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ASYMPTOTIC BEHAVIOR, SPREADING SPEEDS, AND TRAVELING WAVES OF NONMONOTONE DYNAMICAL SYSTEMS*

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Abstract. In this paper, we study the asymptotic behavior, the spreading speed, and the existence/nonexistence of traveling waves of a class of nonmonotone discrete-time dynamical system. As a byproduct, we also obtain some results on the global attractivity of a nontrivial constant fixed point and on the existence of a nonconstant fixed point. We then apply the main results to three model systems: (i) a spatially nonlocal integro-difference equation; (ii) a reaction-diffusion equation with spatial nonlocality and time delay in the reaction term; and (iii) an equation with nonlocal diffusion and delayed nonmonotone nonlinearity in the reaction term. The obtained results for these three equations improve some existing ones by removing the symmetry of the kernel functions and relaxing the conditions on the nonlinear terms.

Key words. asymptotic behavior, global attractivity, nonmonotone dynamical system, spreading speed, traveling waves

AMS subject classifications. 37L15, 92D25

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1. Introduction.

In order to study reaction-diffusion equations, Aronson and Weinberger [2, 3] introduced the concept of the spreading speed and showed that it coincides with the minimal wave speed for traveling wave fronts under appropriate assumptions. Since the solution of the initial value problem of reaction-diffusion equations may also be considered as a solution to some discrete dynamical system in an appropriate space, Weinberger [37] and Lui [21] established the theory of spreading speeds and monostable traveling waves for a monotone *discrete* dynamical system. This theory has been further developed recently in [7, 16, 17, 18, 19, 20, 38, 39, 47] for more general *monotone* semiflows so that it can be applied to a variety of discrete and continuous time evolution equations admitting the comparison principle.

However, many discrete and continuous time population models with spatial structure are not monotone. For example, scalar discrete time integro-difference equations with nonmonotone growth functions and predator-prey type reaction diffusion systems are among such models. The asymptotical behavior of some *nonmonotone* continuous time integral equations and time-delayed reaction diffusion models have been established in [35, 36]. The spreading speeds and the existence of monostable traveling wave fronts were obtained for some nonmonotone *continuous-time* integral equations and time-delayed reaction diffusion models in [4, 36]. Motivated by the ideas of the above works, established in [5, 6, 9, 24, 26, 28, 40] was the existence of monostable traveling wave fronts for several other classes of *nonmonotone* time-delayed reaction-diffusion equations.

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As explained in [2, 3], many spatial-temporal population models can be included (as special cases) in the following form of discrete dynamical system:

$$
\begin{cases} u_{n+1}(x) & = & Q[u_n](x), \\ u_0 & = & \phi, \end{cases} \tag{1.1}
$$

where $Q : C_+ \to C_+$ and $\phi \in C_+ \triangleq BC(\mathbb{R}^N, \mathbb{R}_+)$, which is the set of bounded and continuous functions from $\mathbb{R}^N$ to $\mathbb{R}_+$ equipped with the topology induced by norm $||\phi||_C$ defined in section 2. By applying the results in [18, 19], the asymptotic speeds of spread and traveling waves for some *reflection-equivariant* and *nonmonotone* dynamical systems on $BC([-\tau, 0] \times \mathcal{H}, \mathbb{R}_+)$ with $\mathcal{H} = \mathbb{R}$ or $\mathbb{Z}$ were explored in [42]. The restriction of *reflection-equivalence* implies that the the results and approaches in [18, 19, 42] cannot be applied, at least directly, to those *spatially nonlocal* systems with nonsymmetric kernels.

The main goal of this paper is to develop a new approach which can be applied to spatially nonlocal systems which may not be monotone and allow nonsymmetric spatial kernels. More specifically, without assuming the monotonicity of Q, we study the asymptotic behavior of solutions, the spreading speed, and the existence/nonexistence of traveling wave fronts of (1.1). In particular, we obtain the convergence of solutions of (1.1) by using similar arguments to that of [41, 42, 44, 45, 46]. Combining the proof for the existence of traveling wave fronts in [37, 42] with the Schauder fixed point theorem, we obtain the existence of traveling wave fronts under much weaker conditions. Here, our approach is different from that in [4, 5, 6, 13, 35, 36]. For example, in order to prove the existence of traveling wave fronts, sub- and supersolutions need to be constructed in the above works, and such constructions largely depend on the particular forms of the equations. Our approach aims at revealing the relation between the asymptotic behaviour of general solutions, traveling wavefronts, and the spread phenomenon. As a byproduct of our approach, we are also able to establish the global attractivity of a nontrivial constant steady state and the existence of a nonconstant steady state. Note that by some different approaches, we also discuss the global attractivity of a *nontrivial constant* steady state for a differential equation with spatial nonlocality in an unbounded domain in $C_+ \setminus \{0\}$ under the *compact open topology* in [41, 45]. The main results will be applied to three particular model equations: (i) a spatially nonlocal integro-difference equation; (ii) a reaction-diffusion equation with spatial nonlocality and time delay in the reaction term; and (iii) an equation with nonlocal diffusion as well as delayed nonmonotone nonlinearity in the reaction term. We shall see that the resulting criteria for these systems can be amazingly simple and can even be optimal in some sense, and they improve some existing results by removing the symmetry on the kernel functions and relaxing the conditions on the nonlinear reaction terms in these three equations.

We point out that the recent works [1, 8] also made similar attempts to do what we do in this paper on the traveling wavefront, but there is a significant difference. To be more precise, while our work follows the discrete setting up as in Weinberger [2, 3, 37] and many follow-up works, [1, 8] used the framework of integral equations as in [4] which can also include many frequently encountered equations. In [1, 8], semiwavefronts are the focus, which may also lead to wavefronts in some situations, whereas in our work we directly explore wavefronts as well as the spreading speed and the asymptotic behaviour of solutions. Some of our results on traveling wavefronts for the particular equations may also be obtained by using the method/results in [1, 8] (see subsections 4.2 and 4.3).

2. Preliminaries and basic hypothesis. We first introduce some notation. Let $\mathbb{R}$, $\mathbb{R}_+$, and $\mathbb{R}^N$ be the sets of all reals, and nonnegative reals, N-dimension vectors, respectively, where N is a given positive integer. Denote the Euclidean norm of $\mathbb{R}^N$ by $||\cdot||$. Let $C = BC(\mathbb{R}^N, \mathbb{R})$ be the normed vector space of all bounded and continuous functions from $\mathbb{R}^N$ to $\mathbb{R}$ with the so-called compact open topology, that is, the topology induced by the norm

$$||\phi||_C \triangleq \sum_{n\geq 1} 2^{-n} \sup\{|\phi(x)| : x \in \mathbb{R}^N \text{ with } ||x|| \leq n\}, \quad \phi \in C.$$

Let $C_+ = \{\phi \in C : \phi(x) \geq 0 \text{ for all } x \in \mathbb{R}^N\}$ and $C_+^\circ = \{\phi \in C : \phi(x) > 0 \text{ for all } x \in \mathbb{R}^N\}$. It follows that C_+ is a closed cone in the normed vector space C. Note that $C_+^\circ \neq \text{Int}(C_+)$ due to the noncompactness of the spatial domain $\mathbb{R}^N$.

For a given $y \in \mathbb{R}^N$, define the translation operator T_y by $T_y[\phi](x) = \phi(x-y)$ for all $x \in \mathbb{R}^N$ and $\phi \in C$.

For $a \in \mathbb{R}$, $\breve{a} \in C$ is defined as $\breve{a}(x) = a$ for all $x \in \mathbb{R}^N$. For any $\xi, \eta \in C$, we write $\xi \geq \eta$ if $\xi - \eta \in C_+$, and $\xi > \eta$ if $\xi \geq \eta$ and $\xi \neq \eta$. For simplicity of notation, we shall write $a \triangleq \breve{a}$. For given numbers $r, s > 0$, define $C_r = \{\phi \in C : 0 \leq \phi \leq r\}$ and $C_{r,s} = \{\phi \in C : r \leq \phi \leq s\}$.

In what follows, we very often need to deal with the space C with $N = 1$, i.e., $BC(\mathbb{R}, \mathbb{R})$. For convenience of notation and statements, we denote this space with $N = 1$ by X (i.e., $X = BC(\mathbb{R}, \mathbb{R})$), again equipped also with the corresponding compact open topology, accordingly, and let $X_+ = \{\phi \in X : \phi(x) \geq 0 \text{ for all } x \in \mathbb{R}\}$ and $X_r = \{\phi \in X_+ : \phi(x) \leq r \text{ for all } x \in \mathbb{R}\}$ for all $r > 0$, and let $X_{r,s} = \{\phi \in X_+ : r \leq \phi(x) \leq s \text{ for all } x \in \mathbb{R}\}$ for all $s > r > 0$.

In this paper, we say that a map $Q : C_+ \to C$ is continuous(compact) if $Q|_{C_r}$ is continuous(compact) for any $r > 0$.

Let $r^* > 0$ be given and consider a continuous operator $Q^* : C_+ \to C_+$ possessing the following properties:

(H1) $T_y[Q^*[\phi]] = Q^*[T_y[\phi]]$ for all $(y, \phi) \in \mathbb{R}^N \times C_+$.

(H2) Q^* is order preserving in the sense that $Q^*[\phi] \leq Q^*[\psi]$ for all $\phi, \varphi \in C_+$ with $\phi \leq \psi$.

(H3) $Q^*[0] = 0$, $Q^*[r^*] = r^*$ and $Q^*[\alpha] > \alpha$ for all $\alpha \in (0, r^*)$.

(H4) $Q^*[\alpha] < \alpha$ for all $\alpha \in (r^*, \infty)$.

Following Weinberger [37], for the discrete dynamical system of the form (1.1) with Q replaced by Q^*, we can define the wave speed c^* by choosing a checking function φ that satisfies the following property:

(P1) φ is a continuous and nonincreasing function of one real variable with $\varphi(-\infty) \in (0, r^*)$ and $\varphi([0, \infty)) \equiv 0$.

For any real number c and any unit vector ξ in $\mathbb{R}^N$, we define $H_{c,\xi} : X_{r^*} \to C_+$ and $R_{c,\xi} : C(\mathbb{R}, [0, r^*]) \to C(\mathbb{R}, [0, r^*])$ by

$$H_{c,\xi}[a](x) = a(x \cdot \xi + c)$$

and

$$R_{c,\xi}[a](s) = \max\{\varphi(s), Q^*[H_{c+s,\xi}[a]](0)\},$$

where $x \in \mathbb{R}^N$, $s \in \mathbb{R}$, and $a \in X_{r^*}$. For simplicity, we denote $H_{c,\xi}[a]$ by $a(\cdot \cdot \xi + c)$ and thus $R_{c,\xi}[a](s) = \max\{\varphi(s), Q^*[a(\cdot \cdot \xi + c + s)](0)$ for any $a \in X_{r^*}$ and $s \in \mathbb{R}$.

Given real number c and unit vector ξ in $\mathbb{R}^N$, the iteration scheme

$$a_{n+1}(c,\xi;\cdot) = R_{c,\xi}[a_n(c,\xi;\cdot)],\ \ a_0(c,\xi;\cdot) = \varphi$$

generates a sequence $\{a_n(c,\xi;\cdot)\}$ in X_{r^*}. By the remarks after Lemma 5.1 in [37], this sequence is convergent in X_{r^*}. Let $a(c,\xi;s) = \lim_{n\to\infty} a_n(c,\xi;s)$. We remark that the sequence a_n depends upon the choice of the function φ, and thus a also depends upon this choice. However, by Lemma 5.4 in [37], $a(c,\xi;+\infty)$ is independent of the choice of φ. As in [37], the wave speed c^* with respect to Q^* along the direction ξ is defined as

$$c^*(\xi) \equiv \sup\{c : a(c,\xi;+\infty) = r^*\}, \tag{2.1}$$

where ξ is any unit vector in $\mathbb{R}^N$. If $a(c,\xi,+\infty) = r^*$ for all c, we set $c^*(\xi) = +\infty$.

For any set V of vectors in $\mathbb{R}^N$, we define

$$nV = \{v_1 + v_2 + \cdots + v_n : v_j \in V \text{ for } j = 1,\cdots,n\}.$$

It is easily seen that if V is convex, then $nV = \{nv | v \in V\}$. Also, with respect to Q^*, we define the following convex set:

$$\mathcal{A}^* = \{x \in \mathbb{R}^N | x\cdot\xi \le c^*(\xi) \text{ for all unit vector } \xi \in \mathbb{R}^N\}. \tag{2.2}$$

For $A, B \subseteq \mathbb{R}^N$, we write $A \subset\subset B$ when $ClA \subseteq \text{Int}B$. For $A \subseteq C$, we write

$$A(x) \triangleq \{\phi(x) : \phi \in A\},$$
$$\sup_{x\in\mathbb{R}^N} A(x) \triangleq \sup\{\phi(x) : x \in \mathbb{R}^N \text{ and } \phi \in A\},$$
$$\inf_{x\in\mathbb{R}^N} A(x) \triangleq \inf\{\phi(x) : x \in \mathbb{R}^N \text{ and } \phi \in A\}.$$

3. Main results. Since we do not assume that Q is order preserving, we will make use of two properly chosen auxiliary systems that are order preserving which are closely related to Q in some way. In this section, we always assume that $Q, Q^-, Q^+ : C_+ \to C_+$ are continuous and compact with $Q[0] = Q^-[0] = Q^+[0] = 0$, $Q^-[r^-] = r^-$ and $Q^+[r^+] = r^+$ for two given numbers $r^+ \ge r^-$. For any given $u_0 \in C_+$, the recursion $u_{n+1} = Q[u_n]$ defines a sequence $\{u_n\}_{n=0}^{\infty}$ in C_+.

When Q^+ (Q^-) satisfies (H1)–(H3) with Q^* and r^* replaced by Q^+ (Q^-) and r^+ (r^-), respectively, the wave speed along the direction ξ for Q^+ (Q^-) is also defined by (2.1) with r^* replaced by r^+ (r^-), denoted by $c^*_+(\xi)$ ($c^*_-(\xi)$). Replacing r^* by r^+ (r^-) in (2.2), a convex set $\mathcal{A}^+$ ($\mathcal{A}^-$) is also defined.

THEOREM 3.1. *Assume that $Q[\phi] \le Q^+[\phi]$ for all $\phi \in C_+$, and Q^+ satisfies the assumptions* (H1)–(H3) *with r^* replaced by r^+.*

(i) *Suppose that the set $\mathcal{A}^+$ is nonempty and bounded. Then for any open set $\mathcal{A}_1$ containing $\mathcal{A}^+$ and $u_0 \in C_{r^+}$ that has compact support, it holds that*

$$\lim_{n\to\infty} \max_{x\notin n\mathcal{A}_1} u_n(x) = 0. \tag{3.1}$$

(ii) *If $\mathcal{A}^+$ is empty and $c^*_+(\xi)$ is bounded for all unit vectors ξ in $\mathbb{R}^N$, then the above statement holds when the maximum in* (3.1) *is taken over the whole space $\mathbb{R}^N$.*

Proof. By applying Theorem 6.1 in [37] to Q^+ and combining Proposition 4.1 in [37] with the fact that $0 \leq Q[\phi] \leq Q^+[\phi]$ for all $\phi \in C_+$, we easily see that all conclusions in this theorem hold. This completes the proof. ∎

To proceed further to study the asymptotical behavior of u_n, we formulate the following nonmonotone assumption on the nonlinearity Q:

(H5) There is $h > 0$ such that $Q[h] = h$ and Q satisfies the assumptions (H1) and (H3). Moreover, if $s > r > 0$, then $\inf_{x\in\mathbb{R}^N} Q[C_{r,s}](x) > r$ or $\sup_{x\in\mathbb{R}^N} Q[C_{r,s}](x) < s$.

Clearly, if Q satisfies (H5), then Q satisfies (H4).

THEOREM 3.2. *Assume that $Q[\phi] \geq Q^-[\phi]$ for all $\phi \in C_+$ and Q^- satisfies the assumptions* (H1)–(H3) *with r^* replaced by r^-. Suppose that the interior of $\mathcal{A}^-$ is not empty and let $\mathcal{A}_2$ be any closed bounded subset contained in the interior of $\mathcal{A}^-$. Then, for any $\sigma > 0$ there exists a $r_\sigma > 0$ such that if $u_0(x) \geqq \sigma$ on a ball of radius r_σ, it holds that*

$$\liminf_{n\to\infty} \min_{x\in n\mathcal{A}_2} u_n(x) \geqq r^-. \tag{3.2}$$

Moreover, suppose that $Q[\phi] \leq Q^+[\phi]$ for all $\phi \in C_+$, and Q^+ satisfies the assumptions (H1-H4) with r^* replaced by r^+. If Q satisfies (H5), then

$$\lim_{n\to\infty} \min_{x\in n\mathcal{A}_2} u_n(x) = \lim_{n\to\infty} \max_{x\in n\mathcal{A}_2} u_n(x) = h. \tag{3.3}$$

Proof. Using Theorem 6.2 in [37], we get $\liminf_{n\to\infty} \min_{x\in n\mathcal{A}_2} (Q^-)^n[u_0](x) \geqq r^-$. This together with Proposition 4.1 in [37] leads to (3.2).

We now prove (3.3). Since $\mathcal{A}_2 \subset\subset \mathcal{A}^- \subseteq \mathcal{A}^+$ due to Proposition 5.5 in [37] and the definition of $\mathcal{A}^\pm$, by a similar argument to the proof of (3.2), we have

$$\limsup_{n\to\infty} \max_{x\in n\mathcal{A}_2} u_n(x) \leq r^+.$$

Take a closed bounded subset $\mathcal{D}_0$ with $\mathcal{A}_2 \subset\subset \mathcal{D}_0 \subset\subset \mathcal{A}^-$. Applying the above obtained results to $\mathcal{D}_0$, we easily see

$$r^- \leq \liminf_{n\to\infty} \min_{x\in n\mathcal{D}_0} u_n(x) \leqq \limsup_{n\to\infty} \max_{x\in n\mathcal{D}_0} u_n(x) \leq r^+.$$

For any $\varepsilon \geq 0$, define

$$\mathcal{A}_2^\varepsilon = \{x \in \mathcal{D}_0 : dist(x, \mathcal{A}_2) \leq \varepsilon\},$$

$$U_-(\varepsilon) = \liminf_{n\to\infty} \min\{u_n(x) : x \in n\mathcal{A}_2^\varepsilon\},$$

and

$$U_+(\varepsilon) = \limsup_{n\to\infty} \max\{u_n(x) : x \in n\mathcal{A}_2^\varepsilon\}.$$

Then $U_-(\varepsilon) \leq U_+(\varepsilon)$, and for any $\tau \geq \sigma \geq 0$, we know that $\mathcal{A}_2 \subseteq \mathcal{A}_2^\sigma \subseteq \mathcal{A}_2^\tau \subseteq \mathcal{D}_0$, and thus $U_\pm(\varepsilon) \in [r^-, r^+]$, $U_-(\varepsilon)$ is nonincreasing in $\varepsilon \geq 0$ and $U_+(\varepsilon)$ is nondecreasing in $\varepsilon \geq 0$. Moreover, by $\mathcal{A}_2 \subset\subset \mathcal{D}_0$, there is $\varepsilon_0 > 0$ such that for any $\varepsilon_0 > \tau > \sigma > 0$, $\mathcal{A}_2 \subset\subset \mathcal{A}_2^\sigma \subset\subset \mathcal{A}_2^\tau \subset\subset \mathcal{D}_0$. Due to the monotonicity of $U_\pm$, we easily see that

$U_\pm(\varepsilon)$ are continuous in $\varepsilon \in [0,\varepsilon_0]$, except possibly for ε from a countable set of $[0,\varepsilon_0]$. If $U_-(\varepsilon) < U_+(\varepsilon)$ for any $\varepsilon \in [0,\varepsilon_0]$, then by (H5) and the continuity of $U_\pm$, we may assume, without loss of generality, that for some $\varepsilon \in (0,\varepsilon_0)$, U_- is continuous at ε and $\inf_{x\in\mathbb{R}^N} Q[C_{U_-(\varepsilon),U_+(\varepsilon)}](x) > U_-(\varepsilon)$. According to the definition of $U_-(\tau)$, for any $\tau \in (0,\varepsilon)$, there exists sequences $n_k \to \infty$ and $x_k \in n_k\mathcal{A}_2^\tau$ such that $\lim_{n_k\to\infty} u_{n_k}(x_k) = U_-(\tau)$. Since $\mathcal{A}_2^\tau \subset\subset \mathcal{A}_2^\varepsilon$, we know that for any bounded subset $\mathcal{B}$ of $\mathbb{R}^N$, $x_k + \mathcal{B} \subseteq (n_k - 1)\mathcal{A}_2^\varepsilon$ for all large k, which implies $\liminf_{k\to\infty} \min_{y\in\mathcal{B}} u_{n_k-1}(x_k + y)$ and $\limsup_{k\to\infty} \max_{y\in\mathcal{B}} u_{n_k-1}(x_k + y) \in [U_-(\varepsilon), U_+(\varepsilon)]$. These combined with (H1) imply

$$U_-(\tau) = \lim_{n_k\to\infty} u_{n_k}(x_k) = \lim_{n_k\to\infty} Q[u_{n_k-1}(\cdot + x_k)](0) \\ \geq \inf_{x\in\mathbb{R}^N} Q[C_{U_-(\varepsilon),U_+(\varepsilon)}](x) > U_-(\varepsilon).$$

By the continuity of U_- at ε and letting $\tau \to \varepsilon$, we have $U_-(\varepsilon) \geq \inf_{x\in\mathbb{R}^N} Q[C_{U_-(\varepsilon),U_+(\varepsilon)}](x) > U_-(\varepsilon)$, a contradiction. Thus, $U_-(\varepsilon) = U_+(\varepsilon)$ for some $\varepsilon \in [0,\varepsilon_0]$. This, together with (H5), gives $U_-(0) = U_+(0) = h$, and thus the conclusion follows. This completes the proof. ∎

In the following, we turn to the study of traveling waves of Q by applying some results in [37]. We point out that other various methods have also been used to obtain the traveling waves of some particular equations; see, e.g., [4, 5, 13, 24, 36] and the references therein.

Let φ be a checking function that satisfies the property (P1) in section 2 with r^* replaced by r^-. For any $k \in (0,1)$, any real number c, and any unit vector ξ in $\mathbb{R}^N$, we define the operators $H_{c,\xi} : X_+ \to C_+$, $R_{c,\xi,k} : X_+ \to X_+$, and $R^\pm_{c,\xi,k} : X_+ \to X_+$ by

$$H_{c,\xi}[a](x) = a(x\cdot\xi + c),$$

$$R_{c,\xi,k}[a](s) = \max\{k\varphi(s), Q[H_{c+s,\xi}[a]](0)\},$$

and

$$R^\pm_{c,\xi,k}[a](s) = \max\{k\varphi(s), Q^\pm[H_{c+s,\xi}[a]](0)\},$$

respectively, where $a \in X_+$.

LEMMA 3.1. *Suppose that $Q, Q^\pm$ satisfy all conditions in Theorem 3.2. Also assume that Q and $Q^\pm$ are compact. Then, for any unit vector ξ in $\mathbb{R}^N$ and $c \geq c_+^*(\xi)$, the following statements are true:*

(i) *For any $k \in (0,1)$, $R^-_{c,\xi,k}[a] \leq R_{c,\xi,k}[a] \leq R^+_{c,\xi,k}[a]$ for all $a \in X_+$ and $R^\pm_{c,\xi,k}$ are order-preserving with the pointwise order.*

(ii) *For any $k \in (0,1)$, $R_{c,\xi,k}|_{X_r}$ and $R^\pm_{c,\xi,k}|_{X_r}$ are continuous and compact for all $r > 0$.*

(iii) *For any $k \in (0,1)$, there are two nonincreasing functions $a_k^+, a_k^- \in X_{r^+}$ such that $a_k^- \leq a_k^+$, $a_k^\pm(\infty) = 0$ and $a_k^\pm(-\infty) = r^\pm$, $a_k^\pm = \lim_{n\to\infty}(R^\pm_{c,\xi,k})^n[k\varphi]$, and thus $R_{c,\xi,k}[a_k^\pm] = a_k^\pm$.*

(iv) *For any $k \in (0,1)$, there is $a_k \in X_{r^+}$ such that $R_{c,\xi,k}[a_k] = a_k$, and $a_k^- \leq a_k \leq a_k^+$ with the pointwise order, and thus $a_k(\infty) = 0$, $\liminf_{s\to-\infty} a_k(s) \geq r^-$, and $\limsup_{s\to-\infty} a_k(s) \leq r^+$.*

Proof. First, we notice that (i) and (ii) follow from the continuity and compactness of Q and $Q^{\pm}$ and the monotonicity of $Q^{\pm}$.

(iii) By (ii) and Lemma 5.1 in [37], there are nonincreasing functions $a_k^+, a_k^- \in X_{r^+}$ such that $a_k^+ \geq a_k^-$ with the pointwise order, $\lim_{n\to\infty}(R^{\pm}_{c,\xi,k})^n[k\varphi] = a_k^{\pm}$, and thus $R_{c,\xi,k}[a_k^{\pm}] = a_k^{\pm}$. Proposition 5.2 in [37] gives $a_k^{\pm}(\infty) = 0$ and $a_k^{\pm}(-\infty) = r^{\pm}$.

(iv) Let $Y_k = \{a \in X_{r^+} : a_k^-(s) \leq a(s) \leq a_k^+(s) \text{ for all } s \in \mathbb{R}\}$. By (i) and (iii), we easily see that $R_{c,\xi,k}[Y_k] \subseteq Y_k$. By the convexity of Y_k and the compactness of $R_{c,\xi,k}$, the Schauder fixed point theorem implies $R_{c,\xi,k}[a_k] = a_k$ for some $a_k \in Y_k$. Hence, by the choice of Y_k, we have $a_k^- \leq a_k \leq a_k^+$ with the pointwise order, and thus $a_k(\infty) = 0$, $\liminf_{s\to-\infty} a_k(s) \geq r^-$ and $\limsup_{s\to-\infty} a_k(s) \leq r^+$. This completes the proof. ∎

We say that $W(x\cdot\xi - nc)$ is a traveling wave of the map Q with the wave speed c if $W \in X_+$ is nonconstant and $Q[W(\cdot\cdot\xi - nc)](x) = W(x\cdot\xi - (1+n)c)$ for all integers n. We say that $W(x\cdot\xi - nc)$ connects h to 0 if $W(-\infty) = h$ and $W(\infty) = 0$.

THEOREM 3.3. *Assume that $Q, Q^{\pm}$ satisfies all conditions in Theorem* 3.2. *If* $\mathrm{Int}(\mathcal{A}_-)$ *is not empty and ξ is a unit vector in $\mathbb{R}^N$, then the following statements hold:*

(i) *If $c\xi\cdot\eta < c^*_-(\eta)$ for all unit vector $\eta \in \mathbb{R}^N$, then Q has no traveling wave $W(x\cdot\xi - nc)$ such that $\liminf_{s\to-\infty} W(s) > 0$.*

(ii) *If $Q, Q^{\pm}$ are compact and $c \geq c^*_+(\xi)$, then Q has a traveling wave $W(x\cdot\xi-nc)$ with $W(-\infty) = h$ and $\liminf_{s\to\infty} W(s) = 0$.*

Proof. (i) Obviously, $c\xi \in \mathrm{Int}(\mathcal{A}_-)$, and thus we may choose a closed bounded subset $\mathcal{A}_3$ with $c\xi \in \mathrm{Int}(\mathcal{A}_3) \subseteq \mathcal{A}_3 \subset\subset \mathcal{A}_-$. Assume for the sake of contradiction that Q has a traveling wave $W(x\cdot\xi - nc)$ such that $\liminf_{s\to-\infty} W(s) > 0$. Then by letting $\sigma = \frac{1}{3}\liminf_{s\to-\infty} W(s)$ and by Theorem 3.2, there exists a $r_0 = r_0(\sigma) > 0$ with the property that if $u_0(x) \geqq \sigma$ on a ball with radius r_0, then $\lim_{n\to\infty}\min_{x\in n\mathcal{A}_3} u_n(x) = h$, and thus $\lim_{n\to\infty}\min_{x\in n\mathcal{A}_3} W(x\cdot\xi - nc) = h$. Then by (H5), there is $x^* \in \mathbb{R}^N$ such that $W(x^*\cdot\xi) < h$. By $\liminf_{s\to-\infty} W(s) > 0$ and $c\xi \in \mathrm{Int}(\mathcal{A}_3)$, we obtain that $nc\xi + x^* \in n\mathcal{A}_3$ for all large n, and hence

$$\begin{aligned} W(x^*\cdot\xi) &= \lim_{n\to\infty} W(x^*\cdot\xi + nc - nc) \\ &= \lim_{n\to\infty}\min_{x=nc\xi+x^*} W(x\cdot\xi - nc) \\ &\geq \lim_{n\to\infty}\min_{x\in n\mathcal{A}_3} W(x\cdot\xi - nc) = h. \end{aligned}$$

This is a contradiction which implies that the statement (i) holds.

(ii) By Lemma 3.1(iv), for any $k \in (0,1)$, there is $a(c,\xi,k;\cdot) \in X_{r^+}$ such that $R_{c,\xi,k}[a(c,\xi,k;\cdot)] = a(c,\xi,k;\cdot)$, $a_k^- \leq a(c,\xi,k;\cdot) \leq a_k^+$, $a(c,\xi,k;\infty) = 0$ and $\liminf_{s\to-\infty} a(c,\xi,k;s) \geq r^-$, where $a_k^{\pm}$ are defined as in Lemma 3.1(iii). Thus, for any $k \in (0,1)$, there is $s_k \in \mathbb{R}$ such that $a(c,\xi,k;s_k) = \frac{r^-}{3}$ and $a(c,\xi,k;s+s_k) \geq \frac{r^-}{3}$ for all $s \leq 0$. By the compactness of Q and the fact that $a(c,\xi,k;s) = \max\{k\varphi(s), Q[H_{c+s,\xi}[a(c,\xi,k;\cdot]](0)\}$, there is a subsequence k_l of k in $(0,1)$ such that $\lim_{l\to\infty} k_l = 0$ and $a(c,\xi,k_l;s_{k_l}+\cdot)$ tends to a function $W \in X_{r^+}$ in $BC(\mathbb{R},\mathbb{R})$, as $l\to\infty$. Thus, $W(0) = \lim_{l\to\infty} a(c,\xi,k_l;s_{k_l}) = \frac{r^-}{3}$ and $W(s) \geq \frac{r^-}{3}$ for all $s \leq 0$. In particular, for any given integers n, $a(c,\xi,k_l;\cdot\cdot\xi + s_{k_l} - nc)$ converge uniformly on bounded subsets of $\mathbb{R}^N$ to $W(\cdot\cdot\xi - nc)$, as $l\to\infty$. Therefore, by the fact that $a(c,\xi,k;s) = \max\{k\varphi(s), Q[a(c,\xi,k;\cdot\cdot\xi + s + c)](0)\}$, we know that $W(x\cdot\xi - (1+n)c) = Q[W(\cdot\cdot\xi - nc)](x)$ for all integers n.

Choose a closed bounded subset $\mathcal{A}_4$ with $\emptyset \neq \mathrm{Int}(\mathcal{A}_4) \subseteq \mathcal{A}_4 \subset\subset \mathcal{A}_-$. By (3.3) in Theorem 3.2 and the fact that $W(s) \geq \frac{r^-}{3}$ for all $s \leq 0$, we obtain $\lim_{n\to\infty} \min_{x\in n\mathcal{A}_4} W(x \cdot \xi - nc) = \lim_{n\to\infty} \max_{x\in n\mathcal{A}_4} W(x \cdot \xi - nc) = h$. Take a sequence s_k with $\lim_{k\to\infty} s_k = -\infty$. Then, by the choice of $\mathcal{A}_4$, there exist positive integer n_k and $x_k \in \mathrm{Int}(\mathcal{A}_4)$ such that $s_k = n_k(x_k \cdot \xi - c)$ and $\lim_{k\to\infty} n_k = \infty$. Thus, $\lim_{n\to\infty} W(s_k) = \lim_{n\to\infty} W(n_k x_k \cdot \xi - n_k c) = h$. So, $W(-\infty) = h$, and thus W is a nonconstant function.

We now claim that $\liminf_{s\to-\infty} W(s) = 0$. Otherwise, there exists $\varepsilon > 0$ such that $W(s) > \varepsilon$ for all $s \in \mathbb{R}$. This implies (see also the discussions of the last paragraph in section 5) that $W(x \cdot \xi - nc)$ tends to h with the supremum norm as $n \to \infty$. Thus, there is a positive integer n_0 such that $W(x \cdot \xi - n_0 c) \geq \frac{r^-}{2}$ for all $x \in \mathbb{R}^N$. By letting $x = n_0 c\xi$, we have $W(0) \geq \frac{r^-}{2}$, a contradiction. This completes the proof of the theorem. ∎

THEOREM 3.4. *Suppose that Q and $Q^{\pm}$ satisfy all conditions in Theorem* 3.2. *Assume that there is a bounded nonnegative measure $m(x, dx)$ on $\mathbb{R}^n$ so that the following hold:*

$$Q^+[\phi] \leq \int_{\mathbb{R}^N} \phi(\cdot - y) m(y, dy) \text{ for } \phi \in C_{r^+}; \tag{3.4}$$

and for every $\delta > 0$, there is an $\varepsilon > 0$ such that

$$Q^-[\phi] \geqq (1 - \delta) \int_{\mathbb{R}^N} \phi(\cdot - y) m(y, dy), \quad \text{for } \phi \in C_\varepsilon. \tag{3.5}$$

Then, for any unit vector $\xi \in \mathbb{R}^N$,

$$c^*_-(\xi) = c^*_+(\xi) = \inf_{\mu>0} \log \int_{\mathbb{R}^N} e^{\mu x\cdot\xi} m(x, dx) \triangleq \hat{c}^*(\xi), \tag{3.6}$$

and thus $\mathcal{A}^- = \mathcal{A}^+ \triangleq \hat{\mathcal{A}}$, where the right-hand side is ∞ in (3.6) *if the integral on the right diverges for all positives μ. Moreover, if $\mathrm{Int}(\hat{\mathcal{A}})$ is not empty and $\mathcal{A}_0$ is any closed bounded subset contained in the interior of $\hat{\mathcal{A}}$, then the following statements hold:*

(i) *There exists a $d_0 > 0$ with the property that if $u_0(x) > 0$ on a ball of radius d_0, then $\lim_{n\to\infty} \min_{x\in n\mathcal{A}_0} u_n(x) = \lim_{n\to\infty} \max_{x\in n\mathcal{A}_0} u_n(x) = h$. Furthermore, if the support of $m(\cdot, dx)$ contains a subset $\mathcal{V}$ in $\mathbb{R}^N$ with the property that any bounded subset of $\mathbb{R}^N$ is contained in a translation of the set $n\mathcal{V}$ for some integer n, then for any $u_0 \in C_+ \setminus \{0\}$, $\lim_{n\to\infty} \min_{x\in n\mathcal{A}_0} u_n(x) = \lim_{n\to\infty} \max_{x\in n\mathcal{A}_0} u_n(x) = h$.*

(ii) *If $c\xi \cdot \eta < c^*(\eta)$ for every unit vector η in $\mathbb{R}^N$, then Q has no traveling wave $W(x \cdot \xi - nc)$ such that $\liminf_{s\to-\infty} W(s) > 0$.*

(iii) *Assume that the support of $m(\cdot, dx)$ contains a subset $\mathcal{V}$ in $\mathbb{R}^N$ with the property that any bounded subset of $\mathbb{R}^N$ is contained in a translation of the set $n\mathcal{V}$ for some integer n. If $Q, Q^{\pm}$ are compact and $c \geq c^*(\xi)$, then Q has a traveling wave $W(x \cdot \xi - nc)$ connecting h to* 0.

Before giving the proof, we remark that by the definition, the infimum in (3.6) is taken for values of μ belonging to the abscissa of integral convergence and is set to ∞ if the integral is not convergent for any positive values of μ.

Proof. First, applying Corollary in [37] to $Q^{\pm}$, we easily get (3.6). The proof of (ii) follows from Theorem 3.3. It remains to prove (i) and (iii).

(i) By (3.4)-(3.5) and the fact that Q satisfies (H3), there exist $\delta^* \in (0,1)$ and $\varepsilon^* \in (0, r^-)$ such that

$$(1-\delta^*)\int_{\mathbb{R}^N} m(y,dy) > 1 \text{ and } Q^-[u] \geqq (1-\delta^*)\int_{\mathbb{R}^N} \phi(\cdot - y)m(y,dy) \text{ for all } \phi \in C_{\varepsilon^*}.$$

Define $M : C_+ \to C_+$ by

$$M[\phi](x) = \min\{\varepsilon^*, (1-\delta^*)\int_{\mathbb{R}^N} \min\{\varepsilon^*, \phi(x-y)\}m(y,dy)\} \text{ for } x \in \mathbb{R}^N,\ \ \phi \in C_+.$$

Note that $Q^-[\phi] \geq M[\phi]$ for all $\phi \in C_+$ due to the definition of M. By applying Theorem 6.2 and Lemma 8.8 in [37] to M, there exists a radius d_0 with the property that if $u_0(x) > 0$ on a ball of radius d_0, then

$$\lim_{n\to\infty} \min_{x\in n\mathcal{A}_0} M^n[u_0](x) = \varepsilon^*.$$

Therefore,

$$\liminf_{n\to\infty} \min_{x\in n\mathcal{A}_0} u_n(x) \geq \liminf_{n\to\infty} \min_{x\in n\mathcal{A}_0} (Q^-)^n[u_0](x) \geq \liminf_{n\to\infty} \min_{x\in n\mathcal{A}_0} M^n[u_0](x) = \varepsilon^* > 0.$$

Following the proof of (3.2) in Theorem 3.2, we easily see that the first conclusion of (i) holds. Again, by an argument similar to the proof of Theorem 6.5 in [37], the second conclusion of (i) is also confirmed.

(iii) By Lemma 3.1(iv), for any $k \in (0,1)$, there is $a(c,\xi,k;\cdot) \in X_{r^+}$ such that $R_{c,\xi,k}[a(c,\xi,k;\cdot)] = a(c,\xi,k;\cdot)$, $a_k^-(c,\xi,k;\cdot) \leq a(c,\xi,k;\cdot) \leq a_k^+(c,\xi,k;\cdot)$, $a(c,\xi,k;\infty) = 0$ and $\liminf_{s\to-\infty} a(c,\xi,k;s) \geq r^-$. Thus, for any $k \in (0,1)$, there is $s_k \in \mathbb{R}$ such that $a(c,\xi,k;s_k) = \frac{r^-}{3}$ and $a(c,\xi,k;s+s_k) \leq \frac{r^-}{3}$ for all $s \geq 0$. By the compactness of Q and the fact that

$$a(c,\xi,k;s) = \max\{k\varphi(s), Q[H_{c+s,\xi}[a(c,\xi,k;\cdot)]](0)\},$$

there is a subsequence k_l of k in $(0,1)$ such that $\lim_{l\to\infty} k_l = 0$ and $a(c,\xi,k_l;s_{k_l}+\cdot)$ tends to a function $W \in X_{r^+}$ as $l \to \infty$. Thus, $W(0) = \lim_{l\to\infty} a(c,\xi,k_l;s_{k_l}) = \frac{r^-}{3}$ and $W(s) \leq \frac{r^-}{3}$ for all $s \geq 0$. In particular, for all integers n, $a(c,\xi,k_l;x\cdot\xi + s_{k_l} - nc)$ converge uniformly on bounded subsets of $\mathbb{R}^N$ to $W(x\cdot\xi - nc)$, as $l \to \infty$. Therefore, by the fact that $a(c,\xi,k;s) = \max\{k\varphi(s), Q[a(c,\xi,k;\cdot\cdot\xi + s + c)](0)\}$, we know that $W(x\cdot\xi - (1+n)c) = Q[W(\cdot\cdot\xi - nc)](x)$ for all integers n.

Choose a closed bounded subset $\mathcal{A}_1$ with $\emptyset \neq \mathrm{Int}(\mathcal{A}_1) \subseteq \mathcal{A}_1 \subset\subset \mathcal{A}_- = \hat{\mathcal{A}}$. By the second conclusion of Theorem 3.4(i), we get $\lim_{n\to\infty} \min_{x\in n\mathcal{A}_1} W(x\cdot\xi - nc) = \lim_{n\to\infty} \max_{x\in n\mathcal{A}_1} W(x\cdot\xi - nc) = h$. Take any sequence s_k with $\lim_{k\to\infty} s_k = -\infty$. Then, by the choice of $\mathcal{A}_1$, there exists positive integer n_k and $x_k \in \mathrm{Int}(\mathcal{A}_1)$ such that $s_k = n_k(x_k\cdot\xi - c)$ and $\lim_{k\to\infty} n_k = \infty$. Thus, $\lim_{n\to\infty} W(s_k) = \lim_{n\to\infty} W(n_k x_k\cdot\xi - n_k c) = h$. So, $W(-\infty) = h$ and W is a nonconstant function.

We claim that $\lim_{s\to\infty} W(s) = 0$. Otherwise, there are $W^* \in (0, \frac{r^-}{3}]$ and a sequence b_n such that $\lim_{b_n\to\infty} W(b_n) = W^*$. Let $w_n(x) = W(x\cdot\xi + b_n)$ for all integers n and

$x \in \mathbb{R}^N$. Without loss of generality, we may assume that $\lim_{n\to\infty} w_n = \phi$ for some $\phi \in C_+$ due to the compactness of Q. In particular, $\phi(0) = W^* > 0$, and hence $\phi \in C_+\backslash\{0\}$. It follows from the second conclusion of Theorem 3.4(i) that there are a positive integer N_0 and $x_0 \in N_0\mathcal{A}_0$ such that $Q^{N_0}[\phi](x_0) > \frac{r^-}{2}$. Again, by the continuity of Q, we have $\lim_{n\to\infty} Q^{N_0}[w_n](x_0) > \frac{r^-}{2}$. Hence, there is $n_0 > 1$ such that $x_0 \cdot \xi + b_{n_0} - N_0 c > 0$ and $Q^{N_0}[w_{n_0}](x_0) \geq \frac{r^-}{2}$. By (H1), we have $W(x_0 \cdot \xi + b_{n_0} - N_0 c) = Q^{N_0}[w_{n_0}](x_0) \geq \frac{r^-}{2}$. But, by the facts that $x_0 \cdot \xi + b_{n_0} - N_0 c > 0$ and $W(s) \leq \frac{r^-}{3}$ for all $s \geq 0$, we have $W(x_0 \cdot \xi + b_{n_0} - N_0 c) \leq \frac{r^-}{3}$, a contradiction. This completes the proof. ∎

To describe the *spreading speeds* along a unit vector ξ in $\mathbb{R}^N$, we give the following results by appealing to the proof of Theorem 2.1 in [38] or the proof of Theorems 6.1 and 6.2 in [37].

THEOREM 3.5. *Suppose that Q and $Q^\pm$ satisfy all conditions in Theorem* 3.2. *Assume that there is a bounded nonnegative measure $m(x, dx)$ on $\mathbb{R}^n$ so that* (3.4) *and* (3.5). *Then, for any unit vector $\xi \in \mathbb{R}^N$, $\hat{c}^*(\xi)$ in* (3.6) *satisfies the following spreading properties:*

(i) *If $u_0 \in C_r$ and $u_0|_{x\cdot\xi\geq L} \equiv 0$ for some $(r, L) \in [0, h)\times(0, \infty)$, and if $\hat{c}^*(\xi) < \infty$, then for any $c > \hat{c}^*(\xi)$, it holds that*

$$\lim_{n\to\infty}[\sup\{u_n(x) : x \cdot \xi \geq nc\}] = 0.$$

(ii) *If $u_0 \in C_h$ and $\lim_{K\to\infty}[\inf\{u_0(x) : x \cdot \xi \leq -K\}] > 0$, then for any $c < \hat{c}^*(\xi)$, it holds that*

$$\lim_{n\to\infty}[\sup\{|u_n(x) - h| : x \cdot \xi \leq nc\}] = 0.$$

Proof. By applying the proof of Theorem 2.1(1) in [38] to Q^+ and combining Proposition 4.1 in [37] with the fact that $0 \leq Q[\phi] \leq Q^+[\phi]$ for all $\phi \in C_+$, we easily see that (i) holds. It remains to prove (ii).

(ii) For any $\varepsilon \in [0, \hat{c}^*(\xi) - c)$, define

$$V_-(\varepsilon) = \liminf_{n\to\infty}[\inf\{u_n(x) : x \cdot \xi \leq n(c + \varepsilon)\}]$$

and

$$V_+(\varepsilon) = \limsup_{n\to\infty}[\sup\{u_n(x) : x \cdot \xi \leq n(c + \varepsilon)\}].$$

Applying the proof of Theorem 2.1(2) in [38] to $Q^\pm$, we may obtain for any $\varepsilon \in [0, \hat{c}^*(\xi) - c)$,

$$\liminf_{n\to\infty}[\inf\{(Q^-)^n[u_0](x) : x \cdot \xi \leq n(c + \varepsilon)\}] \geq r^-$$

and

$$\limsup_{n\to\infty}[\sup\{(Q^+)^n[u_0](x) : x \cdot \xi \leq n(c + \varepsilon)\}] \leq r^+.$$

These inequalities, with the fact that $Q^-[\phi] \leq Q[\phi] \leq Q^+[\phi]$ for all $\phi \in C_+$, imply that $r^- \leq V_-(\varepsilon) \leq V_+(\varepsilon) \leq r^+$ for all $\varepsilon \in [0, \hat{c}^*(\xi) - c)$. Note that $V_-(\varepsilon)$ is nonincreasing in $\varepsilon \in [0, \hat{c}^*(\xi) - c)$ and $V_+(\varepsilon)$ is nondecreasing in $\varepsilon \in [0, \hat{c}^*(\xi) - c)$. Due to the monotonicity of $V_\pm$, we easily see that $V_\pm(\varepsilon)$ are continuous in $\varepsilon \in [0, \hat{c}^*(\xi) - c)$, except possibly for ε from a countable set of $[0, \hat{c}^*(\xi) - c)$.

If $V_-(\varepsilon) < V_+(\varepsilon)$ for any $\varepsilon \in (0, \hat{c}^*(\xi) - c)$, then by (H5) and the continuity of $V_\pm$, we may assume, without loss of generality, that for some $\varepsilon_1 \in [0, \hat{c}^*(\xi) - c)$, V_+ is continuous at ε_1 and $\sup_{x\in\mathbb{R}^N} Q[C_{V_-(\varepsilon_1),V_+(\varepsilon_1)}](x) < V_+(\varepsilon_1)$. Suppose $\tau \in (0, \varepsilon_1)$. According to the definition of $V_+(\tau)$, there exists sequences $n_k \to \infty$ and $x_k \cdot \xi \leq n_k(c+\tau)$ such that $\lim_{n_k\to\infty} u_{n_k}(x_k) = V_+(\tau)$. In view of $\tau < \varepsilon_1$, we know that for any bounded subset $\mathcal{B}$ of $\mathbb{R}^N$, $x_k + \mathcal{B} \subseteq (n_k - 1)(c + \varepsilon_1)$ for all large k, which implies $\liminf_{k\to\infty}[\min_{y\in\mathcal{B}} u_{n_k-1}(x_k + y)]$ and $\limsup_{k\to\infty}[\max_{y\in\mathcal{B}} u_{n_k-1}(x_k + y)] \in [V_-(\varepsilon_1), V_+(\varepsilon_1)]$. These combined with (H1) imply

$$\begin{aligned} V_+(\tau) = \lim_{n_k\to\infty} u_{n_k}(x_k) &= \lim_{n_k\to\infty} Q[u_{n_k-1}(\cdot + x_k)](0) \\ &\leq \sup_{x\in\mathbb{R}^N} Q[C_{V_-(\varepsilon_1),V_+(\varepsilon_1)}](x) < V_+(\varepsilon_1). \end{aligned}$$

By the continuity of V_+ at ε_1 and letting $\tau \to \varepsilon_1$, we have $V_+(\varepsilon_1) \leq \sup_{x\in\mathbb{R}^N} Q[C_{V_-(\varepsilon_1), V_+(\varepsilon_1)}](x) < V_+(\varepsilon_1)$, a contradiction. Thus, $V_-(\varepsilon_2) = V_+(\varepsilon_2)$ for some $\varepsilon_2 \in (0, \hat{c}^*(\xi) - c)$. This, together with the monotonicity of $V_\pm$, yields that

$$V_-(\varepsilon) = V_+(\varepsilon) = V_-(0) = V_+(0) \text{ for all } \varepsilon \in [0, \varepsilon_2].$$

Take $x_n \in \mathbb{R}^N$ with $x_n \cdot \xi \leq n(c + \frac{\varepsilon_2}{3})$ for every positive integer n. In view of the definitions of $V_\pm$ and the choices of x_n and ε_2, we easily see that $\lim_{n\to\infty} u_n(x_n) = V_\pm(\frac{\varepsilon_2}{3})$, and for any bounded subset $\mathcal{B}$ of $\mathbb{R}^N$, $x_n + \mathcal{B} \subseteq (n-1)(c+\varepsilon_2)$ for all large n, which implies

$$\liminf_{n\to\infty}[\min_{y\in\mathcal{B}} u_{n-1}(x_n + y)] = \limsup_{n\to\infty}[\max_{y\in\mathcal{B}} u_{n-1}(x_n + y)] = V_\pm(\varepsilon_2) = V_\pm(0).$$

These combined with (H1) imply

$$V_+(0) = \lim_{n\to\infty} u_n(x_n) = \lim_{n\to\infty} Q[u_{n-1}(\cdot + x_n)](0) = Q[V_+(0)](0) \in [r^-, r^+].$$

This, together with (H5), gives $V_-(0) = V_+(0) = h$, and thus the conclusion follows. This completes the proof. ∎

Due to the property of $\hat{c}^*(\xi)$ stated in (i) and (ii) in Theorem 3.5, we follow [38] to call $\hat{c}^*(\xi)$ the asymptotic speed of spread (spreading speed in short) of the discrete-time semiflow $\{Q^n\}_{n=0}^\infty$ on C_+ along the unit vector $\xi \in \mathbb{R}^N$. We also remark that for a given unit vector $\xi \in \mathbb{R}^N$, spreading speed $\hat{c}^*(\xi)$ is unique. The spreading speed $\hat{c}^*(\xi)$ can be vividly explained by the descriptions after Theorem 2.1 in [38]: "if $u_0(x)$ is zero for all large values of $\xi \cdot x$ and uniformly above 0 for all sufficiently negative values of $\xi \cdot x$, then an observer who moves in the direction ξ with a speed faster than $\hat{c}^*(\xi)$ will see the solution go down to at most 0, while an observer who moves in this direction at a speed slower than $\hat{c}^*(\xi)$ sees the solution approach h." As pointed out in [38], "if the model includes a phenomenon such as a prevailing wind, $\hat{c}^*(\xi)$ may be negative in some directions. In this case an observer who stands still sees the solution go down to or below the unstable state 0 because the cloud of growing population gets blown away."

As a byproduct of Theorem 3.4, we easily obtain the following threshold dynamics. Some related results for the delay differential equation with spatial nonlocality and the delayed reaction-diffusion equations in unbounded domains have been obtained in [41, 45], by applying very different methods.

THEOREM 3.6. *Suppose that Q and $Q^{\pm}$ satisfies all conditions in Theorem* 3.4. *Then the following statements are true:*

(i) *If $0 \in \mathrm{Int}(\hat{\mathcal{A}})$, i.e., $\hat{c}^*(\xi) > 0$ for all unit vector ξ in $\mathbb{R}^N$, then h is a globally attractive fixed point of Q in $C_+ \setminus \{0\}$, that is, $\lim_{n\to\infty} Q^n[\phi] = h$ for all $\phi \in C_+ \setminus \{0\}$; in particular, there is no nonconstant $\phi \in C_+$ such that $Q[\phi] = \phi$.*

(ii) *If $0 \notin \mathrm{Int}(\hat{\mathcal{A}})$, i.e., $\hat{c}^*(\xi) \le 0$ for some unit vector ξ in $\mathbb{R}^N$, then there is $\phi \in X_+$ such that $Q[\phi(\cdot \cdot \xi)] = \phi(\cdot \cdot \xi)$, $\phi(\infty) = 0$ and $\phi(-\infty) = h$.*

Remark 3.1. Under the case (i) in Theorem 3.6, by Theorem 3.4(iii), h cannot be globally attractive with respect to the sup norm; and under the case (ii) in Theorem 3.6, h cannot be globally attractive, even with respect to the topology induced by $||\cdot||_C$.

4. Applications. In this section, we shall apply the results obtained in section 3 to three particular model equations: (i) a spatially nonlocal integro-difference equation; (ii) a reaction-diffusion equation with spatial nonlocality and time delay in the reaction term; (iii) an equation with nonlocal diffusion and delayed nonmonotone nonlinearity in the reaction term. We will use $f(\cdot)$ to denote the nonlinear terms in all these three equations, which will be assumed to satisfy the following conditions:

(A1) f is a continuously differentiable function on some right-neighborhood of 0.

(A2) $f'(0) > 1$ and $f(u) \le f'(0)u$ for all $u \in [0, \infty)$.

(A3) f has a unique positive fixed point u^*.

By (A1)–(A3), we easily see that $f(u) > u$ for all $u \in (0, u^*)$ and $f(u) < u$ for all $u \in (u^*, \infty)$. To study the convergent property, we formulate, in addition to (A1)–(A3), another assumption on the nonlinearity f:

(A4) u^* is the only positive fixed point of f^2.

By establishing a relation of the globally stable dynamics of the nonlinear map in the equation and the dynamics of the delay differential equations, we have shown in [44, 45] that (A4) plays a crucial role in the *delay independent (or absolute)* global stability of a positive equilibrium for some delay differential equations with spatial effects. In this section, we shall see that (A4) also plays a similar role in describing the global dynamics of the three models equations.

Note that (A4), together with (A1)–(A2), implies (see Proposition 2.1 in [44]) that

$$\lim_{n\to\infty} dist(f^n([\epsilon, M)), u^*) = 0 \text{ for any } M > \epsilon > 0,$$

which further implies the following property:

(p2) For any interval $[a, b] \subseteq (0, \infty)$ with $a < b$, either $a < \min\{f(u) : u \in [a, b]\}$ or $b > \max\{f(u) : u \in [a, b]\}$.

For a discussion of the equivalence relation of (A4) and (P2), see Lemma 5.3 in [42].

Define

$$f^L(u) = f'(0)u \text{ for all } u \in \mathbb{R}_+.$$

For any $M \ge \max f([0, u^*])$, define

$$f_M(u) = \begin{cases} f(u), & u \in [0, M], \\ f(M), & u > M, \end{cases}$$

and let

$$f_M^+(u) = \max_{v\in[0,u]} f_M(v), \text{ and } f_M^-(u) = \inf_{v\in[u,\infty)} f_M(v) \text{ for } u \in \mathbb{R}_+.$$

Then, we have the following results from Lemma 5.5 in [42].

LEMMA 4.1. *Let $M \geq \max f([0,u^*])$ and let f_M, $f_M^{\pm}$ and f^L be defined as above. Then the following statements are true:*

(i) $f_M^-(x) \leq f_M(x) \leq f_M^+(x) \leq f^L(x)$ *for all* $x \in \mathbb{R}_+$.

(ii) f_M^+ *and* f_M^- *are nondecreasing and continuous on* $\mathbb{R}_+$.

(iii) *There exist positive numbers* $u_{\pm}^*$ *such that* $f_M^{\pm}(u_{\pm}^*) = u_{\pm}^*$ *and* $0 < u_-^* \leq u^* \leq u_+^* \leq M$.

(iv) f_M *and* $f_M^{\pm}$ *satisfy assumptions* (A1)–(A3)*; moreover, if* f *satisfies* (A4)*, then* f_M *also satisfies assumption* (A4).

(v) *For any* $\varepsilon \in (0,1)$, *there is* $\delta \in (0,u_-^*)$ *such that* $f_M^-(x) \geq (1-\varepsilon)f^L(x)$ *for any* $x \in [0,\delta]$.

4.1. An integro-difference equation. Consider the integro-difference equation

$$\begin{cases} u_{n+1}(x) & = & \int_R f(u_n(y))k(x-y)dy \triangleq Q[u_n](x), \\ u_0 & \in & X_+. \end{cases} \tag{4.1}$$

Here, $k:\mathbb{R}\to\mathbb{R}$ is a nonnegative continuous function satisfying $\int_{\mathbb{R}} k(y)dy = 1$, and as in [13], we assume that $\int_{\mathbb{R}} e^{\pm\alpha y}k(y)dy < \infty$ for all $\alpha \in [0,\Delta_{\pm})$, where $\Delta_{\pm} > 0$ is the abscissa of convergence and it may be infinity.

Let

$$c_{\pm}^* = \inf_{\mu\in(0,\Delta_{\pm})} \frac{ln(f'(0)\int_{\mathbb{R}} e^{\pm\mu y}k(y)dy)}{\mu}.$$

Define

$$Q_M[\phi](x) = \int_{\mathbb{R}} f_M(\phi(x-y))k(y)dy, \;\; \phi \in X_+, \;\; x \in \mathbb{R},$$

and

$$Q_M^{\pm}[\phi](x) = \int_{\mathbb{R}} f_M^{\pm}(\phi(x-y))k(y)dy, \;\; \phi \in X_+, \;\; x \in \mathbb{R}.$$

By the assumptions (A1)–(A4), we easily see that $Q_M^- \leq Q_M \leq Q_M^+$, $Q_M^{\pm}$ satisfy the assumptions (H1)–(H4), and Q_M satisfies the assumption (H5) with $r^{\pm}$ and h replaced by $u_{\pm}^*$ and u^*, respectively.

In view of the above preparation and the fact that for any $M \geq \max f([0,u^*])$, $Q[X_M] \subseteq X_M$ and $Q[\phi] = Q_M[\phi]$ for all $\phi \in X_M$, we can apply Theorems 3.1(i) and 3.4 with $m(y,dy) = f'(0)k(y)dy$ to obtain the following result for (4.1).

THEOREM 4.1. *Assume that* (A1)–(A4) *hold. Then the following statements are valid:*

(i) $c_+^*(1) = c_-^*(1)$, $c_+^*(-1) = c_-^*(-1)$, *and they do not depend on the choices of* M, *where* $c_{\pm}^*(\xi)$ *are defined as in* (2.1) *with* $Q_M^{\pm}$. *Denote* $c_+^* = c_{\pm}^*(1)$, $c_-^* = c_{\pm}^*(-1)$.

(ii) *If $c_-^* + c_+^* \geq 0$, then for any c_1, c_2 with $(c_1, c_2) \supseteq [-c_-^*, c_+^*]$ and for any $u_0 \in X_{u_+^*}$ with compact support,*

$$\lim_{n\to\infty} \sup\{u_n(x) : x \notin (nc_1, nc_2)\} = 0.$$

(iii) *If $c_-^* + c_+^* > 0$, $u_0 \in X_+ \setminus \{0\}$, and $c_1, c_2 \in (-c_-^*, c_+^*)$ with $c_2 \geq c_1$, then*

$$\lim_{n\to\infty} \min_{x\in[nc_1,nc_2]} u_n(x) = \lim_{n\to\infty} \max_{x\in[nc_1,nc_2]} u_n(x) = u^*.$$

(iv) *If $c_-^* + c_+^* > 0$, then for any $c \in (-c_-^*, c_+^*)$, (4.1) has no nonconstant traveling wave $U(x - cn)$ with $\liminf_{s\to-\infty} U(s) + \liminf_{s\to\infty} U(s) > 0$.*

(v) *If $c_-^* + c_+^* > 0$, then for any $c \geq c_+^*$, (4.1) has a nonconstant traveling wave $U_+(x - nc)$ connecting u^* to 0; for any $c \leq -c_-^*$, (4.1) has a nonconstant traveling wave $U_-(x - nc)$ connecting 0 to u^*.*

Remark 4.1. The spreading speed and traveling wave fronts of (4.1) were also studied in [13] under the assumption that the kernel $k(\cdot)$ in (4.1) is even. In addition, two conditions, denoted by (C1) and (C2), are required in [13]. One can easily verify that either the condition (C1) or (C2) in [13] implies (A4). We also point out that here we do not require that the kernel $k(\cdot)$ in (4.1) be symmetric. In what follows, we consider three concrete examples of nonlinear functions for $f(u)$ in (4.1) to demonstrate these.

Example 4.1. The first one is the Ricker's function $f(u) = que^{-pu}$, which is widely adopted as birth function for fish population and for blowfly population (see, e.g., [11, 12, 22, 23, 27, 31, 32, 42, 43]). Assume $p > 1$ and $q > 0$ and let $u^* = \frac{1}{q}\ln p$. Then all conclusions (except for the last conclusions of (iii) and (v)) in Theorem 4.1 hold. If p is further confined to $p \in (1, e^2]$, then by applying Theorem 4.1 and making use of the proof of Theorem 4.1 and Remark 4.3 in [44], we can also obtain the last conclusions of (iii) and (v) in Theorem 4.1. The conclusions are summarized in the following theorem.

THEOREM 4.2. *Consider* (4.1) *with $f(u) = que^{-pu}$. Suppose $p > 1$ and $q > 0$ and let $u^* = \frac{1}{p}\ln q$. If $p \in (1, e^2)$, then all the conclusions of Theorem* 4.1 *hold.*

Example 4.2. The second example of the nonlinear function f in (4.1) is the Mackey–Glass hematopoiesis function $f : \mathbb{R} \to \mathbb{R}$ by $f(u) = pu/(q + u^m)$ for all $u \in R_+$. This function was initially used by Mackey and Glass in [25] to model the blood cell production in an ordinary differential equation model, and that model has since been studied and modified by many researchers. Among other topics for these models is the stability of a positive equilibrium, accounting for a long-term stable blood concentration level. See, for example, Kuang [14] and Tang and Zou [34] and the references therein. Applying Theorem 4.1, and taking advantage of the proof of Theorem 4.2 and Remark 4.3 in [44], we obtain the following theorem.

THEOREM 4.3. *Consider* (4.1) *with $f(u) = pu/(q+u^m)$. Suppose that $p > q > 0$ and $m > 0$ and let $u^* = (p-q)^{\frac{1}{n}}$. If $m \leq \frac{2p}{p-q}$, then all the conclusions of Theorem* 4.1 *hold.*

We point out that [13] also obtained the conclusions of Theorems 4.2 and 4.3 when the kernel $k(x)$ is *symmetric* : $k(x) = k(-x)$ for all $x \in \mathbb{R}$. Our approach does not require this symmetry, and hence our results improve the corresponding ones in [13].

In addition to removing symmetry of the kernel, our general results are also applicable for some nonlinear functions when the results in [13] cannot be applied.

To see this, we consider the following example.

Example 4.3. Consider (4.1) with

$$f(u) = \begin{cases} ue^{2-u}, & u \notin [u^{**}, \infty), \\ \frac{2u}{u^{**}}, & u \in [\frac{u^{**}}{2}, u^{**}], \\ 1, & u \in (\frac{u^{**}}{4}, \frac{u^{**}}{2}), \\ \frac{4u}{u^{**}}, & u \in [0, \frac{u^{**}}{4}], \end{cases}$$

where u^{**} is the unique positive solution of $ue^{2-u} = 2$ in $(0, 1)$. Clearly, it is easy to verify that f satisfies the assumptions (A1)–(A4). But $f(x)/x$ is *not* nondecreasing on $[0, 2]$, and thus f does *not* satisfy the assumption (C2) in [13]. Thus, the results in [13] cannot be applied to (4.1) with the above f, but our Theorem 4.1 is applicable to this system.

4.2. A delayed nonlocal reaction-diffusion equation. Consider the following reaction diffusion equation with time delay and spatial nonlocality in the reaction term

(4.2)
$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = u_{xx}(t,x) - \mu u(t,x) + \mu \int_{\mathbb{R}} f(u(t-\tau,y))k(x-y)\mathrm{d}y, & (t,x) \in (0,\infty)\times\mathbb{R}, \\ u(\theta,x) = \varphi(\theta,x), \quad (\theta,x) \in [-\tau,0]\times\mathbb{R}, \end{cases}$$

where $\mu > 0$, $\tau \geq 0$ and $f : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies (A1)–(A4) stated in the beginning of this section and the initial data φ belongs to $BC([-\tau, 0] \times \mathbb{R}, \mathbb{R}_+)$.

For the kernel, we assume that $k : \mathbb{R} \to [0, \infty)$ is continuous and satisfies $\int_{\mathbb{R}} k(y)\mathrm{d}y = 1$ and $\int_{\mathbb{R}} e^{\rho y} k(y)\mathrm{d}y < \infty$ for $\rho \in \mathbb{R}$.

A prototype of the kernel is $k(x) = \frac{1}{\sqrt{4\pi\alpha}} e^{-\frac{x^2}{4\alpha}}$ with which (4.2) is the model derived in So, Wu, and Zou [33] to describe the growth of the matured population of a single species. Moreover, Yi, Chen, and Wu in [41] established the global attractivity of (4.2) under some conditions, and Yi, Chen, and Wu in [42] also obtained some results on the asymptotic speeds of spread and traveling waves of (4.2) when $k(\cdot)$ possesses the following symmetry: $k(x) = k(-x)$ for all $x \in \mathbb{R}$. Here, we do not assume this symmetry. We should mention that for nonsymmetric kernel functions, Gomez, Prado, and Trofimchuk [8, section 5.1] have also obtained some results on existence/nonexistence of traveling wave for (4.2) by using a sub- and super-solution method.

For any $c \in \mathbb{R}$, let

$$\lambda_1 = \frac{-c - \sqrt{c^2 + 4\mu}}{2}, \quad \lambda_2 = \frac{-c + \sqrt{c^2 + 4\mu}}{2}$$

and denote

$$m_c(y, dy) = \frac{\mu f'(0)}{\lambda_2 - \lambda_1} \left[\int_{-\infty}^{0} e^{-\lambda_1 z} k(z + y + \tau c) dz + \int_{0}^{\infty} e^{-\lambda_2 z} k(z + y + \tau c) dz \right] dy.$$

Furthermore, for any $g \in C(\mathbb{R}_+, \mathbb{R}_+)$, define the operators K_c, L_c, $Q[\cdot; c, g] : X_+ \to X_+$ by

$$K_c[\phi](x) = \int_{\mathbb{R}} \phi(y) k(x - y + \tau c) dy,$$

$$L_c[\phi](x) = \frac{\mu}{\lambda_2 - \lambda_1} \left[\int_{-\infty}^{x} e^{\lambda_1 (x-y)} \phi(y) dy + \int_{x}^{\infty} e^{\lambda_2 (x-y)} \phi(y) dy \right],$$

$$Q[\phi; c, g](x) = L_c[K_c[g(\phi(\cdot))]](x).$$

The following lemma itemizes some properties of these maps.

LEMMA 4.2. *Let $c \in \mathbb{R}$, $g \in C(\mathbb{R}_+, \mathbb{R}_+)$, and $M \geq \max f([0, u^*])$. Then the following statements hold:*

(i) *K_c, L, $Q[\cdot; c, g]$ are all continuous and compact maps, that is, $K_c|_{X_r}, L|_{X_r}$, $Q[\cdot; c, g]|_{X_r}$ are all continuous and compact.*
(ii) *$Q[\phi; c, f_M^-] \leq Q[\phi; c, f_M] \leq Q[\phi; c, f_M^+]$ for all $\phi \in X_+$.*
(iii) *$Q[X_M; c, f] \subseteq X_M$ and $Q[\phi; c, f] = Q[\phi; c, f_M]$ for all $\phi \in X_M$.*
(iv) *$Q[\cdot; c, f_M^{\pm}]$ satisfies the assumptions* (H1)–(H4) *with $r^{\pm}$ replaced by $u_{\pm}^*$, respectively.*
(v) *$Q[\cdot; c, f_M]$ satisfies the assumptions* (H5) *with h replaced by u^*.*
(vi) *$Q[\cdot; c, f_M^{\pm}]$ and $m_c(y, dy)$ satisfy the inequalities in* (3.4) *and* (3.5).

Proof. (i) By using a similar proof to that of Lemma 2.2(iv) in [45], we easily see that K_c is a continuous and compact map.

Define $J_1, J_2 : X_+ \to X_+$ by

$$J_1[\phi](x) = \int_{-\infty}^{x} e^{\lambda_1(x-y)}\phi(y)dy, \quad J_2[\phi](x) = \int_{x}^{\infty} e^{\lambda_2(x-y)}\phi(y)dy, \text{ for } \phi \in X_+, \ x \in \mathbb{R}.$$

To prove the continuity and compactness of L_c, it suffices to show that $J_1|_{X_1}$ and $J_2|_{X_1}$ are continuous and compact, due to the linear property.

We first prove the continuity of $J_1|_{X_1}$. Let $\varepsilon > 0$ and $\phi, \phi_n \in X_1$ with $n \in \mathbb{N}$ and $\lim_{n\to\infty} \phi_n = \phi$. By Lemma 2.1(i) in [45], we only need to show that

$$\lim_{n\to\infty} (\sup\{|J_1(\phi_n)(x) - J_1(\phi)(x)| : x \in I\}) = 0$$

for any bounded and closed interval $I \equiv [a, b] \subseteq \mathbb{R}$. Indeed, for any $\varepsilon > 0$, by taking $T_\varepsilon = 1 + |\frac{\ln(\frac{-\varepsilon\lambda_1}{6})}{\lambda_1}|$ and using Lemma 2.1(i) in [45], there exists $n_\varepsilon > 1$ such that $|\phi_n(x) - \phi(x)| < \frac{-\varepsilon\lambda_1}{6}$ for all $x \in [a - T_\varepsilon, b + T_\varepsilon]$ and $n \geq n_\varepsilon$. It follows from the definition of J_1 that, for any $x \in I$ and $n \geq n_\varepsilon$,

$$\begin{aligned}
&|J_1[\phi_n](x) - J_1[\phi](x)| \\
&= \int_{-\infty}^{0} |\phi_n(x+y) - \phi(x+y)| e^{-\lambda_1 y} \mathrm{d}y \\
&\leq \int_{-T_\varepsilon}^{0} |\phi_n(x+y) - \phi(x+y)| e^{-\lambda_1 y} \mathrm{d}y + \int_{-\infty}^{-T_\varepsilon} |\phi_n(x+y) - \phi(x+y)| e^{-\lambda_1 y} \mathrm{d}y \\
&\leq \frac{-\varepsilon\lambda_1}{6} \int_{-T_\varepsilon}^{0} e^{-\lambda_1 y} \mathrm{d}y + \int_{-\infty}^{-T_\varepsilon} e^{-\lambda_1 y} \mathrm{d}y \\
&\leq \frac{-\varepsilon\lambda_1}{6} \frac{1}{-\lambda_1} + \frac{1}{-\lambda_1} e^{\lambda_1 T_\varepsilon} \\
&< \varepsilon.
\end{aligned}$$

This means that $J_1|_{X_1}$ is continuous.

We next prove the compactness of $J_1|_{X_1}$. Note that $J_1[X_1] \subseteq X_{\frac{-1}{\lambda_1}}$. This together with Lemma 2.1(ii) in [45] and the Arzelà–Ascoli theorem implies that it suffices to show that $J_1[X_1]|_I$ is a family of equicontinuous functions in $C(I, \mathbb{R})$ for any bounded and closed interval $I \equiv [a, b] \subseteq \mathbb{R}$. Indeed, for any $\varepsilon \in (0, \frac{-4}{\lambda_1})$, by taking $\delta_\varepsilon = \frac{1}{\lambda_1} \ln(1 + \frac{\varepsilon\lambda_1}{4})$, it follows from the definition of J_1 that, for any $\phi \in X_1$ and $x, \tilde{x} \in I$

with $\tilde{x} - x \in [0, \delta_\varepsilon)$,

$$\begin{aligned}|J_1[\phi](\tilde{x}) - J_1[\phi](x)| &= |\int_{-\infty}^{\tilde{x}} \phi(y) e^{\lambda_1(\tilde{x}-y)} \mathrm{d}y - \int_{-\infty}^{x} \phi(y) e^{\lambda_1(x-y)} \mathrm{d}y| \\ &\le (e^{\lambda_1 x} - e^{\lambda_1 \tilde{x}}) \int_{-\infty}^{x} \phi(y) e^{-\lambda_1 y} \mathrm{d}y + \int_{x}^{\tilde{x}} \phi(y) e^{\lambda_1(\tilde{x}-y)} \mathrm{d}y \\ &\le \frac{-2}{\lambda_1}[1 - e^{\lambda_1(\tilde{x}-x)}] \\ &< \frac{-2}{\lambda_1}[1 - e^{\lambda_1 \delta_\varepsilon}] \\ &< \varepsilon.\end{aligned}$$

This implies that $J_1|_{X_1}$ is compact.

By similar arguments, we can show that $J_2|_{X_1}$ is also continuous and compact, and so is L_c.

Now, by the boundedness, compactness, and continuity of g, K_c, and L_c, we easily know that $Q[\cdot; c, g]$ is a continuous and compact map.

(ii) The proof follows from Lemma 4.1(i) and the definition of $Q[\cdot; c, \cdot]$.

(iii) By Lemma 4.1(iii), $f([0, M]) \subseteq [0, M]$ and $f_M([0, M]) \subseteq [0, M]$, and hence $Q[X_M; c, f] \subseteq X_M$ and $Q[X_M; c, f_M] \subseteq X_M$. This, combined with the definition of $Q[\cdot; c, \cdot]$ and the fact that $f_M|_{[0,M]} \equiv f|_{[0,M]}$, implies that $Q[\phi; c, f] = Q[\phi; c, f_M]$ for all $\phi \in X_M$.

(iv) The proof follows from Lemma 4.1(ii)–(iv) and the definitions of $Q[\cdot; c, f_M^{\pm}]$.

(v) By Lemma 4.1 and the definition of $Q[\cdot; c, f_M]$, we easily verify that $Q[\cdot; c, f_M]$ satisfies the assumptions (H1) and (H3). Fix $s > r > 0$, and without loss of generality we may assume that $f_M([r, s]) > r$ due to Lemma 4.1(iv). Thus, there is $\delta = \delta_{r,s} > 0$ such that $f_M([r, s]) \ge r + \delta$. By the monotonicity of K_c, L_c, we obtain $K_c[X_{f_M([r,s])}] \ge K_c[r + \delta] = r + \delta$ and $L_c[K_c[X_{f_M([r,s])}]] \ge L_c[r + \delta] = r + \delta$. This, combined with the fact that $f_M(X_{r,s}) \subseteq X_{f_M([r,s])}$ and the definition of $Q[\cdot; c, f_M]$, gives $\inf_{x\in\mathbb{R}} Q[X_{r,s}; c, f_M](x) \ge r + \delta > r$. Therefore, $Q[\cdot; c, f_M]$ satisfies the assumption (H5).

(vi) Note that $\int_{\mathbb{R}} \phi(x - y)m(y, dy) = Q[\phi; c, f^L](x)$ for all $(x, \phi) \in \mathbb{R} \times X_+$. It follows from Lemma 4.1(i) and the definitions of $Q[\phi; c, f^L]$ and $Q[\phi; c, f_M]$ that $Q[\cdot; c, f_M^+]$ and $m_c(y, dy)$ satisfy the inequality in (3.4). In view of Lemma 4.1(v), for any $\delta \in (0, 1)$ there is $\varepsilon \in (0, u_-^*)$ such that $f_M^-(x) \ge (1 - \delta)f^L(x)$ for any $x \in [0, \varepsilon]$. Thus, $f_M^-(\phi) \ge (1 - \delta)f^L(\phi)$ for any $\phi \in X_\varepsilon$, which together with the monotonicity and linearity of K_c, L_c implies that, for any $\phi \in X_\varepsilon$, $L_c[K_c[f_M(\phi)]] \ge L_c[K_c[(1 - \delta)f^L(\phi)]] = (1 - \delta)L_c[K_c[f^L(\phi)]]$. So, $Q[\cdot; c, f_M^-]$ and $m_c(y, dy)$ satisfy the inequality in (3.5).

The proof of the lemma is completed. □

Note that if $Q[\phi; c, f] = \phi$ for some $\phi \in X_+$, then $\phi \in X_{\sup f([0,u^*])}$. Thus, by Lemma 4.2(iii), we shall tacitly approve that $f = f_{\sup f([0,u^*]}$ in this subsection.

Let

$$\hat{k}(\rho) = \int_{\mathbb{R}} e^{\rho y} k(y) dy \;\text{ and }\; l(c, \rho) = \frac{\mu f'(0) e^{-\rho\tau c}}{\rho c + \mu - \rho^2} \hat{k}(\rho) \;\text{ for all }\; c, \rho \in \mathbb{R}$$

and

$$l^{\pm}(c, \rho) = \int_{\mathbb{R}} e^{\pm\rho y} m_c(y, dy) \;\text{ for }\; c \in \mathbb{R},\; \rho \in \mathbb{R}_+.$$

Then we have the following.

LEMMA 4.3. *The following statements are true:*

(i) $l^{\pm}(c,\rho) = l(c,\pm\rho)$ *for all* $c, \rho \in \mathbb{R}$ *with* $0 < \rho < \frac{\pm c+\sqrt{c^2+4\mu}}{2}$, *while* $l^{\pm}(c,\rho) = \infty$ *for all* $c, \rho \in \mathbb{R}$ *with* $\rho \geq \frac{\pm c+\sqrt{c^2+4\mu}}{2}$.

(ii) $\pm\frac{\partial l^{\pm}(c,\rho)}{\partial c} < 0$ *for all* $c, \rho \in \mathbb{R}$ *with* $0 < \rho < \frac{\pm c+\sqrt{c^2+4\mu}}{2}$.

Proof. We only consider the case of "+," since the other case can be dealt with by similar arguments.

(i) For any $c \in \mathbb{R}, \rho > 0$, it follows from the definitions of l, l^+, Fubini's theorem, and the linear transformations of variables that

$$\begin{aligned} l^+(c,\rho) &= \frac{\mu f'(0)}{\lambda_2-\lambda_1}\int_{\mathbb{R}} e^{\rho y}[\int_{-\infty}^{0} e^{-\lambda_1 z}k(z+y+\tau c)dz + \int_0^{\infty} e^{-\lambda_2 z}k(z+y+\tau c)dz]dy \\ &= \frac{\mu f'(0)e^{-\rho\tau c}\hat{k}(\rho)}{\lambda_2-\lambda_1}[\int_{-\infty}^{0} e^{-(\lambda_1+\rho)z}dz + \int_0^{\infty} e^{-(\lambda_2+\rho)z}dz], \end{aligned}$$

which yields the statement (i).

(ii) By the statement (i) and a simple computation, we have

$$\begin{aligned} \frac{\partial l^+(c,\rho)}{\partial c} &= \mu f'(0)\hat{k}(\rho)\frac{\partial}{\partial c}\,\frac{e^{-\rho\tau c}}{\rho c+\mu-\rho^2} \\ &= -\mu\rho f'(0)\hat{k}(\rho)e^{-\rho\tau c}\frac{1+\tau(\rho c+\mu-\rho^2)}{(\rho c+\mu-\rho^2)^2} \\ &< 0, \end{aligned}$$

where $0 < \rho < \frac{c+\sqrt{c^2+4\mu}}{2}$.

The proof of the lemma is completed. ∎

To proceed further to study the convergence property, we give the following assumptions on the nonsymmetry kernel function $k(\cdot)$.

(K1) $\hat{k}(\rho) \geq 1$ for all $\rho \in \mathbb{R}$.

(K2) $\hat{k}(\rho)^{\frac{1}{\rho}}\hat{k}(-\tilde{\rho})^{\frac{1}{\tilde{\rho}}} \geq 1$ for all $\rho, \tilde{\rho} > 0$.

Note that the assumption (K1) implies (K2), while the assumption (K1) holds if $k(x) = k(-x)$ for all $x \in \mathbb{R}$.

Let

$$p_{\pm}(c) = \inf_{\rho>0}\frac{1}{\rho}\log l^{\pm}(c,\rho) \ \text{ for } \ c \in \mathbb{R}$$

and

$$c_+^* = \inf\{c \in \mathbb{R} : p_+(c) \leq 0\} \ \text{ and } \ c_-^* = \sup\{c \in \mathbb{R} : p_-(c) \leq 0\}.$$

LEMMA 4.4. *The following statements are true:*

(i) $p_{\pm}(c) \leq 0$ *for all* $\pm c > \pm c_{\pm}^*$ *and* $p_{\pm}(c) > 0$ *for all* $c < \pm c_{\pm}^*$.

(ii) *If* (K2) *holds, then* $p_+(c) + p_-(c) > 0$ *for all* $c \in \mathbb{R}$.

(iii) *If* $p_+(c) + p_-(c) > 0$ *for all* $c \in \mathbb{R}$, *then* $c_+^* \geq c_-^*$*; in particular,* $c_+^* = -c_-^*$ *when* $k(x) = k(-x)$ *for all* $x \in \mathbb{R}$.

Proof. (i) We only consider the case of "+," since the other case can be dealt with by similar arguments. Note that $p_+(c)$ is nonincreasing in c due to the definition of p_+

and Lemma 4.2(ii). This, together with the definition of c_+^*, implies that $p_+(c) \leq 0$ for all $c > c_+^*$ and $p_+(c) > 0$ for all $c < c_+^*$.

(ii) Let $a(\rho, c) = \frac{1}{\rho} \log \frac{\mu f'(0)}{\mu - \rho c - \rho^2}$ with $\rho \in I_a \triangleq (0, \frac{-c+\sqrt{c^2+4\mu}}{2})$ and $b(\rho, c) = \frac{1}{\rho} \log \frac{\mu f'(0)}{\mu + \rho c - \rho^2}$ with $\rho \in I_b \triangleq (0, \frac{c+\sqrt{c^2+4\mu}}{2})$ for all $c \in \mathbb{R}$. Then $\lim_{\rho \to 0^+} a(\rho, c) = \lim_{\rho \to 0^+} b(\rho, c) = \infty$. Suppose that $c \geq 0$. Clearly, $\inf_{\rho \in I_b} b(\rho, c) \geq -\frac{1}{\rho} \log(1 + \frac{c}{\mu}\rho) \geq -\frac{c}{\mu}$. By $\lim_{\rho \to 0^+} a(\rho, c) = \infty$, there exists $\delta \in I_a$ such that $\inf_{\rho \in I_a} a(\rho, c) = \inf\{a(\rho, c) : \rho \in I_a \text{ and } \rho \geq \delta\}$. By letting $I_a^0 = \{\rho \in I_a \cap [\delta, \infty) : \frac{\partial a(\rho, c)}{\partial \rho} = 0\}$, we have $\inf_{\rho \in I_a} a(\rho, c) = \inf_{\rho \in I_a^0} a(\rho, c) \geq \inf\{\frac{c+2\rho}{\mu - c\rho - \rho^2} : \rho \in I_a \cap [\delta, \infty)\} > \frac{c}{\mu}$. Thus, $\inf_{\rho \in I_a} a(\rho, c) + \inf_{\rho \in I_b} b(\rho, c) > 0$. Similarly, $\inf_{\rho \in I_a} a(\rho, c) + \inf_{\rho \in I_b} b(\rho, c) > 0$ for all $c \leq 0$. These together with the assumption (K2) and the definitions of $p_\pm$ and $l^\pm$ imply that for any $c \in \mathbb{R}$, we have

$$
\begin{aligned}
p_+(c) + p_-(c) &= \inf_{\rho > 0} [a(\rho, c) + \frac{1}{\rho} \log(\hat{k}(\rho)) - \tau c] + \inf_{\rho \in > 0} [b(\rho, c) + \frac{1}{\rho} \log(\hat{k}(-\rho)) + \tau c] \\
&\geq \inf_{\rho > 0} a(\rho, c) + \inf_{\rho > 0} b(\rho, c) + \inf_{\rho > 0} \frac{1}{\rho} \log(\hat{k}(\rho)) + \inf_{\rho > 0} \frac{1}{\rho} \log(\hat{k}(-\rho)) \\
&> \inf_{\rho > 0} \frac{1}{\rho} \log(\hat{k}(\rho)) + \inf_{\rho > 0} \frac{1}{\rho} \log(\hat{k}(-\rho)) \\
&\geq 0.
\end{aligned}
$$

(iii) We shall prove $c_+^* \geq c_-^*$; otherwise, $c_+^* < c_-^*$. Thus, by (i) and taking $c \in (c_+^*, c_-^*)$, we have $p_\pm(c) \leq 0$, a contradiction.

Since $k(x) = k(-x)$ for all $x \in \mathbb{R}$, we have $\hat{k}(\rho) = \hat{k}(-\rho) \geq 1$ for all $\rho > 0$. Thus, $l^+(c, \rho) = l^-(-c, \rho)$ for all $(c, \rho) \in \mathbb{R} \times (0, \infty)$. These, combined with the definition of $c_\pm^*$, yield $c_+^* = -c_-^*$.

The proof of the lemma is completed. ∎

Let

$$
\tilde{l}(c, \rho) = \frac{\mu f'(0) e^{-\rho \tau c}}{\rho c + \mu - \rho^2} \hat{k}(-\rho), \quad \tilde{p}_\pm(c) = \inf_{\rho > 0} \frac{1}{\rho} \log \tilde{l}(c, \pm\rho) \text{ for } c, \rho \in \mathbb{R}
$$

and

$$
\tilde{c}_+^* = \inf\{c \in \mathbb{R} : \tilde{p}_+(c) \leq 0\} \text{ and } \tilde{c}_-^* = \sup\{c \in \mathbb{R} : \tilde{p}_-(c) \leq 0\}.
$$

Similarly, we have the following results.

LEMMA 4.5. *The following statements are true:*

(i) $\tilde{p}_\pm(c) \leq 0$ *for all* $\pm c > \pm\tilde{c}_\pm^*$ *and* $\tilde{p}_\pm(c) > 0$ *for all* $c < \pm\tilde{c}_\pm^*$.

(ii) *If (K2) holds, then* $\tilde{p}_+(c) + \tilde{p}_-(c) > 0$ *for all* $c \in \mathbb{R}$.

(iii) *If* $\tilde{p}_+(c) + \tilde{p}_-(c) > 0$ *for all* $c \in \mathbb{R}$, *then* $\tilde{c}_+^* \geq \tilde{c}_-^*$; *in particular,* $\tilde{c}_+^* = -\tilde{c}_-^*$ *when* $k(x) = k(-x)$ *for all* $x \in \mathbb{R}$.

Note that $l(-c, \pm\rho) = l(c, \mp\rho)$ for all $c \in \mathbb{R}$ and $\rho > 0$. Thus, by Lemma 4.4(iii) and the definitions of $c_\pm^*$ and $\tilde{c}_\pm^*$, we easily obtain the following lemma.

LEMMA 4.6. *If* $p_+(c) + p_-(c) > 0$ *for all* $c \in \mathbb{R}$, *then* $\tilde{c}_+^* = -c_-^* \geq \tilde{c}_-^* = -c_+^*$.

With the above preparation, we are in the position to state and prove the main results for (4.2).

THEOREM 4.4. *Assume that* (A1)–(A4) *and* (K2) *hold. Then the following statements are valid:*

(i) *For any* $c \geq c_+^*$, (4.2) *has a traveling wave* $\phi(x-ct)$ *with* $\phi(\infty)=0$ *and* $\phi(-\infty)=u^*$.

(ii) *For any* $c \leq c_-^*$, (4.2) *has a traveling wave* $\phi(x-ct)$ *with* $\phi(-\infty)=0$ *and* $\phi(\infty)=u^*$.

(iii) *For any* $c \in (c_-^*, c_+^*)$, (4.2) *has no nonconstant traveling wave* $\phi(x-ct)$.

Proof. By some simple computations, we know that for any $\phi \in X_+$, $\phi = Q[\phi; c, f]$ if and only if $u(t,x)=\phi(x-ct)$ satisfies (4.2).

(i) Suppose that $c \geq c_+^*$. If $c > c_+^*$, then by Lemma 4.4, we have $p_+(c) \leq 0$ and $p_+(c) > -p_-(c)$. Applying Theorem 3.6(ii) to $Q[\cdot; c, f]$ with $m_c(y, dy)$ defined in the beginning of this subsection, we conclude that there is a nonconstant $\phi_c \in X_M$ such that $Q[\phi_c; c, f] = \phi_c$ with $\phi_c(\infty)=0$ and $\phi_c(-\infty)=u^*$.

Suppose that $c = c_+^*$. We shall prove $p_+(c_+^*) \leq 0$; otherwise, $p_+(c_+^*) > 0$. Note that by using the discussions in the previous paragraph, there exist two sequences $\{c_m \in (c_+^*, \infty) : m \in \mathbb{N}\}$ and $\{\phi_m \in X_M : \phi_m(\infty) = 0 \text{ and } \phi_m(-\infty) = u^*\}_{m\in\mathbb{N}}$ such that $\lim_{m\to\infty} c_m = c_+^*$ and $Q[\phi_m; c_m, f] = \phi_m$ for all $m \in \mathbb{N}$. Let $B = \{\phi_m : m \in \mathbb{N}\}$. Then $B \subseteq X_M$ and $K_{c_m}[\phi_m] \in X_M$, where $M = \sup f([0, u^*])$. By a proof similar to that for the compactness of L_c in Lemma 4.2(i), we can show that $\cup_{m\in\mathbb{N}} L_{c_m}[X_M]$ is precompact, and hence B is precompact. Without loss of generality, we may assume that the limiting of ϕ_m exists, and $\sup\{x \in \mathbb{R} : \phi_m(x) = \frac{u^*}{3}\} = 0$ due to (H1). Hence, $\phi_m(0) = \frac{u^*}{3}$ and $\phi_m(x) \leq \frac{u^*}{3}$ for all $x \in [0, \infty)$. Let $\phi = \lim_{m\to\infty} \phi_m$. Then $\phi(0) = \frac{u^*}{3}$, $\phi(x) \leq \frac{u^*}{3}$ for all $x \in [0,\infty)$ and $Q[\phi; c_+^*, f] = \phi$. This together with Theorem 3.4(i) and $p_+(c_+^*) > 0$ implies $\limsup_{x\to\infty} \phi(x) = u^*$, a contradiction. Thus, $p_+(c_+^*) \leq 0$. By the above arguments for the case of $c > c_+^*$, we easily obtain a nonconstant $\phi_c \in X_M$ such that $Q[\phi_c; c, f] = \phi_c$ with $\phi_c(\infty) = 0$ and $\phi_c(-\infty) = u^*$.

(iii) Suppose that $c \in (c_-^*, c_+^*)$. By Lemma 4.4, we have $p_-(c) > 0$ and $p_+(c) > 0$. Applying Theorem 3.6(i), we know that $Q[\cdot; c, f]$ has no nonconstant $\phi \in X_+$ such that $Q[\phi; c, f] = \phi$, that is, (4.2) has no nonconstant traveling wave $\phi(x-ct)$.

To complete the proof of (ii), we apply the conclusion in (i) to the following auxiliary system, which is exactly the same as (4.2) when $k(\cdot)$ is symmetric:

(4.3)
$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = u_{xx}(t,x) - \mu u(t,x) + \mu \int_{\mathbb{R}} f(u(t-\tau,y))k(y-x)\mathrm{d}y, & (t,x) \in (0,\infty)\times\mathbb{R}, \\ u(\theta,x) = \varphi(\theta,x), \quad (\theta,x) \in [-\tau,0]\times\mathbb{R}. \end{cases}$$

This together with Lemmas 4.5 and 4.6 leads to the following conclusion:

(S) For any $c \geq \tilde{c}_+^* = -c_-^*$, (4.3) has a nonconstant traveling wave $\tilde{\phi}(x-ct)$ with $\tilde{\phi}(\infty) = 0$ and $\tilde{\phi}(-\infty) = u^*$.

It is clear that the statement (S) implies the statement (ii) of the theorem with c, ϕ replaced by $-c, \tilde{\phi}(-\cdot)$, respectively.

The proof of the theorem is completed. ∎

4.3. An equation with nonlocal diffusion and delayed reaction. Consider the following equation with nonlocal diffusion and delayed reaction

$$(4.4)\quad \begin{cases} \frac{\partial u}{\partial t}(t,x) = d[\int_{\mathbb{R}} u(t,y)k(x-y)\mathrm{d}y - u(t,x)] - \mu u(t,x) + \mu f(u(t-\tau,x)) \\ \qquad\qquad\qquad\qquad (t,x) \in (0,\infty)\times\mathbb{R}, \\ u(\theta,x) = \varphi(\theta,x), \quad (\theta,x) \in [-\tau,0]\times\mathbb{R}, \end{cases}$$

where $d, \mu > 0$, $\tau \geq 0$ and the initial data φ belongs to $BC([-\tau,0]\times\mathbb{R},\mathbb{R}_+)$.

It is known that nonlocal diffusion may demonstrate essential differences from random diffusion in some context. Taking a bounded spatial domain as an example, the solution semiflow of an equation with the random diffusion (represented by a Laplacian operator) is compact, while the solution semiflow of an equation with nonlocal diffusion (such as PDE in (4.4)) is not compact, and hence many existing methods/approaches cannot be applied, at least directly. See, e.g., [10, 15] and the references therein for more details.

For (4.4), Pan, Li, and Lim [29, 30] investigated the existence of traveling wave fronts using the approach of upper-lower solutions and monotone iteration developed in [40]. In order for the approach to be applicable, some monotonicity conditions were posed in [29, 30]; in addition, the kernel function $k(\cdot)$ was assumed to be symmetric. Here, we only assume that $f : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies (A1)–(A4) stated at the beginning of this section; and the kernel $k : \mathbb{R} \to (0, \infty)$ is continuous and satisfies $\int_{\mathbb{R}} k(y)\mathrm{d}y = 1$ and $\int_{\mathbb{R}} e^{\rho y} k(y)\mathrm{d}y < \infty$ for $\rho \in \mathbb{R}$. Thus, we *do not* require the symmetry for $k(\cdot)$ and *neither* do we assume the quasi-monotonicity posed in [29, 30].

Following the scheduling in [37], Yagisita [47] and Fang and Zhao [7] obtained some general results on the existence/nonexistence of traveling waves which are applicable to (4.4) with the quasimonotone nonlinearity f for $\tau > 0$. By applying the existence/nonexistence of traveling waves and asymptotic behaviour of solutions in nonlocal and nonmonotone convolution equations with nonsymmetric kernels in [8] and by using the discussions of subsection 6.1 in [1], one may also establish results in this section. This suggests that the approach of [8] may be equally effective for equation (4.4), as far as traveling wavefronts go.

For any $c \in \mathbb{R}$ and $g \in C(\mathbb{R}_+, \mathbb{R}_+)$, let $\lambda = \lambda(c) \triangleq \frac{d+\mu}{c}$ for all $c \neq 0$ and define $K[\cdot; c, g]$, $L[\cdot; c]$, and $Q[\cdot; c, g] : X_+ \to X_+$ by

$$
\begin{aligned}
&K[\phi; c, g](x) = \frac{1}{d+\mu}\left[d\int_{\mathbb{R}} \phi(y)k(x-y)\,dy + \mu g(\phi(x+c\tau))\right],\\
&L[\phi; c](x) = \begin{cases} \lambda\int_x^{\infty} e^{\lambda(x-y)}\phi(y)\,dy, & c > 0,\\ \phi(x), & c = 0,\\ -\lambda\int_{-\infty}^{x} e^{\lambda(x-y)}\phi(y)\,dy, & c < 0,\end{cases}\\
&Q[\phi; c, g](x) = L[K[\phi; c, g]; c](x).
\end{aligned}
$$

Also let

$$
m_c(y, dy) = \begin{cases} \frac{1}{c}\left[d\int_0^{\infty} e^{-\lambda z}k(z+y)\,dz + \mu f'(0)\int_0^{\infty} e^{-\lambda z}\delta_{-c\tau}(y+z)\,dz\right]\,dy, & c > 0,\\ \frac{1}{d+\mu}[dk(y) + \mu f'(0)\delta_0(y)]dy, & c = 0,\\ \frac{-1}{c}\left[d\int_{-\infty}^{0} e^{-\lambda z}k(z+y)\,dz + \mu f'(0)\int_{-\infty}^{0} e^{-\lambda z}\delta_{-c\tau}(y+z)\,dz\right]dy, & c < 0.\end{cases}
$$

The following lemma summarizes some properties for the maps $K[\cdot; c, g]$, $L[\cdot; c]$, $Q[\cdot; c, g]$.

LEMMA 4.7. *Let $r > 0$, $c \in \mathbb{R}$, $g \in C(\mathbb{R}_+, \mathbb{R}_+)$ and for any $M \geq \max f([0, u^*])$, the following statements are true:*

(i) *$K[\cdot; c, g]|_{X_r}, L[\cdot; c]|_{X_r}, Q[\cdot; c, g]|_{X_r}$ are all continuous maps for any $c \in \mathbb{R}$.*
(ii) *$L[X_r \times J]$ is precompact for any bounded closed interval $J \subseteq \mathbb{R} \setminus \{0\}$, and hence $Q[\cdot; c, g]|_{X_r}$ is compact map for any $c \neq 0$.*
(iii) *$Q[\phi; c, f_M^-] \leq Q[\phi; c, f_M] \leq Q[\phi; c, f_M^+]$ for all $\phi \in X_+$.*
(iv) *$Q[X_M; c, f] \subseteq X_M$ and $Q[\phi; c, f] = Q[\phi; c, f_M]$ for all $\phi \in X_M$.*
(v) *$Q[\cdot; c, f_M^{\pm}]$ satisfies the assumptions* (H1)–(H4) *with $r^{\pm}$ replaced by $u^*_{\pm}$, respectively.*
(vi) *$Q[\cdot; c, f_M]$ satisfies the assumptions (H5) with h replaced by u^*.*

(vii) *$Q[\cdot;c,f_M^{\pm}]$ and $m_c(y,dy)$ satisfy the inequalities in* (3.4) *and* (3.5).

Proof. (i) By the continuity of g and using a similar proof to that of Lemma 2.2(iv) in [45], we easily see that $K[\cdot;c,g]|_{X_r}$ is a continuous map. By the continuity of J_1 and J_2 in subsection 4.2, we also easily see that $L[\cdot;c]|_{X_r}$ and $Q[\cdot;c,g]|_{X_r}$ are continuous.

(ii) Let J be a bounded closed interval in $\mathbb{R}\setminus\{0\}$. Without loss of generality, we may assume $J\subseteq(0,\infty)$. By (i) and the definition of $Q[\cdot;c,g]$, it suffices to prove the compactness of $L[X_r\times J]$. Note that $L[X_r\times J]\subseteq X_r$. This together with Lemma 2.1(ii) in [45] and the Arzelà–Ascoli theorem implies that it suffices to show that $L[X_r\times J]|_I$ is a family of equicontinuous functions in $C(I,\mathbb{R})$ for any bounded and closed interval $I\equiv[a,b]\subseteq\mathbb{R}$. Indeed, for any $\varepsilon\in(0,1)$, by taking $\delta_\varepsilon=\frac{1}{d+\mu}\log(\frac{3r}{3r-\varepsilon})\cdot(\inf J)$, it follows from the definition of L that, for any $\phi\in X_r$, $c\in J$ and $x,\tilde{x}\in I$ with $x-\tilde{x}\in[0,\delta_\varepsilon)$,

$$\begin{aligned}|L[\phi;c](\tilde{x})-L[\phi;c](x)| &= |\int_{\tilde{x}}^{\infty}\lambda\phi(y)e^{\lambda(\tilde{x}-y)}\mathrm{d}y-\int_{x}^{\infty}\lambda\phi(y)e^{\lambda(x-y)}\mathrm{d}y|\\ &\le(e^{\lambda x}-e^{\lambda\tilde{x}})\int_{x}^{\infty}\lambda\phi(y)e^{-\lambda y}\mathrm{d}y|+\int_{\tilde{x}}^{x}\lambda\phi(y)e^{\lambda(\tilde{x}-y)}\mathrm{d}y\\ &\le 2r[1-e^{\lambda(\tilde{x}-x)}]\\ &<\varepsilon,\end{aligned}$$

confirming that (ii) holds.

(iii) The proof follows from Lemma 4.1(i) and the definitions of $Q[\cdot;c,f_M]$ and $Q[\cdot;c,f_M^{\pm}]$.

(iv) By Lemma 4.1(iii), $f([0,M])\subseteq[0,M]$ and $f_M([0,M])\subseteq[0,M]$, and hence we have $Q[X_M;c,f]\subseteq X_M$ and $Q[X_M;c,f_M]\subseteq X_M$. This, combined with the definition of $Q[\cdot;c,\cdot]$ and the fact that $f_M|_{[0,M]}\equiv f|_{[0,M]}$, implies that $Q[\phi;c,f]=Q[\phi;c,f_M]$ for all $\phi\in X_M$.

(v) The proof follows from Lemma 4.1 and the definition of $Q[\cdot;c,f_M^{\pm}]$.

(vi) By Lemma 4.1 and the definition of $Q[\cdot;c,f_M]$, we easily verify that $Q[\cdot;c,f_M]$ satisfies the assumptions (H1) and (H3). Fix $s>r>0$; without loss of generality, we may assume that $f_M([r,s])>r$ due to Lemma 4.1(iv). Thus, there is $\delta=\delta_{r,s}>0$ such that $f_M([r,s])\ge r+\delta$. By the monotonicity of $K[\cdot;c,f_M],L[\cdot,c]$, we conclude that $K[X_{r,s};c,f_M]\ge r+\frac{\mu}{d+\mu}\delta>r$ and $L[K[X_{r,s};c,f_M];c]\ge L[r+\frac{\mu}{d+\mu}\delta;c]=r+\frac{\mu}{d+\mu}\delta>r$. This, combined with the definition of $Q[\cdot;c,f_M]$, gives $\inf_{x\in\mathbb{R}}Q[X_{r,s};c,f_M](x)\ge r+\frac{\mu}{d+\mu}\delta>r$. Therefore, $Q[\cdot;c,f_M]$ satisfies the assumption (H5).

(vii) Note that $\int_{\mathbb{R}}\phi(x-y)m(y,dy)=Q[\phi;c,f^L](x)$ for all $(x,\phi)\in\mathbb{R}\times X_+$. It follows from Lemma 4.1(i) and the definitions of $Q[\phi;c,f^L]$ and $Q[\phi;c,f_M]$ that $Q[\cdot;c,f_M^+]$ and $m_c(y,dy)$ satisfy the inequality in (3.4). In view of Lemma 4.1(v), for any $\delta\in(0,1)$ there is $\varepsilon\in(0,u_-^*)$ such that $f_M^-(x)\ge(1-\delta)f^L(x)$ for any $x\in[0,\varepsilon]$. Thus, $f_M^-(\phi)\ge(1-\delta)f^L(\phi)$ for any $\phi\in X_\varepsilon$, which together with the monotonicity and linearity of $K[\cdot;c,f_M]$ and $L[\cdot,c]$ implies that for any $\phi\in X_\varepsilon$, $L[K[\phi;c,f_M];c]\ge L[K[\phi;c,(1-\delta)f^L];c]=(1-\delta)L[K[\phi;c,f^L];c]$. So, $Q[\cdot;c,f_M^-]$ and $m_c(y,dy)$ satisfy the inequality in (3.5).

The proof is completed. ∎

Note that if $Q[\phi;c,f]=\phi$ for some $\phi\in X_+$, then $\phi\in X_{\sup f([0,u^*])}$. Thus, by Lemma 4.7(iv), we shall tacitly approve $f=f_{\sup f([0,u^*]}$ in the rest of this subsection.

Let

$$\hat{k}(\rho) = \int_{\mathbb{R}} e^{\rho y} k(y)\, dy \;\text{ and }\; l(c,\rho) = \frac{1}{c\rho + d + \mu}[d\hat{k}(\rho) + \mu f'(0) e^{-\rho c \tau}] \text{ for } \rho,\, c \in \mathbb{R}$$

and

$$l^{\pm}(c,\rho) = \int_{\mathbb{R}} e^{\pm \rho y} m_c(y, dy) \;\text{ for }\; c \in \mathbb{R},\;\; \rho \in \mathbb{R}_+.$$

Then we have the following.

LEMMA 4.8. *The following statements are true:*

(i) $l^{\pm}(c,\rho) = l(c, \pm\rho)$ *for all* $(c,\rho) \in J_{\pm} \triangleq \{(a,b) \in \mathbb{R}\times(0,\infty) : d+\mu \pm ab > 0\}$*, while* $l^{\pm}(c,\rho) = \infty$ *for all* $(c,\rho) \notin J_{\pm}$.

(ii) $\pm \frac{\partial l^{\pm}(c,\rho)}{\partial c} < 0$ *for all* $(c,\rho) \in J_{\pm}$.

Proof. We only consider the case of $c > 0$, since the other case can be dealt with by similar arguments.

(i) For any $\rho > 0$, it follows from the definitions of $l, l^{\pm}$, Fubini's theorem, and the linear transformations of variables that

$$\begin{aligned} l^{\pm}(c,\rho) &= \int_{\mathbb{R}} e^{\pm\rho y} m_c(y, dy) \\ &= \frac{1}{c}\int_{\mathbb{R}} e^{\pm \rho y} \left[d \int_0^{\infty} e^{-\lambda z} k(z+y)\, dz + \mu f'(0) \int_0^{\infty} e^{-\lambda z} \delta_{-c\tau}(y+z)\, dz \right] dy \\ &= \left[\frac{d}{c}\hat{k}(\pm\rho) + \frac{\mu f'(0) e^{\mp \rho\tau c}}{c} \right] \int_0^{\infty} e^{-(\lambda \pm \rho) z}\, dz, \end{aligned}$$

which yields the statement (i).

(ii) The proof follows from the statement (i) and the definition of l, and the proof of the lemma is completed. ∎

Let

$$p_{\pm}(c) = \inf_{\rho > 0} \frac{1}{\rho} \log l^{\pm}(c,\rho) \;\text{ for }\; c \in \mathbb{R},$$

and define

$$c_+^* = \inf\{c \in \mathbb{R} : p_+(c) \le 0\} \;\text{ and }\; c_-^* = \sup\{c \in \mathbb{R} : p_-(c) \le 0\}.$$

Then we have the following.

LEMMA 4.9. *The following statements are true:*

(i) $p_{\pm}(c) \le 0$ *for all* $\pm c > \pm c_{\pm}^*$ *and* $p_{\pm}(c) > 0$ *for all* $c < \pm c_{\pm}^*$.

(ii) $p_+(c) + p_-(c) > 0$ *for all* $c \in \mathbb{R}$*, provided that* (K1) *holds (see subsection* 4.2*).*

(iii) $c_+^* > 0 > c_-^*$*, provided that* (K1) *holds; in particular,* $c_+^* = -c_-^* > 0$ *when* $k(x) = k(-x)$ *for all* $x \in \mathbb{R}$.

Proof. (i) We only consider the case of "+," since the other case can be dealt with by similar arguments. Note that $p_+(c)$ is nonincreasing in c due to the definition of p_+ and Lemma 4.8(ii). This, together with the definition of c_+^*, implies that $p_+(c) \le 0$ for all $c > c_+^*$ and $p_+(c) > 0$ for all $c < c_+^*$.

(ii) Suppose that $c = 0$. By $k(\mathbb{R}) \subseteq (0,\infty)$ and the continuity of k, there is $\delta > 0$ such that $k(x) \ge \delta$ for all $x \in [-2, 2]$. Thus, we have

$$\liminf_{\rho\to\infty} \frac{\log \hat{k}(\rho)}{\rho} \ge \liminf_{\rho\to\infty} \frac{\log[\frac{\delta(e^{2\rho} - e^{-2\rho})}{\rho}]}{\rho} \ge \liminf_{\rho\to\infty} \frac{\log[\frac{\delta}{2\rho} e^{2\rho}]}{\rho} = 2.$$

Similarly, $\liminf_{\rho\to\infty} \frac{\log \hat{k}(-\rho)}{\rho} \geq 2$. Thus, we have

$$\liminf_{\rho\to\infty} \frac{\log l_{\pm}(0,\rho)}{\rho} = \liminf_{\rho\to\infty} \frac{\log[\frac{d\hat{k}(\pm\rho)+\mu f'(0)}{d+\mu}]}{\rho} \geq \liminf_{\rho\to\infty} \frac{\log \hat{k}(\pm\rho)}{\rho} \geq 2.$$

This, combined with the fact that $\frac{1}{\rho}\log l_{\pm}(0,\rho) \geq \frac{1}{\rho}\log[\frac{d+\mu f'(0)}{d+\mu}] > 0$ for all $\rho > 0$, implies that $p_{\pm}(0) > 0$ and hence $p_{+}(0) + p_{+}(0) > 0$.

Now, it suffices to consider the case of $c > 0$ since the remaining case $c < 0$ can be dealt with similarly. Let $a(c,\rho) = \frac{1}{\rho}\log\frac{d+\mu f'(0)e^{-\tau c\rho}}{d+\mu+c\rho}$ with $\rho \in (0,\infty)$ and $b(c,\rho) = \frac{1}{\rho}\log\frac{d+\mu f'(0)e^{\tau c\rho}}{d+\mu-c\rho}$ with $\rho \in J \triangleq (0, \frac{d+\mu}{c})$. Noticing that

$$\lim_{\rho\to 0^+} b(c,\rho) = \lim_{\rho\to(\frac{d+\mu}{c})^-} b(c,\rho) = \infty,$$

we know that $\inf_{\rho\in J} b(\rho,c) = \inf_{\rho\in J_0} b(\rho,c)$, where $J_0 = \{\rho \in J \cap [\delta.\infty) : \frac{\partial b(c,\rho)}{\partial\rho} = 0\}$ for some $\delta \in J$. It is easy to verify that $\frac{\partial b(c,\rho)}{\partial\rho} = 0$ if and only if $b(c,\rho) = \frac{\mu\tau c f'(0)e^{\tau c\rho}}{d+\mu f'(0)e^{\tau c\rho}} + \frac{c}{d+\mu-c\rho}$. Thus,

$$\begin{aligned}
\inf_{\rho\in J} b(\rho,c) &= \inf_{\rho\in J_0}[\frac{\mu\tau c f'(0)e^{\tau c\rho}}{d+\mu f'(0)e^{\tau c\rho}} + \frac{c}{d+\mu-c\rho}] \\
&\geq \inf\{[\frac{\mu\tau c f'(0)e^{\tau c\rho}}{d+\mu f'(0)e^{\tau c\rho}} + \frac{c}{d+\mu-c\rho}] : \rho \in [\delta, \frac{d+\mu}{c})\} \\
&> \lim_{\rho\to 0+}[\frac{\mu\tau c f'(0)e^{\tau c\rho}}{d+\mu f'(0)e^{\tau c\rho}} + \frac{c}{d+\mu-c\rho}] \\
&= \frac{\mu\tau c f'(0)}{d+\mu f'(0)} + \frac{c}{d+\mu}.
\end{aligned}$$

Similarly, in view of $\lim_{\rho\to 0+} a(c,\rho) = \infty$ and $\lim_{\rho\to\infty} a(c,\rho) = 0$, letting $J^0 = \{\rho \in (0,\infty) : \frac{\partial a(c,\rho)}{\partial\rho} = 0\}$, we similarly obtain

$$\begin{aligned}
\inf_{\rho>0} a(\rho,c) &\geq \min\{0, \inf_{\rho\in J^0}[-\frac{\mu\tau c f'(0)e^{-\tau c\rho}}{d+\mu f'(0)e^{-\tau c\rho}} - \frac{c}{d+\mu+c\rho}]\} \\
&\geq \lim_{\rho\to 0+} -[\frac{\mu\tau c f'(0)e^{-\tau c\rho}}{d+\mu f'(0)e^{-\tau c\rho}} + \frac{c}{d+\mu+c\rho}] \\
&= -[\frac{\mu\tau c f'(0)}{d+\mu f'(0)} + \frac{c}{d+\mu}].
\end{aligned}$$

Note that by the assumption (K1), we have $\frac{1}{\rho}\log(l^{+}(c,\rho)) \geq a(c,\rho)$ and $\frac{1}{\rho}\log(l^{-}(c,\rho)) \geq b(c,\rho)$ for all $(c,\rho) \in \mathbb{R}\times(0,\infty)$. It follows from the definitions of $p_{\pm}$ and $l^{\pm}$ that for any $c \in \mathbb{R}$,

$$\begin{aligned}
p_{+}(c) + p_{-}(c) &= \inf_{\rho>0}[\frac{1}{\rho}\log(l^{+}(c,\rho))] + \inf_{\rho\in>0}[\frac{1}{\rho}\log(l^{-}(c,\rho))] \\
&\geq \inf_{\rho>0} a(\rho,c) + \inf_{\rho>0} b(\rho,c) \\
&> 0.
\end{aligned}$$

(iii) By the proof of the case $c = 0$ in (ii), we have $p_\pm(0) > 0$, which together with Lemma 4.8(ii) and the definition of c_+^* and c_-^* implies $c_+^* \geq 0 \geq c_-^*$.

We only prove $c_+^* > 0$, since $c_-^* < 0$ can be proved by similar arguments. Otherwise, $c_+^* = 0$, and thus by (i), we have $p_+(c) \leq 0$ for all $c \in (0, \infty)$. Clearly, by (K1), there exist $M_1 > 1$ and $\delta_1 > 0$ such that $l^+(c,\rho) \geq \frac{d+\mu f'(0)e^{-\rho c\tau}}{c\rho+d+\mu} \geq \frac{d+\mu f'(0)e^{-\rho\tau}]}{\rho+d+\mu} \geq M_1 > 1$ for all $(c,\rho) \in [0,1]\times(0,\delta_1]$. Thus, $\frac{\log[l^+(c,\rho)]}{\rho} \geq \frac{M_1}{\delta_1}$ for all $(c,\rho) \in [0,1]\times(0,\delta_1]$. Note that $l^+(c,\rho) \geq \frac{d\hat{k}(\rho)}{\rho+d+\mu}$ for all $(c,\rho) \in [0,1] \times (0,\infty)$. By the proof of the case $c = 0$ in (ii), $\liminf_{\rho\to\infty} \frac{\log[\frac{d\hat{k}(\rho)}{\rho+d+\mu}]}{\rho} = \liminf_{\rho\to\infty} \frac{\log \hat{k}(\rho)}{\rho} \geq 2$. Thus, there exists $\gamma_1 > \delta_1$ such that $\frac{\log[l^+(c,\rho)]}{\rho} \geq 1$ for all $(c,\rho) \in [0,1] \times [\gamma_1,\infty)$. So, $\frac{\log[l^+(c,\rho)]}{\rho} \geq \min\{1, \frac{M_1}{\delta_1}\} > 0$ for all $(c,\rho) \in [0,1] \times ((0,\delta_1] \cup [\gamma_1,\infty))$. This together with the definition of p_+ and the fact that $p_+(c) \leq 0$ for all $c \in (0,1]$ implies that there exist sequences $\{c_m\}$ in $(0,1)$ and $\{\rho_m\}$ in $[\delta_1,\gamma_1]$ such that $\lim_{m\to\infty} c_m = 0$ and $\liminf_{m\to\infty} \frac{\log[l^+(c_m,\rho_m)]}{\rho_m} \leq 0$. Without loss of generality, we may assume that $\lim_{m\to\infty} \rho_m = \rho^*$ for some $\rho^* \in [\delta_1,\gamma_1]$. Hence, by (K1), $\liminf_{m\to\infty} \frac{\log[l^+(c_m,\rho_m)]}{\rho_m} \geq \frac{\log[\frac{d \liminf_{m\to\infty}[\hat{k}(\rho_m)]+\mu f'(0)}{d+\mu}]}{\rho^*} \geq \frac{\log[\frac{d+\mu f'(0)}{d+\mu}]}{\rho^*} > 0$, a contradiction.

Since $k(x) = k(-x)$ for all $x \in \mathbb{R}$, we have $\hat{k}(\rho) = \hat{k}(-\rho) \geq 1$ for all $\rho > 0$. Thus, $l^+(c,\rho) = l^-(-c,\rho)$ for all $(c,\rho) \in \mathbb{R} \times (0,\infty)$. These, combined with the definition of $c_\pm^*$, yield $c_+^* = -c_-^*$. This completes the proof of the lemma. ∎

Let

$$\tilde{l}(c,\rho) = \frac{1}{c\rho+d+\mu}[d\hat{k}(-\rho) + \mu f'(0)e^{-\rho c\tau}], \quad \tilde{p}_\pm(c) = \inf_{\rho>0} \frac{1}{\rho} \log \tilde{l}(c,\pm\rho) \text{ for } c,\rho \in \mathbb{R},$$

and

$$\tilde{c}_+^* = \inf\{c \in \mathbb{R} : \tilde{p}_+(c) \leq 0\}, \quad \tilde{c}_-^* = \sup\{c \in \mathbb{R} : \tilde{p}_-(c) \leq 0\}.$$

Note that $l(-c,\pm\rho) = l(c,\mp\rho)$ for all $c \in \mathbb{R}$ and $\rho > 0$. Thus, by the definitions of $c_\pm^*$ and $\tilde{c}_\pm^*$, we easily easily obtain the following lemma.

LEMMA 4.10. *If (K1) holds, then* $\tilde{c}_+^* = -c_-^* > 0 > \tilde{c}_-^* = -c_+^*$.

For any $(a,b,\delta) \in (0,1) \times (1,\infty) \times (0,\infty)$, define $A[\cdot;a,b], B[\cdot;a,b,\delta] : X_+ \to X_+$ by

$$A[\phi;a,b] = a\Gamma[\phi] + (1-a)b\phi, \quad B[\phi;a,b,\delta] = a\Gamma[\phi] + (1-a)h_{b,\delta}(\phi),$$

where $\Gamma[\phi](x) = \int_{\mathbb{R}} \phi(y)k(x-y)dy$, $h_{b,\delta}(\phi)(x) = h_{b,\delta}(\phi(x))$ and

$$h_{b,\delta}(u) = \begin{cases} bu, & u \in (0, \frac{\delta}{b}), \\ \delta, & u \geq \frac{\delta}{b}. \end{cases}$$

LEMMA 4.11. *Assume that* (K1) *holds and let* $(a,b,\delta) \in (0,1) \times (1,\infty) \times (0,\infty)$ *and* $c_\pm^*[b] = \inf_{\rho>0} \frac{\log(b\hat{k}(\pm\rho))}{\rho}$. *Then the following statements are true:*

(i) $c_\pm^*[b] > 0$.

(ii) $\lim_{n\to\infty} \min_{x\in[nc_1,nc_2]}(A[\cdot;a,b])^n[\phi](x) = \infty$ *for all* $\phi \in X_+ \setminus \{0\}$ *and* $c_1, c_2 \in (-c_-^*[b], c_+^*[b])$ *with* $c_1 \leq c_2$.

(iii) *For any* $\phi \in X_+ \setminus \{0\}$ *and* $c_1, c_2 \in (-c_-^*[b], c_+^*[b])$ *with* $c_1 \leq c_2$, *there exist* $\varepsilon > 0$ *and an integer* $N_0 > 0$ *such that* $\max_{x\in[nc_1,nc_2]} B^n[\phi](x) > \varepsilon$ *for all* $n \geq N_0$.

(iv) $\lim_{n\to\infty} \min_{x\in[nc_1,nc_2]} |B[\cdot;a,b,\delta])^n[\phi](x)-\delta| = 0$ *for all* $\phi \in X_+\setminus\{0\}$ *and* $c_1, c_2 \in (-c_-^*[b], c_+^*[b])$ *with* $c_1 \le c_2$.

Proof. (i) By (K1) and $b>1$, we have $\frac{\log(b\hat{k}(\pm\rho))}{\rho} \ge \frac{\log b}{\rho} > 0$ for all $\rho>0$. This together with the fact that $\liminf_{\rho\to\infty} \frac{\log(b\hat{k}(\pm\rho))}{\rho} \ge 2$ due to the proof of the case $c=0$ in Lemma 4.9(ii) implies $c_\pm^*[b]>0$, that is, (i) holds.

(ii) Suppose that $\phi\in X_+\setminus\{0\}$ and $c_1,c_2\in(-c_-^*[b],c_+^*[b])$ with $c_1\le c_2$. Take $b_0>1$ such that $\frac{a}{b_0}+b-ab>1$. Let $D[\phi](x)=\Gamma[h_{b_0,\delta}(\phi)](x)$ for all $(x,\phi)\in\mathbb{R}\times X_+$ and $A=A[\cdot;a,b]$. By applying Theorem 4.1(iii) to D with f and u^* replaced, respectively, by $h_{b_0,\delta}$ and δ, we have

$$\lim_{n\to\infty} \min_{x\in[nc_1,nc_2]} |D^n[\phi](x)-\delta| = 0.$$

Thus, there is an integer $N_1>0$ such that $\min_{x\in[nc_1,nc_2]} D^n[\phi](x) \ge \frac{\delta}{2}$ for all $n\ge N_1$. So, $b_0^n\Gamma^n[\phi](x) \ge D^n[\phi](x) \ge \frac{\delta}{2}$ and hence $\Gamma^n[\phi](x) \ge \frac{\delta}{2}\frac{1}{b_0^n}$ for all $n\ge N_1$ and $x\in[nc_1,nc_2]$. Now, we remark that for any $n\ge N_1$ and $x\in[nc_1,nc_2]$,

$$\begin{aligned} A^n[\phi](x) &= \sum_{l=N_1}^{n} X_n^l a^l [b-ab]^{n-l}\Gamma^l[\phi](x) \\ &\ge \sum_{l=N_1}^{n} X_n^l a^l [b-ab]^{n-l}\Gamma^l[\phi](x) \\ &\ge \frac{\delta}{2}\sum_{l=N_1}^{n} X_n^l [\frac{a}{b_0}]^l [b-ab]^{n-l}. \end{aligned}$$

In view of $\frac{a}{b_0}+b-ab>1$ and $\lim_{n\to\infty} \frac{X_n^l[\frac{a}{b_0}]^l[b-ab]^{n-l}}{[\frac{a}{b_0}+b-ab]^n} = 0$ for each nonnegative integer l, we have $\lim_{n\to\infty} \sum_{l=N_1}^{n} X_n^l[\frac{a}{b_0}]^l[b-ab]^{n-l} = \infty$, and hence $\lim_{n\to\infty} \min_{x\in[nc_1,nc_2]} A^n[\phi](x) = \infty$.

(iii) Suppose that $\phi\in X_+\setminus\{0\}$ and $c_1,c_2\in(-c_-^*[b],c_+^*[b])$ with $c_1\le c_2$. Without loss of generality, we may assume that $c_1<0$, $c_2>0$ and $\phi\in X_\delta$ has compact support due to the monotonicity of $B(\cdot):=B[\cdot;a,b,\delta]$. Letting $c_3\in(-c_[^*b],c_1)$, $c_4\in(c_2,c_+^*[b])$ and applying the statement (ii), there exists a integer $N_1>0$ such that $A^n[\phi](x)\ge\delta$ for all $n\ge N_1$ and $x\in[nc_3,nc_4]$. In view of choices of c_3,c_4 and ϕ, there is an integer $N_2>N_1$ such that $T_{-y}[A^n[\phi]]\ge\phi$ and $A^n[\phi](y)\ge\delta$ for all $n\ge N_2$ and $y\in[nc_1,nc_2]$. Take $\alpha>0$ such that $A^n[\alpha\phi]\le\delta$ for all nonnegative integers $n\le 2N_2-1$. Thus,

$$T_{-y}[B^n[\alpha\phi]]\ge\alpha\phi \text{ and } B^n[\alpha\phi](y)\ge\alpha\delta \text{ for all integers } n\in[N_2,2N_2-1] \text{ and } y\in[nc_1,nc_2].$$

By the monotonicity of B, we apply the above inequalities repeatedly to obtain that, for any integer $n\ge N_2$,

$$T_{-y}[B^n[\alpha\phi]]\ge\alpha\phi \text{ and } B^n[\alpha\phi](y)\ge\alpha\delta \text{ for all } y\in[nc_1,nc_2].$$

In particular, $B^n[\alpha\phi](y)\ge\alpha\delta$ for all $n\ge N_2$ and $y\in[nc_1,nc_2]$, which yields (iii) with $N_0=N_2$ and $\varepsilon=\alpha\delta$.

Finally, by (iii) and by using an argument similar to the proof of Theorem 3.2, we easily get (iv).

The proof of the lemma is completed. ∎

THEOREM 4.5. *Assume that* (A1)–(A4) *and* (K1) *hold. Then the following statements are valid:*

(i) *For any* $c \in [c_+^*, \infty)$, (4.4) *has a traveling wave* $\phi(x-ct)$ *with* $\phi(\infty) = 0$ *and* $\phi(-\infty) = u^*$.
(ii) *For any* $c \in (-\infty, c_-^*]$, (4.4) *has a traveling wave* $\phi(x-ct)$ *with* $\phi(-\infty) = 0$ *and* $\phi(\infty) = u^*$;
(iii) *For any* $c \in (c_-^*, c_+^*)$, (4.4) *has no nonconstant traveling wave* $\phi(x-ct)$.

Proof. By some simple computations, we know that for any $\phi \in X_+$, $\phi = Q[\phi; c, f]$ if and only if $u(t,x) = \phi(x-ct)$ satisfies (4.4). We remark that $c_+^* > 0 > c_-^*$ due to Lemma 4.9(iii)

(i) Suppose $c > c_+^*$. Then by Lemma 4.9, we have $p_+(c) \leq 0$ and $p_+(c) > -p_-(c)$. Applying Theorem 3.6(ii) to $Q[\cdot; c, f]$ with $m_c(y, dy)$ defined as in this subsection, we obtain that there is a nonconstant $\phi_c \in X_M$ such that $Q[\phi_c; c, f] = \phi_c$ with $\phi_c(\infty) = 0$ and $\phi_c(-\infty) = u^*$.

Next, we consider the case $c = c_+^* > 0$. We shall prove $p_+(c_+^*) \leq 0$. Otherwise, $p_+(c_+^*) > 0$. Note that there exist two sequences $\{c_m\}$ and $\{\phi_m\}$ satisfying $c_m \in (c_+, \infty)$, $\phi_m \in X_M$, $\phi_m(\infty) = 0$, and $\phi_m(-\infty) = u^*$ such that $\lim_{m\to\infty} c_m = c_+^*$ and $Q[\phi_m; c_m, f] = \phi_m$ for all $m \in \mathbb{N}$, where $M = \sup f([0, u^*])$. Obviously, $K[\phi_m, c_m, f] \in X_M$ for $m \in \mathbb{N}$. By Lemma 4.7(ii), we know that $\cup_{m\in\mathbb{N}} L(X_M], c_m)$ is precompact, and hence $\{\phi_m\}$ is precompact. Without loss of generality, we may assume that the limiting of ϕ_m exists, and $\sup\{x \in \mathbb{R} : \phi_m(x) = \frac{u^*}{3}\} = 0$ due to (H1). Hence, $\phi_m(0) = \frac{u^*}{3}$ and $\phi_m(x) \leq \frac{u^*}{3}$ for all $x \in [0, \infty)$. Let $\phi = \lim_{m\to\infty} \phi_m$. Then $\phi(0) = \frac{u^*}{3}$, $\phi(x) \leq \frac{u^*}{3}$ for all $x \in [0, \infty)$ and $Q[\phi; c_+^*, f] = \phi$. This together with Theorem 3.4(i) and $p_+(c_+^*) > 0$ implies $\limsup_{x\to\infty} \phi(x) = u^*$, a contradiction. Thus, $p_+(c_+^*) \leq 0$. By the above arguments for the case of $c > c_+^*$, we easily obtain a nonconstant $\phi_c \in X_M$ such that $Q[\phi_c; c, f] = \phi_c$ with $\phi_c(\infty) = 0$ and $\phi_c(-\infty) = u^*$.

(iii) Suppose that $c \in (c_-^*, c_+^*) \setminus \{0\}$. By Lemma 4.9(i), we have $p_-(c) > 0$ and $p_+(c) > 0$. Applying Theorem 3.6(i), we know that $Q[\cdot; c_+^*, f]$ has no nonconstant $\phi \in X_+$ such that $Q[\phi; c, f] = \phi$, that is, (4.4) has no nonconstant traveling wave $\phi(x-ct)$.

Suppose that $c = 0$ and (4.4) has a nonconstant traveling wave $\phi(x-ct)$ with $c = 0$ (i.e., standing wave); then $Q[\phi; 0, f] = \phi$ and $\phi \in X_{\max f([0,u^*])}$. We claim that $\inf_{s\in\mathbb{R}} \phi(s) = 0$; otherwise, $\inf_{s\in\mathbb{R}} \phi(s) > 0$. Hence, $\underline{u} \geq \bar{u} > 0$, where $\underline{u} = \inf_{s\in\mathbb{R}} \phi(s)$ and $\bar{u} = \sup_{s\in\mathbb{R}} \phi(s)$. Then $\underline{u} \geq \frac{d}{d+\mu}\underline{u} + \frac{\mu}{d+\mu} \inf f([\underline{u}, \bar{u}])$ and $\bar{u} \leq \frac{d}{d+\mu}\bar{u} + \frac{\mu}{d+\mu} \sup f([\underline{u}, \bar{u}])$, that is, $f([\underline{u}, \bar{u}]) \supseteq [\underline{u}, \bar{u}]$. Hence, (A4) implies $\underline{u} = \bar{u} = u^*$, that is, $\phi = u^*$, contradiction. By the assumptions of f, there exist $\delta > 0$ and $b \in (1, f'(0))$ such that $f(x) \geq bx$ for all $x \in [0, \delta]$ and $f(x) \geq b\delta$ for all $x \in f([0, u^*]) \setminus [0, \delta]$. Letting $a = \frac{d}{d+\mu}$, we have $\phi = (Q[\cdot; 0, f])^n[\phi] \geq (Q[\cdot; 0, h_{b,\delta}])^n[\phi] = (B[\cdot; a, b, \delta])^n[\phi]$. By Lemma 4.11(iii), there is $\alpha \in (0, \min\{c_-^*[b], c_+^*[b]\})$ such that $\lim_{n\to\infty} \min_{|x|\leq n\alpha} B[\cdot; a, b, \delta])^n[\phi](x) \geq \alpha\delta$. Hence, $\phi \geq \alpha\delta > 0$, a contradiction. This proves the statement (iii).

To complete the proof of (ii), we consider the following auxiliary system:

$$(4.5)\quad \begin{cases} \frac{\partial u}{\partial t}(t,x) &= d[\int_{\mathbb{R}} u(t,y)k(y-x)\mathrm{d}y - u(t,x)] - \mu u(t,x) + \mu f(u(t-\tau,x)) \\ & \qquad (t,x)\in(0,\infty)\times\mathbb{R}, \\ u(\theta,x) &= \varphi(\theta,x), \quad (\theta,x)\in[-\tau,0]\times\mathbb{R}, \end{cases}$$

This equation can be considered as the spatial reflection of (4.4), and when the kernel $k(\cdot)$ is even, (4.5) is exactly the same as (4.4). Applying the conclusion in (i) to (4.5) and making use of Lemma 4.10 gives the following result:

(E) For any $c \in [\tilde{c}_+^*, \infty) = [-c_-^*, \infty)$, (4.5) has a nonconstant traveling wave $\tilde{\phi}(x-ct)$ with $\tilde{\phi}(\infty) = 0$ and $\tilde{\phi}(-\infty) = u^*$.

It is clear that the statement (E) implies the statement (ii) of the theorem with c, ϕ replaced by $-c, \tilde{\phi}(-\cdot)$, respectively.

The proof of the lemma is completed. ∎

5. Discussion. First, we briefly discuss the conclusion (i) in Theorem 3.6. Theorem 3.6(i) confirms that under some assumptions, h is globally attractive in $BC(\mathbb{R},[0,\infty)) \setminus \{0\}$ with respect to the *compact open topology.* One naturally wonders if it is also globally attractive in $BC(\mathbb{R},[0,\infty)) \setminus \{0\}$ with respective to the *usual supremum norm*. The answer to this question is no, and we explain why below by making a connection to the existence of traveling wave front solutions of Q. By Theorem 3.4(iii) and taking $c \geq \hat{c}^*(\xi)$, Q has a traveling wave $W(x\cdot\xi - nc)$ connecting h to 0 such that W is continuous on $\mathbb{R}$. This implies that the positive equilibrium h cannot attract all positive solutions with respect to the supremum norm, because the positive solution $u(t,x) = W(x\cdot\xi - nc)$ cannot approach h in the supremum norm as $t \to \infty$ due to the fact that $W(\infty) = 0$.

However, the positive equilibrium h can be attractive with respect to the *supremum norm* in a *subset* of $BC(\mathbb{R},[0,\infty)) \setminus \{0\}$. To see this, define

$$C_+^> = \{\phi \in BC(\mathbb{R},[0,\infty)) : \text{ there exists } \varepsilon_\phi > 0 \text{ such that } \phi(x) > \varepsilon_\phi \text{ for all } x \in \mathbb{R}\},$$

and let

$$||\phi||_{sup} = \sup\{|\phi(\theta,x)| : (\theta,x) \in [-1,0]\times\mathbb{R}\},\ \phi \in BC(\mathbb{R},\mathbb{R}).$$

That is, $C_+^>$ consists of those bounded continuous functions that are bounded below by a positive constant. If $\phi \in C_+^>$, then there exists $\varepsilon_\phi \in (0, r^-)$ such that $\phi(x) > \varepsilon_\phi$ for all $x \in \mathbb{R}$. Thus, by the monotonicity of Q^- and Proposition 4.1 in [37], we have

$$Q^n[\phi] \geq (Q^-)^n[\phi] \geq (Q^-)^n[\varepsilon_\phi] > 0.$$

By appealing to the arguments in the proof of Theorem 3.2 with some slight modifications, we conclude that $\lim_{n\to\infty} ||Q^n[\phi] - h||_{sup} = 0$.

Second, we point out that in subsections 4.2 and 4.3, we have only obtained some results on the existence/nonexistence of traveling wavefronts for (4.2) and (4.4), without concluding anything on the spreading speeds and asymptotic behaviours of these two equations. This is because both systems (4.2) and (4.4) contain a time delay $\tau > 0$, and accordingly, $BC([-\tau,0]\times\mathbb{R},\mathbb{R})$ should be adopted as the phase space, equipped with the topology induced by the norm

$$|||\phi||| \triangleq \sum_{n\geq 1} 2^{-n} \sup\{|\varphi(\theta,x)| : \theta \in [-\tau,0],\ x \in \mathbb{R} \text{ with } |x| \leq n\} \quad \text{for} \quad \varphi \in BC([-\tau,0]\times\mathbb{R},\mathbb{R}).$$

This is a different space from the one dealt with in section 3, and hence, one cannot directly apply the results on the spreading speeds and asymptotic behaviors to these two systems. It is possible to follow the same framework in section 3 to establish some similar results on the spreading speeds and asymptotic behavior for these two systems, but we decide not to do so in this already lengthy paper and will leave it for a possible future project. However, as far as traveling wave solutions are concerned, one only needs to consider the corresponding profile equations containing a parameter c and the delay τ, which has the same phase space as discussed in section 3, and as such, one can directly apply the results on traveling wavefronts in section 3 to the operators Q properly formulated from the respective profile equations in these two sections.

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