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Invariance entropy for uncertain control systems[☆]

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ABSTRACT

We introduce a notion of invariance entropy for *uncertain* control systems. This entropy extends the invariance entropy for *deterministic* control systems introduced by Colonius and Kawan (2009). We show that for *uncertain* control systems, the invariance feedback entropy, proposed by Tomar, Rungger, and Zamani (2020), is bounded from below by our invariance entropy. We generalize the formula for the calculation of entropy of invariant partitions obtained by Tomar, Kawan, and Zamani (2022) to quasi-invariant-partitions. Under some reasonable assumptions, we obtain two explicit formulas of invariance entropy for *uncertain* control systems and invariance feedback entropy for finite controlled invariant sets.

1. Introduction

Entropy for a dynamical system is an intrinsic quantity that measures the complexity of the system. There are two popular dynamical entropies. One is measure-theoretic entropy, which was introduced by Kolmogorov [1], and improved by his student Sinai [2] who practically brought it to the contemporary form. The other is topological entropy, which was proposed via open covers by Adler et al. [3] and was redefined by Dinaburg [4] and Bowen [5] independently in the language of metric spaces. The classical variational principle states that topological entropy is equal to the supremum of measure-theoretic entropy over all invariant measures. We refer the reader to Refs. [6–8] for more details about the history of dynamical entropies.

Invariance (or stabilization) is another important notion that describes a widely needed property for control systems. For control systems subject to information constraints, an interesting question involving invariance is how much information practically needs to be communicated between the coder and controller in order to make a given set invariant under information constraint(s). Early work on this topics are Delchamps [9] and Wong and Brockett [10], in which they respectively investigated quantized feedbacks and the influence of restricted communication channels for stabilization. Topological feedback entropy for *deterministic* control systems was first introduced by Nair et al. [11] via invariant open covers to measure the smallest data rate between an encoder and a controller under which it is possible to render a compact subset of the state space invariant. Then Colonius and Kawan [12] introduced the notion of invariance entropy for *deterministic* control systems via spanning sets to describe the exponential growth rate of the minimal number of different control functions sufficient to render a compact subset of the state space invariant. It was shown by Colonius et al. [13] that invariance entropy is an alternative characterization of topological feedback entropy up to technical assumptions.

Analogously, people have been making effort to consider variants of invariance entropy from a dynamical point of view. These variants include, among others, invariance pressure [14–19], measure-theoretic versions of invariance entropy [20–23], dimension types of invariance entropy [24], complexity of invariance entropy [25–27], and stabilization entropy [28] for *stochastic* systems. We refer the reader to the monograph written by Kawan [29] for more details about invariance entropy of *deterministic* control systems.

Since uncertain control systems are described by difference inclusions of the form: $\xi(n+1) \in F(\xi(n), \nu(n))$, where $\xi(n) \in X$(state alphabet) is the state signal and $\nu(n) \in U$(input alphabet) is the input signal, the current theories of invariance entropy and topological feedback entropy for *deterministic* control systems [11,12,29] are unable to explain the minimal data rate needed to make subsets invariant for *uncertain* control systems (see Example 1 in [30] or in [31]). Rungger and Zamani [30] introduced invariance feedback entropy (IFE) to quantify the state information required by any controller to render a subset of the state space invariant. Later, Tomar et al. [31] and Tomar and Zamani [32] further investigated the properties of IFE. Gao

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et al. [33] presented an algorithm for computing invariant covers for linear uncertain control systems. Recently, Tomar et al. [34] provided algorithms for the numerical computation of invariance entropy for *deterministic* control systems and IFE for *uncertain* control systems, respectively. Particularly, they obtained a beautiful formula for the calculation of the entropy with respect to an invariant partition, which states that this entropy is equivalent to the maximum mean cycle weight (MMCW) of the weighted graph associated with the partition. The algorithms allow us to compute upper bounds for invariance entropy of *deterministic* control systems and IFE of *uncertain* control systems.

In this paper, we focus on *uncertain* control systems (UCS). In Section 2, we first introduce a new notion of invariance entropy for UCS, which is a natural generalization of invariance entropy for *deterministic* control systems. Roughly speaking, the invariance entropy for UCS measures the exponential growth rate of numbers of "branches" of "trees". Such trees are formed by control functions that are necessary to make the target set invariant. Different from *deterministic* control systems, in which every state can be made invariant by only one control function, for *uncertain* control systems, we may need more than one control function to make a state invariant. So we need to modify the notion of spanning set to define the invariance entropy. Then we present some basic properties of invariance entropy for UCS and show that this invariance entropy is less than or equal to IFE (see Theorem 2.8). In Section 3, we derive some formulas for the calculation of our invariance entropy and IFE. We show that the invariance entropy for a controlled invariant set is equal to the logarithm of the spectral radius of its admissible matrix under some technical assumption (see Theorem 3.11). We also extend the formula for the calculation of entropy of invariant partitions, obtained by Tomar et al. [34], to quasi-invariant-partitions. That is, the entropy for a quasi-invariant-partition is equal to the maximal mean weight of this partition (see Theorem 3.3), and show that if the spectral radius of the adjacency matrix of this partition is 1, then the entropy for this partition is equal to the logarithm of the spectral radius of its weighted adjacency matrix (see Theorem 3.7). Finally, we show that for a finite controlled invariant set, if it has an atom partition, then the invariance feedback entropy of this set is equal to the entropy of this atom partition (see Theorem 3.14).

2. Invariance entropy

This section consists of two subsections. In the first subsection we introduce a notion of invariance entropy for UCS and present its basic properties. Then in the second subsection we discuss its relation to invariance feedback entropy and show that invariance entropy is a tight lower bound for invariance feedback entropy.

2.1. Invariance entropy

We adopt some terminology and notions from [12]. By $f : A \rightrightarrows B$ we denote a *set-valued map* from A into B, whereas $f : A \to B$ denotes an ordinary map. A set-valued map f is called *strict* if $f(a) \neq \emptyset$ for every $a \in A$. The composition of two maps $f : A \rightrightarrows B$ and $g : C \rightrightarrows A$ is defined as $(f \circ g)(x) = f(g(x))$. We call a triple $\Sigma := (X, U, F)$ a *system* where the state set X and the control set U are nonempty and the set-valued map $F : X \times U \rightrightarrows X$ is strict. A subset Q of X is called *controlled invariant* with respect to a system $\Sigma = (X, U, F)$, if for every $x \in Q$ there exists $u \in U$ such that $F(x, u) \subset Q$. Fixing $u \in U$ and $Q \subset X$, let $Q_u = \{x \in Q | F(x, u) \subset Q\}$.

Given $n \in \mathbb{N} = \{1, 2, \ldots\}$ and a nonempty set A, let $A^n := \{s = s_0 s_1 \cdots s_{n-1} : s_i \in A, i = 0, 1, \ldots, n-1\}$. Given $i \in \mathbb{N}$ and $s \in A^n$, we denote by $s|_{[0,i]} := s_{[0,i]} := s_0 s_1 \cdots s_i$ the sequence which occurs in s between coordinates 0 and i, and write $s(i) := s_i$.

A subset $S \subset U^n$ is said to be *an admissible family of length* n for Q if

(a). $\omega'_0 = \omega''_0$ for any $\omega', \omega'' \in S$ with $\omega' = (\omega'_0, \ldots, \omega'_{n-1})$ and $\omega'' = (\omega''_0, \ldots, \omega''_{n-1})$;

(b). there exists $x \in Q$ such that for any $\omega \in S$,

$$F(I^i_\omega(x), \omega_i) \subset \bigcup_{\substack{\omega' \in S,\\ \omega'_{[0,i]} = \omega_{[0,i]}}} Q_{\omega'_{i+1}},$$

$$I^{i+1}_\omega(x) := F(I^i_\omega(x), \omega_i) \cap Q_{\omega_{i+1}} \neq \emptyset, \forall\, i = 0, 1, \ldots, n-2,$$

$$I^n_\omega(x) := F(I^{n-1}_\omega(x), \omega_{n-1}) \subset Q,$$

where $I^0_\omega(x) := x$.

Let

$$AF^n(Q) := \{S \subset U^n : S \text{ is an admissible family of length } n \text{ for } Q\}$$

and

$$AF(Q) := \bigcup_{n=1}^{\infty} AF^n(Q).$$

Given $S \in AF(Q)$, we denote by Q_S the set of points that satisfy condition (b).

Let $K \subset Q$ be a nonempty set. A set $\mathscr{S} \subset U^n$ is called an (n, K, Q)-*spanning set* of (K, Q) if

$$K \subset \bigcup_{S \subset \mathscr{S} \text{ and } S \in AF^n(Q)} Q_S.$$

By $r_{inv}(n, K, Q)$ we denote the minimal number of elements in such a spanning set, i.e.,

$$r_{inv}(n, K, Q) := \inf\{\sharp\mathscr{S} : \mathscr{S} \text{ is an } (n, K, Q)\text{-spanning set of } (K, Q)\},$$

where $\sharp\mathscr{S}$ denotes the cardinality of $\mathscr{S}$. For convenience, we write $r_{inv}(n, Q)$ instead of $r_{inv}(n, Q, Q)$.

Definition 2.1. Given a pair (K, Q), we define the *invariance entropy* of (K, Q) by

$$h_{inv}(K, Q) := h_{inv}(K, Q; \Sigma) := \limsup_{n\to\infty} \frac{\log r_{inv}(n, K, Q)}{n},$$

where log signifies the logarithm base 2.

Remark 2.2 (*1*)**.** When F is a single-valued map, the definition of invariance entropy coincides with that of invariance entropy for *deterministic* control systems (see [29, Definition 2.2]).

(2) It is well-known that topological feedback entropy [11] and invariance entropy [12,13] for *deterministic* control systems both characterize the minimal date rate for the control task of rendering a compact subset of the state space invariant. Invariance feedback entropy [31] characterizes the smallest data rate for making a subset invariant for *uncertain* control systems. The notion of invariance entropy for *deterministic* control systems motivates us to introduce invariance entropy for *uncertain* control systems. Our invariance entropy for *uncertain* control systems and invariance entropy for *deterministic* control systems respectively measure how fast the number of open-loop control functions grows which are needed to make a subset invariant for *uncertain* and *deterministic* control systems. Invariance feedback entropy is bounded below by our invariance entropy, and they coincide with each other under some special cases. This is one of the results from the paper and is proven below; see Theorem 2.8 and Example 2.10.

In the rest of this subsection, we list some properties of invariance entropy for *uncertain* control systems, including finiteness, time discretization, finite stability, and invariance under conjugacy. Since the proofs are analogous to those for *deterministic* control systems (see [29]), we omit them here.

Proposition 2.3. *Let $\Sigma = (X, U, F)$ be a system and $Q \subset X$ be a controlled invariant set. Then the following assertions hold:*

1. *The number $r_{inv}(n, Q)$ is either finite for all $n \in \mathbb{N}$ or for none.*
2. *The function $n \mapsto \log r_{inv}(n, Q)$, $\mathbb{N} \to [0, +\infty]$, is subadditive and thus*

$$h_{inv}(Q) = \lim_{n\to\infty} \frac{1}{n} \log r_{inv}(n, Q).$$

Remark 2.4. We see from Proposition 2.3 that $r_{inv}(n, Q)$ is finite for some n if and only if $r_{inv}(n, Q)$ is finite for all n if and only if $h_{inv}(Q)$ is finite.

Proposition 2.5 (*Time Discretization*)**.** *Let $\Sigma = (X, U, F)$ be a system and $Q \subset X$ be a controlled invariant set. If $K \subset Q$ and $m \in \mathbb{N}$, then*

$$h_{inv}(K, Q) = \limsup_{n\to\infty} \frac{1}{nm} \log r_{inv}(nm, K, Q).$$

Proposition 2.6 (*Subsets Rule or Finite Stability*)**.** *Let $\Sigma = (X, U, F)$ be a system and $Q \subset X$ be a controlled invariant set. If $K \subset Q$ and $K = \cup_{i=1}^{m} K_i$, then $h_{inv}(K, Q) = \max_{i=1,\ldots,m} h_{inv}(K_i, Q)$.*

Consider two systems $\Sigma_i = (X_i, U_i, F_i)$, $i = 1, 2$. Let $\pi : X_1 \to X_2$ be a continuous map and $r : U_1 \to U_2$ a map. We say (π, r) is a *semi-conjugacy from Σ_1 to Σ_2* if

$$F_2(\pi(x), r(u)) \subset \pi(F_1(x, u)), \ \ \forall\, x \in X_1,\ \ u \in U_1.$$

Proposition 2.7. *Let $\Sigma_1 = (X_1, U_1, F_1)$ and $\Sigma_2 = (X_2, U_2, F_2)$ be two systems, $Q \subset X_1$ be controlled invariant, and (π, r) a semi-conjugacy from Σ_1 to Σ_2. Then for any $K \subset Q$,*

$$h_{inv}(K, Q; \Sigma_1) \geq h_{inv}(\pi(K), \pi(Q); \Sigma_2).$$

2.2. Invariance feedback entropy

Let us recall the concept of invariance feedback entropy proposed by Tomar et al. [31]. Assume that $\Sigma = (X, U, F)$ is a system and $Q \subset X$ is controlled invariant. A pair $(\mathcal{A}, G)$ is called an *invariant cover* of Q if $\mathcal{A}$ is a finite cover of Q and G is a map $G : \mathcal{A} \to U$ such that for every $A \in \mathcal{A}$, $F(A, G(A)) \subset Q$.

Suppose $(\mathcal{A}, G)$ is an invariant cover of Q. Let $n \in \mathbb{N}$ and $\mathcal{S} \subset \mathcal{A}^n$ be a set of sequences in $\mathcal{A}$. For $\alpha \in \mathcal{S}$ and $t \in [0, n-1)$ we define

$$P(\alpha|_{[0,t]}) := \{A \in \mathcal{A} | \exists \hat{\alpha} \in \mathcal{S} \text{ s.t. } \hat{\alpha}|_{[0,t]} = \alpha|_{[0,t]} \text{ and } A = \hat{\alpha}_{t+1}\}.$$

This means that the set $P(\alpha|_{[0,t]})$ contains the cover elements A so that the sequence $\alpha|_{[0,t]}A$ can be extended to a sequence in $\mathcal{S}$. If $t = n-1$ then $\alpha|_{[0,n-1]} = \alpha$ and define

$$P(\alpha) := \left\{A \in \mathcal{A} | \exists \hat{\alpha} \in \mathcal{S} \text{ s.t. } A = \hat{\alpha}_0\right\},$$

which is actually independent of $\alpha \in \mathcal{S}$ and corresponds to the "initial" cover elements A in $\mathcal{S}$, i.e., there exists $\hat{\alpha} \in \mathcal{S}$ with $A = \hat{\alpha}_0$. A set $\mathcal{S} \subset \mathcal{A}^n$ is called *(n, Q)-spanning* in $(\mathcal{A}, G)$ if

(1). the set $P(\alpha)$ with $\alpha \in \mathcal{S}$ covers Q;
(2). for every $\alpha \in \mathcal{S}$ and $t \in [0, n-1)$, we have

$$F(\alpha(t), G(\alpha(t))) \subseteq \bigcup_{A' \in P(\alpha|_{[0,t]})} A'.$$

The *expansion number* $N(\mathcal{S})$ associated with $\mathcal{S}$ is defined by

$$N(\mathcal{S}) := \max_{\alpha\in\mathcal{S}} \prod_{t=0}^{n-1} \sharp P\left(\alpha|_{[0,t]}\right).$$

Let

$$r_{inv}(n, Q, \mathcal{A}, G) := \min\{N(\mathcal{S}) | \mathcal{S} \text{ is } (n, Q)\text{-spanning in} (\mathcal{A}, Q)\}.$$

Since $\log r_{inv}(\cdot, Q, \mathcal{A}, G)$ is subadditive (see Lemma 1 in [31]), the following limit exists

$$h_{inv}(\mathcal{A}, G) := \lim_{n\to\infty} \frac{1}{n} \log r_{inv}(n, Q, \mathcal{A}, G),$$

and is called *entropy* of $(\mathcal{A}, G)$. The *invariance feedback entropy* of Q is defined as

$$h_{inv}^{fb}(Q) := \inf_{(\mathcal{A},G)} h_{inv}(\mathcal{A}, G),$$

where the infimum is taken over all invariant covers of Q.

The following theorem states that invariance entropy is bounded above by invariance feedback entropy.

Theorem 2.8. *If $\Sigma = (X, U, F)$ is a system and $Q \subset X$ is a controlled invariant set, then*

$$h_{inv}(Q) \leq h_{inv}^{fb}(Q).$$

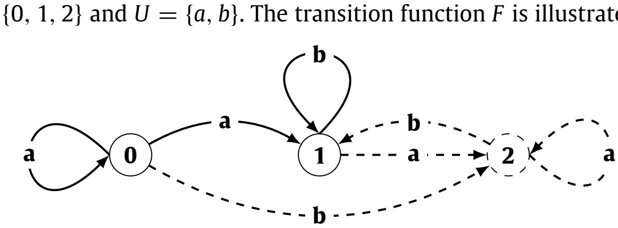
Proof. Suppose that $(\mathcal{A}, G)$ is an invariant cover of Q, $n \in \mathbb{N}$, and $\mathcal{S} \subset \mathcal{A}^n$ is (n, Q)-spanning in $(\mathcal{A}, Q)$. Let $\mathscr{S} = \{G(\alpha) | \alpha \in \mathcal{S}\}$, where $G(\alpha) := G(\alpha_0) \cdots G(\alpha_{n-1})$. It is obvious that

$$Q \subset \bigcup_{S\subset\mathscr{S} \text{ and } S\in AF^n(Q)} Q_S.$$

Hence, $\mathscr{S}$ is an (n, Q)-spanning set of Q and $\sharp\mathscr{S} \leq \sharp\mathcal{S}$. It follows that $r_{inv}(n, Q) \leq \sharp\mathcal{S}$. Applying Lemma 2 in [31], we have $r_{inv}(n, Q) \leq N(\mathcal{S})$, which implies that $r_{inv}(n, Q) \leq r_{inv}(n, Q, \mathcal{A}, G)$. Thus we obtain the desired inequality. □

The following two examples illustrate that both $h_{inv}(Q) < h_{inv}^{fb}(Q)$ and $h_{inv}(Q) = h_{inv}^{fb}(Q)$ can be possible.

Example 2.9. Let $\Sigma = (X, U, F)$ be a system, where $X = \{0, 1, 2\}$ and $U = \{a, b\}$. The transition function F is illustrated by



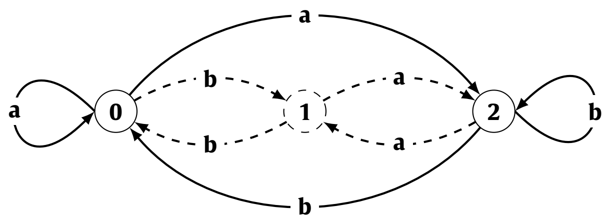
The set of interest is $Q := \{0, 1\}$. Then $h_{inv}(Q) = 0$ and $h_{inv}^{fb}(Q) = 1$.

Proof. Let $\mathscr{S} = \{a^i b^{n-i} | i = 0, 1, \ldots, n\}$. It is not difficult to check that $\mathscr{S}$ is an (n, Q)-spanning set of Q. So $h_{inv}(Q) = 0$. Put $\mathcal{A} = \{\{0\}, \{1\}\}$, and define $G : \mathcal{A} \to U$ by $G(\{0\}) = a$ and $G(\{1\}) = b$. We shall show that $h_{inv}^{fb}(Q) = 1$. Suppose that $\mathcal{S} \subset \mathcal{A}^n$ is (n, Q)-spanning in $(\mathcal{A}, G)$. Then $\alpha = \underbrace{\{0\}\ldots\{0\}}_{n} \in \mathcal{S}$. This yields that $N(\mathcal{S}) = 2^n$ and

$$r_{inv}(n, Q, \mathcal{A}, G) = 2^n.$$

It then follows that $h(\mathcal{A}, G) = 1$. Since $(\mathcal{A}, G)$ is the only invariant cover of Q, we obtain $h_{inv}^{fb}(Q) = 1$. □

Example 2.10. Let $\Sigma = (X, U, F)$ be a system, where $X = \{0, 1, 2\}$ and $U = \{a, b\}$. The transition function F is illustrated by



The set of interest is $Q := \{0, 2\}$. Then $h_{inv}(Q) = h_{inv}^{fb}(Q) = 1$.

Proof. The fact that $h_{inv}^{fb}(Q) = 1$ is from Example 1 in [31]. We shall show that $h_{inv}(Q) = 1$. Suppose that $\mathscr{S} \subset U^n$ is an (n, Q)-spanning set. Since $Q_a = \{0\}$, $Q_b = \{2\}$, $F(0, a) = F(2, b) = Q = \{0, 2\}$, we have $U^n \subset \mathscr{S}$. It follows that $\sharp\mathscr{S} = 2^n$. Hence $r_{inv}(n, Q) = 2^n$ and $h_{inv}(Q) = 1$. $\square$

3. Calculations for invariance entropy and IFE

This section deals with the calculations for invariance entropy and invariance feedback entropy.

3.1. Calculation for entropy of quasi-invariant-partitions

Let $\Sigma = (X, U, F)$ be a system, $Q \subset X$ a controlled invariant set, and $(\mathcal{A}, G)$ an invariant cover of Q. Before going further, we borrow some notations from [31] and introduce some new concepts. For every $A \in \mathcal{A}$, let $D_{\mathcal{A}}(A) := \{A' \in \mathcal{A} : F(A, G(A)) \cap A' \neq \emptyset\}$ and $w_{\mathcal{A}}(A) := \log \sharp D_{\mathcal{A}}(A)$. When there is no ambiguity, we write $D(A)$ and $w(A)$ instead of $D_{\mathcal{A}}(A)$ and $w_{\mathcal{A}}(A)$, respectively. Given $m \in \mathbb{N}$, a sequence $(A_i)_{i=0}^m$ is called *admissible* for $(\mathcal{A}, G)$ if $F(A_i, G(A_i)) \cap A_{i+1} \neq \emptyset$ for every $0 \le i < m$. Set

$$W_m(\mathcal{A}, G) := \{(A_i)_{i=0}^{m-1} | (A_i)_{i=0}^{m-1} \text{ is admissible for } (\mathcal{A}, G)\}.$$

A sequence $c = (A_i)_{i=0}^{k-1}$ is called *an irreducible sequence of period* k for $(\mathcal{A}, G)$ if c^∞ is admissible for $(\mathcal{A}, G)$ and $A_i \neq A_j$ for distinct i, j. (By "c^∞" we mean $ccc\cdots$.) The *period* of c is defined as k (denoted by $l(c)$) and the *mean weight* for c is defined as

$$\overline{w}(c) := \frac{1}{k}\sum_{i=0}^{k-1} w(A_i).$$

The *maximum mean weight* $\overline{w}^*(\mathcal{A}, G)$ is defined by $\overline{w}^*(\mathcal{A}, G) := \max_c \overline{w}(c)$, where the maximum is taken over all irreducible periodic sequences for $(\mathcal{A}, G)$.

The *adjacency matrix* $M_{\mathcal{A},G} = (M_{AB})$ of $(\mathcal{A}, G)$ is defined by

$$M_{AB} := \begin{cases} 1, & F(A, G(A)) \cap B \neq \emptyset; \\ 0, & otherwise. \end{cases}$$

We define the *weighted adjacency matrix* $W_{\mathcal{A},G} = (W_{AB})$ with $A, B \in \mathcal{A}$ of $(\mathcal{A}, G)$ as

$$W_{AB} := \begin{cases} \sharp D(A), & F(A, G(A)) \cap B \neq \emptyset; \\ 0, & otherwise. \end{cases}$$

Recall that the l_∞-norm for a $n \times n$ matrix M is defined by $\|M\|_\infty := \max_{1\le i,j\le n} |a_{ij}|$.

An invariant cover $(\mathcal{A}, G)$ of Q is said to be an *invariant partition* if $\mathcal{A}$ is a partition of Q. A weaker version of invariant partition is as follows.

Definition 3.1. An invariant cover $(\mathcal{A}, G)$ of Q is called a *quasi-invariant-partition* of Q if the following two conditions hold.

(1). For any $A \in \mathcal{A}$,

$$A \setminus \bigcup_{B\in\mathcal{A}, B\neq A} B \neq \emptyset. \tag{3.1}$$

(2). For any $A \in \mathcal{A}$ and $B \in D(A)$,

$$F(A, G(A)) \bigcap (B \setminus \bigcup_{C\in D(A),\ C\neq B} C) \neq \emptyset. \tag{3.2}$$

Tomar et al. in [34] derived a formula for computing entropy of an invariant partition. Here we generalize this result to quasi-invariant-partitions.

Theorem 3.2. *Suppose that $\Sigma = (X, U, F)$ is a system, $Q \subset X$ is controlled invariant, and $(\mathcal{A}, G)$ is a quasi-invariant-partition of Q. Then*

$$h_{inv}(\mathcal{A}, G) = \lim_{m\to\infty} \frac{1}{m} \max_{\alpha\in W_m(\mathcal{A},G)} \sum_{i=0}^{m-2} w(\alpha(i)).$$

Proof.

Claim 1. *$W_m(\mathcal{A}, G)$ is an (m, Q)-spanning set of $(\mathcal{A}, G)$ for $m \ge 2$.*

For every $\alpha \in W_m(\mathcal{A}, G)$, we have $P(\alpha) = \{\alpha(0) : \alpha \in W_m(\mathcal{A}, G)\} = \mathcal{A}$, $P(\alpha|_{[0,t]}) = D(\alpha(t))$, and

$$F(\alpha(t), G(\alpha(t))) \subset \bigcup_{A'\in D(\alpha(t))} A'.$$

Hence Claim 1 holds.

Claim 2. *For any (m, Q)-spanning set $\mathcal{S}$ in $(\mathcal{A}, G)$, it holds that $W_m(\mathcal{A}, G) \subset \mathcal{S}$.*

We apply the inductive argument to show this claim. For every (m, Q)-spanning set $\mathcal{S}$ and every $\alpha \in \mathcal{S}$ we have $P(\alpha) = \mathcal{A}$ by the condition (3.1). Suppose that $\mathcal{S}$ is a $(2, Q)$-spanning set of $(\mathcal{A}, G)$. So the claim holds for $m = 2$. Assume that the claim holds for $2 \le i \le m - 1$. Let $\mathcal{S}$ be an (m, Q)-spanning set of $(\mathcal{A}, Q)$. Then $\mathcal{S}' = \{\alpha|_{[0,m-2]} : \alpha \in \mathcal{S}\}$ is an $(m - 1, Q)$-spanning set and $W_{m-1}(\mathcal{A}, G) \subset \mathcal{S}'$. Hence $W_m(\mathcal{A}, G) \subset \mathcal{S}$ by the condition (3.2). So the claim holds for every $m \ge 2$.

Combining Claim 1 with Claim 2, we have

$$r_{inv}(m, Q, \mathcal{A}, G) = N(W_m(\mathcal{A}, G)). \tag{3.3}$$

It follows that

$$\begin{aligned} h_{inv}(\mathcal{A}, G) &= \lim_{m\to\infty} \frac{1}{m} \log N(W_m(\mathcal{A}, G)) \\ &= \lim_{m\to\infty} \frac{1}{m} \log\big(\sharp\mathcal{A} \max_{\alpha\in W_m(\mathcal{A},G)} \prod_{i=0}^{m-2} \sharp D(\alpha(i))\big) \\ &= \lim_{m\to\infty} \frac{1}{m} \max_{\alpha\in W_m(\mathcal{A},G)} \sum_{i=0}^{m-2} w(\alpha(i)). \quad \square \end{aligned}$$

Theorem 3.3. *Let $\Sigma = (X, U, F)$ be a system, $Q \subset X$ a controlled invariant set, and $(\mathcal{A}, G)$ a quasi-invariant-partition of Q. Then*

$$h_{inv}(\mathcal{A}, G) = \overline{w}^*(\mathcal{A}, G),$$

where $\overline{w}^*(\mathcal{A}, G)$ is the maximum mean weight.

Proof. We first show that $h_{inv}(\mathcal{A}, G) \ge \overline{w}^*(\mathcal{A}, G)$. Suppose that $c = (A_i)_{i=0}^{k-1}$ is an irreducible sequence of period k. For any $m \in \mathbb{N}$, let

$$\beta_{c,m} := \underbrace{A_0 \cdots A_{k-1} \cdots A_0 \cdots A_{k-1}}_{m} A_0.$$

Then $\beta_{c,m} \in W_{mk+1}(\mathcal{A}, G)$. Utilizing Theorem 3.2, we have

$$\begin{aligned} h_{inv}(\mathcal{A}, G) &= \lim_{m\to\infty} \frac{1}{mk+1} \max_{\alpha\in W_{mk+1}(\mathcal{A},G)} \sum_{i=0}^{mk-1} w(\alpha(i)) \\ &\geq \lim_{m\to\infty} \frac{1}{mk+1} \sum_{i=0}^{mk-1} w(\beta_{c,m}(i)) \\ &= \lim_{m\to\infty} \frac{m}{mk+1} \sum_{i=0}^{k-1} w(A_i) \\ &= \frac{1}{k} \sum_{i=0}^{k-1} w(A_i) = \overline{w}(c). \end{aligned}$$

The desired inequality immediately follows from the arbitrariness of c.

We now show the inverse inequality. For any $m \geq \sharp\mathcal{A}+3$ and $\alpha_1 \in W_m(\mathcal{A}, G)$, we have $\alpha_1(0)\alpha_1(1)\cdots\alpha_1(m-2) \in W_{m-1}(\mathcal{A}, G)$. Using the pigeonhole principle, we can pick an irreducible sequence $c_1 = (A_{1,i})_{i=0}^{k_1-1}$ of period k_1 in $(\mathcal{A}, G)$ and there exists $p_1 \in [0, \sharp\mathcal{A}]$ such that

$$\alpha_1(p_1)\alpha_1(p_1+1)\cdots\alpha_1(p_1+k_1) = A_{1,0}A_{1,1}\cdots A_{1,k_1-1}.$$

Thus

$$\begin{aligned} &w(\alpha_1(p_1)) + w(\alpha_1(p_1+1)) + \cdots w(\alpha_1(p_1+k_1-1)) \\ &\quad = k_1\overline{w}(c_1) \leq k_1\overline{w}^*(\mathcal{A}, G). \end{aligned}$$

Let

$$\alpha_2 = \alpha_1(0)\cdots\alpha_1(p_1-1)\alpha_1(p_1+k_1)\cdots\alpha_1(m-2).$$

Clearly, $\alpha_2 \in W_{m-k_1-1}(\mathcal{A}, G)$. Applying the pigeonhole principle repeatedly, we can find a sequence of irreducible sequences of period $\{c_j\}_{j=1}^q$ in $(\mathcal{A}, G)$, a sequence $\{\alpha_{j+1}\}_{j=1}^q$ with $\alpha_{j+1} \in W_{m-\sum_{i=1}^j k_i-1}(\mathcal{A}, G)$ and two sequence numbers $\{k_j\}_{j=1}^q$ with $l(c_j) = k_j$ and $\{p_j\}_{j=1}^q$ with $p_j \in [0, \sharp\mathcal{A}]$ such that

$$\alpha_j(p_j)\alpha_j(p_j+1)\cdots\alpha_j(p_j+k_j) = A_{j,0}A_{j,1}\cdots A_{j,k_j},$$

$$w(\alpha_j(p_j)) + w(\alpha_j(p_j+1)) + \cdots w(\alpha_j(p_j+k_j-1)) = k_j\overline{w}(c_j) \leq k_j\overline{w}^*(\mathcal{A}, G),$$

and $\alpha_{q+1} \in W_{m-\sum_{j=1}^q k_j-1}(\mathcal{A}, G)$, where $m - \sum_{j=1}^q k_j - 1 \in [0, \sharp\mathcal{A}]$. Write $L = m - \sum_{j=1}^q k_j - 1$. Then we have

$$\begin{aligned} \sum_{i=0}^{m-2} w(\alpha(i)) &= \sum_{j=1}^{q}\sum_{i=0}^{k_j-1} w(\alpha_j(p_j+i)) + \sum_{i=0}^{L-1} w(\alpha_{q+1}(i)) \\ &\leq \sum_{j=1}^{q} k_j\overline{w}^*(\mathcal{A}, G) + L\max_{A\in\mathcal{A}} w(A) \\ &= (m-1-L)\overline{w}^*(\mathcal{A}, G) + L\max_{A\in\mathcal{A}} w(A) \\ &\leq (m-1)\overline{w}^*(\mathcal{A}, G) + \sharp\mathcal{A}\max_{A\in\mathcal{A}} w(A). \end{aligned}$$

Therefore,

$$\begin{aligned} h_{inv}(\mathcal{A}, G) &= \lim_{m\to\infty} \frac{1}{m} \max_{\alpha\in W_m(\mathcal{A},G)} \sum_{i=0}^{m-2} w(\alpha(i)) \\ &\leq \lim_{m\to\infty} \frac{m-1}{m}\overline{w}^*(\mathcal{A}, G) + \lim_{m\to\infty} \frac{1}{m}\sharp\mathcal{A}\max_{A\in\mathcal{A}} w(A) \\ &= \overline{w}^*(\mathcal{A}, G). \end{aligned}$$

This completes the proof. □

Remark 3.4. (1) Passing $(\mathcal{A}, G)$ to an invariant partition of Q, we recover Theorem 1 in [34].

(2) The proof of [34, Theorem 1] is based on graph-theoretic constructions, and this proof is also valid for Theorem 3.2 and 3.3.

The connection between invariant partition and quasi-invariant partition can be characterized by the following corollary.

Corollary 3.5. *Let $\Sigma = (X, U, F)$ be a system and $Q \subset X$ be controlled invariant. For every quasi-invariant-partition $(\mathcal{A}, G)$ of Q, there exists an invariant partition $(\mathcal{A}', G')$ of Q such that $\sharp\mathcal{A} = \sharp\mathcal{A}'$ and $h_{inv}(\mathcal{A}', G') \leq h_{inv}(\mathcal{A}, G)$.*

Proof. Let $(\mathcal{A}, G)$ be a quasi-invariant partition with $\mathcal{A} = \{A_1, \dots, A_p\}$. Define a new cover $\mathcal{A} = \{A'_1, \dots, A'_p\}$ by $A'_1 = A_1$, $A'_j = A_j \setminus \cup_{i=1}^{j-1} A_i$, for any $2 \leq j \leq p$, and $G'(A'_j) := G(A_j)$, $j = 1, \dots, p$. Then $(\mathcal{A}', G')$ is an invariant partition of Q. Suppose $c' = (A'_i)_1^{q-1}$ is an irreducible periodic sequence in $(\mathcal{A}', G')$ so that $h_{inv}(\mathcal{A}', G') = \overline{w}(c')$. Since $A'_i \subset A_i$ for any $1 \leq i \leq p$, it follows that $c := (A_i)_1^{q-1}$ is an irreducible periodic sequence in $(\mathcal{A}, G)$ and $\overline{w}(c') \leq \overline{w}(c)$. It follows from Theorem 3.3 that $h_{inv}(\mathcal{A}, G) \geq \overline{w}(c) \geq h_{inv}(\mathcal{A}', G')$. □

In the following example, we construct a system that has a quasi-invariant-partition $(\mathcal{A}_1, G_1)$, where we find two invariant partitions $(\mathcal{A}_2, G_2)$ and $(\mathcal{A}_3, G_3)$ such that

$$h_{inv}(\mathcal{A}_2, G_2) = h_{inv}(\mathcal{A}_1, G_1) \text{ and } h_{inv}(\mathcal{A}_3, G_3) < h_{inv}(\mathcal{A}_1, G_1).$$

Example 3.6. Let $\Sigma = (X, U, F)$ be a system, where $X = \{0, 1, 2, 3\}$ and $U = \{a, b\}$. The transition function F is illustrated by Fig. 1.

The set of interest is $Q := \{0, 1, 2\}$. Let $A_{11} = \{0, 1\}$, $A_{12} = \{1, 2\}$, $A_{21} = \{2\}$, $A_{31} = \{0\}$, $A_{32} = \{1\}$, and $A_{33} = \{2\}$; and set $\mathcal{A}_1 = \{A_{11}, A_{12}\}$, $\mathcal{A}_2 = \{A_{11}, A_{21}\}$, and $\mathcal{A}_3 = \{A_{31}, A_{32}, A_{33}\}$. Define $G_1 : \mathcal{A}_1 \to U$ by $G_1(A_{11}) = a$ and $G_1(A_{12}) = b$; $G_2 : \mathcal{A}_2 \to U$ by $G_2(A_{11}) = a$ and $G_2(A_{21}) = b$; and $G_3 : \mathcal{A}_3 \to U$ by $G_3(A_{31}) = a$, $G_3(A_{32})$=a, and $G_3(A_{33}) = b$. Then

$$h_{inv}(\mathcal{A}_1, G_1) = h_{inv}(\mathcal{A}_2, G_2) = 1,\ \ h_{inv}(\mathcal{A}_3, G_3) = \frac{1}{2}.$$

Proof. It is easy to see that $(\mathcal{A}_1, G_1)$ is a quasi-invariant-partition, and $(\mathcal{A}_2, G_2)$ and $(\mathcal{A}_3, G_3)$ are invariant partitions. We now compute entropy for $(\mathcal{A}_1, G_1)$, $(\mathcal{A}_2, G_2)$, and $(\mathcal{A}_3, G_3)$. From Fig. 1, we see $A_{11} = \{0, 1\}$ is an irreducible sequence of period 1 for both $(\mathcal{A}_1, G_1)$ and $(\mathcal{A}_2, G_2)$ and $\overline{w}(A_{11}) = 1$. Since $h_{inv}(\mathcal{A}_1, G_1) \leq 1$ and $h_{inv}(\mathcal{A}_2, G_2) \leq 1$, it follows from Theorem 3.3 that $h_{inv}(\mathcal{A}_1, G_1) = 1$ and $h_{inv}(\mathcal{A}_2, G_2) = 1$. Fig. 1 also tells us that $(\mathcal{A}_3, G_3)$ only has irreducible sequences of period 2: $A_{31}A_{32}$, $A_{31}A_{33}$, $A_{32}A_{31}$, and $A_{33}A_{31}$. Applying Theorem 3.3, a direct computation shows that $h_{inv}(\mathcal{A}_3, G_3) = \frac{1}{2}$. □

Theorem 3.3 says that the entropy for a quasi-invariant-partition is equal to its maximum mean weight. Next we relate the entropy of a quasi-invariant-partition to its associated adjacency matrix and weighted adjacency matrix. In particular, we give an alternative calculation formula of the entropy of a quasi-invariant-partition when the spectral radius of the adjacency matrix of this quasi-invariant-partition is equal to 1.

Theorem 3.7. *Let $\Sigma = (X, U, F)$ be a system and $Q \subset X$. If $(\mathcal{A}, G)$ is a quasi-invariant-partition of Q, then*

$$\begin{aligned} \log\rho(W_{\mathcal{A},G}) - \log\rho(M_{\mathcal{A},G}) &\leq h_{inv}(\mathcal{A}, G) \\ &\leq \min\{\log\|W_{\mathcal{A},G}\|_\infty, \log\rho(W_{\mathcal{A},G})\}, \end{aligned}$$

where $\rho(W_{\mathcal{A},G})$ is the spectral radius of $W_{\mathcal{A},G}$ (the maximum of absolute values of its eigenvalues). Particularly, if $\rho(M_{\mathcal{A},G}) = 1$, then $h_{inv}(\mathcal{A}, G) = \log\rho(W_{\mathcal{A},G})$.

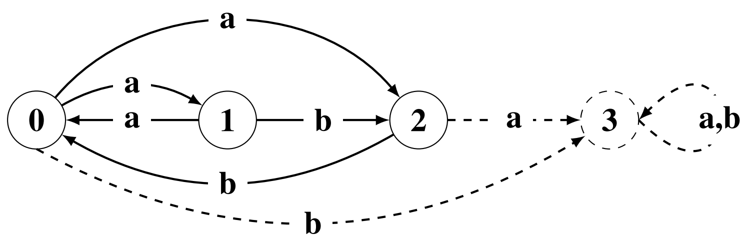


Fig. 1. The transition map of Example 3.6.

Proof. We first show the right hand side inequality. It is clear that $h_{inv}(\mathcal{A}, G) \leq \log \|W_{\mathcal{A},G}\|_\infty$. Since $(\mathcal{A}, G)$ is a quasi-invariant-partition, it follows from (3.3) in the proof of Theorem 3.2 that

$$r_{inv}(n, Q, \mathcal{A}, G) = \sharp\mathcal{A} \cdot \max_{\alpha \in W_n(\mathcal{A},G)} \prod_{i=0}^{n-2} \sharp D(\alpha_i).$$

For any $\alpha \in W_n(\mathcal{A}, G)$, we have

$$\prod_{t=0}^{n-2} \sharp D(\alpha_t) = W_{\alpha_0\alpha_1} \cdot W_{\alpha_1\alpha_2} \cdots W_{\alpha_{n-2}\alpha_{n-1}} \leq \|W^{n-1}_{\mathcal{A},G}\|_\infty.$$

This implies that

$$r_{inv}(n, Q, \mathcal{A}, G) \leq \sharp\mathcal{A} \cdot \|W^{n-1}_{\mathcal{A},G}\|_\infty.$$

Hence,

$$\begin{aligned} h_{inv}(\mathcal{A}, G) &= \lim_{n\to\infty} \frac{1}{n} \log r_{inv}(n, Q, \mathcal{A}, G) \\ &\leq \lim_{n\to\infty} \frac{1}{n} \log \sharp\mathcal{A} \cdot \|W^{n-1}_{\mathcal{A},G}\|_\infty \\ &= \lim_{n\to\infty} \frac{n-1}{n} \log \|W^{n-1}_{\mathcal{A},G}\|_\infty^{\frac{1}{n-1}}. \end{aligned}$$

Employing Theorem 5.7.10 in [35], we obtain $h_{inv}(\mathcal{A}, Q) \leq \log \rho(W_{\mathcal{A},G})$.

We now show the left hand side inequality. For any $\beta \in W_n(\mathcal{A}, G)$, we have

$$\prod_{t=0}^{n-2} \sharp D(\beta_i) = W_{\beta_0\beta_1} \cdot W_{\beta_1\beta_2} \cdots W_{\beta_{n-2}\beta_{n-1}}$$

and

$$M_{\beta_0\beta_1} \cdot M_{\beta_1\beta_2} \cdots M_{\beta_{n-2}\beta_{n-1}} = 1.$$

Then

$$\begin{aligned} W_{\beta_0\beta_{n-1}} &\leq M_{\beta_0\beta_{n-1}} \cdot \max_{\alpha \in W_n(\mathcal{A},G)} \prod_{i=0}^{n-2} \sharp D(\alpha_i) \\ &\leq \|M^{n-1}_{\mathcal{A},G}\|_\infty \cdot \max_{\alpha \in W_n(\mathcal{A},G)} \prod_{i=0}^{n-2} \sharp D(\alpha_i). \end{aligned}$$

It follows that

$$\|W^{n-1}_{\mathcal{A},G}\|_\infty \leq \|M^{n-1}_{\mathcal{A},G}\|_\infty \cdot \max_{\alpha \in W_n(\mathcal{A},G)} \prod_{i=0}^{n-2} \sharp D(\alpha_i).$$

Thus

$$\begin{aligned} \lim_{n\to\infty} \frac{1}{n} \log r_{inv}(n, Q, \mathcal{A}, G) &\geq \lim_{n\to\infty} \frac{1}{n} \log \sharp\mathcal{A} \cdot \frac{\|W^{n-1}_{\mathcal{A},G}\|_\infty}{\|M^{n-1}_{\mathcal{A},G}\|_\infty} \\ &= \log \rho(W_{\mathcal{A},G}) - \log \rho(M_{\mathcal{A},G}). \quad \square \end{aligned}$$

Remark 3.8. (1) Since the norm $\|\cdot\|_\infty$ is not spectrally dominant,[1] we take the minimum of $\log \|W_{\mathcal{A},G}\|_\infty$ and $\log \rho(W_{\mathcal{A},G})$ in the right hand side of the inequality in Theorem 3.7. See Example 3.15.

(2) If $(\mathcal{A}, G)$ is a quasi-invariant-partition of Q, then we have the following three assertions.

(i) For any $A \in \mathcal{A}$ there exists $B \in \mathcal{A}$ so that $M_{AB} = 1$, which implies $\|M^n_{\mathcal{A},G}\|_\infty \geq 1, \ \forall\, n \in \mathbb{N}$. So $\rho(M_{\mathcal{A},G}) \geq 1$ and $\log \rho(M_{\mathcal{A},G}) \geq 0$;

(ii) Since $W_{AB} \geq M_{AB}$, $W_{AB} = \sharp D(A) M_{AB}$, and $M_{AB} \in \{0, 1\}$ for any $A, B \in \mathcal{A}$, we have

$$\|W^n_{\mathcal{A},G}\|_\infty \geq \min_{A\in\mathcal{A}}\{(\sharp D(A))^{n-1}\} \cdot \|M^n_{\mathcal{A},G}\|_\infty, \ \ \forall\, n \in \mathbb{N}.$$

Thus

$$\log \rho(W_{\mathcal{A},G}) - \log \rho(M_{\mathcal{A},G}) \geq \min_{A\in\mathcal{A}}\{w(A)\}.$$

(iii) When $\rho(M_{\mathcal{A},G}) > 1$, we can apply Theorem 3.3 to compute the entropy of $(\mathcal{A}, G)$. Otherwise, when $\rho(M_{\mathcal{A},G}) = 1$, the entropy of $(\mathcal{A}, G)$ is $\log \rho(W_{\mathcal{A},G})$. See Example 3.15.

3.2. Calculation of invariance entropy and IFE for some control systems

Let $\Sigma = (X, U, F)$ be a system, $Q \subset X$, and $V \subset U$. We say that V is a *cover* of Q if $Q \subset \cup_{a\in V} Q_a$, where $Q_a = \{x \in Q \,|\, F(x, a) \subset Q\}$. The *admissible matrix* $M_{Q,V} = (M_{ab})_{a,b\in V}$ of Q associated with a cover V is defined by

$$M_{ab} := \begin{cases} 1, & \exists\, x \in Q_a \ s.t.\ F(x, a) \cap Q_b \neq \emptyset; \\ 0, & otherwise. \end{cases}$$

Recall that the l_1 norm of an $n \times n$ matrix M is $\|M\|_1 := \sum_{i,j=1}^n |a_{ij}|$.

Proposition 3.9. *Let $\Sigma = (X, U, F)$ be a system, $Q \subset X$ a controlled invariant set, and $V \subset U$ a cover of Q. Then $h_{inv}(Q) \leq \log \rho(M_{Q,V})$.*

Proof. Since V is a cover of Q, it follows that

$$\mathscr{S}_n = \{u_0u_1 \cdots u_{n-1} : M_{u_0u_1} \cdot M_{u_1u_2} \cdots M_{u_{n-2}u_{n-1}} = 1\}$$

is an (n, Q)-spanning set for every $n \geq 2$. Thus $r_{inv}(n, Q) \leq \sharp\mathscr{S}_n = \|M^{n-1}_{Q,V}\|_1$. This gives

$$\begin{aligned} h_{inv}(Q) &= \limsup_{n\to\infty} \frac{1}{n} \log r_{inv}(n, Q) \\ &\leq \limsup_{n\to\infty} \frac{1}{n} \log \|M^{n-1}_{Q,V}\|_1 \\ &= \limsup_{n\to\infty} \frac{n-1}{n} \log \|M^{n-1}_{Q,V}\|_1^{\frac{1}{n-1}}. \end{aligned}$$

By Gelfand formula [35, Corollary 5.6.14], we have the result. $\square$

[1] A matrix norm $|||\cdot|||$ is said to be *spectrally dominant* if $|||M||| \geq \rho(M)$ for every $n \times n$ matrix M; see [35, p. 373].

Corollary 3.10. *Let $\Sigma = (X, U, F)$ be a system and $Q \subset X$ be controlled invariant. Then*

$$h_{inv}(Q) \le \inf_{V \subset U \text{ covers } Q} \log \rho(M_{Q,V}).$$

Kawan in [29, p. 87] posed a question:“ Are there conditions which guarantee the existence of ‘generators’, that is, the invariance entropy of Q for *deterministic* control systems is equal to the entropy of an invariant cover?” We in [24] presented an affirmative answer for this question. The following theorem provides a sufficient condition that guarantees the existence of ‘generators’ for *uncertain* control systems.

Theorem 3.11. *Let $\Sigma = (X, U, F)$ be a system, $Q \subset X$ a controlled invariant set, and $V \subset U$ a finite cover of Q. If in addition*

(C.1) $Q_a \cap Q_b = \emptyset$ for distinct $a, b \in V$;
(C.2) there exists $K \subset Q_a$ such that $Q_b \subset F(K, a)$ for every $M_{ab} = 1$;
(C.3) $Q_c = \emptyset$ for every $c \in U \setminus V$.

Then $h_{inv}(Q) = \log \rho(M_{Q,V})$.

Proof. To simplify the notation, set $M = M_{Q,V}$. It is easy to see from the conditions (C.1) and (C.3) that the set $\mathscr{S}_2 = \{u_0u_1 : M_{u_0u_1} = 1\}$ is the only $(2, Q)$-spanning set of Q with minimal cardinality. Assume that

$$\mathscr{S}_n = \{u_0u_1 \cdots u_{n-1} : M_{u_0u_1} \cdot M_{u_1u_2} \cdots M_{u_{n-2}u_{n-1}} = 1\}$$

is the only (n, Q)-spanning set of Q with minimal cardinality. We will show that $\mathscr{S}_{n+1} = \{u_0u_1 \cdots u_n : M_{u_0u_1} \cdot M_{u_1u_2} \cdots M_{u_{n-1}u_n} = 1\}$ is the only $(n+1, Q)$-spanning set of Q with minimal cardinality. Let $\mathscr{S}$ be an $(n+1, Q)$-spanning set. If there exists $w_0w_1 \cdots w_n \in \mathscr{S}$ such that $M_{w_iw_{i+1}} = 0$ for some i (otherwise, $\mathscr{S} \subset \mathscr{S}_{n+1}$), then we can discard $w_0w_1 \cdots w_n$ and obtain a new $(n+1, Q)$-spanning set: $\mathscr{S}_1 = \mathscr{S} \setminus \{w_0w_1 \cdots w_n\}$. Repeating this procedure, we can obtain an $(n+1, Q)$-spanning set $\mathscr{S}^* \subset \mathscr{S} \cap \mathscr{S}_{n+1}$ and $\sharp\mathscr{S}^* \le \sharp\mathscr{S}$. Let

$$\mathscr{S}^*|_n = \{w \in U^n : \exists u \in \mathscr{S}^* \ s.t. \ w_i = u_i, i = 0, \dots, n-1\}$$

is an (n, Q)-spanning set. By assumption, we see that $\mathscr{S}_n = \mathscr{S}^*|_n$. For every $u \in \mathscr{S}_n$, let $B_u := \{b \in V | M_{u_{n-1}b} = 1\}$ and $\mathscr{S}_u := \{ub | b \in B_u\}$. Condition (C.2) tells us that $\mathscr{S}_u \subset \mathscr{S}^*$, and thus $\mathscr{S}_{n+1} \subset \mathscr{S}^*$. It immediately follows that $\mathscr{S}_{n+1}$ is the only $(n+1, Q)$-spanning set of Q with minimal cardinality and $r_{inv}(n, Q) = \sharp\mathscr{S}_n$ for $n \ge 2$. A standard induction on n then yields

$$\sharp\mathscr{S}_n = \sum_{u_0 \in V} \sum_{u_{n-1} \in V} \left(M^{n-1}\right)_{u_0,u_{n-1}}.$$

Hence $r_{inv}(n, Q) = \sharp\mathscr{S}_n = \|M^{n-1}\|_1$, which together with Gelfand formula, shows that $h_{inv}(Q) = \log \rho(M)$. □

Remark 3.12. (1) The formula for invariance entropy in Theorem 3.11 is analogous to that for topological entropy of Markov subshifts (see for example Theorem 3.48 in [36]).

(2) Example 3.15 will present an uncertain control system that satisfies the conditions of Theorem 3.11. Let $\mathscr{A} = \{Q_a : a \in V\}$, $G(Q_a) = a$ for every $a \in V$. If F is a single-valued map, then $(\mathscr{A}, 1, G)$ is a maximally irreducible invariant cover. Then by [24, Theorem 5.1], the invariance entropy is equal to the entropy of $(\mathscr{A}, 1, G)$, and is also equal to $\log \rho(M_{Q,V})$ by Theorem 3.11. Hence Theorem 3.11 generalizes [24, Theorem 5.1] when $n = 1$.

(3) Suppose that $V \subset U$ satisfies the conditions of Theorem 3.11. Let $\mathcal{A}_V = \{Q_a : a \in V\}$ and define $G_V : \mathcal{A}_V \to U$ by $G(Q_a) = a$ for every $a \in V$. Then $(\mathcal{A}_V, G_V)$ is an invariant partition of Q. From Theorem 3.11, we see $h_{inv}(Q) = \log \rho(M_{Q,V})$. It is natural to wonder if $h^{fb}_{inv}(Q) = h_{inv}(\mathcal{A}_V, G_V)$. However, these conditions of Theorem 3.11 are not sufficient (see Example 3.15). But the invariance feedback entropy of Q can be determined by this invariant partition, as shown in the following.

Under the conditions of Theorem 3.11, we say $(\mathcal{B}, G_{\mathcal{B}})$ is a *refinement* of $(\mathcal{A}_V, G_V)$ if $\mathcal{B}$ is cover of Q, every element B of $\mathcal{B}$ is contained in some element A of $\mathcal{A}_V$, and $G_{\mathcal{B}}(B) = G_V(A)$ for every $B \in \mathcal{B}$. Let $\mathscr{B}(\mathcal{A}_V, G_V)$ denote all the refinements of $(\mathcal{A}_V, G_V)$. We call $(\mathcal{C}, G_{\mathcal{C}}) \in \mathscr{B}(\mathcal{A}_V, G_V)$ an *atom refinement* of $(\mathcal{A}_V, G_V)$ if $\mathcal{C} = \{\{x\} : x \in Q\}$ and $\sharp(F(x, a) \cap Q_b)$ is at most 1 for every $a, b \in V$ and $x \in Q_a$. Note that the atom refinement of $(\mathcal{A}_V, G_V)$ is unique when it exists.

Corollary 3.13. *Under the conditions of Theorem 3.11,*

$$h^{fb}_{inv}(Q) = \inf\{h_{inv}(\mathcal{B}, G_{\mathcal{B}}) : (\mathcal{B}, G_{\mathcal{B}}) \in \mathscr{B}(\mathcal{A}_V, G_V)\}.$$

Proof. It suffices to show that any invariant cover $(\mathcal{B}, G_{\mathcal{B}})$ is a refinement of $(\mathcal{A}_V, G_V)$. By (C.3) in Theorem 3.11, For every $B \in \mathcal{B}$, $G_{\mathcal{B}}(B) \in V$. Since $B \subset \cup_{A \in \mathcal{A}_V} A$, it follows from (C.1) in Theorem 3.11 that there exists only one element $A \subset \mathcal{A}_V$ such that $B \subset A$. Hence $G_{\mathcal{B}}(B) = G_V(A)$ and it follows that $(\mathcal{B}, G_{\mathcal{B}})$ is a refinement of $(\mathcal{A}_V, G_V)$. □

Theorem 3.14. *Under the conditions of Theorem 3.11, if moreover $\sharp Q$ is finite and $(\mathcal{C}, G_{\mathcal{C}})$ is the atom refinement of $(\mathcal{A}_V, G_V)$, then $h^{fb}_{inv}(Q) = h_{inv}(\mathcal{C}, G_{\mathcal{C}})$.*

Proof. Since $(\mathcal{C}, G_{\mathcal{C}})$ is an invariant partition of Q, it immediately follows from Theorem 3.3 that $h_{inv}(\mathcal{C}, G_{\mathcal{C}}) = \overline{w}^*(\mathcal{C}, G_{\mathcal{C}})$. Take an irreducible periodic sequence c in $(\mathcal{C}, G_{\mathcal{C}})$ such that $\overline{w}^*(\mathcal{C}, G_{\mathcal{C}}) = \overline{w}(c)$. We can without loss of generality assume that $c = (C_i)_{i=0}^{k-1}$, where $k \le \sharp Q$. Fixing $m \in \mathbb{N}$ and a refinement $(\mathcal{B}, G_{\mathcal{B}})$ of $(\mathcal{A}, G)$, let

$$\beta_{c,m} := \underbrace{C_0 \cdots C_{k-1} \cdots C_0 \cdots C_{k-1}}_{m} C_0$$

and $\mathcal{S}$ be an $(mk+1, Q)$-spanning set in $(\mathcal{B}, G_{\mathcal{B}})$. Since $P(\alpha)$ covers Q for any $\alpha \in \mathcal{S}$ and $\sharp C_0 = 1$, there exists $\alpha^0 \in \mathcal{S}$ so that $C_0 \subset \alpha^0(0)$ and

$$F(C_0, G_{\mathcal{C}}(C_0)) \subset F(\alpha^0(0), G_{\mathcal{B}}(\alpha^0(0))) \subset \bigcup_{A' \in P\left(\alpha^0|_{[0,0]}\right)} A'.$$

Thus there exists $\alpha^1 \in \mathcal{S}$ such that $C_0 \subset \alpha^1(0), C_1 \subset \alpha^1(1)$. Repeating this process, we can find $\alpha^{mk+1} \in \mathcal{S}$ such that

$$C_i \subset \alpha^{mk+1}(jk+i), \ j = 0, \dots, m-1, \ i = 0, \dots, k-1, \ C_0 \subset \alpha^{mk+1}(mk).$$

Since $(\mathcal{C}, G_{\mathcal{C}})$ is the atom refinement, we have

$$\sharp D_{\mathcal{C}}(\{x\}) \le D^*_x \tag{3.4}$$

for any $x \in Q$, where

$$D^*_x := \min_{\substack{(\mathcal{B}, G_{\mathcal{B}}) \in \mathscr{B}(\mathcal{A}, G) \\ A \in \mathcal{B} \\ \{x\} \subset A}} \min\{\sharp\mathcal{F} : \mathcal{F} \subset D_{\mathcal{B}}(A), \ F(A, G_{\mathcal{B}}(A)) \subset \bigcup_{A' \in \mathcal{F}} A'\}.$$

Replacing $\{x\}$ in (3.4) by C_i, we have

$$\sharp D_{\mathcal{C}}(C_i) \le \sharp P(\alpha^{mk+1}|_{[0,jk+i]}), \ j = 0, \dots, m-1, \ i = 0, \dots, k-1.$$

Thus

$$N(\mathcal{S}) \ge \prod_{t=0}^{mk} \sharp P(\alpha^{mk+1}|_{[0,t]}) \ge \prod_{t=0}^{mk-1} \sharp P(\alpha^{mk+1}|_{[0,t]}) \ge \left(\prod_{i=0}^{k-1} \sharp D_{\mathcal{C}}(C_i)\right)^m.$$

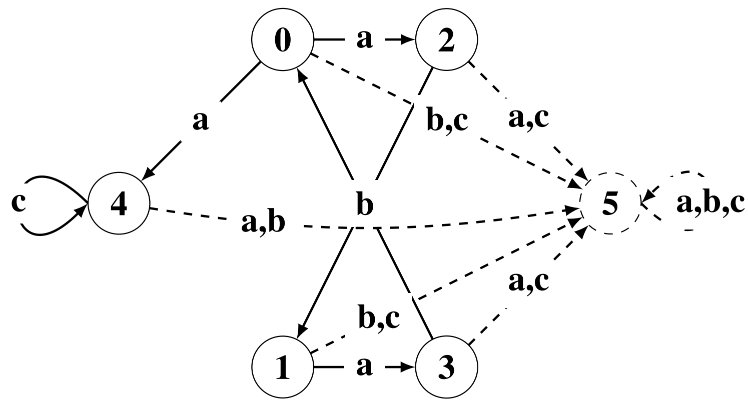


Fig. 2. The transition map of Example 3.15.

Since $\mathcal{S}$ is arbitrary, it holds that

$$r_{inv}(mk+1, Q, \mathcal{B}, G_{\mathcal{B}}) \geq \left(\prod_{i=0}^{k-1} \sharp D_{\mathcal{C}}(C_i)\right)^m.$$

Therefore

$$\begin{aligned} h_{inv}(\mathcal{B}, G_{\mathcal{B}}) &= \lim_{m\to\infty} \frac{1}{mk+1} \log r_{inv}(mk+1, Q, \mathcal{B}, G_{\mathcal{B}}) \\ &\geq \lim_{m\to\infty} \frac{1}{mk+1} \log \left(\prod_{i=0}^{k-1} \sharp D_{\mathcal{C}}(C_i)\right)^m \\ &= \overline{w}(c) = h_{inv}(\mathcal{C}, G_{\mathcal{C}}). \end{aligned}$$

This together with Corollary 3.13 yields the desired equality. $\square$

As the end of this paper, we construct an uncertain control system to apply Theorems 3.11 and 3.14 to compute the invariance entropy and invariance feedback entropy for a subset of the sate space. Note that the equality in Theorem 3.7 can be possible.

Example 3.15. Let $\Sigma = (X, U, F)$ be a system, where $X = \{0, 1, 2, 3, 4, 5\}$ and $U = \{a, b, c\}$. The transition function F is illustrated by Fig. 2.

The set of interest is $Q := \{0, 1, 2, 3, 4\}$. Let

$$\begin{aligned} &\mathcal{A}_1 = \{A_{10}, A_{11}, A_{12}\},\ A_{10} = \{0, 1\},\ A_{11} = \{2, 3\},\ A_{12} = \{4\}, \\ &\mathcal{A}_2 = \{A_{20}, A_{21}, A_{22}, A_{23}, A_{24}\},\ A_{20} = \{0\},\ A_{21} = \{1\}, \\ &\qquad A_{22} = \{2\},\ A_{23} = \{3\},\ A_{24} = \{4\}, \\ &\mathcal{A}_3 = \{A_{30}, A_{31}, A_{32}, A_{33}\},\ A_{30} = \{0\},\ A_{31} = \{1\}, \\ &\qquad A_{32} = \{2, 3\},\ A_{33} = \{4\}. \end{aligned}$$

Define $G_i : \mathcal{A}_i \to U$, $i = 1, 2, 3$ by

$$\begin{aligned} &G_1(A_{10}) = a,\ G_1(A_{11}) = b,\ G_1(A_{12}) = c, \\ &G_2(A_{20}) = a,\ G_2(A_{21}) = a,\ G_2(A_{22}) = b,\ G_2(A_{23}) = b,\ G_2(A_{24}) = c, \\ &G_3(A_{30}) = a,\ G_3(A_{31}) = a,\ G_3(A_{32}) = b,\ G_3(A_{33}) = c. \end{aligned}$$

Then

$$h_{inv}(Q) = 0,\ h_{inv}(\mathcal{A}_1, G_1) = \frac{1}{2},\ h_{inv}(\mathcal{A}_2, G_2) = \frac{1}{4},$$

$$h_{inv}(\mathcal{A}_3, G_3) = 1,\ h_{inv}^{fb}(Q) = \frac{1}{4}.$$

Proof. Clearly $Q_a = \{0, 1\}$, $Q_b = \{2, 3\}$, $Q_c = \{4\}$, and thus U is a cover of Q. It is obvious that conditions (C.1) and (C.3) in Theorem 3.11 hold. Since $Q_b \subset F(Q_a, a)$, $Q_c \subset F(0, a)$, $Q_a \subset F(Q_b, b)$, and $Q_c \subset F(Q_c, c)$, condition (C.2) in Theorem 3.11 holds. From Fig. 2, we have

$$M_{Q,U} = \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array} \begin{array}{l} \\ a \\ b. \\ c \end{array}$$

A brief computation shows that $\rho(M_{Q,U}) = 1$. It then follows from Theorem 3.11 that $h_{inv}(Q) = 0$.

We now compute $h_{inv}(\mathcal{A}_i, G_i)$, $i = 1, 2, 3$. Since

$$M_{\mathcal{A}_1,G_1} = \begin{array}{c} \begin{array}{ccc} A_{10} & A_{11} & A_{12} \end{array} \\ \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array} \begin{array}{l} \\ A_{10} \\ A_{11}, \\ A_{12} \end{array} \quad W_{\mathcal{A}_1,G_1} = \begin{array}{c} \begin{array}{ccc} A_{10} & A_{11} & A_{12} \end{array} \\ \begin{pmatrix} 0 & 2 & 2 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array} \begin{array}{l} \\ A_{10} \\ A_{11}, \\ A_{12} \end{array}$$

$$M_{\mathcal{A}_2,G_2} = \begin{array}{c} \begin{array}{ccccc} A_{20} & A_{21} & A_{22} & A_{23} & A_{24} \end{array} \\ \begin{pmatrix} 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array} \begin{array}{l} \\ A_{20} \\ A_{21} \\ A_{22}, \\ A_{23} \\ A_{24} \end{array}$$

$$W_{\mathcal{A}_2,G_2} = \begin{array}{c} \begin{array}{ccccc} A_{20} & A_{21} & A_{22} & A_{23} & A_{24} \end{array} \\ \begin{pmatrix} 0 & 0 & 2 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array} \begin{array}{l} \\ A_{20} \\ A_{21} \\ A_{22}, \\ A_{23} \\ A_{24} \end{array}$$

$$M_{\mathcal{A}_3,G_3} = \begin{array}{c} \begin{array}{cccc} A_{30} & A_{31} & A_{32} & A_{33} \end{array} \\ \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array} \begin{array}{l} \\ A_{30} \\ A_{31} \\ A_{32}, \\ A_{33} \end{array}$$

$$W_{\mathcal{A}_3,G_3} = \begin{array}{c} \begin{array}{cccc} A_{30} & A_{31} & A_{32} & A_{33} \end{array} \\ \begin{pmatrix} 0 & 0 & 2 & 2 \\ 0 & 0 & 1 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array} \begin{array}{l} \\ A_{30} \\ A_{31} \\ A_{32}, \\ A_{33} \end{array}$$

it follows by a straightforward calculation that

$$\rho(M_{\mathcal{A}_1,G_1}) = 1,\quad \rho(W_{\mathcal{A}_1,G_1}) = \sqrt{2},\quad \|W_{\mathcal{A}_1,G_1}\|_\infty = 2,$$

$$\rho(M_{\mathcal{A}_2,G_2}) = 1,\quad \rho(W_{\mathcal{A}_2,G_2}) = \sqrt[4]{2},\quad \|W_{\mathcal{A}_2,G_2}\|_\infty = 2,$$

$$\rho(M_{\mathcal{A}_3,G_3}) = \sqrt{2},\quad \rho(W_{\mathcal{A}_3,G_3}) = \sqrt{6},\quad \|W_{\mathcal{A}_3,G_3}\|_\infty = 2.$$

It then follows from Theorem 3.7 that $h_{inv}(\mathcal{A}_1, G_1) = \frac{1}{2}$ and $h_{inv}(\mathcal{A}_2, G_2) = \frac{1}{4}$. Noting that $\overline{w}^*(\mathcal{A}_3, G_3) = \overline{w}(c)$, where $c = A_{30}A_{32}$, we have by Theorem 3.3 $h_{inv}(\mathcal{A}_3, G_3) = 1$.

It is not difficult to check that (A_2, G_2) is the atom refinement, and therefore Theorem 3.14 asserts that $h^{fb}_{inv}(Q) = \frac{1}{4}$. $\square$

4. Conclusion and future work

We introduced a notion of invariance entropy for *uncertain* control systems and showed the relation between our invariance entropy and invariance feedback entropy. We presented two formulas for our invariance entropy and invariance feedback entropy under some reasonable assumptions. This answers the open questions posed in [34].

The future research will look into dimension types of invariance entropy, measure-theoretic invariance entropy, variational principle, and invariance pressure for *uncertain* control systems.

CRediT authorship contribution statement

Xingfu Zhong: Conceptualization, Methodology, Writing – original draft, Writing – review & editing. **Yu Huang:** Conceptualization, Supervision, Writing – review & editing. **Xingfu Zou:** Supervision, Writing – review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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