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# STABILITY IN A LINEAR DELAY SYSTEM WITHOUT INSTANTANEOUS NEGATIVE FEEDBACK*

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**Abstract.** It is shown that every solution of a linear differential system with constant coefficients and time delays tends to zero if a certain matrix derived from the coefficient matrix is a nonsingular $M$-matrix and the diagonal delays satisfy the so-called 3/2 condition.

**Key words.** linear system, pure-delay type, stability, diagonal dominant, $M$-matrix

**AMS subject classifications.** 34K20, 34K60

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## 1. Introduction.

Consider a system of delayed linear differential equations with constant coefficients of the form

$$\dot{x}_i(t) = -\sum_{j=1}^{n} a_{ij}x_j(t-\tau_{ij}), \qquad i = 1, 2, \ldots, n, \tag{1.1}$$

with

$$\tau_{ij} \geq 0 \quad \text{for all} \quad 1 \leq i, j \leq n. \tag{1.2}$$

System (1.1) arises as linearization about an equilibrium point of many nonlinear systems with time delays. The interested reader can refer to Stépán [14] and the references therein for multiple-delay examples, such as machine tool vibration and human-machine systems.

When $\tau_{ij} = 0$ for all $i, j = 1, 2, \ldots, n$, it is well known that (1.1) is asymptotically stable if and only if the matrix $A = (a_{ij})$ is a positively stable matrix, meaning that all eigenvalues of $A$ have positive real parts. When some of the delays $\tau_{ij}$ are nonzero, (1.1) is asymptotically stable if and only if all the roots of its characteristic equation have negative real parts (cf. Hale and Verduyn Lunel [6]). In general, it is extremely difficult to analyze the characteristic equation of (1.1) when there are multiple (nonzero) delays. In Hofbauer and So [8], the authors considered the case when $\tau_{ii} = 0$ for $i = 1, 2, \ldots, n$, and they established the following result.

THEOREM 1.1. *Assume that* $\tau_{ii} = 0$ *for all* $i = 1, 2, \ldots, n$. *Then* (1.1) *is asymptotically stable for all choices of delays of the form* (1.2) *if and only if* $a_{ii} > 0$ *for* $i = 1, 2, \ldots, n$, $\det A \neq 0$, *and* $A$ *is weakly diagonally dominant (i.e., all the principal minors of* $\hat{A} = (\hat{a}_{ij})$ *are nonnegative, where* $\hat{a}_{ii} = a_{ii}$ *and* $\hat{a}_{ij} = -|a_{ij}|$ *for* $j \neq i$*).*

In such a case (i.e., when there is no diagonal delay), Györi [5] also obtained a similar result for a quasi-monotone matrix $A$ (i.e., $a_{ij} \leq 0$ for $i \neq j$). Motivated by the

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study of neural networks of Hopfield type, there is a recent extension of Theorem 1.1 by Sue Ann Campbell [2] of the University of Waterloo to include both types of diagonal terms, both with and without delays. A result similar to Theorem 1.1 was obtained (with the conditions on a suitably derived matrix) using the same proof as in [8].

When $\tau_{ii} \neq 0,\ i = 1, \ldots, n$, instantaneous feedback is absent, and (1.1) becomes a system of "pure-delay type." For such a "pure-delay-type" system, the stability problem becomes much harder, as pointed out by Gopalsamy and He [4], He [7], and Kuang [10]. However, it is reasonable to expect that a similar stability criterion holds as long as the diagonal delays are sufficiently small. This paper will provide an answer to this question. More precisely, by employing a new technique (without analyzing the characteristic equation or constructing a Liapunov functional), we will extend the sufficiency part of Theorem 1.1 to the case when $\tau_{ii}$ $(i = 1, 2, \ldots, n)$ are not necessarily all zero. For convenience, we recall the concept of a nonsingular $M$-matrix (cf. Fiedler [3]).

DEFINITION 1.2. *The $n \times n$ matrix $B = (b_{ij})$ is a nonsingular M-matrix if* (i) $b_{ij} \leq 0$ *for $j \neq i$ and* (ii) *all principal minors of $B$ are positive.*

There are many equivalent formulations of this concept (cf. Fiedler [3, Theorem 5.1, p. 114]. In particular, if $B$ is a nonsingular $M$-matrix, then $B^{-1}$ is a positive matrix.

We associate with the $n \times n$ matrix $A = (a_{ij})$ a new matrix $\tilde{A} = (\tilde{a}_{ij})$ defined by

$$\tilde{a}_{ii} = a_{ii} \quad \text{for} \quad i = 1, 2, \ldots, n \tag{1.3}$$

and

$$\tilde{a}_{ij} = -\frac{1 + \frac{1}{9} a_{ii}\tau_{ii}(3 + 2a_{ii}\tau_{ii})}{1 - \frac{1}{9} a_{ii}\tau_{ii}(3 + 2a_{ii}\tau_{ii})} |a_{ij}| \quad \text{for} \quad i \neq j, \quad j = 1, 2, \ldots, n. \tag{1.4}$$

Now we can state our main result.

THEOREM 1.3. *Assume that*

$$a_{ii}\tau_{ii} < \frac{3}{2} \quad \textit{for all} \quad i = 1, 2, \ldots, n. \tag{1.5}$$

*If $\tilde{A}$ is a nonsingular M-matrix, then every solution $(x_1(t), x_2(t), \ldots, x_n(t))$ of* (1.1) *tends to* $0$ *as $t \to \infty$.*

*Remark* 1.1. Condition (1.5) will be referred to as the 3/2 condition. When $\tau_{ii} = 0$ for all $i = 1, \ldots, n$, the 3/2 condition is automatically satisfied and $\tilde{A} = \hat{A}$. According to Bapat and Raghavan [1, Theorem 7.8.6], if $\hat{A}$ is a nonsingular $M$-matrix, then $A$ itself is nonsingular. Hence, in the case of no diagonal delays, a matrix $A$ satisfying the hypotheses of Theorem 1.3 will also satisfy the criterion in Theorem 1.1. The stability criterion in Theorem 1.3 is concrete and easily verifiable for any given (numerical) system.

*Remark* 1.2. There are many 3/2 stability results for scalar (linear or nonlinear, autonomous or nonautonomous, one or several delays) equations in the literature. See, for example, [11, 16, 15, 9, 12, 13]. It would be interesting to see if these results can be extended to systems.

*Remark* 1.3. In [8], besides the linear equation (1.1), the authors also considered global stability of Lotka–Volterra equations (with $\tau_{ii} = 0$). We are currently investigating the possibility of a 3/2 result for Lotka–Volterra systems when $\tau_{ii} > 0$.

**2. Proof of Theorem 1.3.** The proof of Theorem 1.3 consists of the following two lemmas. The first lemma establishes the boundedness of solutions of (1.1).

LEMMA 2.1. *Under the conditions of Theorem* 1.3, *every (forward) solution of* (1.1) *is bounded.*

*Proof.* Let $(x_1(t), x_2(t), \ldots, x_n(t))$ be a solution of (1.1) on $[t_0, \infty)$. Without loss of generality, $t_0$ can be taken to be 0. For the sake of contradiction, assume that $\max\{|x_i(t)| : i = 1, 2, \ldots, n\}$ is unbounded on $[t_0, \infty)$. By rearranging the indices $i$, we may assume that

$$\limsup_{t\to\infty} |x_i(t)| = \infty \quad \text{for} \quad i = 1, 2, \ldots, k(\le n) \tag{2.1}$$

and

$$|x_i(t)| \le M \quad \text{for} \quad t \ge t_0 - \max_{h,k}\{\tau_{hk}\}, \qquad i = k+1, \ldots, n. \tag{2.2}$$

Let $N$ be the smallest integer such that $N > t_0 + \tau_{ii}$ for all $i$. There is an integer $N_1 > N$ such that for each $i = 1, \ldots, k$, the maximum of the function $|x_i(t)|$ on the interval $[t_0, N_1]$ is attained at a point in $[N, N_1]$. Fix $i = 1, \ldots, k$. For each integer $m \ge 1$, let $t_{im} \in [N, N_1 + m]$ be such that $|x_i(t_{im})| = \max\{|x_i(t)| : t \in [t_0, N_1 + m]\}$. We may assume that $\{t_{im}\}_{m=1}^{\infty}$ is a nondecreasing sequence. By going to subsequences if necessary, we have $k$ sequences $\{t_{im}\}_{m=1}^{\infty}, i = 1, 2, \ldots, k$, such that

$$\begin{cases} t_{im} \uparrow \infty, \quad |x_i(t_{im})| \uparrow \infty \quad \text{as} \quad m \to \infty, \\ |x_i(t)| \le |x_i(t_{im})| \quad \text{for} \quad t_0 \le t \le t_m, \end{cases} \qquad \text{for} \quad i = 1, 2, \ldots, k, \tag{2.3}$$

where $t_m = \max\{t_{im} : i = 1, 2, \ldots, k\}$. Again by going to subsequences if necessary, we may assume that for each $i = 1, \ldots, k$, all the terms in the sequence $\{x_i(t_{im})\}_{m=1}^{\infty}$ are of the same sign. Without loss of generality (i.e., by using $-x_i(t)$ instead of $x_i(t)$ and $-a_{ij}$ instead of $a_{ij}$ for $j \ne i$, if necessary), we may assume that $|x_i(t_{im})| = x_i(t_{im})$. Then

$$|x_i(t)| \le x_i(t_{im}) \quad \text{for} \quad t_0 \le t < t_m \qquad \text{and} \qquad \dot{x}_i(t_{im}) \ge 0, \qquad i = 1, 2, \ldots, k.$$

It follows from (1.1) that

$$0 \le -\sum_{j=1}^{n} a_{ij} x_j(t_{im} - \tau_{ij}) \le -a_{ii} x_i(t_{im} - \tau_{ii}) + \sum_{j\ne i}^{k} |a_{ij}| x_j(t_{jm}) + M \sum_{j=k+1}^{n} |a_{ij}|$$

or

$$x_i(t_{im} - \tau_{ii}) \le \frac{1}{a_{ii}} \left[ \sum_{j\ne i}^{k} |a_{ij}| x_j(t_{jm}) + M \sum_{j=k+1}^{n} |a_{ij}| \right], \qquad i = 1, 2, \ldots, k. \tag{2.4}$$

Set

$$\alpha_i = \frac{1}{a_{ii}} \left[ \sum_{j\ne i}^{k} |a_{ij}| x_j(t_{jm}) + M \sum_{j=k+1}^{n} |a_{ij}| \right], \qquad i = 1, 2, \ldots, k. \tag{2.5}$$

We will now show

$$a_{ii} x_i(t_{im}) + \sum_{j\ne i}^{k} \tilde{a}_{ij} x_j(t_{jm}) \le M \sum_{j=k+1}^{n} |\tilde{a}_{ij}| \quad \text{for} \quad i = 1, 2, \ldots, k. \tag{2.6}$$

If $x_i(t_{im}) \le \alpha_i$, then (2.6) follows from a simple calculation. If $x_i(t_{im}) > \alpha_i$, by (2.4) there exists $\xi_{im} \in [t_{im} - \tau_{ii}, t_{im}]$ such that $x_i(\xi_{im}) = \alpha_i$. From (1.1) we have

$$\dot{x}_i(t) \le a_{ii}[-x_i(t-\tau_{ii}) + \alpha_i] \le a_{ii}\left(|x_i(t_{im})| + \alpha_i\right) \quad \text{for} \quad N \le t \le t_m. \tag{2.7}$$

For $t \in [\xi_{im}, t_{im})$, integrating (2.7) from $t - \tau_{ii}$ to $\xi_{im}$, we have

$$\alpha_i - x_i(t-\tau_{ii}) \le a_{ii}\left(|x_i(t_{im})| + \alpha_i\right)(\xi_{im} + \tau_{ii} - t) \quad \text{for} \quad \xi_{im} \le t \le t_{im}.$$

Substituting this into the first inequality in (2.7), we obtain

$$\dot{x}_i(t) \le a_{ii}^2\left(|x_i(t_{im})| + \alpha_i\right)(\xi_{im} + \tau_{ii} - t) \quad \text{for} \quad \xi_{im} \le t \le t_{im}.$$

Combining this and (2.7), we have

$$\dot{x}_i(t) \le a_{ii}\left(|x_i(t_{im})| + \alpha_i\right)\min\{1, a_{ii}(\xi_{im} + \tau_{ii} - t)\} \quad \text{for} \quad \xi_{im} \le t \le t_{im}. \tag{2.8}$$

We consider the following two cases.

*Case* 1. $t_{im} - \xi_{im} \le 2\tau_{ii}/3$. In this case, by (2.8) we have

$$\begin{aligned} x_i(t_{im}) - x_i(\xi_{im}) &\le a_{ii}^2\left(|x_i(t_{im})| + \alpha_i\right)\int_{\xi_{im}}^{t_{im}}(\xi_{im} + \tau_{ii} - t)dt \\ &= a_{ii}^2\left(|x_i(t_{im})| + \alpha_i\right)\left[\tau_{ii}(t_{im} - \xi_{im}) - \frac{1}{2}(t_{im} - \xi_{im})^2\right] \\ &\le \left(|x_i(t_{im})| + \alpha_i\right)\left[\frac{2}{3}(a_{ii}\tau_{ii})^2 - \frac{2}{9}(a_{ii}\tau_{ii})^2\right] \\ &= \frac{4}{9}(a_{ii}\tau_{ii})^2\left(|x_i(t_{im})| + \alpha_i\right) \\ &\le \frac{1}{9}a_{ii}\tau_{ii}(3 + 2a_{ii}\tau_{ii})\left(|x_i(t_{im})| + \alpha_i\right), \end{aligned}$$

since the function $y \mapsto \tau_{ii}y - \frac{1}{2}y^2$ is increasing on the interval $[0, \frac{2\tau_{ii}}{3}]$.

*Case* 2. $t_{im} - \xi_{im} > 2\tau_{ii}/3$. In this case, let $t_{im} - \eta_{im} = 2\tau_{ii}/3$ so that $\eta_{im} \in (\xi_{im}, t_{im}]$. Then by (2.8) we have

$$\begin{aligned} &x_i(t_{im}) - x_i(\xi_{im}) \\ &\le \left(|x_i(t_{im})| + \alpha_i\right)\left[a_{ii}(\eta_{im} - \xi_{im}) + a_{ii}^2\int_{\eta_{im}}^{t_{im}}(\xi_{im} + \tau_{ii} - t)dt\right] \\ &= \left(|x_i(t_{im})| + \alpha_i\right)\left[a_{ii}(\eta_{im} - \xi_{im})(1 - a_{ii}(t_{im} - \eta_{im})) + a_{ii}^2\int_{\eta_{im}}^{t_{im}}(\eta_{im} + \tau_{ii} - t)dt\right] \\ &= \left(|x_i(t_{im})| + \alpha_i\right)\left[a_{ii}(\eta_{im} - \xi_{im})\left(1 - \frac{2}{3}a_{ii}\tau_{ii}\right)\right. \\ &\qquad \left. + a_{ii}^2\tau_{ii}(t_{im} - \eta_{im}) - \frac{1}{2}a_{ii}^2(t_{im} - \eta_{im})^2\right] \\ &\le \left(|x_i(t_{im})| + \alpha_i\right)\left[\frac{1}{3}a_{ii}\tau_{ii} + \frac{2}{9}(a_{ii}\tau_{ii})^2\right] \\ &= \frac{1}{9}a_{ii}\tau_{ii}(3 + 2a_{ii}\tau_{ii})\left(|x_i(t_{im})| + \alpha_i\right), \end{aligned}$$

since $\eta_{im} - \xi_{im} \le \frac{\tau_{ii}}{3}$ in this case.

Combining Cases 1 and 2, we have

$$a_{ii}x_i(t_{im}) \le \frac{1+\frac{1}{9}a_{ii}\tau_{ii}(3+2a_{ii}\tau_{ii})}{1-\frac{1}{9}a_{ii}\tau_{ii}(3+2a_{ii}\tau_{ii})}\left[\sum_{j\neq i}^{k}|a_{ij}|x_j(t_{jm})+M\sum_{j=k+1}^{n}|a_{ij}|\right], \quad i=1,2,\ldots,k,$$

which implies (2.6) is true.

Let $\tilde{A}_k=(\tilde{a}_{ij})_{k\times k}$ denote the $k$th leading principal submatrix of $\tilde{A}$. Then $\tilde{A}_k$ is a nonsingular $M$-matrix of order $k$, and so $\tilde{A}_k^{-1}>0$. Hence, it follows from (2.6) that

$$(x_1(t_{1m}),x_2(t_{2m}),\ldots,x_k(t_{km}))^T \le M\tilde{A}_k^{-1}\left(\sum_{j=k+1}^{n}|\tilde{a}_{1j}|,\sum_{j=k+1}^{n}|\tilde{a}_{2j}|,\ldots,\sum_{j=k+1}^{n}|\tilde{a}_{kj}|\right)^T, \quad m=1,2,\ldots.$$

We conclude that

$$\limsup_{m\to\infty}|x_i(t_{im})|<\infty, \quad i=1,2,\ldots,k.$$

This contradicts the fact that $|x_i(t_{im})|\to\infty$ as $m\to\infty$ for $i=1,2,\ldots,k$, and the proof is complete.

Next, using the boundedness of solutions, we can prove the convergence of all solutions of (1.1).

LEMMA 2.2. *Under the conditions of Theorem* 1.3*, every solution of* (1.1) *tends* 0 *as* $t\to\infty$.

*Proof.* Let $(x_1(t),x_2(t),\ldots,x_n(t))$ be a solution of (1.1) on $[t_0,\infty)$. We will prove that

$$\lim_{t\to\infty}x_i(t)=0, \quad i=1,2,\ldots,n. \tag{2.9}$$

We distinguish the two cases.

*Case* A. All of the functions $\sum_{j=1}^{n}a_{ij}x_j(t-\tau_{ij})$, $i=1,2,\ldots,n$, are nonoscillatory. Then the functions $\dot{x}_i(t)$ $(i=1,2,\ldots,n)$ are eventually sign-definite, and so by Lemma 2.1, the limit $c_i=\lim_{t\to\infty}x_i(t)$ exists. By (1.1), $\dot{x}_i(t)$ converges as $t\to\infty$. Since $\dot{x}_i(t)$ is bounded, $x_i(t)$ is uniformly continuous and convergent. Therefore, $\lim_{t\to\infty}\dot{x}_i(t)=0$ for $i=1,2,\ldots,n$, and we have

$$\sum_{j=1}^{n}a_{ij}c_j=0 \quad\text{for}\quad i=1,2,\ldots,n.$$

It follows that

$$a_{ii}|c_i|-\sum_{j\neq i}|a_{ij}||c_j|\le 0 \quad\text{for}\quad i=1,2,\ldots,n. \tag{2.10}$$

Set $\hat{A}=(\hat{a}_{ij})$, where $\hat{a}_{ii}=a_{ii}$ and $\hat{a}_{ij}=-|a_{ij}|$ for $j\neq i$. Then $\hat{A}\ge\tilde{A}$ and $\hat{A}$ has nonpositive off-diagonal entries. In view of [3, Theorem 2.5.4], the matrix $\hat{A}$ is also a nonsingular $M$-matrix. Since (2.10) can be expressed as the matrix inequality $\hat{A}(|c_1|,\ldots,|c_n|)^T\le(0,\ldots,0)^T$, by applying the positive matrix $\hat{A}^{-1}$ to both sides, we conclude that $c_1=c_2=\cdots=c_n=0$.

*Case* B. At least one of the functions $\sum_{j=1}^{n} a_{ij}x_j(t-\tau_{ij})$ $(i = 1,2,\ldots,n)$ is oscillatory. Set

$$U_i = \limsup_{t\to\infty} |x_i(t)|, \qquad i = 1,2,\ldots,n.$$

By Lemma 2.1, we have $U_i < \infty$, $i = 1,2,\ldots,n$. It suffices to prove that $U_1 = \cdots = U_n = 0$. By rearranging the indices, we may assume that $\sum_{j=1}^{n} a_{ij}x_j(t-\tau_{ij})$, $i = 1,\ldots,k$, are oscillatory and $\sum_{j=1}^{n} a_{ij}x_j(t-\tau_{ij})$, $i = k+1,\ldots,n$, are nonoscillatory. It follows from (1.1) that $\dot{x}_i(t)$ $(i = 1,2,\ldots,k)$ are oscillatory and

$$\lim_{t\to\infty} \dot{x}_i(t) = 0 \quad \text{for} \quad i = k+1,\ldots,n. \tag{2.11}$$

Hence, for any $\epsilon > 0$, there exist $k$ sequences $\{t_{im}\}$ $i = 1,2,\ldots,k$, such that

$$\begin{cases} t_{im} \uparrow \infty, \quad |x_i(t_{im})| \to U_i \quad \text{as} \quad m \to \infty, \\ |\dot{x}_i(t_{im})| = 0, \quad U_i - \epsilon < |x_i(t)| < U_i + \epsilon \quad \text{for} \quad t \ge t_1, \end{cases} \qquad i = 1,2,\ldots,k, \tag{2.12}$$

where $t_1 = \min\{t_{i1} : i = 1,2,\ldots,k\}$. By going to subsequences if necessary, we may assume $|x_i(t_{im})| = x_i(t_{im})$ (use $-x_i(t)$ instead of $x_i(t)$ and $-a_{ij}$ instead of $a_{ij}$ for $j \ne i$, if necessary). By (1.1), as long as $m$ is sufficiently large, we have

$$0 = -\sum_{j=1}^{n} a_{ij}x_j(t_{im}-\tau_{ij}) \le -a_{ii}x_i(t_{im}-\tau_{ii}) + \sum_{j\ne i}^{n} |a_{ij}|(U_j+\epsilon)$$

or

$$x_i(t_{im}-\tau_{ii}) \le \frac{1}{a_{ii}} \sum_{j\ne i}^{n} |a_{ij}|(U_j+\epsilon), \qquad i = 1,2,\ldots,k. \tag{2.13}$$

Set

$$\beta_i = \frac{1}{a_{ii}} \sum_{j\ne i}^{n} |a_{ij}|(U_j+\epsilon), \qquad i = 1,2,\ldots,k. \tag{2.14}$$

We will now show

$$a_{ii}x_i(t_{im}) + \sum_{j\ne i} \tilde{a}_{ij}(U_j+\epsilon) \le \frac{2\epsilon a_{ii}\tau_{ii}(3+2a_{ii}\tau_{ii})}{9 - a_{ii}\tau_{ii}(3+2a_{ii}\tau_{ii})}, \qquad i = 1,2,\ldots,k. \tag{2.15}$$

If $x_i(t_{im}) \le \beta_i$, then (2.15) obviously holds. If $x_i(t_{im}) > \beta_i$, by (2.13) there exists $\xi_{im} \in [t_{im}-\tau_{ii}, t_{im}]$ such that $x_i(\xi_{im}) = \beta_i$. Using (1.1), for $m$ sufficiently large we have

$$\dot{x}_i(t) \le a_{ii}[-x_i(t-\tau_{ii}) + \beta_i] \le a_{ii}\left[(U_i+\epsilon) + \beta_i\right] \quad \text{for} \quad \xi_{im} - \tau_{ii} \le t \le t_{im}. \tag{2.16}$$

For $t \in [\xi_{im}, t_{im})$, integrating (2.16) from $t - \tau_{ii}$ to $\xi_{im}$, we have

$$\beta_i - x_i(t-\tau_{ii}) \le a_{ii}\left[(U_i+\epsilon)+\beta_i\right](\xi_{im}+\tau_{ii}-t) \quad \text{for} \quad \xi_{im} \le t \le t_{im}.$$

Substituting this into the first inequality in (2.16), we obtain

$$\dot{x}_i(t) \le a_{ii}^2\left[(U_i+\epsilon)+\beta_i\right](\xi_{im}+\tau_{ii}-t) \quad \text{for} \quad \xi_{im} \le t \le t_{im}.$$

Combining this and (2.16), we have

$$\dot{x}_i(t) \le a_{ii}\left[(U_i+\epsilon)+\beta_i\right]\min\{1, a_{ii}(\xi_{im}+\tau_{ii}-t)\}, \qquad \xi_{im}\le t\le t_{im}. \tag{2.17}$$

We consider the following two cases.

*Case* 1. $t_{im}-\xi_{im}\le 2\tau_{ii}/3$. In this case, by (2.17) we have

$$\begin{aligned}
x_i(t_{im})-x_i(\xi_{im}) &\le a_{ii}^2\left[(U_i+\epsilon)+\beta_i\right]\int_{\xi_{im}}^{t_{im}}(\xi_{im}+\tau_{ii}-t)dt\\
&= a_{ii}^2\left[(U_i+\epsilon)+\beta_i\right]\left[\tau_{ii}(t_{im}-\xi_{im})-\frac{1}{2}(t_{im}-\xi_{im})^2\right]\\
&\le \left[(U_i+\epsilon)+\beta_i\right]\left[\frac{2}{3}(a_{ii}\tau_{ii})^2-\frac{2}{9}(a_{ii}\tau_{ii})^2\right]\\
&= \frac{4}{9}(a_{ii}\tau_{ii})^2\left[(U_i+\epsilon)+\beta_i\right]\\
&\le \frac{1}{9}a_{ii}\tau_{ii}(3+2a_{ii}\tau_{ii})\left[(U_i+\epsilon)+\beta_i\right]\\
&= \frac{1}{9}a_{ii}\tau_{ii}(3+2a_{ii}\tau_{ii})\left[(U_i-\epsilon)+\beta_i+2\epsilon\right]
\end{aligned}$$

by the 3/2 condition (1.5).

*Case* 2. $t_{im}-\xi_{im}>2\tau_{ii}/3$. In this case, let $t_{im}-\eta_{im}=2\tau_{ii}/3$. Then $\eta_{im}\in(\xi_{im},t_{im}]$. By (2.17), we have

$$\begin{aligned}
&x_i(t_{im})-x_i(\xi_{im})\\
&\le \left[(U_i+\epsilon)+\beta_i\right]\left[a_{ii}(\eta_{im}-\xi_{im})+a_{ii}^2\int_{\eta_{im}}^{t_{im}}(\xi_{im}+\tau_{ii}-t)dt\right]\\
&= \left[(U_i+\epsilon)+\beta_i\right]\left[a_{ii}(\eta_{im}-\xi_{im})(1-a_{ii}(t_{im}-\eta_{im}))+a_{ii}^2\int_{\eta_{im}}^{t_{im}}(\eta_{im}+\tau_{ii}-t)dt\right]\\
&= \left[(U_i+\epsilon)+\beta_i\right]\left[a_{ii}(\eta_{im}-\xi_{im})\left(1-\frac{2}{3}a_{ii}\tau_{ii}\right)+a_{ii}^2\tau_{ii}(t_{im}-\eta_{im})-\frac{1}{2}a_{ii}^2(t_{im}-\eta_{im})^2\right]\\
&\le \left[(U_i+\epsilon)+\beta_i\right]\left[\frac{1}{3}a_{ii}\tau_{ii}+\frac{2}{9}(a_{ii}\tau_{ii})^2\right]\\
&= \frac{1}{9}a_{ii}\tau_{ii}(3+2a_{ii}\tau_{ii})\left[(U_i+\epsilon)+\beta_i\right]\\
&= \frac{1}{9}a_{ii}\tau_{ii}(3+2a_{ii}\tau_{ii})\left[(U_i-\epsilon)+\beta_i+2\epsilon\right],
\end{aligned}$$

since $\eta_{im}-\xi_{im}\le\frac{\tau_{ii}}{3}$.

Combining Cases 1 and 2 with (2.12), we have

$$\begin{aligned}
&a_{ii}x_i(t_{im})\\
&\le \frac{1+\frac{1}{9}a_{ii}\tau_{ii}(3+2a_{ii}\tau_{ii})}{1-\frac{1}{9}a_{ii}\tau_{ii}(3+2a_{ii}\tau_{ii})}\sum_{j\ne i}|a_{ij}|(U_i+\epsilon)+\frac{2\epsilon a_{ii}\tau_{ii}(3+2a_{ii}\tau_{ii})}{9-a_{ii}\tau_{ii}(3+2a_{ii}\tau_{ii})}, \qquad i=1,2,\ldots,k.
\end{aligned}$$

This shows (2.15) is true. Letting $m\to\infty$ and $\epsilon\to 0$ in (2.15), we obtain

$$a_{ii}U_i+\sum_{j\ne i}\tilde{a}_{ij}U_j\le 0 \quad\text{for}\quad i=1,2,\ldots,k. \tag{2.18}$$

On the other hand, for each $i = k+1, \ldots, n$, let $\{s_{im}\}_{m=1}^{\infty} \uparrow \infty$ be such that $\lim_{m\to\infty} x_i(s_{im}) = U_i$. By (2.11), we have $\lim_{m\to\infty} \dot{x}_i(s_{im}+\tau_{ii}) = 0$. Using (1.1), we have

$$\begin{aligned} 0 &= \dot{x}_i(s_{im}+\tau_{ii}) + a_{ii}x_i(s_{im}) + \sum_{j\neq i} a_{ij}x_j(s_{im}+\tau_{ii}-\tau_{ij}) \\ &\geq \dot{x}_i(s_{im}+\tau_{ii}) + a_{ii}x_i(s_{im}) + \sum_{j\neq i} \tilde{a}_{ij}|x_j(s_{im}+\tau_{ii}-\tau_{ij})|, \end{aligned}$$

since $\tilde{a}_{ij} \leq -|a_{ij}| \leq 0$. Letting $m \to \infty$, we obtain

$$a_{ii}U_i + \sum_{j\neq i} \tilde{a}_{ij}U_j \leq 0 \quad \text{for} \quad i = k+1, \ldots, n. \tag{2.19}$$

By (2.17) and (2.18) and using the fact that $\tilde{A}$ is a nonsingular $M$-matrix (so that $\tilde{A}^{-1}$ is a positive matrix), we have $U_1 = U_2 = \cdots = U_n = 0$. The proof is now complete.

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