the definition of c^* that there exist $\epsilon_0 \in (0, \frac{\epsilon}{2})$ and $\lambda > 0$ such that

$$-d\lambda^2 + (c+\kappa\xi_1)\lambda + \mu_+ = b_+e^{-\delta\tau}e^{-\lambda(c+\kappa\xi_1)\tau} \text{ with } c = c^* - \kappa\xi_1 + \varepsilon_0.$$

Let $M = e^{\lambda\tau|c|}\sup_{x\in\mathbb{R}^N}(||\varphi(\cdot,x)||e^{\lambda x\cdot\xi})$ and define $Z : [-\tau,\infty)\times\mathbb{R}^N \to \mathbb{R}$ by

$$Z(t,x) = w^{\varphi}(t,x) - Me^{-\lambda(x\cdot\xi - ct)} \; \forall (t,x) \in [-\tau,\infty)\times\mathbb{R}^N.$$

It follows from (4.2) that

$$\begin{cases} \dfrac{\partial Z}{\partial t}(t,x) \leq d\Delta Z(t,x) + \kappa Z_{x_1} - \mu_+ Z(t,x) \\ \qquad\qquad + e^{-\delta\tau}b_+Z(t-\tau, x+\kappa\tau e_1), (t,x) \in (0,\infty)\times\mathbb{R}^N, \\ Z(t,x) \leq 0, (t,x) \in [-\tau, 0]\times\mathbb{R}^N. \end{cases}$$

By applying the step arguments and the Phragmén–Lindelöf type maximum principle [23], we may obtain that $Z(t,x) \leq 0$ for all $(t,x) \in [-\tau,\infty)\times\mathbb{R}^N$. This and the definition of Z directly lead to our results. The proof is now completed. ∎

4.2. Upward convergence. In this subsection, we are ready to derive upward convergence of nontrivial solutions of (4.2) in space-time region.

THEOREM 4.5. *Assuming that $\frac{b_+}{\mu_+}e^{-\delta\tau} \leq e^2$. If $\kappa < c^*$ and $\varphi \in C_+\backslash\{0\}$, then for any $c \in (\max\{\kappa, 0\}, c^*)$, there holds*

$$\lim_{\alpha\to\infty}[\inf\{w^{\varphi}(t,x) : (t,x) \in \Omega^*_{c,\alpha}\}] = \lim_{\alpha\to\infty}[\sup\{w^{\varphi}(t,x) : (t,x) \in \Omega^*_{c,\alpha}\}] = w^*,$$

where $\Omega^*_{c,\alpha}$ is defined as in Proposition 4.1 and $w^* = \log(\frac{b_+e^{-\delta\tau}}{\mu_+})$. Hence, if $|\kappa| < c^*$, then $\lim_{x_1\to\infty}\psi(\cdot,x) = w^*$ for all $\psi \in \omega(\varphi)$.

Proof. Suppose that $\kappa < c^*$, $\varphi \in C_+\backslash\{0\}$, and $c \in (\max\{\kappa,0\}, c^*)$. By Proposition 4.1, there exist $\varepsilon^*, \alpha^* > 0$ such that

$$w(t,x) := w^{\varphi}(t,x) \leq M^* := \max\left\{||\varphi||_{L^\infty}, \frac{b_+}{\mu_+e^{\delta\tau+1}}\right\}$$

for all $(t,x) \in [-\tau,\infty)\times\mathbb{R}^N$ and $w(t,x) \geq \varepsilon^*$ for all $(t,x) \in \Omega^*_{\frac{c+c^*}{2},\alpha^*}$. For any $\varepsilon \in [0, \frac{c^*-c}{2}]$, define

$$W_-(\varepsilon) = \lim_{\alpha\to\infty}[\inf\{w(t,x) : (t,x) \in \Omega^*_{c+\varepsilon,\alpha}\}]$$

and

$$W_+(\varepsilon) = \lim_{\alpha\to\infty}[\sup\{w(t,x) : (t,x) \in \Omega^*_{c+\varepsilon,\alpha}\}].$$

Then $\varepsilon^* \leq W_-(\varepsilon) \leq W_+(\varepsilon) \leq M^*$ for all $\varepsilon \in [0, \frac{c^*-c}{2}]$. Since $W_\pm(\varepsilon)$ are monotone in $\varepsilon \in [0, \frac{c^*-c}{2}]$, we may assume, without loss of generality, that W_- and W_+ are continuous at some $\varepsilon_1 \in [0, \frac{c^*-c}{2}]$.

To complete the proof, we only prove $W_-(\varepsilon_1) = W_+(\varepsilon_1) = w^*$. Otherwise, $W_-(\varepsilon_1) < W_+(\varepsilon_1)$ or $W_-(\varepsilon_1) = W_+(\varepsilon_1) \neq w^*$. By letting $f_\gamma(w) = \frac{b_+-\gamma}{\mu_++\gamma} we^{-w-\delta\tau}$ and applying Lemma 2.3 in [44], we easily see that either (I) $f_0([W_-(\varepsilon_1), W_+(\varepsilon_1)]) \subseteq (0, W_+(\varepsilon_1))$ or (II) $f_0([W_-(\varepsilon_1), W_+(\varepsilon_1)]) \subseteq (W_-(\varepsilon_1), \infty)$. It suffices to consider case (II) since case (I) can be dealt by similar arguments. In view of (II) and the definition of f_γ, we know that

$$f_{\gamma_0}([W_-(\varepsilon_1) - \gamma_0, W_+(\varepsilon_1) + \gamma_0]) \subseteq (W_-(\varepsilon_1) + \gamma_0, \infty) \text{ for some } \gamma_0 > 0.$$

According to the definitions $b_+, \mu_+, W_\pm(\varepsilon_1)$, we may obtain that there exists $\alpha_0 > \alpha^*$ such that

$$\begin{cases} W_+(\varepsilon_1) + \gamma_0 \geq w(t,x) \geq W_-(\varepsilon_1) - \gamma_0 & \forall (t,x) \in \Omega^*_{c+\varepsilon_1,\alpha_0,}, \\ \mu(x_1) < \mu_+ + \gamma_0, b(x_1 + \kappa\tau) > b_+ - \gamma_0 & \forall x_1 \in [\alpha_0, \infty). \end{cases}$$

Take $\varrho > \max\{1, c^*, |\kappa|, \tau\}$ such that $\int_0^\varrho \int_{||y||\leq\varrho} k(s, y; \mu_+ + \gamma_0)\mathrm{d}y\mathrm{d}s > \frac{W_+(\varepsilon_1)+\frac{2\gamma_0}{3}}{W_+(\varepsilon_1)+\gamma_0}$. Fix $\zeta \in (0, \varepsilon_1)$ and choose $\alpha_1 > 1 + \max\{\alpha_0 + 3\varrho^2, \frac{7\varrho^2}{\varepsilon_1-\zeta}\}$ with

$$\frac{\mu_- M^*}{2(\mu_+ + \gamma_0)} \exp\left(-\sqrt{\frac{\mu_+ + \gamma_0}{d}}(\alpha_1 - \alpha_0 - |\kappa|\varrho)\right) < \frac{\gamma_0}{3}.$$

As a result, we may verify that for any $(t,x) \in \Omega^*_{c+\zeta,\alpha_1}$, there holds

$$||x + \kappa e_1 s - y|| \leq \varrho \textbf{ implies } y_1 \geq \alpha_0 \text{ and } (t - s - \tau, y + \kappa\tau e_1) \in \Omega^*_{c+\varepsilon_1,\alpha_0} \ \forall s \in [0, \varrho].$$

By letting $T_{\mu_++\gamma_0}(s)[\phi](x) = e^{-(\mu_++\gamma_0)s} T(s)[\phi](x+\kappa e_1 s)$ $\forall (s,x,\phi) \in \mathbb{R}_+ \times \mathbb{R}^N \times X$, it follows from (4.2), Fubini's theorem, and Lemma 2.1-(iv) in [39] that for any $(t,x) \in \Omega^*_{c+\zeta,\alpha_1}$, there holds

$$\begin{aligned}
w(t,x) &= w^{w_{t-\varrho}}(\varrho, x) \\
&= T_{\mu_++\gamma_0}(\varrho)[w(t-\varrho,\cdot)](x) + \int_0^\varrho T_{\mu_++\gamma_0}(\varrho - s)\Big[[\mu_+ + \gamma_0 - \mu(\cdot\, e_1)]w(s+t-\varrho,\cdot) \\
&\quad + \frac{\mu_+ + \gamma_0}{b_+ - \gamma_0} b(\cdot\, e_1 + \kappa\tau) f_{\gamma_0}(w(s+t-\varrho-\tau, \cdot + \kappa\tau e_1))\Big](x)\mathrm{d}s \\
&\geq \frac{1}{\mu_+ + \gamma_0} \int_0^\varrho \int_{\mathbb{R}^N} k(\varrho - s, x + \kappa e_1(\varrho - s) - y; \mu_+ + \gamma_0)(\mu_+ + \gamma_0 - \mu(y_1)) \\
&\quad \times w(s+t-\varrho, y)\mathrm{d}y\mathrm{d}s + \frac{1}{\mu_+ + \gamma_0}\int_0^\varrho \int_{\mathbb{R}^N} k(\varrho - s, x + \kappa e_1(\varrho - s) - y; \mu_+ + \gamma_0) \\
&\quad \times \frac{\mu_+ + \gamma_0}{b_+ - \gamma_0} b(y_1 + \kappa\tau) f_{\gamma_0}(w(s+t-\varrho-\tau, y + \kappa\tau e_1))\mathrm{d}y\mathrm{d}s \\
&\geq -\frac{\mu_- M^*}{\mu_+ + \gamma_0} \int_0^\varrho \int_{(-\infty,\alpha_0]\times\mathbb{R}^{N-1}} k(\varrho - s, x + \kappa e_1(\varrho - s) - y; \mu_+ + \gamma_0)\mathrm{d}y\mathrm{d}s \\
&\quad + (W_-(\varepsilon_1) + \gamma_0) \int_0^\varrho \int_{||x+\kappa e_1 s - y||\leq\varrho} k(s, x + \kappa e_1 s - y; \mu_+ + \gamma_0)\mathrm{d}y\mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
&\geq W_-(\varepsilon_1)+\frac{2\gamma_0}{3}-\frac{\mu_-M^*}{\mu_++\gamma_0}\int_{-\infty}^{\alpha_0}\int_0^{\varrho}\frac{(\mu_++\gamma_0)e^{-(\mu_++\gamma_0)s}}{\sqrt{4d\pi s}}\\
&\quad\times\exp\left(-\frac{(x_1-|\kappa|\varrho-y_1)^2}{4ds}\right)\mathrm{d}s\mathrm{d}y_1\\
&\geq W_-(\varepsilon_1)+\frac{2\gamma_0}{3}-\frac{\mu_-M^*}{2(\mu_++\gamma_0)}\int_{-\infty}^{\alpha_0}\sqrt{\frac{\mu_++\gamma_0}{d}}\exp\left(-\sqrt{\frac{\mu_++\gamma_0}{d}}|x_1-|\kappa|\varrho-y_1|\right)\mathrm{d}y_1\\
&\geq W_-(\varepsilon_1)+\frac{2\gamma_0}{3}-\frac{\mu_-M^*}{2(\mu_++\gamma_0)}\exp\left(-\sqrt{\frac{\mu_++\gamma_0}{d}}(\alpha_1-|\kappa|\varrho-\alpha_0)\right)\\
&\geq\frac{\gamma_0}{3}+W_-(\varepsilon_1).
\end{aligned}
$$

This, together with the definition of $W_-(\zeta)$, implies $W_-(\zeta)\geq W_-(\varepsilon_1)+\frac{\gamma_0}{3}$ $\forall\zeta\in(0,\varepsilon_1)$, a contradiction with the continuity of $W_-(\cdot)$ at ε_1. This completes the proof. ∎

4.3. Existence/nonexistence and attractivity of forced steady states. In this subsection, we shall establish the existence, nonexistence, and attractivity of forced steady states of (4.2). First, we shall prove the existence and attractivity of forced steady states of (4.2) when $|\kappa|<c^*$.

THEOREM 4.6. *Assume that* $|\kappa|<c^*$. *Then* (4.2) *has a positive steady state* w_+. *Moreover, if either (*$\kappa=0$ *and* $\frac{b_+}{\mu_+}e^{-\delta\tau}\leq e^2$*) or (*$\kappa\neq 0$ *and* $\frac{b_+e^{-\delta\tau}}{\mu_+}\leq 2e$*), then* (4.2) *has a unique positive steady state* w_+ *which attracts all solutions of* (4.2) *with the initial value* $\varphi\in C_+\setminus\{0\}$*, in the sense that* $\lim_{t\to\infty}||(w^{\varphi})_t-w_+||=0$ *for all* $\varphi\in C_+\setminus\{0\}$.

Proof. First, we shall establish the existence of w_+. Let $\mathcal{M}=\frac{b^+}{\mu^+}e^{-\delta\tau-1}$, and let $\varepsilon_{c,\frac{\mathcal{M}}{3}},\alpha_{c,\frac{\mathcal{M}}{3}}$ defined as in Proposition 4.1 with $c=\frac{|k|+c^*}{2}$. Then $\omega(\frac{\mathcal{M}}{3};g_{\mathcal{M}})\geq\varepsilon_{\frac{\mathcal{M}}{3}}$ for all $x\in\alpha_{c,\frac{\mathcal{M}}{3}}e_1+\mathbb{R}_+^N$. Note that $0\leq\Phi(t,\psi;g_{\mathcal{M}})\leq\Phi(t,\varphi;g_{\mathcal{M}})\leq\Phi(t,\varphi)$ for all $(t,\varphi,\psi)\in\mathbb{R}_+\times C_{\mathcal{M}}\times C_{\mathcal{M}}$ with $\psi\leq\varphi$. Let

$$\mathcal{A}=\left\{\varphi\in C_+:\omega\left(\frac{\mathcal{M}}{3};g_{\mathcal{M}}\right)\leq\varphi\leq\mathcal{M}\right\}.$$

Clearly, $\mathcal{A}$ is a nonempty, closed, and convex subset in C such that $\mathcal{A}\subseteq C_+^\circ$ and $\Phi(t,\mathcal{A})\subseteq\mathcal{A}$ for all $t\geq 0$. This, together with the compactness of Φ and Theorem 3.4.7 in [8], implies that Φ has a positive steady state $w_+\in\mathcal{A}$, located in $C_+^\circ\cap C_{\mathcal{M}}$.

Next, the uniqueness will be a consequence of the global attractiveness of w_+ in $C_+\setminus\{0\}$. So, we need to show that w_+ attracts all solutions of (4.2) with the initial value $\varphi\in C_+\setminus\{0\}$. Fix $\varphi\in C_+\setminus\{0\}$. Let $\gamma_0:=1+\mathcal{M}+||\varphi||_{L^\infty([-\tau,0]\times\mathbb{R}^N,\mathbb{R})}$. It follows from Theorem 4.2 and the conditions of $\mu(\cdot),b(\cdot)$ there exist $\mu_0,b_0,\lambda_0>0$, and $\rho_0>1$ such that

$$
\begin{cases}
w_+(x)\leq 1,\psi(\cdot,x)\leq 1\ \forall x\in(-\infty,-\rho_0+\kappa\tau]\times\mathbb{R}^{N-1},\psi\in\omega(\varphi),\\
\mu(x_1-\rho_0)>\mu_0,b(x_1+\kappa\tau-\rho_0)<b_0\ \forall x_1\leq 0,\\
\lambda_0-\mu_0+e^{-\delta\tau}b_0e^{\lambda_0\tau}=0.
\end{cases}
$$

Let

$$a_-^*=\sup\{a\in\mathbb{R}:\psi(\theta,x)\geq aw_+(x)\ \forall(\theta,x,\psi)\in[-\tau,0]\times([-\rho_0,\infty)\times\mathbb{R}^{N-1})\times\omega(\varphi)\}$$

and

$$a_+^*=\inf\{a\in\mathbb{R}:\psi(\theta,x)\leq aw_+(x)\ \forall(\theta,x,\psi)\in([-\tau,0]\times([-\rho_0,\infty))\times\mathbb{R}^{N-1})\times\omega(\varphi)\}.$$

Then $0 < a_-^* \leq 1 \leq a_+^* < \infty$ due to Proposition 4.1 and Theorem 4.5. For any $\psi \in \omega(\varphi)$, set

$$v(t,x) = \Phi[t+\tau,\psi](0,x-\rho_0 e_1) - a_-^* w_+(x-\rho_0 e_1) + \gamma_0 e^{-\lambda_0 t} \;\forall (t,x) \in [-\tau,\infty)\times\mathbb{R}^N.$$

By (4.2) and the choices of a_-^*, ρ_0, γ_0, and λ_0, it follows that $v(t,x)$ satisfies the following equation:

$$\begin{cases} \dfrac{\partial v}{\partial t}(t,x) \geq d\Delta v_{xx}(t,x) + \kappa v_{x_1}(t,x) - \mu(x_1-\rho_0)v(t,x) \\ \qquad\qquad +e^{-\delta\tau} b(x_1+\kappa\tau-\rho_0)p(t,x)v(t-\tau,x+\kappa\tau e_1), \\ \qquad\qquad\qquad (t,x)\in(0,\infty)\times((-\infty,0]\times\mathbb{R}^{N-1}), \\ v(t,x)\geq 0, \qquad (t,x)\in[0,\infty)\times\mathbb{R}_+^N, \\ v(\theta,x)\geq 0, \qquad (\theta,x)\in[-\tau,0]\times\mathbb{R}^N, \end{cases}$$

where

$$p(t,x) := \int_0^1 (1 - sw^\psi(t,x-\rho_0 e_1+\kappa\tau e_1) - (1-s)a_-^* w_+(x-\rho_0 e_1+\kappa\tau e_1)) \\ \times e^{-sw^\psi(t,x-\rho_0 e_1+\kappa\tau e_1)-(1-s)a_-^* w_+(x-\rho_0 e_1+\kappa\tau e_1)} \mathrm{d}s \in [0,1]$$

for all $(t,x) \in \mathbb{R}_+ \times (-\infty,0]\times\mathbb{R}^{N-1}$. By the step arguments and the Phragmén–Lindelöf type maximum principle [23], it follows that $v(t,x) \geq 0$ for all $(t,x) \in [-\tau,\infty)\times\mathbb{R}^N$. By the arbitrariness of ψ, we then have

$$\omega(\varphi) \geq a_-^* w_+ - \gamma_0 e^{-\lambda_0 t} \;\forall t \in \mathbb{R}_+,$$

and hence, $\omega(\varphi) \geq a_-^* w_+$. Similarly, we can obtain $\omega(\varphi) \leq a_+^* w_+$. Thus, it suffices to prove that $a_+^* = a_-^* = 1$. Assume, by contradiction, that $\{a_+^*, a_-^*\} \neq \{1\}$. By Lemma 3.3-(iii) in [46], we easily check that either (I) $ke^{(1-k)u} > a_-^* \;\forall (u,k)\in(0,2]\times[a_-^*,a_+^*]$ or (II) $ke^{(1-k)u} < a_+^* \;\forall (u,k)\in(0,2]\times[a_-^*,a_+^*]$.

We only consider the case of (I) since the case of (II) can be dealt with in a similar way. Note that $w_+, \omega(\varphi) \leq 2$, and thus (I) implies $a_-^* \in (0,1)$ and $\psi(\theta,x)e^{-\psi(\theta,x)} > a_-^* w_+(x)e^{-w_+(x)} \;\forall (\theta,x,\psi)\in[-\tau,0]\times\mathbb{R}^N\times\omega(\varphi)$. By letting

$$Z^\psi(t,x) = w^\psi(t,x) - a_-^* w_+(x) \;\forall (t,x,\psi) \in [-\tau,\infty)\times\mathbb{R}^N\times\omega(\varphi),$$

it follows from (4.2) and Theorem 4.5 that there exist $\epsilon_0, \alpha_0 > 0$ such that

$$\begin{cases} \dfrac{\partial Z^\psi}{\partial t}(t,x) > d\Delta Z^\psi(t,x) + \kappa Z_{x_1}^\psi(t,x) - \mu_- Z^\psi(t,x), (t,x)\in(0,\infty)\times\mathbb{R}^N, \\ Z^\psi(t,x) \geq 0, (t,x)\in[-\tau,\infty)\times\mathbb{R}^N, \\ Z^\psi(t,x) \geq \epsilon_0, (t,x)\in[-\tau,\infty)\times(\alpha_0 e_1+\mathbb{R}_+^N). \end{cases}$$

As a result, $Z^\psi(0,x) \geq \epsilon_0 \;\forall x \geq \alpha_0 e_1$ and

$$Z^\psi(t,x) \geq e^{-\mu_- t} T(t)[Z^\psi(0,\cdot)](x+\kappa e_1) \;\forall (t,x,\psi)\in(0,\infty)\times\mathbb{R}^N\times\omega(\varphi).$$

These, together with the second paragraph in the proof of Proposition 4.1, implies $Z^\psi(1,x) \geq \epsilon_1 \;\forall (x,\psi) \in ([-\rho_0,\alpha_0]\times\mathbb{R}^{N-1})\times\omega(\varphi)$ for some $\epsilon_1 > 0$. Hence, by the arbitrariness of ψ and the properties of $\omega(\varphi)$, we easily see that

$$\psi - a_-^* w_+ \geq \min\{\epsilon_0,\epsilon_1\}, x \in [-\rho_0,\infty)\times\mathbb{R}^{N-1},$$

a contradiction to the definition of a_-^*. This completes the proof. ∎

By slightly adapting the proof of Theorems 4.5 and 4.6, we may verify the following corollary.

COROLLARY 4.7. *Assume that* $|\kappa| < c^*$ *and* $\varphi \in C_+ \setminus \{0\}$. *If either* ($\kappa = 0$ *and* $\frac{b_+}{\mu_+} e^{-\delta\tau} \le e^2$) *or* ($\kappa \neq 0$ *and* $\frac{b_+ e^{-\delta\tau}}{\mu_+} \le 2e$), *then*

$$\lim_{t\to\infty} \sup \left\{ ||(w^{\psi})_t - w_+|| : \psi \in C_+ \text{ with } \varphi \le \psi \le \frac{b_+}{\mu_+} e^{-\delta\tau} \right\} = 0.$$

Second, we shall give the existence of forced steady states of (4.2) when $\kappa \ge c^*$.

THEOREM 4.8. *Assume that* $\kappa \ge c^*$. *Then* (4.2) *has positive steady states* w_+ *in* C_+°.

Proof. Let $\tilde{b}(s) = \min\{b(s), \mu_+ e^{1+\delta\tau}\}$ for all $s \in \mathbb{R}$, and let $\tilde{w}^{\varphi,\tilde{\kappa}}(t,x)$ be the solution of (4.2) by replacing $(b(\cdot),\kappa)$ with $(\tilde{b}(\cdot),\tilde{\kappa})$. Clearly, $c^* \ge c^*(0,\mu_+, \frac{\tilde{b}_+}{\mu_+} g_{\mathcal{M}}) > 0$ with $\mathcal{M} := \frac{b_+}{\mu_+ e^{\tau\delta+1}}$. Choose $\alpha_0 > 1$ such that

$$g_{\alpha_0,\mathcal{M}}(u) \le g_{\mathcal{M}} \le e^{-\delta\tau} u e^{-u} \ \forall u \in [0,\mathcal{M}]$$

and $g_{\alpha_0,\mathcal{M}}(u)$ is nondecreasing in $u \in \mathbb{R}_+$, where $g_{\alpha_0,\mathcal{M}} : \mathbb{R}_+ \to \mathbb{R}_+$ is defined by

$$g_{\alpha_0,\mathcal{M}}(u) = \begin{cases} e^{-\delta\tau} u e^{-\alpha_0 u}, & u \in \left[0, \dfrac{1}{\alpha_0}\right], \\ \dfrac{1}{\alpha_0 e^{1+\delta\tau}} & u \in \left(\dfrac{1}{\alpha_0}, \infty\right). \end{cases}$$

By applying Theorem 4.6 to $\tilde{w}^{\varphi,0}(t,x)$, there exists $W_+ \in C(\mathbb{R}^N,(0,1])$ such that

$$\tilde{w}^{W_+,0}(t,x) = W_+(x) \ \forall (t,x) \in [-\tau,\infty) \times \mathbb{R}^N$$

and $\tilde{w}^{1,0}(t,x) \to W_+$, locally uniform in x, as $t \to \infty$. According to the monotonicity of $\tilde{b}(\cdot),\mu(\cdot)$, we easily check that for any $t > 0$, $\tilde{w}^{1,0}(t,x)$, and hence $W_+(x)$ are nondecreasing in $x_1 \in \mathbb{R}$, which implies $\frac{\partial W_+}{\partial x_1} \ge 0$ for all $\mathbb{R}^N$. As a result, we know that

$$w^{\varphi}(t,x) \ge w^{\frac{W_+}{\alpha_0}}(t,x; g_{\alpha_0,\mathcal{M}}, \tilde{b}(\cdot)) \ge \frac{1}{\alpha_0} W_+(x)$$

for all $(t,x,\varphi) \in [-\tau,\infty) \times \mathbb{R}^N \times \mathcal{A}$ with $\mathcal{A} := \{\varphi \in C_+ : \frac{W_+}{\alpha_0} \le \varphi \le \mathcal{M}\}$, where $w^{\frac{W_+}{\alpha_0}}(t,x; g_{\alpha_0,\mathcal{M}}, \tilde{b}(\cdot))$ represents the solution of (4.3) with $(\phi, b, g_{\mathcal{M}})$ be replaced by $(\frac{W_+}{\alpha_0}, \tilde{b}, g_{\alpha_0,\mathcal{M}})$. Using the same discussions as in the first paragraph of the proof of Theorem 4.6, we may get the existence of forced steady states w_+. This completes the proof. ◻

Finally, we shall give the nonexistence of forced steady states of (4.2) when $\kappa < -c^*$.

THEOREM 4.9. *Let* $\kappa < -c^*$. *Then* (4.2) *has no nontrival steady state* $w_+ \in C_+ \setminus \{0\}$.

Proof. Suppose that (4.2) has nonnegative steady state $w_+ \in C_+ \setminus \{0\}$. Then $w_+ \in C(\mathbb{R}^N, (0, \frac{b_+}{\mu_+ e^{\delta\tau+1}}))$. Take $c = \frac{c^*+\kappa}{2}$. It follows from the definition of c^* that there exists $\lambda > 0$ such that

$$-d\lambda^2 + (c-\kappa)\lambda + \mu_+ = b_+ e^{-\delta\tau} e^{-\lambda(c-\kappa)\tau}.$$

Let $h(\gamma,\epsilon)=d(\gamma)^2+\kappa\gamma-\mu_-+\epsilon+e^{-\delta\tau}(b_-+\epsilon)e^{\gamma\kappa\tau}$ for all $(\gamma,\epsilon)\in\mathbb{R}_+\times(0,\frac{\mu_--b_-e^{-\delta\tau}}{1+e^{-\delta\tau}})$. Then $h(0,0)<0$ and $\lim_{\gamma\to\infty}h(\gamma,0)>0$, which imply $h(\gamma_0,\epsilon_0)=0$ for some $(\gamma_0,\epsilon_0)\in(\lambda,\infty)\times(0,\frac{\mu_--b_-e^{-\delta\tau}}{1+e^{-\delta\tau}})$. By using the proof of Theorem 4.2, there exists $M_0=\sup_{(t,x)\in[-\tau,\infty)\times\mathbb{R}^N}|w^\varphi(t,x)|$ such that $w_+(x)\le M_0e^{\gamma_0x_1}$, and hence, $w_+(x)\le M_0e^{\lambda x_1}$ for all $x\in\mathbb{R}^N$.

Define $Z:[-\tau,\infty)\times\mathbb{R}^N\to\mathbb{R}$ by

$$Z(t,x)=w_+(x)-M_0e^{\lambda(x_1+ct)}\ \forall(t,x)\in[-\tau,\infty)\times\mathbb{R}^N.$$

It follows from (4.2) that

$$\begin{cases}\dfrac{\partial Z}{\partial t}(t,x)\le d\Delta Z(t,x)+\kappa Z_{x_1}-\mu_+Z(t,x)\\ \qquad\qquad +e^{-\delta\tau}b_+Z(t-\tau,x+\kappa\tau e_1),(t,x)\in(0,\infty)\times\mathbb{R}^N,\\ Z(t,x)\le 0,(t,x)\in[-\tau,0]\times\mathbb{R}^N.\end{cases}$$

By applying the step arguments and the Phragmén–Lindelöf type maximum principle [23], we may obtain that $Z(t,x)\le 0$ for all $(t,x)\in[-\tau,\infty)\times\mathbb{R}^N$. By letting $t\to\infty$, we easily get $w_+=0$, a contradiction. This completes the proof. ∎

4.4. Asymptotic propagation and spreading speed. In this subsection, we shall turn our attention to (1.3) with general direction ν and reorganize results of the above three subsections to obtain propagation dynamics for (1.3). To this end, we first note that the c^* defined in (4.5) only depends on the limit equation with μ^+ and b^+ and is independent of the direction ν. In what follows, we will see this c^* also plays an important role for (1.3) with general direction ν.

THEOREM 4.10. *Let $u^\varphi(t,x)$ be the solution of* (1.3). *Then the following statements hold.*

(i) **Asymptotic extinction.** *If $\varphi\in C_+$, then*

$$\lim_{\alpha\to\infty}(\sup\{u^\varphi(t,x):(t,x)\in\mathbb{R}_+\times\mathbb{R}^N\ \textit{with}\ x\cdot\nu\le\kappa t-\alpha,t\ge\alpha\})=0.$$

Moreover, if $\xi\in S^{N-1}$ and $\varphi(\cdot,x)$ is zero for all sufficiently positive $x\cdot\xi$, then

$$\lim_{t\to\infty}\left[\sup\{u^\varphi(t,x):(t,x)\in\mathbb{R}_+\times\mathbb{R}^N\ \textit{with}\ x\cdot\xi\ge t(\varepsilon+c^*)\}\right]=0\ \forall\varepsilon>0.$$

Additionally, if φ has a compact support, then

$$\lim_{t\to\infty}(\sup\{u^\varphi(t,x):||x||\ge t(c^*+\varepsilon)\})=0.$$

(ii) **Upward convergence.** *If $\varphi\in C_+\setminus\{0\}$, $\kappa<c^*$, and $\frac{b_+}{\mu_+}e^{-\delta\tau}\le e^2$, then, for any $\varepsilon\in(0,\min\{c^*,c^*-\kappa\})$,*

$$\lim_{\alpha\to\infty}\left(\sup\left\{\left|u^\varphi(t,x)-\log\left(\frac{b_+e^{-\delta\tau}}{\mu_+}\right)\right|:t\ge\alpha,\right.\right.$$
$$\left.\left.\|x\|\le t(c^*-\varepsilon),x\cdot\nu\ge(\alpha+\kappa t)\right\}\right)=0.$$

Hence,

$$\lim_{t\to\infty}\left(\sup\left\{\left|u^\varphi(t,x)-\log\left(\frac{b_+}{\mu_+}e^{-\delta\tau}\right)\right|:||x||\right.\right.$$
$$\left.\le t(c^*-\varepsilon),\ \textit{and}\ x\cdot\nu\ge t(\varepsilon+\kappa)\}\right)=0.$$

(iii) **Existence and nonexistence.** *When* $\kappa > -c^*$ *and* $\frac{b_+}{\mu_+}e^{-\delta\tau} \leq e^2$, *then* (1.3) *has a forced wave* $w_+(x-\kappa\nu t) \in C_+ \setminus \{0\}$ *with* $\lim_{x\cdot\nu\to-\infty} w_+(x) = 0$ *and* $\lim_{x\cdot\nu\to\infty} w_+(x) = \log(\frac{b_+}{\mu_+}e^{-\delta\tau})$; *while when* $\kappa < -c^*$, (1.3) *has no nontrivial waves of the form* $w_+(x-\kappa\nu t)$ *with speed* κ.

(iv) **Spreading phenomenon.** *If* $\varphi \in C_+ \setminus \{0\}$ *and* $|\kappa| < c^*$ *with either* ($\kappa = 0$ *and* $\frac{b_+}{\mu_+}e^{-\delta\tau} \leq e^2$) *or* ($\kappa \neq 0$ *and* $\frac{b_+e^{-\delta\tau}}{\mu_+} \leq 2e$), *then*

$$\lim_{t\to\infty}(\sup\{|u^\varphi(t,x) - w_+(x-\kappa\nu t)| : ||x|| \leq t(c^*-\varepsilon)\}) = 0 \ \forall\varepsilon > 0.$$

(v) **Uniformly extinction.** *If* $\kappa > c^*$ *and* $\varphi \in C_+ \setminus \{0\}$ *with* $\varphi(\cdot, x)$ *being zero for all sufficiently positive* $x\cdot\nu$, *then*

$$\lim_{t\to\infty}(\sup\{u^\varphi(t,x) : x \in \mathbb{R}^N\}) = 0.$$

Proof. Without loss of generality, we may assume $\nu = e_1$. For any given $\varphi \in C_+$, we should realize that $u^\varphi(t,x) = w^{\tilde\varphi}(t, x-\kappa e_1 t)$ for all $(t,x) \in [-\tau,\infty) \times \mathbb{R}^N$, where $\tilde\varphi(\theta,x) = \varphi(\theta, x+\kappa\theta e_1)$ $\forall(\theta,x) \in [-\tau,0] \times \mathbb{R}^N$.

Clearly, (i) follows from Theorems 4.2–4.4, (ii) follows from Theorem 4.5, and (iii) follows from Theorems 4.6–4.9.

(iv) To finish the proof, it suffices to prove that

$$\lim_{t\to\infty}(\sup\{|w^\varphi(t,x) - w_+(x)| : ||x+\kappa e_1 t|| \leq t(c^*-\varepsilon)\}) = 0 \ \forall\varepsilon > 0, \varphi \in C_+ \setminus \{0\}.$$

Suppose that $\varphi \in C_+ \setminus \{0\}$, $\gamma \in (0, \log(\frac{b_+e^{-\delta\tau}}{\mu_+}))$, and let $c = c^* - \varepsilon$ $\forall\varepsilon > 0$. Without loss of generality, we may assume $c > |\kappa|$ and $\varphi \leq \frac{b_+e^{-\delta\tau}}{\mu_+}$. According to Theorems 4.2 and 4.5, we know that there exists $\alpha_0 > 0$ such that

$$\begin{cases} w_+(x) + w^\varphi(t,x) < \gamma \ \forall t, -x\cdot e_1 \in [\alpha_0,\infty), \\ |w_+(x) - \log\left(\frac{b_+e^{-\delta\tau}}{\mu_+}\right)| + |w^\varphi(t,x) - \log\left(\frac{b_+e^{-\delta\tau}}{\mu_+}\right)| < \gamma \ \forall t, x\cdot e_1 \in [\alpha_0,\infty) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{with } ||x+\kappa e_1 t|| \leq tc. \end{cases}$$

This leads to

$$|w_+(x) - w^\varphi(t,x)| < \gamma \ \forall t, |x\cdot e_1| \in [\alpha_0,\infty) \text{ with } ||x+\kappa e_1 t|| \leq tc. \tag{4.6}$$

In view of the uniqueness of w_+, we easily see $w_+(x) = w_+(x_1e_1) := \phi_+(x_1)$ for all $x \in \mathbb{R}^N$. Take $\delta_0 = \frac{c^*-c}{3}$. Again, by applying Theorems 4.5, there exists $\alpha_1 > \max\{\alpha_0, \frac{2}{\delta_0}\}$ such that

$$w_+(x), (w^\varphi)_t(\cdot,x) \geq W_0 := \frac{1}{3}\log\left(\frac{b_+e^{-\delta\tau}}{\mu_+}\right) \ \forall(t,x) \in \Omega^*_{c+\delta_0,\alpha_1}.$$

Choose $\phi \in C(\mathbb{R}^N, [0,W_0]) \setminus \{0\}$ with

$$supp(\phi) \subseteq B_1(e_1+\alpha_1 e_1) := \{x \in \mathbb{R}^N : ||x-(1+\alpha_1)e_1|| < 1\}.$$

By Corollary 4.7, there exists $\alpha_2 > 0$ such that $||(w^\psi)_t(\cdot,x) - w_+(x)|| < \gamma$ for all $(t,x,\psi) \in [\alpha_2,\infty) \times [-\alpha_0,\alpha_0]e_1 \times C_+$ with $\phi \leq \psi \leq \frac{b_+}{\mu_+}e^{-\delta\tau}$.

Let $\alpha_3 = \frac{2(1+\delta_0+c+|\kappa|)(1+\tau+\alpha_1+\alpha_2)}{\delta_0}$. According to the definition of $\Omega^*_{c+\delta_0,\alpha_1}$, we easily check that

$$\left.\begin{array}{l}(t,x)\in[\alpha_3,\infty)\times\mathbb{R}^N, ||x+\kappa te_1||\le ct, |x\cdot e_1|\le\alpha_0\\ (\tilde{\theta},\tilde{x})\in[-\tau,0]\times B_1(e_1+\alpha_1e_1)\end{array}\right\}$$
$$\Rightarrow(t+\tilde{\theta}-\alpha_2, x-x_1e_1+\tilde{x})\in\Omega^*_{c+\delta_0,\alpha_1}.$$

This, together with the choice of ϕ, implies that

$$(w^\varphi)_{t-\alpha_2}(\cdot,\cdot+x-x_1e_1)\ge\phi\ \forall(t,x)\in[\alpha_3,\infty)\times\mathbb{R}^N$$

with $|x_1|\le\alpha_0$ and $||x+\kappa te_1||\le ct$. It follows from (4.2) and the choice of α_2 that for any $(t,x)\in[\alpha_3,\infty)\times\mathbb{R}^N$, with $|x_1|\le\alpha_0$ and $||x+\kappa te_1||\le ct$, there holds

$$\begin{aligned}&||w^\varphi(t,x)-w_+(x)||\\&=||w^{(w^\varphi)_{t-\alpha_2}}(\alpha_2,x)-w_+(x)||\\&=||w^{(w^\varphi)_{t-\alpha_2}(\cdot,\cdot+x-x_1e_1)}(\alpha_2,x_1e_1)-w_+(x_1e_1)||<\gamma,\end{aligned}$$

which, together with (4.6), implies (iv).

(v) follows from Theorems 4.2 and 4.3 with $\varepsilon=\frac{\kappa-c^*}{2}$. This completes the proof. ∎

As a direct corollary, we easily obtain the propagation dynamics for (1.3) on $\mathbb{R}$.

COROLLARY 4.11. *Let $u^\varphi(t,x)$ be the solution of* (1.3) *with $N=1$. Then the following statements hold.*

(i) *If $\varphi\in C_+$, then*

$$\lim_{\alpha\to\infty}(\sup\{u^\varphi(t,x):(t,x)\in\mathbb{R}_+\times\mathbb{R}\text{ with }x\le\kappa t-\alpha,t\ge\alpha\})=0.$$

Moreover, if $\varphi(\cdot,x)$ is zero for all sufficiently positive x, then

$$\lim_{t\to\infty}[\sup\{u^\varphi(t,x):(t,x)\in\mathbb{R}_+\times\mathbb{R}\text{ with }x\ge t(\varepsilon+c^*)\}]=0\ \forall\varepsilon>0.$$

(ii) *If $\varphi\in C_+\setminus\{0\}$, $\kappa<c^*$, and $\frac{b_+}{\mu_+}e^{-\delta\tau}\le e^2$, then, for all $\varepsilon\in(0,\min\{c^*,c^*-\kappa\})$,*

$$\lim_{\alpha\to\infty}\left(\sup\left\{\left|u^\varphi(t,x)-\log\left(\frac{b_+}{\mu_+}e^{-\delta\tau}\right)\right|:t\ge\alpha,\alpha+\kappa t\le x\le t(c^*-\varepsilon)\right\}\right)=0.$$

Hence,

$$\lim_{t\to\infty}\left(\sup\left\{\left|u^\varphi(t,x)-\log\left(\frac{b_+}{\mu_+}e^{-\delta\tau}\right)\right|:t(\varepsilon+\kappa)\le x\le t(c^*-\varepsilon)\right\}\right)=0.$$

(iii) *Equation* (1.3) *has forced waves $w_+(x-\kappa t)\in C_+\setminus\{0\}$ with $\lim_{x\to-\infty}w_+(x)=0$ and $\lim_{x\to\infty}w_+(x)=\log(\frac{b_+}{\mu_+}e^{-\delta\tau})$, when $\kappa>-c^*$ and $\frac{b_+}{\mu_+}e^{-\delta\tau}\le e^2$. Also,* (4.2) *has no nontrivial waves $w_+(x-\kappa t)$, when $\kappa<-c^*$.*

(iv) *If $\varphi\in C_+\setminus\{0\}$ and $|\kappa|<c^*$ with either ($\kappa=0$ and $\frac{b_+}{\mu_+}e^{-\delta\tau}\le e^2$) or ($\kappa\ne0$ and $\frac{b_+e^{-\delta\tau}}{\mu_+}\le2e$), then*

$$\lim_{t\to\infty}(\sup\{|u^\varphi(t,x)-w_+(x-\kappa t)|:x\le t(c^*-\varepsilon)\})=0\ \forall\varepsilon>0.$$

(v) *If* $\kappa > c^*$ *and* $\varphi \in C_+ \setminus \{0\}$ *with* $\varphi(\cdot, x)$ *being zero for all sufficiently positive* x*, then*

$$\lim_{t\to\infty}(\sup\{u^\varphi(t,x) : x \in \mathbb{R}\}) = 0.$$

Note that all of the above results are for the case (W-1). As mentioned in the end of the introduction, by applying the mirror transform, the case (B-1) can be converted to case (W-1), and accordingly, parallel conclusions can be drawn by applying Theorem 4.10 for the case (B-1). This immediately leads to the following corollary.

COROLLARY 4.12. *Assume that* $\mu(\cdot)$ *is increasing,* $b(\cdot)$ *is decreasing and* (B-1) *holds, and let*

$$c_-^* = \inf\left\{\sigma \in \mathbb{R}_+ : b_- e^{-\delta\tau} e^{-\sigma\tau\rho} \le \mu_- + \sigma\rho - d\rho^2, \exists \rho \in \left(0, \frac{\sigma + \sqrt{\sigma^2 + 4d\mu}}{2d}\right)\right\}.$$

If $u^\varphi(t,x)$ *is the solution of* (1.3)*, then the following statements hold:*

(i) *If* $\varphi \in C_+$*, then*

$$\lim_{\alpha\to\infty}(\sup\{u^\varphi(t,x) : (t,x) \in \mathbb{R}_+ \times \mathbb{R}^N \text{ with } x\cdot\nu \ge \kappa t + \alpha, t \ge \alpha\}) = 0.$$

If $\xi \in S^{N-1}$ *and* $\varphi(\cdot, x)$ *is zero for all sufficiently negative* $x \cdot \xi$*, then*

$$\lim_{t\to\infty}\left[\sup\{u^\varphi(t,x) : (t,x) \in \mathbb{R}_+ \times \mathbb{R}^N \text{ with } x\cdot\xi \le -t(\varepsilon + \underline{c}^*)\}\right] = 0 \ \forall \varepsilon > 0.$$

Additionally, if φ *has a compact support, then*

$$\lim_{t\to\infty}(\sup\{u^\varphi(t,x) : ||x|| \ge t(\underline{c}^* + \varepsilon)\}) = 0.$$

(ii) *If* $\varphi \in C_+ \setminus \{0\}$, $\kappa > -c_-^*$*, and* $\frac{b_-}{\mu_-}e^{-\delta\tau} \le e^2$*, then, for all* $\varepsilon \in (0, \min\{c_-^*, c_-^* + \kappa\})$,

$$\lim_{\alpha\to\infty}\left(\sup\left\{\left|u^\varphi(t,x) - \log\left(\frac{b_-}{\mu_-}e^{-\delta\tau}\right)\right| : t \ge \alpha, ||x|| \le t(c_-^* - \varepsilon),\right.\right.$$
$$\left.\left.\text{and } x\cdot\nu \le (\kappa t - \alpha)\right\}\right) = 0.$$

Hence,

$$\lim_{t\to\infty}\left(\sup\left\{\left|u^\varphi(t,x) - \log\left(\frac{b_-}{\mu_-}e^{-\delta\tau}\right)\right| : ||x|| \le t(\underline{c}^* - \varepsilon),\right.\right.$$
$$\left.\left.\text{and } x\cdot\nu \le t(\kappa - \varepsilon)\right\}\right) = 0.$$

(iii) *Equation* (1.3) *has forced waves* $w_-(x - \kappa\nu t) \in C_+ \setminus \{0\}$ *with* $\lim_{x\cdot\nu\to\infty} w_-(x) = 0$ *and* $\lim_{x\cdot\nu\to-\infty} w_-(x) = \log(\frac{b_-}{\mu_-}e^{-\delta\tau})$*, when* $\kappa < c_-^*$ *and* $\frac{b_-}{\mu_-}e^{-\delta\tau} \le e^2$*. Also,* (4.2) *has no nontrivial waves* $w_-(x - \kappa\nu t)$*, when* $\kappa > c_-^*$.

(iv) *If* $\varphi \in C_+ \setminus \{0\}$ *and* $|\kappa| < c_-^*$ *with either* ($\kappa = 0$ *and* $\frac{b_-}{\mu_-}e^{-\delta\tau} \le e^2$) *or* ($\kappa \ne 0$ *and* $\frac{b_- e^{-\delta\tau}}{\mu_-} \le 2e$)*, then*

$$\lim_{t\to\infty}(\sup\{|u^\varphi(t,x) - w_-(x - \kappa\nu t)| : ||x|| \le t(\underline{c}^* - \varepsilon)\}) = 0 \ \forall \varepsilon > 0.$$

(v) *If* $\kappa < -c_-^*$ *and* $\varphi \in C_+ \setminus \{0\}$ *with* $\varphi(\cdot, x)$ *is zero for all sufficiently negative* $x \cdot \nu$*, then*

$$\lim_{t\to\infty}(\sup\{u^\varphi(t,x) : x \in \mathbb{R}^N\}) = 0.$$

5. Discussion on ecological implications. In this section, we explore the implications of the main results in Theorem 4.10 in terms of population spreading. Since the parameter ε in Theorem 4.10 can be arbitrarily small, we can explain the results more conveniently by letting $\varepsilon \to 0$. Then, by Theorem 4.10-(i)–(ii), *for large t (asymptotically),* the population only persists in the time varying region $\Omega_n(t)$ given by

$$\Omega_n(t) = \{x \in \mathbb{R}^n : ||x|| < c^* t \text{ and } x \cdot \nu > \kappa t\}. \tag{5.1}$$

For the 1-D space $\mathbb{R}$, this time dependent region becomes

$$\Omega_1(t) = \{x \in \mathbb{R} : |x| < c^* t \text{ and } x > \kappa t\}, \tag{5.2}$$

which is illustrated in Figure 1 for the cases (a) $0 < \kappa < c^*$; (b) $-c^* < \kappa < 0$; (c) $\kappa < -c^*$. From the Figure 1, we can conveniently distinguish the three cases for κ as below:

(1-a) For the case $0 < \kappa < c^*$, the population spreads to the right with speed c^* with domain expansion rate $c^* - \kappa$.

(1-b) For the $-c^* < \kappa < 0$, the population spreads to both directions, with a *right speed* c^* and a *left speed* $-\kappa$; the domain expansion rate is still given by $c^* - \kappa$.

(1-c) For the cases $\kappa < -c^*$, the population spreads to both directions with the same speed c^*; the domain expansion rate is $2c^*$.

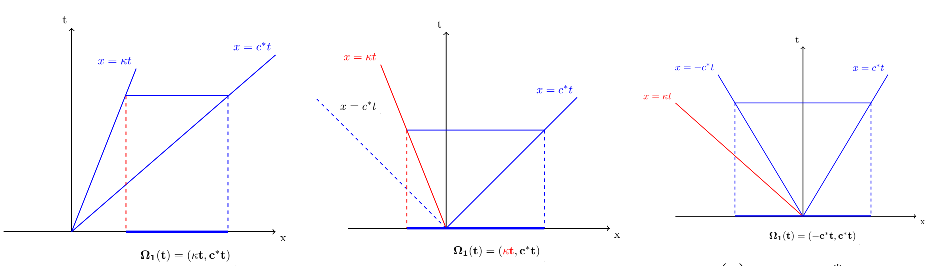
For the 2-D space $\mathbb{R}^2$, denoting $\nu = (\cos\theta_0, \sin\theta_0)$, the time dependent persistence region becomes

$$\Omega_2(t) = \{(x, y) \in \mathbb{R} : x^2 + y^2 < c^* t \text{ and } x\cos\theta_0 + y\sin\theta_0 > \kappa t\}. \tag{5.3}$$

For convenience of demonstration and yet without loss of generality, we choose $\nu = e_1 = (1, 0)$; accordingly, we illustrate the corresponding $\Omega_2(t)$ in Figure 2, also for the three cases (a) $0 < \kappa < c^*$; (b) $-c^* < \kappa < 0$; and (c) $\kappa < -c^*$. From Figure 2, we can observe the following:

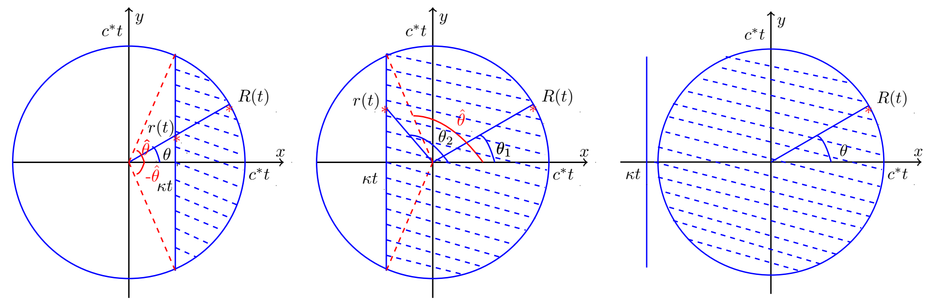
(2-a) For the case $0 < \kappa < c^*$, the population spreads with speed c^* in every direction θ between $-\hat{\theta}$ and $\hat{\theta}$ where $\hat{\theta} = \arccos(\kappa/c^*)$; the population does not spread in other directions. Moreover, by calculating the area of the shaded region, we can obtain a time dependent *domain expansion rate* as

$$2t\left[(c^*)^2 \arccos\left(\frac{\kappa}{c^*}\right) - \kappa\sqrt{(c^*)^2 - \kappa^2}\right].$$



(a) $0 < \kappa < c^*$. (b) $-c^* < \kappa < 0$. (c) $\kappa < -c^*$.

FIG. 1. *Time varying persistence region $\Omega_1(t)$ for the 1-D space, demonstrated by the solid bold line on x-axis.*



(a) $0 < \kappa < c^*$. (b) $-c^* < \kappa < 0$. (c) $\kappa < -c^*$.

FIG. 2. *When the direction $v = e_1 = (1,0)$, time varying existence region $\Omega_2(t)$ for the 2-D space is demonstrated by the shaded portion inside the circle with radius c^*t.*

(2-b) For the case $-c^* < \kappa < 0$, we present the following:

(i) The population spreads with the same speed c^* in every direction θ_1 between $-\hat{\theta}$ and $\hat{\theta}$ where $\hat{\theta} = \pi - \arccos(|\kappa|/c^*)$.

(ii) In every direction $\theta_2 \in (-\pi, -\hat{\theta}) \cup (\hat{\theta}, \pi)$, the population also spreads but with a *direction dependent speed* $c(\theta_2) = |\kappa|/\cos\theta_2$.

(iii) The *domain expansion rate* is also time dependent and is given by

$$2t\left[(c^*)^2 \arccos\left(\frac{\kappa}{c^*}\right) + |\kappa|\sqrt{(c^*)^2 - \kappa^2}\right],$$

where $\arccos\left(\frac{\kappa}{c^*}\right) > \frac{\pi}{2}$.

(2-c) For the cases $\kappa < -c^*$, the population spreads with the same speed c^* in all directions and the *habitat domain expands at the rate* $2\pi(c^*)^2 t$.

Acknowledgment. We would like to thank the anonymous referees for their valuable comments, which led to an improvement in our presentation.

REFERENCES

[1] H. Berestycki and J. Fang, *Forced waves of the Fisher–KPP equation in a shifting environment*, J. Differential Equations, 264 (2018), pp. 2157–2183, https://doi.org/10.1016/j.jde.2017.10.016.

[2] H. Berestycki and L. Rossi, *Reaction-diffusion equations for population dynamics with forced speed.* I. *The case of the whole space*, Discrete Contin. Dyn. Syst., 21 (2008), pp. 41–67, https://doi.org/10.3934/dcds.2008.21.41.

[3] H. Berestycki and L. Rossi, *Reaction-diffusion equations for population dynamics with forced speed.* II. *Cylindrical-type domains*, Discrete Contin. Dyn. Syst., 25 (2009), pp. 19–61, https://doi.org/10.3934/dcds.2009.25.19.

[4] Y. Du, L. Wei, and L. Zhou, *Spreading in a shifting environment modeled by the diffusive logistic equation with a free boundary*, J. Dynam. Differential Equations, 30 (2018), pp. 1389–1426, https://doi.org/10.1007/s10884-017-9614-2.

[5] J. Fang, Y. Lou, and J. Wu, *Can pathogen spread keep pace with its host invasion?*, SIAM J. Appl. Math., 76 (2016), pp. 1633–1657, https://doi.org/10.1137/15M1029564.

[6] J. Fang, X. Yu, and X.-Q. Zhao, *Traveling waves and spreading speeds for time–space periodic monotone systems*, J. Funct. Anal., 272 (2017), pp. 4222–4262, https://doi.org/10.1016/j.jfa.2017.02.028.

[7] S. Gourley and J. Wu, *Delayed non-local diffusive systems in biological invasion and disease spread*, Nonlinear Dynam. Evol. Equ., 48 (2006), pp. 137–200, https://doi.org/10.1090/fic/048/06.

[8] J. K. Hale, *Asymptotic Behaviour of Dissipative Systems*, American Mathematical Society, Providence, RI, 1989.

[9] H. Hu, T. Yi, and X. Zou, *On spatial-temporal dynamics of a Fisher-KPP equation with a shifting environment*, Proc. Amer. Math. Soc., 148 (2020), pp. 213–221, https://doi.org/10.1090/proc/14659.

[10] H. Hu and X. Zou, *Existence of an extinction wave in the Fisher equation with a shifting habitat*, Proc. Amer. Math. Soc., 145 (2017), pp. 4763–4771, https://doi.org/10.1090/proc/13687.

[11] Y. Kuang, *Delay Differential Equations*. Academic Press, New York, 1993.

[12] K.-Y. Lam and X. Yu, *Asymptotic spreading of KPP reactive fronts in heterogeneous shifting environments*, J. Math. Pures Appl., 167 (2022), pp. 1–47, https://doi.org/10.1016/j.matpur.2022.09.001.

[13] B. Li, S. Bewick, J. Shang, and W. F. Fagan, *Persistence and spread of a species with a shifting habitat edge*, SIAM J. Appl. Math., 74 (2014), pp. 1397–1417, https://doi.org/10.1137/130938463.

[14] W.-T. Li, J.-B. Wang, and X.-Q. Zhao, *Spatial dynamics of a nonlocal dispersal population model in a shifting environment*, J. Nonlinear Sci., 28 (2018), pp. 1189–1219, https://doi.org/10.1007/s00332-018-9445-2.

[15] X. Liang, Y. Yi, and X.-Q. Zhao, *Spreading speeds and traveling waves for periodic evolution systems*, J. Differential Equations, 231 (2006), pp. 57–77, https://doi.org/10.1016/j.jde.2006.04.010.

[16] X. Liang and X.-Q. Zhao, *Asymptotic speeds of spread and traveling waves for monotone semiflows with applications*, Comm. Pure Appl. Math., 60 (2007), pp. 1–40, https://doi.org/10.1002/cpa.20154.

[17] X. Liang and X.-Q. Zhao, *Spreading speeds and traveling waves for abstract monostable evolution systems*, J. Funct. Anal., 259 (2010), pp. 857–903, https://doi.org/10.1016/j.jfa.2010.04.018.

[18] R. Lui, *Biological growth and spread modeled by systems of recursions.* I. *Mathematical theory*, Math. Biosci., 93 (1989), pp. 269–295, https://doi.org/10.1016/0025-5564(89)90026-6.

[19] R. H. Martin and H. L. Smith, *Abstract functional-differential equations and reaction-diffusion systems*, Trans. Amer. Math. Soc., 321 (1990), pp. 1–44, https://doi.org/10.1090/S0002-9947-1990-0967316-X.

[20] R. H. Martin and H. L. Smith, *Reaction-diffusion systems with time delays: Monotonicity, invariance, comparison and convergence*, J. Reine Angew. Math., 413 (1991), pp. 1–35, https://doi.org/10.1515/crll.1991.413.1.

[21] M. Mei, J. W.-H. So, M. Y. Li, and S. S. Shen, *Asymptotic stability of travelling waves for Nicholson's blowflies equation with diffusion*, Proc. Roy. Soc. Edinburgh Sect. A, 134 (2004), pp. 579–594, https://doi.org/10.1017/S0308210500003358.

[22] A. B. Potapov and M. A. Lewis, *Climate and competition: The effect of moving range boundaries on habitat invasibility*, Bull. Math. Biol., 66 (2004), pp. 975–1008, https://doi.org/10.1016/j.bulm.2003.10.010.

[23] M. H. Protter and H. F. Weinberger, *Maximum Principles in Differential Equations*, Prentice-Hall, 1967.

[24] W. Shen, *Variational principle for spreading speeds and generalized propagating speeds in time almost periodic and space periodic KPP models*, Trans. Amer. Math. Soc., 362 (2010), pp. 5125–5168, https://doi.org/10.1090/S0002-9947-10-04950-0.

[25] W. Shen and A. Zhang, *Spreading speeds for monostable equations with nonlocal dispersal in space periodic habitats*, J. Differential Equations, 249 (2010), pp. 747–795, https://doi.org/10.1016/j.jde.2010.04.012.

[26] W. Shen and A. Zhang, *Stationary solutions and spreading speeds of nonlocal monostable equations in space periodic habitats*, Proc. Amer. Math. Soc., 140 (2012), pp. 1681–1696, https://doi.org/10.1090/S0002-9939-2011-11011-6.

[27] H. L. Smith, *Monotone Dynamical Systems: An Introduction to the Theory of Competitive and Cooperative Systems: An Introduction to the Theory of Competitive and Cooperative Systems*, American Mathematical Society, Providence, RI, 1995.

[28] J. W.-H. So, J. Wu, and X. Zou, *A reaction–diffusion model for a single species with age structure.* I *Travelling wavefronts on unbounded domains*, R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci., 457 (2001), pp. 1841–1853, https://doi.org/10.1098/rspa.2001.0789.

[29] J. W.-H. So and Y. Yang, *Dirichlet problem for the diffusive Nicholson's blowflies equation*, J. Differential Equations., 150 (1998), pp. 317–348, https://doi.org/10.1006/jdeq.1998.3489.

[30] J. W.-H. So and X. Zou, *Traveling waves for the diffusive Nicholson's blowflies equation*, Appl. Math. Comput., 122 (2001), pp. 385–392, https://doi.org/10.1016/S0096-3003(00)00055-2.

[31] C. Travis and G. Webb, *Existence and stability for partial functional differential equations*, Trans. Amer. Math. Soc., 200 (1974), pp. 395–418, https://doi.org/10.1090/S0002-9947-1974-0382808-3.

[32] H. F. Weinberger, *Long-time behavior of a class of biological models*, SIAM J. Math. Anal., 13 (1982), pp. 353–396, https://doi.org/10.1137/0513028.

[33] C. Wu, Y. Wang, and X. Zou, *Spatial-temporal dynamics of a Lotka-Volterra competition model with nonlocal dispersal under shifting environment*, J. Differential Equations, 267 (2019), pp. 4890–4921, https://doi.org/10.1016/j.jde.2019.05.019.

[34] J. Wu, *Theory and Applications of Partial Functional Differential Equations*, Applied Mathematical Sciences, Springer-Verlag, New York, 1996.

[35] D. Xu and X.-Q. Zhao, *A nonlocal reaction-diffusion population model with stage structure*, Can. Appl. Math. Q., 11 (2003), pp. 303–319.

[36] Y. Yang and J. So, *Dynamics for the diffusive Nicholson's blowflies equation*, in Dynamical systems and differential equations, Vol. II, W. Chen and S. Hu eds., Springfield MO: Southwest Missouri State University, 1998, pp. 333–352.

[37] T. Yi and Y. Chen, *Study on monostable and bistable reaction–diffusion equations by iteration of travelling wave maps*, J. Differential Equations, 263 (2017), pp. 7287–7308, https://doi.org/10.1016/j.jde.2017.08.017.

[38] T. Yi, Y. Chen, and J. Wu, *Threshold dynamics of a delayed reaction diffusion equation subject to the Dirichlet condition*, J. Biol. Dynam., 3 (2009), pp. 331–341, https://doi.org/10.1080/17513750802425656.

[39] T. Yi, Y. Chen, and J. Wu, *Global dynamics of delayed reaction–diffusion equations in unbounded domains*, Z. Angew. Math. Phys., 63 (2012), pp. 793–812, https://doi.org/10.1007/s00033-012-0224-x.

[40] T. Yi, Y. Chen, and J. Wu, *Unimodal dynamical systems: Comparison principles, spreading speeds and travelling waves*, J. Differential Equations, 254 (2013), pp. 3538–3572, https://doi.org/10.1016/j.jde.2013.01.031.

[41] T. Yi, Y. Chen, and J. Wu, *Asymptotic propagations of asymptotical monostable type equations with shifting habitats*, J. Differential Equations., 269 (2020), pp. 5900–5930, https://doi.org/10.1016/j.jde.2020.04.025.

[42] T. Yi and X.-Q. Zhao, *Propagation dynamics for monotone evolution systems without spatial translation invariance*, J. Funct. Anal., 279 (2020), 108722, https://doi.org/10.1016/j.jfa.2020.108722.

[43] T. Yi and X.-Q. Zhao, *Spatio-temporal Dynamics for Non-Monotone Semiflows with Limiting Systems Having Spreading Speeds*, preprint, arXiv:2305.00051, 2023.

[44] T. Yi and X. Zou, *Global attractivity of the diffusive Nicholson blowflies equation with Neumann boundary condition: A non-monotone case*, J. Differential Equations., 245 (2008), pp. 3376–3388, https://doi.org/10.1016/j.jde.2008.03.007.

[45] T. Yi and X. Zou, *Map dynamics versus dynamics of associated delay reaction–diffusion equations with a Neumann condition*, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci., 466 (2010), pp. 2955–2973, https://doi.org/10.1098/rspa.2009.0650.

[46] T. Yi and X. Zou, *On Dirichlet problem for a class of delayed reaction–diffusion equations with spatial non-locality*, J. Dynam. Differential Equations, 25 (2013), pp. 959–979, https://doi.org/10.1007/s10884-013-9324-3.

[47] T. Yi and X. Zou, *Asymptotic behavior, spreading speeds, and traveling waves of nonmonotone dynamical systems*, SIAM J. Math. Anal., 47 (2015), pp. 3005–3034, https://doi.org/10.1137/14095412X.

[48] T. Yi and X. Zou, *Propagation and heterogeneous steady states in a delayed nonlocal R-D equation without spatial translation-invariance*, J. Differential Equations, 268 (2020), pp. 1600–1632, https://doi.org/10.1016/j.jde.2019.09.004.

[49] Y. Yuan and X. Zou, *Spatial-temporal dynamics of a diffusive Lotka–Volterra competition model with a shifting habitat* II*: Case of faster diffuser being a weaker competitor*, J. Dynam. Differential Equations, 33 (2021), pp. 2091–2132, https://doi.org/10.1007/s10884-020-09885-w.

[50] X.-Q. Zhao, *Dynamical Systems in Population Biology*, 2nd ed., Springer-Verlag, New York, 2017.