# Convergence of Discrete–Time Neural Networks with Delays[*]

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To Professor Zhicheng Wang on the occasion of his retirement

**Abstract.**  An LMI (Linear Matrix Inequality) approach and an embedding technique are employed to derive some sufficient conditions for the global exponential stability of discrete-time neural networks with time-dependent delays and constant parameters. For networks with time-dependent parameters but constant delays, by using the property of internally chain transitive sets, it is shown that these conditions are also sufficient for the convergence of the networks.

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## 1. Introduction

When a neural network is updated discretely, the model describing the network is in the form of system of difference equations (See, e.g., Hopfield [6]). Also, in numerical simulations and practical implementation of a continuous-time neural network, discretization is needed, which leads again to a system of difference equations. Therefore, it is of both theoretical and practical importance to study the dynamics of discrete-time neural networks.

Recently, there has been increasing interest in the effects of delays on neural dynamics of continuous-time networks. See, for example, [1, 13, 14, 18]. It has been

noticed that delays sometimes are harmless in the sense that the appearance of delays does not change the stability of the neural networks [17], and sometimes the delays do change the dynamics quite a lot [13, 14].

In this paper, we consider a discrete-time neural network model with constant parameters and variable delays

$$x_i(n+1) = a_i x_i(n) + \sum_{j=1}^{m} w_{ij} g_j(x_j(n - k(n))) + I_i, n = 0, 1, \ldots, \qquad (1.1)$$

and a model with time-dependent parameters and constant delay

$$x_i(n+1) = a_i(n) x_i(n) + \sum_{j=1}^{m} w_{ij}(n) g_j(x_j(n - k)) + I_i(n), n = 0, 1, \ldots. \qquad (1.2)$$

In (1.1), $k(n)$ are positive integers with $0 \leq k(n) \leq k$ (not necessarily increasing), $a_i \in (0,1)$, $i \in \{1, 2, \ldots, m\} := N(1, m)$. While, in (1.2), $a_i(n) \to a_i, w_{ij}(n) \to w_{ij}, I_i(n) \to I_i$ as $n \to \infty$, $i, j \in N(1, m)$.

We will apply an LMI approach and an embedding technique to derive some delay-dependent and delay-independent conditions under which system (1.1) admits a unique equilibrium and which is globally exponentially stable.

We point out the LMI approach has been used by Liao, Chen and Snachez in [11] to establish some stability criteria for delayed continuous-time neural networks and by de Souza and Trofino in [4] for discrete-time periodic systems. In this paper, we attempt to establish some LMI based stability criteria for the discrete time neural network model (1.1), which can be easily tested by efficient and reliable algorithms [2].

Motivated also by [3, 16], where an embedding technique was used in [3] for a simple discrete-time neural netwok model having specific performance, and in [16] for the attractivity of some Hopfield type continuous-time neural networks with delays, we apply the embedding technique to system (1.1) to derive some sufficient conditons for its exponential stability.

Note that system (1.2) has the autonomous system (1.1) as its limiting system. In this paper, we will obtain a convergence result for the asymptotically autonomous system (1.2) by relating (1.2) to (1.1) with interally chain transitive sets (for the notion of interally chain transitive set, see, e.g., [19]).

The rest of the paper is organized as follows. In Section 2 we establish some criteria for exponential stability of (1.1) by combining the LMI approach, Liapunov function method, embedding technique and the comparison method for discrete mono-tone systems. Section 3 is devoted to the convergence of the asymptotic discrete-time neural networks (1.2), in which the chain transitive set and the strong attractivity theorem [19] play a crucial role.

## 2. Exponential stability of (1.1)

We use the following notations: $\lambda_M(W)$: the largest eigenvalue of the symmetric matrix $W$; $\lambda_m(W)$: the smallest eigenvalue of the symmetric matrix $W$; $W^T$: the transpose of the matrix $W$; $W^{-1}$: the inverse of the matrix $W$; $||x|| = \left(\sum_{i=1}^m x_i^2\right)^{\frac{1}{2}}$: the Euclidean norm of the vector $x = (x_1, x_2, \ldots, x_m)^T \in \mathbb{R}^m$ and $||W|| = \left(\max \lambda(W^T W)\right)^{\frac{1}{2}}$: the matrix norm induced by the Euclidean vector norm.

The initial conditions associated with (1.1) are of the form

$$x_i(s) = \phi_i(s), i = N(1, m), s \in N(-k, 0). \tag{2.1}$$

Throughout this paper, we assume

**(H)** For each $i \in N(1, m)$, $g_i : \mathbb{R} \to \mathbb{R}$ is globally Lipschitz continuous with

$$\sup_{u,v \in \mathbb{R}, u \neq v} \frac{|g_i(u) - g_i(v)|}{|u - v|} = l_i,$$

and $|g_i(u)| \leq M_i, u \in \mathbb{R}, M_i > 0$.

If we let $x = (x_1, x_2, \ldots, x_m)^T$, $A = diag(a_1, a_2, \ldots, a_m)$, $W = (w_{ij})_{n \times n}$, $I = (I_1, I_2, \ldots, I_m)^T$, and $g(x(n)) = (g_1(x_1(n)), g_2(x_2(n)), \ldots, g_m(x_m(n)))^T$, then (1.1) can be written in form of matrices:

$$x(n+1) = Ax(n) + Wg[x(n - k(n))] + I, n \in N(0). \tag{2.2}$$

As usual, a vector $x^* = (x_1^*, x_2^*, \ldots, x_m^*)^T$ is said to be an equilibrium of (2.2) if it satisfies

$$x^* = Ax^* + Wg(x^*) + I.$$

Based on our assumption on the activation functions, it is easily seen that (2.2) admits at least one equilibrium.

### 2.1. LMI based criteria

In what follows, $S > (\geq) 0$ means the matrix $S$ is symmetric and positive definite (semi-positive definite). From the theory of matrices, we have the following lemma.

**Lemma 2.1.** (i) *If $A > 0$, $B \geq 0$, $\alpha > 0$, then $A + B > 0$, $\alpha A > 0$;*

(ii) $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} > 0$ *if and only if $A_{11} > 0$ and $A_{22} - A_{21}A_{11}^{-1}A_{12} > 0$;*

(iii) *For any real matrices $A, B, C$ and a scalar $\epsilon > 0$ with $C > 0$, the inequality*

$$A^T B + B^T A \leq \epsilon A^T C A + \epsilon^{-1} B^T C^{-1} B$$

*holds.*

*Proof.* (i) and (ii) can be found in [5] and (iii) is available in [15].          □

Suppose $x^*$ is an equilibrium of system (2.2) and let $y(n) = x(n) - x^*$ and $f(y(n)) = g(x(n)) - g(x^*)$. Then the stability of equilibrium $x^*$ of (2.2) corresponds to that of the zero solution of the following system

$$y(n+1) = Ay(n) + Wf(y(n-k(n))). \tag{2.3}$$

It follows from (H) that $f$ has the property:

$$||f(y)|| \le ||L|| ||y||, \tag{2.4}$$

with $L = diag(l_1, l_2, \ldots, l_m)$.

**Theorem 2.1.** *Assume that the time-dependent delay $k(n)$ is bounded satisfying $0 \le k(n) \le k$ and $\Delta k(n) = k(n+1) - k(n) < 1$. If there exist two scalars $q > 1, \epsilon > 0$ and two matrices $P > 0, R > 0$ such that*

$$\begin{pmatrix} R & W^T P A \\ \frac{q}{\epsilon} APW & P - qAPA - LQL \end{pmatrix} > 0, \tag{2.5}$$

*then the equilibrium $x^*$ of (2.2) is exponentially stable. More precisely, for any solution $x(n)$ of (2.2), the following inequality holds*

$$||x(n) - x^*||^2 \le q^{-n} C_1 \sup_{s \in N(-k,0)} ||x(s) - x^*||^2, \tag{2.6}$$

*where*

$$C_1 = \frac{\lambda_M(P) + \delta \lambda_M(Q) ||L||^2}{\lambda_m(P)} \text{ with } \delta = \frac{1 - (1/q)^k}{q-1},$$

*and*

$$Q = q^{1+k} \epsilon R + q^{1+k} W^T P W > 0.$$

*Proof.* Define $V(n) = V(y(n))$ by

$$V(n) = q^n y^T(n) P y(n) + \sum_{s=n-k(n)}^{n-1} q^s f^T(y(s)) Q f(y(s)). \tag{2.7}$$

Then, we have

$$\begin{aligned}
\Delta V(n) &= V(n+1) - V(n) \\
&= q^{n+1} y^T(n+1) P y(n+1) - q^n y^T(n) P y(n) \\
&\quad + \sum_{s=n+1-k(n+1)}^{n} q^s f^T(y(s)) Q f(y(s)) - \sum_{s=n-k(n)}^{n-1} q^s f^T(y(s)) Q f(y(s)) \\
&\le q^{n+1} \left( Ay(n) + Wf(y(n-k(n))) \right)^T P \left( Ay(n) + Wf(y(n-k(n))) \right) \\
&\quad - q^n y^T(n) P y(n) + q^n f^T(y(n)) Q f(y(n)) \\
&\quad - q^{n-k(n)} f^T(y(n-k(n))) Q f(y(n-k(n))),
\end{aligned}$$

which further gives

$$\begin{aligned}
\Delta V(n) \leq{} & q^{n+1} y^T(n) APAy(n) - q^n y^T(n) Py(n) + q^n f^T(y(n)) Qf(y(n)) \\
& + q^{n+1}[y^T(n) APW f(y(n-k(n))) + f^T(y(n-k(n))) W^T PAy(n)] \\
& + f^T(y(n-k(n)))(q^{n+1} W^T PW - q^{n-k(n)} Q) f(y(n-k(n))).
\end{aligned}$$

Letting $B = W^T PAy$, $C = R$, it follows from Lemma 2.1 (iii) that

$$\begin{aligned}
& y^T(n) APW f(y(n-k(n))) + f^T(y(n-k(n))) W^T PAy(n) \\
& \leq \epsilon f^T(y(n-k(n))) Rf(y(n-k(n))) + \tfrac{1}{\epsilon} y^T(n) APW R^{-1} W^T PAy(n),
\end{aligned}$$

which shows

$$\begin{aligned}
\Delta V(n) \leq{} & -q^n y^T(n) \left( P - qAPA - LQL - \frac{q}{\epsilon} APW R^{-1} W^T PA \right) y(n) \\
& - q^{n-k(n)} f^T(y(n-k(n))) \left( Q - q^{1+k(n)} (\epsilon R + W^T PW) \right) f^T(y(n-k(n))).
\end{aligned}$$

Recalling that $Q = q^{1+k}(\epsilon R + W^T PW)$, we know from Lemma 2.1 that $Q > 0$ and $Q - q^{1+k(n)}(\epsilon R + W^T PW) > 0$. This shows that

$$\Delta V(n) \leq -q^n y^T(n) \Omega y(n),$$

where $\Omega = P - qAPA - LQL - \frac{q}{\epsilon} APW R^{-1} W^T PA$. Condition (2.5) and Lemma 2.1–(ii) imply that $\Omega > 0$ and hence

$$\Delta V(n) \leq 0.$$

Therefore, we have

$$\begin{aligned}
V(n) \leq V(0) &= y^T(0) Py(0) + \sum_{s=-k(0)}^{-1} q^s f^T(y(s)) Qf(y(s)) \\
&\leq \lambda_M(P) ||y(0)||^2 + \sum_{s=-k}^{-1} \lambda_M(Q) ||L||^2 ||y(s)||^2 q^s \\
&\leq \left( \lambda_M(P) + \delta \lambda_M(Q) ||L||^2 \right) \sup_{s \in N(-k,0)} ||y(s)||^2.
\end{aligned}$$

On the other hand, from the definition of $V(n)$ it follows that

$$V(y(n)) \geq q^n \lambda_m(P) ||y(n)||^2.$$

We then obtain

$$||y(n)||^2 \leq q^{-n} \frac{\lambda_M(P) + \delta \lambda_M(Q) ||L||^2}{\lambda_m(P)} \sup_{s \in N(-k,0)} ||y(s)||^2,$$

which gives (2.6) and thus the proof is complete. $\square$

By defining a different Liapunov function, we have the following theorem.

**Theorem 2.2.** *Assume that there are two matrices* $P > 0$, $\Sigma > 0$ *and a scalar* $\sigma \in (0,1)$ *such that*

$$\begin{pmatrix} \Sigma & W^T P A \\ APW & \sigma P - APA \end{pmatrix} > 0 \qquad (2.8)$$

*and*

$$\lambda_M(\Sigma + W^T PW)||L||^2 < \lambda_m(P)(1 - \sigma). \qquad (2.9)$$

*Then every solution of (2.2) is exponentially stable satisfying*

$$||x(n) - x^*||^2 \leq C_2 \sigma^{\bar{\gamma}n} \sup_{s \in N(-k,0)} ||x(s) - x^*||^2, \qquad (2.10)$$

*where*

$$C_2 = \frac{\lambda_M(P)}{\lambda_m(P)(1 - C_3(\bar{\gamma}))}, \ C_3(\gamma) = \frac{\lambda_M(\Sigma + W^T PW)||L||^2}{\lambda_m(P)\sigma^{\gamma k}(\sigma^\gamma - \sigma)}$$

*and*

$$\bar{\gamma} = \sup\{\gamma \in (0,1) : 0 < C_3(\gamma) \leq 1 - \frac{1}{2}(1 - C_3(0))\}.$$

*Proof.* Define $V(n) = V(y(n)) = y^T(n)Py(n)$, then we have

$$\lambda_m(P)||y(n)||^2 \leq V(n) \leq \lambda_M(P)||y(n)||^2 \qquad (2.11)$$

and

$$\begin{aligned} \Delta V(n) &= V(n+1) - V(n) = y^T(n+1)Py(n+1) - y^T(n)Py(n) \\ &= (Ay(n) + Wf(y(n - k(n))))^T P \left(Ay(n) + Wf(y(n - k(n)))\right) \\ &\quad - y^T(n)Py(n) \\ &= y^T(n)(APA - P)y(n) + f^T(y(n - k(n)))W^T PAy(n) \\ &\quad + y^T(n)APWf(y(n - k(n))) + f^T(y(n - k(n)))W^T PWf(y(n - k(n))) \end{aligned}$$

Using Lemma 2.1(iii) with $\epsilon = 1, B = W^T PAy, C = \Sigma$, we further have

$$\begin{aligned} \Delta V(n) &\leq y^T(n)[-P + APA + APW\Sigma^{-1}W^T PA]y(n) \\ &\quad + f^T(y(n - k(n)))[\Sigma + W^T PW]f(y(n - k(n))) \\ &\leq y^T(n)[-P + APA + APW\Sigma^{-1}W^T PA]y(n) \\ &\quad + \lambda_M(\Sigma + W^T PW)||L||^2||y(n - k(n))||^2 \\ &= -(1 - \sigma)y^T(n)Py(n) - y^T(n)[\sigma P - APA - APW\Sigma^{-1}W^T PA]y(n) \\ &\quad + \lambda_M(\Sigma + W^T PW)||L||^2||y(n - k(n))||^2 \end{aligned}$$

Notice that condition (2.8) and Lemma 2.1(ii) imply that

$$\sigma P - APA - APW\Sigma^{-1}W^T PA > 0.$$

This shows that

$$\Delta V(n) \leq -(1 - \sigma)V(n) + \lambda_M(\Sigma + W^T PW)||L||^2||y(n - k(n))||^2, n \in N(1),$$

and hence we have

$$V(n) \le \sigma^n V(0) + \lambda_M(\Sigma + W^T P W)||L||^2 \sum_{s=0}^{n-1} \sigma^{n-1-s}||y(s-k(s))||^2. \qquad (2.12)$$

From (2.11), it follows that

$$V(0) \le \lambda_M(P)||y(0)||^2 \le \lambda_M(P)|| \sup_{s \in N(-k,0)} ||x(s) - x^*||^2.$$

Thus, (2.11), together with (2.12), shows that

$$||y(n)||^2 \le \sigma^n \frac{\lambda_M(P)}{\lambda_m(P)} \sup_{s \in N(-k,0)} ||x(s) - x^*||^2 + C_4 \sum_{s=0}^{n-1} \sigma^{n-1-s}||y(s-k(s))||^2, \quad (2.13)$$

where

$$C_4 = \frac{\lambda_M(\Sigma + W^T P W)||L||^2}{\lambda_m(P)}.$$

Condition (2.9) guarantees that $\bar{\gamma} \in (0,1)$ exists and $C_3(\bar{\gamma}) \le \frac{1}{2}(1 + C_3(0)) < 1$. Multiplying both sides of (2.13) by $\sigma^{-\bar{\gamma}n}$, we have

$$
\begin{aligned}
\sigma^{-\bar{\gamma}n}||y(n)||^2 \quad \le \quad & \sigma^{(1-\bar{\gamma})n} \frac{\lambda_M(P)}{\lambda_m(P)} \sup_{s \in N(-k,0)} ||x(s) - x^*||^2 \\
& + C_4 \sum_{s=0}^{n-1} \sigma^{n-1-s-\bar{\gamma}n}||y(s-k(s))||^2 \\
\le \quad & \frac{\lambda_M(P)}{\lambda_m(P)} \sup_{s \in N(-k,0)} ||x(s) - x^*||^2 + C_4 \sigma^{-1+(1-\bar{\gamma})n} \times \\
& \sum_{s=0}^{n-1} \sigma^{-(1-\bar{\gamma})s} \sigma^{-\bar{\gamma}k(s)} \sigma^{-\bar{\gamma}(s-k(s))}||y(s-k(s))||^2.
\end{aligned}
$$

Letting

$$z(n) := \sup_{s \in [-k,n]} \sigma^{-\bar{\gamma}s}||y(s)||^2 \qquad (2.14)$$

and noticing that $k(n) \le k$ and $\sigma \in (0,1)$, we obtain

$$
\begin{aligned}
\sigma^{-\bar{\gamma}n}||y(n)||^2 \quad \le \quad & \frac{\lambda_M(P)}{\lambda_m(P)} \sup_{s \in N(-k,0)} ||x(s) - x^*||^2 + C_4 \sigma^{-\bar{\gamma}k} \frac{1}{\sigma^{\bar{\gamma}} - \sigma} z(n) \\
= \quad & \frac{\lambda_M(P)}{\lambda_m(P)} \sup_{s \in N(-k,0)} ||x(s) - x^*||^2 + C_3(\bar{\gamma})z(n),
\end{aligned}
$$

which shows that

$$
\begin{aligned}
z(n) &= \sup_{s \in [-k,n]} \sigma^{-\bar{\gamma}s} ||y(s)||^2 \\
&\leq \sup_{s \in [-k,n]} \left( \frac{\lambda_M(P)}{\lambda_m(P)} \sup_{s \in N(-k,0)} ||x(s) - x^*||^2 + C_3(\bar{\gamma})z(s) \right) \\
&= \frac{\lambda_M(P)}{\lambda_m(P)} \sup_{s \in N(-k,0)} ||x(s) - x^*||^2 + C_3(\bar{\gamma})z(n).
\end{aligned}
$$

Therefore,

$$
(1 - C_3(\bar{\gamma}))z(n) \leq \frac{\lambda_M(P)}{\lambda_m(P)} \sup_{s \in N(-k,0)} ||x(s) - x^*||^2.
$$

This indicates that

$$
||y(n)||^2 \sigma^{-\bar{\gamma}n} \leq z(n) \leq C_2 \sup_{s \in N(-k,0)} ||x(s) - x^*||^2,
$$

or

$$
||y(n)||^2 \leq \sigma^{\bar{\gamma}n} C_2 \sup_{s \in N(-k,0)} ||x(s) - x^*||^2.
$$

Thus the proof is complete. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ $\square$

**Remark 2.1.** *Theorem 2.1 and Theorem 2.2 show that the equilibrium of (2.2) is unique.*

**Remark 2.2.** *In Theorem 2.1, a technical assumption is required: $\Delta k(n) < 1$, which implies that the time-dependent delays are non-increasing. Moreover, the condition (2.5) is delay-dependent through the expression of $Q$. Theorem 2.2 only requires that $k(n)$ be bounded and the condition (2.8) in Theorem 2.2 is independent of delay.*

**Remark 2.3.** *Based on Theorem 2.1 and Theorem 2.2, we can determine the upper bound of $q$ in (2.6) and the lower bound of $\sigma$ in (2.10) so that the neural network (2.2) has a rapid convergence. This requires solving the following optimization problems:*

$$
\begin{cases} Max\ q \\ Subject\ to\ P > 0, R > 0\ and\ (2.5)\ is\ satisfied \end{cases} \tag{2.15}
$$

*and*

$$
\begin{cases} Min\ \sigma \\ Subject\ to\ P > 0, \Sigma > 0\ and\ (2.8)\ and\ (2.9)\ are\ satisfied, \end{cases} \tag{2.16}
$$

*respectively. Note that (2.15) and (2.16) can be solved easily by the LMI Toolbox such as the Scilab developed by INRIA and ENPC in France, which is available at: www-rocq.inria.fr/scilab/.*

**Example 2.1.** *Consider*

$$\begin{cases} x_1(n+1) = & 1/2x_1(n) + 1/4\tanh(x_1(n-1)) + 1/8\tanh(x_2(n-1)) \\ x_2(n+1) = & 1/2x_2(n) + 1/4\tanh(x_1(n-1)) + 1/16\tanh(x_2(n-1)). \end{cases} \quad (2.17)$$

In this example, $k(n) = k = 1$, $L = I$, $W = \begin{pmatrix} 1/4 & 1/8 \\ 1/4 & 1/16 \end{pmatrix}$. Taking $R = I$,
$P = \begin{pmatrix} 8/5 & 0 \\ 0 & 9/5 \end{pmatrix}$, $\epsilon = 1/2$ and $q = 6/5$, we then find that $R > 0$, $P > 0$, $\epsilon > 0$,
$q > 1$ and (2.5) holds. This shows, by Theorem 2.1, that the zero solution of (2.17) is globally exponentially stable with the exponential decay rate less than $1/q = 5/6$.

## 2.2. Embedding technique deduced stability

We employ the embedding technique for exponential stability of (1.1) in this subsection. Note that we only need to consider the stability of the zero solution of system (2.3). In this subsection, we assume that for each $i \in N(1,m)$, $f_i$ satisfies

$$0 \le \frac{f_i(u) - f_i(v)}{u - v} \le l_i, \text{ for } u \ne v.$$

Denote $W^+ = (w_{ij}^+), W^- = (w_{ij}^-)$ with $w_{ij}^+ = \max\{w_{ij}, 0\}, w_{ij}^- = \max\{-w_{ij}, 0\}$ and $h(-s) = -f(s)$. It follows from $W = W^+ - W^-$ that (2.3) can be embedded into a $2m$-dimensional system

$$\begin{bmatrix} u(n+1) \\ v(n+1) \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} \begin{bmatrix} u(n) \\ v(n) \end{bmatrix} + \begin{bmatrix} W^+ & W^- \\ W^- & W^+ \end{bmatrix} \begin{bmatrix} f(u(n-k(n))) \\ h(v(n-k(n))) \end{bmatrix}. \quad (2.18)$$

Let

$$z(n) = \begin{bmatrix} u(n) \\ v(n) \end{bmatrix}, B = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}, C = \begin{bmatrix} W^+ & W^- \\ W^- & W^+ \end{bmatrix}, F(z(n)) = \begin{bmatrix} f(u(n)) \\ h(v(n)) \end{bmatrix},$$

then (2.18) can be rewritten as

$$z(n+1) = Bz(n) + CF(z(n-k(n))). \quad (2.19)$$

For system (2.19) we have the following comparison theorem.

**Theorem 2.3.** *Let $\phi(n)$ and $\psi(n)$ be two solutions of (2.19) with initial data $\phi(s)$, $\psi(s)$, $s \in N(-k, 0)$. Then $\phi(n) \le \psi(n)$ provided that $\phi(s) \le \psi(s)$ for $s \in N(-k, 0)$. Moreover, if $\phi(n)$ satisfies*

$$\phi(n+1) \ge B\phi(n) + CF(\phi(n-k(n))), \ n \ge 0,$$

*and $z(n)$ is the solution of (2.19) with initial data $z(s), s \in N(-k, 0)$, then $z(s) \le \phi(s), s \in N(-k, 0)$ implies $z(n) \le \phi(n), n \ge 1$.*

*Proof.* Taking advantage of the fact that both $B$ and $C$ are non-negative matrices, we can easily complete the proof by using the method of induction. $\quad\square$

A consequence of Theorem 2.3 is the following

**Corollary 2.1.** *Let* $\{y(n)\}, n \in N(1)$ *be the solution of (2.3) with initial data* $\{y(s)\}$, $s \in N(-k,0)$ *and* $\phi(n) = \begin{bmatrix} u(n) \\ v(n) \end{bmatrix}, n \in N(1)$ *the solution of (2.19) with initial data* $\phi(s) = \begin{bmatrix} u(s) \\ v(s) \end{bmatrix}, s \in N(-k,0)$. *If* $-v(s) \leq y(s) \leq u(s), s \in N(-k,0)$, *then we have* $-v(n) \leq y(n) \leq u(n)$ *for* $n \in N(1)$.

We now introduce the definitions of *Class $K_0$* and *Class K* for matrices.

**Definition 2.1.** *Let* $A \in \{A = (a_{ik}), i, k = 1, \ldots, n;\ a_{ik} \leq 0, i \neq k\}$. *The matrix A is said to be of class $K_0$ (respectively, K) if there is a vector* $x > 0$ *such that* $Ax \geq 0$ *(respectively, $Ax > 0$).*

Denoting $L = diag(l_1, \ldots, l_m)$, $D = \begin{bmatrix} L & 0 \\ 0 & L \end{bmatrix}$ and the identity matrix with dimension $m$ by $I_m$, and using the property of matrices of class $K_0$ and class $K$, we may establish our exponential stability result as follows.

**Theorem 2.4.** *Assume that there is a* $\sigma \in (0,1)$ *such that*

$$\Omega_1 := \sigma I_{2m} - B - \sigma^{-k}CD$$

*is of class $K_0$. Then the zero solution of (2.3) is (globally) exponentially stable in the sense that for every solution $y(n)$ of (2.3), there exist $\xi_0, \eta_0 \in \mathbb{R}^m$ with $\xi_0 > 0$ and $\eta_0 > 0$ such that*

$$-\sigma^n \eta_0 \leq y(n) \leq \sigma^n \xi_0.$$

*Proof.* $\Omega_1 \in K_0$ implies that there exists a vector $(\xi, \eta)^T \in \mathbb{R}^{2m}$ with $\xi \in \mathbb{R}^m, \eta \in \mathbb{R}^m$ and $\xi > 0, \eta > 0$ such that

$$\Omega_1 \begin{bmatrix} \xi \\ \eta \end{bmatrix} \geq 0.$$

Let $y(n)$ be the solution of (2.3) with initial data $y(s), s \in N(-k,0)$. We then can find a positive constant $q$ such that $-q\eta \leq y(s) \leq q\xi$ for $s \in N(-k,0)$. Denoting the solution of (2.19) with initial data $\phi(s) = \begin{bmatrix} u(s) \\ v(s) \end{bmatrix} = q \begin{bmatrix} \xi \\ \eta \end{bmatrix}$, $s \in N(-k,0)$, by $\phi(n) = \begin{bmatrix} u(n) \\ v(n) \end{bmatrix}$, we then have $-v(n) \leq y(n) \leq u(n)$ for $n \in N(1)$. The fact that $\Omega_1 \begin{bmatrix} q\xi \\ q\eta \end{bmatrix} \geq 0$ implies that $z(n) = \begin{bmatrix} q\sigma^n\xi \\ q\sigma^n\eta \end{bmatrix}, n \in N(-k)$, satisfies

$$z(n+1) \geq Bz(n) + CDz(n - k(n)), \text{ for } n \geq 0,$$

which shows that $z(n) = \begin{bmatrix} q\sigma^n\xi \\ q\sigma^n\eta \end{bmatrix}, n \in N(1)$ is a solution of the following inequality

$$z(n+1) \geq Bz(n) + CF(z(n - k(n))), \text{ for } n \geq 0,$$

with initial data $z(s) = \begin{bmatrix} q\sigma^s\xi \\ q\sigma^s\eta \end{bmatrix}, s \in N(-k, 0)$. The fact that $\phi(s) = q\begin{bmatrix} \xi \\ \eta \end{bmatrix} \leq \begin{bmatrix} q\sigma^s\xi \\ q\sigma^s\eta \end{bmatrix}, s \in N(-k, 0)$, and the comparison Theorem 2.3 imply that

$$\phi(n) \leq z(n) = \begin{bmatrix} q\sigma^n\xi \\ q\sigma^n\eta \end{bmatrix}, n \in N(1).$$

This indicates that

$$-\sigma^n\eta_0 := -q\sigma^n\eta \leq y(n) \leq q\sigma^n\xi =: \sigma^n\xi_0$$

for all $n \in N(1)$. Thus $y(n) \to 0$ exponentially as $n \to \infty$ and the proof is complete. $\square$

**Corollary 2.2.** *If*
$$\Omega_1' := I_{2m} - B - CD$$
*is of class $K$, then the zero solution of (2.3) is (globally) exponentially stable.*

*Proof.* $\Omega_1'$ is of class $K$ implies that there is a $\sigma \in (0, 1)$ such that $\Omega_1$ defined as in Theorem 2.4 is of class $K_0$. Therefore the proof follows from Theorem 2.4. $\square$

Denoting $|W| = (|w_{ij}|)$, we have

**Corollary 2.3.** *If there exists a $\sigma \in (0, 1)$ such $\Omega_2 := \sigma I_m - A - \sigma^{-k}|W|L$ is of class $K_0$ or equivalently if $\Omega_2' := I_m - A - |W|L$ is of class $K$, then the zero solution of (2.3) is (globally) exponentially stable.*

*Proof.* Since $\Omega_2 := \sigma I_m - A - \sigma^{-k}|W|L$ is of class $K_0$, there exists a positive vector $\xi \in \mathbb{R}^m$ such that $\Omega_2\xi \geq 0$. This implies that

$$\Omega_1\begin{bmatrix} \xi \\ \xi \end{bmatrix} \geq 0,$$

which shows that $\Omega_1$ is of class $K_0$ and then the conclusion follows from Theorem 2.4. $\square$

**Remark 2.4.** *In [5], the matrices of class $K_0$ (K) are called $M-$matrices (nonsingular $M-$matrices). Many other equivalent definitions are also available in [5]. For example, a matrix $M$ is of class $K$ if* **(a)** *all principal minors of $M$ are positive; or* **(b)** *every real eigenvalue of $M$ is positive.*

Next, we give an example to demonstrate the exponential stability of a two-neuron network.

**Example 2.2.** *Consider*

$$\begin{cases} x_1(n+1) = & 1/2x_1(n) + 1/4\tanh(x_1(n-2)) - 1/4\tanh(x_2(n-2)) \\ x_2(n+1) = & 1/4x_2(n) - 1/8\tanh(x_1(n-2)) + 1/2\tanh(x_2(n-2)). \end{cases} \quad (2.20)$$

In this example, $m = 2$, $L = I_2$ and $A = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/4 \end{pmatrix}$, $W = \begin{pmatrix} 1/4 & -1/4 \\ -1/8 & 1/2 \end{pmatrix}$. It is easy to verify that $\Omega_2' = I_2 - A - |W|L = \begin{pmatrix} 1/4 & -1/4 \\ -1/8 & 1/4 \end{pmatrix}$ is of class $K$ and thus, by Corollary 2.3, every component of each solution of (2.20) exponentially converges to zero.

## 3. Convergence of asymptotic neural networks (1.2)

In this section, we study the convergence of system (1.2). Note that system (1.2) can be rewritten as

$$x(n+1) = A(n)x(n) + W(n)g(x(n-k)) + I(n), n \in N(0), \quad (3.1)$$

with

$$A(n) \to A, W(n) \to W, I(n) \to I, \qquad \text{as } n \to \infty. \quad (3.2)$$

Note that the global exponential stability achieved in Section 2 for system (2.2) shows that the equilibrium of (2.2) is unique under the stability conditions. Then we have

**Theorem 3.1.** *Assume that all conditions in Theorem 2.1 or in Theorem 2.2 or Theorem 2.4 are satisfied, then all solutions of (3.1) will converge to the unique equilibrium of the limiting system (2.2).*

*Proof.* Let $X := \{z : z = (z_{-k}, z_{-k+1}, \ldots, z_{-1}, z_0)\}$, where for each $i \in N(-k, 0)$, $z_i = (z_{i,1}, z_{i,2}, \ldots, z_{i,m}) \in R^m$. Define $||z|| = \max_{-k \le i \le 0} ||z_i||$ with $||z_i|| = \max_{1 \le j \le m} |z_{i,j}|$, then $(X, || \cdot ||)$ is a Banach space. Let $d(u, v) = ||u - v||$ for $u, v \in X$ be the norm induced metric, then $(X, d)$ is a metric space. In the following, the space $X$ is referred as the metric space $(X, d)$.

For any $z = (z_{-k}, z_{-k+1}, \ldots, z_0) \in X$, we define $S_n : X \to X, n = 0, 1, \ldots,$ and $S : X \to X$, by

$$S_n z = (z_{-k+1}, z_{-k+2}, \ldots, z_0, A(n)z_0 + W(n)g(z_{-k}) + I(n))$$

and

$$Sz = (z_{-k+1}, z_{-k+2}, \ldots, z_0, Az_0 + Wg(z_{-k}) + I),$$

respectively. Let

$$T_0 = I, \ T_n z = S_{n-1} \circ S_{n-2} \circ \cdots \circ S_1 \circ S_0 z, \qquad n = 1, 2, \ldots,$$

Then it is seen from (3.2) that $T_n$ is asymptotic to $S$.

Let $Z^0 = (x(-k), x(-k+1), \ldots, x(0))$, where

$$x(-j) = (x_1(-j), x_2(-j), \ldots, x_m(-j)), \qquad j = k, k-1, \ldots, 0,$$

are the initial conditions of (3.1). Let $Z^n = T_n Z^0$, then $\{Z^n : n \geq 0\}$ is an orbit of the discrete process $T_n$ and $\{x(n) : x(n) = (Z^n)_{k+1}, n \geq 0\}$ is the solution vector of (3.1) with initial conditions given by $Z^0$. By the variation of constants formula, we can easily show that the solution of (3.1) will be bounded for given bounded initial data. Therefore, it follows from Lemma 1.2.2 of [19] that the omega limit set of any orbit of $\{T_n\}$ is internally chain transitive for $S$. Under the given assumptions, we know that the equilibrium, denoted by $x^*$, of (2.2) is unique and is globally stable. This implies that $A := \{(x^*, x^*, \ldots, x^*)\}$ is the global attractor of $S$, i.e., $A$ is an attractor and $W^s(A) = X$. By the strong attractivity theorem: Theorem 1.2.1 of [19], we conclude that the omega limit set of any orbit of $\{T_n\}$ is the set $A$. This implies that $x(n) = (Z^n)_{k+1} \to x^*$ as $n \to \infty$, which shows that all solutions of (3.1) converge to the unique equilibrium of the limiting system (2.2). The proof is complete. $\qquad\square$

# References

[1] P. Baldi and A. F. Atiya, How delays affect neural dynamics and learning, IEEE Trans. Neural Networks, 5 (1994) 612–621.

[2] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, Linear matrix inequalities in system and control theory, volume 15 of SIAM Studies in Applied Mathematics, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1994.

[3] T. Chu, Convergence in discrete-time neural networks with specific performance, Physical Review E., 63:051904, 2001.

[4] C. E. de Souza and A. Trofino, An LMI approach to stabilization of linear discrete-time periodic systems, Internat. J. Control, 73:8 (2000) 696–703.

[5] M. Fiedler, Special matrices and their applications in numerical mathematics, Martinus Nijhoff Publishers, Dordrecht, 1986. Translated from the Czech by Petr Přikryl and Karel Segeth.

[6] J. J. Hopfield, Neurons with graded response have collective computational properties like those of two-stage neurons, Proc. Nat. Acad. Sci. U.S.A., 81 (1984) 3088–3092.

[7] S. Hu and J. Wang, Global stability of a class of discrete-time recurrent neural networks, IEEE Trans. Circuits Systems I Fund. Theory Appl., 49:8 (2002) 1104–1117.

[8] S. Jagannathan and F. L. Lewis, Multilayer discrete-time neural-net controller with guaranteed performance, IEEE Trans. Neural Networks, 7:1 (1996) 107–130.

[9] L. Jin and M. M. Gupta, Globally asymptotical stability of discrete time analog neural networks, IEEE Trans. Neural Networks, 7 (1996) 1024–1031.

[10] L. Jin, P. N. Nikiforuh, and M. M. Gupta, Absolute stability conditions for discrete-time recurrent neural networks, IEEE Trans. Neural Networks, 5 (1994) 945–955.

[11] X. Liao, G. Chen, and E. N. Sanchez, LMI-based approach for asymptotically stability analysis of delayed neural networks, IEEE Trans. Circuits Systems I Fund. Theory Appl., 49:7 (2002) 1033–1039.

[12] D .Liu and A. N. Michel, Asymptotic stability of discrete-time systems with saturation nonlinearities with applications to digital filters, IEEE Trans. Circuits Systems I Fund. Theory Appl., 39 (1992) 798–807.

[13] K. Pakdaman, C. Grotta-Ragazzo, and Malta, Transient region duration in countinuous-time neural networks with delay, Physical Review E., 58 (1998) 3623–3627.

[14] K. Pakdaman, C. Grotta-Ragazzo, C. P. Malta, O. Arino, and J.-F Vibert, Effect of delay on the boundary of the basin of attraction in a system of two neurons, Neural Networks, 11 (1998) 509–519.

[15] E. N. Sanchez and J. P. Perez, Input-to-state stability (ISS) analysis for dynamic neural networks, IEEE Trans. Circuits Systems I Fund. Theory Appl., 46:11 (1999) 1395–1398.

[16] P. van den Driessche, J. Wu, and X. Zou, Stabilization role of inhibitory self-connections in a delayed neural network, Phys. D, 150:1-2 (2001) 84–90.

[17] L. Wang and X. Zou, Harmless delays in Cohen-Grossberg neural networks, Phys. D, 170:2 (2002) 162–173.

[18] J. Wu, Introduction to neural dynamics and signal transmission delay, volume 6 of de Gruyter Series in Nonlinear Analysis and Applications, Walter de Gruyter & Co., Berlin, 2001.

[19] X.-Q. Zhao, Dynamical systems in population biology, CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, 16. Springer-Verlag, New York, 2003.