



# Harmless delays in Cohen–Grossberg neural networks<sup>☆</sup>

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## Abstract

Without assuming monotonicity and differentiability of the activation functions and any symmetry of interconnections, we establish some sufficient conditions for the globally asymptotic stability of a unique equilibrium for the Cohen–Grossberg neural network with multiple delays. Lyapunov functionals and functions combined with the Razumikhin technique are employed. The criteria are all independent of the magnitudes of the delays, and thus the delays under these conditions are harmless.

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## 1. Introduction

Cohen and Grossberg [3] proposed and studied an artificial feedback neural network, which is described by a system of ordinary differential equations

$$\dot{x}_i = -a_i(x_i) \left( b_i(x_i) - \sum_{j=1}^n t_{ij}s_j(x_j) \right), \quad i = 1, \dots, n, \quad (1.1)$$

where  $n \geq 2$  is the number of neurons in the network,  $x_i$  denotes the state variable associated to the  $i$ th neuron,  $a_i$  represents an amplification function, and  $b_i$  is an appropriately behaved function. The  $n \times n$  connection matrix  $T = (t_{ij})$  tells how the neurons are connected in the network, and the activation function  $s_j$  shows how the  $j$ th neuron reacts to the input. Functions  $a_i$ ,  $b_i$  and  $s_i$  are subject to certain conditions to be specified later. It is seen that (1.1) includes the Hopfield neural network as a special case, which is of the form

$$C_i \dot{x}_i = -\frac{x_i}{R_i} + \sum_{j=1}^n t_{ij}s_j(x_j) + J_i, \quad i = 1, 2, \dots, n, \quad (1.2)$$

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where the positive constants  $C_i$  and  $R_i$  are the neuron amplifier input capacitances and resistances, respectively;  $J_i$  is the constant input from outside of the network and  $x_i$ ,  $s_j$  and  $T = (t_{ij})$  are the same as in (1.1).

Due to their promising potential for the tasks of classification, associative memory, parallel computations, and their ability to solve difficult optimization problems, (1.1) and (1.2) have greatly attracted the attention of the scientific community. Various generalizations and modifications of (1.1) and (1.2) have also been proposed and studied, among which is the incorporation of time delay into the model. In fact, due to the finite speeds of the switching and transmission of signals in a network, time delays do exist in a working network and thus should be incorporated into the model equations of the network. For more detailed justifications for introducing delays into model equations of neural networks, see [8] and the recent book [14].

Marcus and Westervelt [8] first introduced a single delay into (1.2) and considered the following system of delay differential equations:

$$C_i \dot{x}_i = -\frac{x_i}{R_i} + \sum_{j=1}^n t_{ij} s_j(x_j(t - \tau)) + J_i, \quad i = 1, 2, \dots, n. \quad (1.3)$$

It was observed both experimentally and numerically in [8] that delay could destroy an otherwise stable network and cause sustained oscillations and thus could be harmful. System (1.3) has also been studied by Wu [15], Wu and Zou [16], Gopalsamy and He [6], and van den Driessche and Zou [13] studied a further generalized version with multiple delays

$$\dot{x}_i = -b_i x_i + \sum_{j=1}^n t_{ij} s_j(x_j(t - \tau_{ij})) + J_i, \quad i = 1, 2, \dots, n. \quad (1.4)$$

For the Cohen–Grossberg model (1.1), Ye et al. [18] also introduced delays by considering the following system of delay differential equations:

$$\dot{x}_i(t) = -a_i(x_i) \left( b_i(x_i) - \sum_{k=0}^K \sum_{j=1}^n t_{ij}^{(k)} s_j(x_j(t - \tau_k)) \right), \quad i = 1, 2, \dots, n, \quad (1.5)$$

where  $n \times n$  matrixes  $T_k = (t_{ij}^{(k)})$  represent the interconnections which are associated with delay  $\tau_k$  and the delays  $\tau_k$ ,  $k = 0, 1, \dots, K$ , are arranged such that  $0 = \tau_0 < \tau_1 < \dots < \tau_K$ .

Established in the pioneering work of Cohen and Grossberg [3] and Hopfield [7] was the “globally asymptotic stability” of systems (1.1) and (1.2), respectively. It was proved that given any initial conditions, the solution of the system (1.1) (or (1.2)) will converge to some equilibrium of the corresponding system. Such a “global stability” in [3,7] was obtained by considering some potential functions under the assumption that the connection matrix  $T$  is *symmetric*. When it comes to the delayed systems (1.3)–(1.5), it is natural to expect that this global stability remains if the delays are sufficiently small. Indeed, such an expectation was confirmed in [17,18] under a certain type of *symmetry* requirement. When a network is designed for the purpose of associative memories, it is required that the system have a set of stable equilibria, each of which corresponds to an addressable memory. The global stability confirmed in [3,7,17,18] is necessary and crucial for associative memory networks. However, an obvious drawback of the above work is the lack of description or even estimates for the basin of attraction of each stable equilibrium. In other words, given a set of initial conditions, one knows that the solution will converge to some equilibrium, but *does not* know exactly to which equilibrium it will converge. In terms of associative memories, one does not know precisely what initial conditions are needed in order to retrieve a particular pattern stored in the network. Also, the work of Ye et al. [17,18] cannot tell what would happen when the delays increase. We have mentioned above that large delay could destroy the stability of an equilibrium in a network. Even if the delay does not change the stability

of an equilibria, it could affect the basin of attraction of the stable equilibrium. For such a topic, see the recent work of Pakdaman et al. [9,10], or Wu [14].

On the other hand, in applications of neural networks to parallel computations, signal processing and other problems involving the optimization, it is required that there be a well-defined computable solution for all possible initial states. In other words, it is required that the network have a *unique* equilibrium that is *globally attractive*. In fact, earlier applications of neural networks to optimization problems have suffered from the existence of a complicated set of equilibria (see [12]). Thus, the global attractivity of a unique equilibrium for the model system is of great importance for both practical and theoretical purposes, and has been the major concern of many authors. We refer to Bélair [1], Cao and Wu [2], Gopalsamy and He [6], van den Driessche and Zou [13] for the delayed Hopfield model (1.3) or (1.4). As for the delayed Cohen–Grossberg model (1.5), to the best of the authors’ knowledge, no similar result has been established yet, and this fact motivates this work. Thus, the purpose of this paper is to obtain some criteria for the global attractivity of a unique equilibrium of the following system

$$\dot{x}_i(t) = -a_i(x_i) \left[ b_i(x_i) - \sum_{k=0}^K \sum_{j=1}^n t_{ij}^{(k)} s_j(x_j(t - \tau_k)) + J_i \right], \quad i = 1, 2, \dots, n, \tag{1.6}$$

where  $J_i, i = 1, 2, \dots, n$ , denote the constant inputs from outside of the system. We do not confine ourselves to the symmetric connections and thus allow much broader connection topologies for the network. Moreover, unlike most of the previous authors, we will not assume monotonicity and differentiability for the activation functions. Our main results show that under some conditions on the connection strengths and structures, the delays could be *harmless* in the sense that the solutions of system (1.6) always converge to the unique equilibrium, irrespective of the amplitudes of the delays.

This paper is organized as follows. In Section 2, we introduce some notations and assumptions. In Section 3, we establish our main results on the globally asymptotic stability of (1.6). Some examples and numerical simulations are given in Section 4 to demonstrate the main results, and a summary is given in Section 5.

## 2. Preliminaries

Let  $\mathcal{R}$  denote the set of real numbers and

$$\mathcal{R}^n = \underbrace{\mathcal{R} \times \mathcal{R} \times \dots \times \mathcal{R}}_n.$$

If  $x \in \mathcal{R}^n$ , then  $x^T = (x_1, \dots, x_n)$  denotes the transpose of  $x$ ,  $\|x\|_2 = (x^T x)^{1/2}$ . Let  $\mathcal{R}^{n \times n}$  denote the set of  $n \times n$  real matrices. For  $Z \in \mathcal{R}^{n \times n}$ , the spectral norm of  $Z$  is defined as

$$\|Z\|_2 = (\max\{|\lambda| : \lambda \text{ is an eigenvalue of } Z^T Z\})^{1/2}.$$

The initial conditions associated with (1.6) are given in the form

$$x_i(s) = \phi_i(s) \in C([-\tau, 0], \mathcal{R}), \quad i = 1, 2, \dots, n, \tag{2.1}$$

where  $\tau = \max\{\tau_k, 0 \leq k \leq K\} = \tau_K$ .

For functions  $a_i(x)$  and  $b_i(x), i = 1, 2, \dots, n$ , we have the following assumptions:

- (H<sub>1</sub>) For each  $i \in \{1, 2, \dots, n\}$ ,  $a_i$  is bounded, positive and continuous, furthermore we assume  $0 < \underline{\alpha}_i \leq a_i(u) \leq \bar{\alpha}_i$ .
- (H<sub>2</sub>) For each  $i \in \{1, 2, \dots, n\}$ ,  $b_i \in C^1(\mathcal{R}, \mathcal{R})$  and  $b'_i > 0$ .

For the activation functions  $s_i(x)$ ,  $i = 1, 2, \dots, n$ , they are typically assumed to be sigmoid which implies that they are monotone and smooth, that is, they are required to satisfy the following:

- (A<sub>1</sub>)  $s_i \in C^1(\mathcal{R})$ ,  $s'_i(x) > 0$  for  $x \in \mathcal{R}$  and  $s'_i(0) = \sup_{x \in \mathcal{R}} s'_i(x) > 0$ ,  $i = 1, 2, \dots, n$ .
- (A<sub>2</sub>)  $s_i(0) = 0$  and  $\lim_{x \rightarrow \pm\infty} s_i(x) = \pm 1$ .

As pointed out in [13], for some purpose of networks, non-monotonic and not necessarily smooth functions might be better candidates for neuron activation functions in designing and implementing an artificial neural network. Note that in many electronic circuits, amplifiers that have neither monotonically increasing nor continuously differentiable input–output functions are frequently adopted (e.g., piecewise linear functions). This suggests a modification of (A<sub>1</sub>) and (A<sub>2</sub>) to the following:

- (H<sub>3</sub>) For each  $i \in \{1, 2, \dots, n\}$ ,  $s_i : \mathcal{R} \rightarrow \mathcal{R}$  is globally Lipschitz with Lipschitz constant  $L_i$ , i.e.  $|s_i(u_i) - s_i(v_i)| \leq L_i |u_i - v_i|$  for all  $u_i, v_i \in \mathcal{R}$ .
- (H<sub>4</sub>) For each  $i \in \{1, 2, \dots, n\}$ ,  $|s_i(x)| \leq M_i$ ,  $x \in \mathcal{R}$  for some constant  $M_i > 0$ .

Note that unlike in [18],  $T = \sum_{k=0}^K T_k$  is not required to be symmetric in this paper, which means that our results will be applicable to networks with much broader connection structures.

### 3. Main results

First, we show that system (1.6) does have an equilibrium.

**Proposition 3.1.** *If (H<sub>1</sub>), (H<sub>2</sub>) and (H<sub>4</sub>) hold, then for every input  $J$ , there exists an equilibrium for system (1.6).*

**Proof.** Let the input  $J$  be given. By (H<sub>1</sub>),  $x^*$  is an equilibrium of (1.6) if and only if  $x^* = (x_1^*, \dots, x_n^*)^T$  is a solution of the system

$$b_i(x_i) - \sum_{k=0}^K \sum_{j=1}^n t_{ij}^{(k)} s_j(x_j) + J_i = 0, \quad i = 1, \dots, n. \tag{3.1}$$

From (H<sub>4</sub>), we obtain

$$\left| \sum_{k=0}^K \sum_{j=1}^n t_{ij}^{(k)} s_j(x_j) - J_i \right| \leq \sum_{k=0}^K \sum_{j=1}^n |t_{ij}^{(k)}| M_j + |J_i| =: P_i.$$

By (H<sub>2</sub>), we know that  $b_i^{-1}$  exists and is increasing. Now consider the equivalent (to (3.1)) system

$$x_i = h_i(x_1, x_2, \dots, x_n) := b_i^{-1} \left( \sum_{k=0}^K \sum_{j=1}^n t_{ij}^{(k)} s_j(x_j) - J_i \right), \quad i = 1, 2, \dots, n.$$

Then we have

$$|h_i(x_1, x_2, \dots, x_n)| \leq \max\{|b_i^{-1}(|P_i|)|, |b_i^{-1}(-|P_i|)|\} := D_i \quad \text{for } i = 1, 2, \dots, n.$$

It follows that  $h = (h_1, h_2, \dots, h_n)^T$  maps a bounded set  $D = D_1 \times D_2 \times \dots \times D_n$  to itself. By the Brouwer’s fixed point theorem [5, Theorem 3.2],  $h(x)$  has a fixed point  $x^*$  in  $D$ , which gives an equilibrium of (1.6). The proof is complete. □

Let  $x^*$  be an equilibrium of (1.6) and  $u(t) = x(t) - x^*$ . Substituting  $x(t) = u(t) + x^*$  into (1.6) leads to

$$\dot{u}_i(t) = -a_i(u_i + x_i^*) \left[ b_i(u_i + x_i^*) - \sum_{k=0}^K \sum_{j=1}^n t_{ij}^{(k)} s_j(u_j(t - \tau_k) + x_j^*) + J_i \right] \tag{3.2}$$

for  $i = 1, 2, \dots, n$ . Using the relation  $J_i = -b_i(x_i^*) + \sum_{k=0}^K \sum_{j=1}^n t_{ij}^{(k)} s_j(x_j^*)$ , system (3.2) can be written as

$$\dot{u}_i(t) = -\alpha_i(u_i(t)) \left( \beta_i(u_i(t)) - \sum_{k=0}^K \sum_{j=1}^n t_{ij}^{(k)} g_j(u_j(t - \tau_k)) \right), \quad i = 1, 2, \dots, n, \tag{3.3}$$

where  $\alpha_i(u_i(t)) = a_i(u_i(t) + x_i^*)$ ,  $\beta_i(u_i(t)) = b_i(u_i(t) + x_i^*) - b_i(x_i^*)$ ,  $g_j(u_j(t)) = s_j(u_j(t) + x_j^*) - s_j(x_j^*)$ . If we let  $u = (u_1, \dots, u_n)^T \in \mathcal{R}^n$ ,  $T_k = (t_{ij}^{(k)})_{n \times n}$ ,  $A(u) = \text{diag}\{\alpha_1(u_1), \dots, \alpha_n(u_n)\}$ ,  $B(u) = (\beta_1(u_1), \dots, \beta_n(u_n))^T \in \mathcal{R}^n$ ,  $g(u) = (g_1(u_1), \dots, g_n(u_n))^T$ , then system (3.3) can be further expressed in the matrix form

$$\dot{u} = -A(u) \left( B(u) - \sum_{k=0}^K T_k g(u(t - \tau_k)) \right). \tag{3.4}$$

It is obvious that  $x^*$  is globally asymptotically stable for (1.6) if and only if the trivial solution  $u = 0$  of (3.3) or (3.4) is globally asymptotically stable. Moreover, the *uniqueness* of the equilibrium of (1.6) follows from its globally asymptotic stability.

**Theorem 3.1.** *Suppose (H<sub>1</sub>)–(H<sub>4</sub>) are satisfied. If*

$$b'_i(u) \geq \gamma_i, \quad i = 1, 2, \dots, n \text{ for some } \gamma_i > 0, \tag{3.5}$$

$$\theta := \min_{1 \leq i \leq n} \left\{ \frac{\alpha_i}{\bar{\alpha}_i} \gamma_i - L_i \sum_{k=0}^K \sum_{j=1}^n |t_{ji}^{(k)}| \right\} > 0, \tag{3.6}$$

*then, for every input  $J$ , system (1.6) has a unique equilibrium  $x^*$  which is globally asymptotically stable, independent of the delays.*

**Proof.** Let

$$V(t) = V(u)(t) = \sum_{i=1}^n \left[ \frac{1}{\bar{\alpha}_i} |u_i(t)| + \sum_{k=0}^K \sum_{j=1}^n |t_{ij}^{(k)}| L_j \int_{t-\tau_k}^t |u_j(s)| ds \right]. \tag{3.7}$$

Then the upper-right derivative of  $V$  along a solution of (3.3) is given by

$$\begin{aligned} D^+V(t) &= \sum_{i=1}^n \left[ \frac{1}{\bar{\alpha}_i} \dot{u}_i(t) \text{sgn}(u_i(t)) + \sum_{k=0}^K \sum_{j=1}^n |t_{ij}^{(k)}| L_j (|u_j(t)| - |u_j(t - \tau_k)|) \right] \\ &= \sum_{i=1}^n \left\{ -\frac{1}{\bar{\alpha}_i} a_i(u_i(t)) \text{sgn}(u_i(t)) \left[ \beta_i(u_i(t)) - \sum_{k=0}^K \sum_{j=1}^n t_{ij}^{(k)} g_j(u_j(t - \tau_k)) \right] \right. \\ &\quad \left. + \sum_{k=0}^K \sum_{j=1}^n |t_{ij}^{(k)}| L_j (|u_j(t)| - |u_j(t - \tau_k)|) \right\} \end{aligned}$$

$$\begin{aligned} &\leq \sum_{i=1}^n \left\{ -\frac{1}{\bar{\alpha}_i} a_i(u_i(t)) \operatorname{sgn}(u_i(t)) \beta_i(u_i(t)) + \sum_{k=0}^K \sum_{j=1}^n |t_{ij}^{(k)}| |g_j(u_j(t - \tau_k))| \right. \\ &\quad \left. + \sum_{k=0}^K \sum_{j=1}^n |t_{ij}^{(k)}| |L_j(|u_j(t)| - |u_j(t - \tau_k)|) \right\} \\ &\leq \sum_{i=1}^n \left[ -\frac{\alpha_i}{\bar{\alpha}_i} \gamma_i |u_i(t)| + \sum_{k=0}^K \sum_{j=1}^n |t_{ij}^{(k)}| |L_j| |u_j(t)| \right] = -\sum_{i=1}^n \left( \frac{\alpha_i}{\bar{\alpha}_i} \gamma_i - L_i \sum_{k=0}^K \sum_{j=1}^n |t_{ji}^{(k)}| \right) |u_i(t)| \\ &\leq -\min_{1 \leq i \leq n} \left( \frac{\alpha_i}{\bar{\alpha}_i} \gamma_i - L_i \sum_{k=0}^K \sum_{j=1}^n |t_{ji}^{(k)}| \right) \sum_{i=1}^n |u_i(t)|, \end{aligned}$$

that is

$$D^+V(t) \leq -\theta \sum_{i=1}^n |u_i(t)| \leq 0. \tag{3.8}$$

In the above estimate, we have used

$$\operatorname{sgn}(u_i(t)) \beta_i(u_i(t)) \geq \gamma_i |u_i(t)|, \tag{3.9}$$

which follows from condition (3.5), since

$$\begin{aligned} \operatorname{sgn}(u_i(t)) \beta_i(u_i(t)) &= \operatorname{sgn}(u_i(t)) \beta_i(u_i(t)) = \operatorname{sgn}(u_i(t)) (b_i(u_i(t) + x_i^*) - b_i(x_i^*)) = \operatorname{sgn}(u_i(t)) b_i'(\xi) u_i(t) \\ &= b_i'(\xi) |u_i(t)| \geq \gamma_i |u_i(t)|, \end{aligned}$$

where  $\xi$  lies between  $u_i(t)$  and  $u_i(t) + x_i^*$ . By virtue of (3.7) and (3.8), we know that  $\sum_{i=1}^n |u_i(t)|$  is bounded for all  $t \geq 0$ . Thus the solutions of (3.3) exist for all  $t \geq 0$ . From (3.8), we have

$$V(t) + \theta \int_0^t \sum_{i=1}^n |u_i(s)| \, ds \leq V(0) < \infty, \tag{3.10}$$

which shows that  $\sum_{i=1}^n |u_i(t)| \in L^1[0, \infty)$ . Applying Lemma 1.2.2 of [6, pp. 4–5], we then know that the trivial solution of (3.3) or (3.4) is globally asymptotically stable, and hence  $x^*$  is globally asymptotically stable for (1.6). The proof is complete.  $\square$

**Theorem 3.2.** Assume that (H<sub>1</sub>)–(H<sub>4</sub>) and (3.5) hold. If

$$\mu := \min_{1 \leq i \leq n} \left( 2 \frac{\alpha_i}{\bar{\alpha}_i} \gamma_i - \sum_{k=0}^K \sum_{j=1}^n (|t_{ij}^{(k)}| + L_i^2 |t_{ji}^{(k)}|) \right) > 0, \tag{3.11}$$

then the equilibrium  $x^*$  of system (1.6) is globally asymptotically stable.

**Proof.** Let  $V(t) = V(u)(t)$  be defined by

$$V(u)(t) = \sum_{i=1}^n \left( \frac{1}{\bar{\alpha}_i} u_i^2(t) + \sum_{k=0}^K \sum_{j=1}^n |t_{ij}^{(k)}| \int_{t-\tau_k}^t g_j^2(u_j(s)) \, ds \right). \tag{3.12}$$

Then,

$$\begin{aligned}
 D^+V(t) &= \sum_{i=1}^n \left( \frac{1}{\bar{\alpha}_i} 2u_i \dot{u}_i + \sum_{k=0}^K \sum_{j=1}^n |t_{ij}^{(k)}| (g_j^2(u_j(t)) - g_j^2(u_j(t - \tau_k))) \right) \\
 &= \sum_{i=1}^n \left( -\frac{2}{\bar{\alpha}_i} \alpha_i(u_i) u_i \beta_i(u_i) + \frac{2}{\bar{\alpha}_i} \alpha_i(u_i) u_i \sum_{k=0}^K \sum_{j=1}^n t_{ij}^{(k)} g_j(u_j(t - \tau_k)) \right. \\
 &\quad \left. + \sum_{k=0}^K \sum_{j=1}^n |t_{ij}^{(k)}| (g_j^2(u_j(t)) - g_j^2(u_j(t - \tau_k))) \right) \\
 &\leq \sum_{i=1}^n \left( -\frac{2\alpha_i}{\bar{\alpha}_i} \gamma_i u_i^2 + 2 \sum_{k=0}^K \sum_{j=1}^n |t_{ij}^{(k)}| |u_i g_j(u_j(t - \tau_k))| + \sum_{k=0}^K \sum_{j=1}^n |t_{ij}^{(k)}| (g_j^2(u_j(t)) - g_j^2(u_j(t - \tau_k))) \right).
 \end{aligned}$$

Using the fact that  $2|ab| \leq (a^2 + b^2)$ , we can get

$$\begin{aligned}
 D^+V(t) &\leq \sum_{i=1}^n \left( -\frac{2\alpha_i}{\bar{\alpha}_i} \gamma_i u_i^2 + \sum_{k=0}^K \sum_{j=1}^n |t_{ij}^{(k)}| (u_i^2 + g_j^2(u_j(t))) \right) \\
 &\leq \sum_{i=1}^n \left( -\frac{2\alpha_i}{\bar{\alpha}_i} \gamma_i u_i^2 + \sum_{k=0}^K \sum_{j=1}^n (|t_{ij}^{(k)}| u_i^2 + L_j^2 |t_{ij}^{(k)}| u_j^2) \right) \\
 &\leq -\sum_{i=1}^n \left( \frac{2\alpha_i}{\bar{\alpha}_i} \gamma_i - \sum_{k=0}^k \sum_{j=1}^n (|t_{ij}^{(k)}| + L_i^2 |t_{ji}^{(k)}|) \right) u_i^2(t) \\
 &\leq -\min_{1 \leq i \leq n} \left\{ \frac{2\alpha_i}{\bar{\alpha}_i} \gamma_i - \sum_{k=0}^K \sum_{j=1}^n (|t_{ij}^{(k)}| + L_i^2 |t_{ji}^{(k)}|) \right\} \sum_{i=1}^n u_i^2(t) = -\mu \sum_{i=1}^n u_i^2(t) \leq 0.
 \end{aligned}$$

The rest of the proof is similar to the proof of [Theorem 3.1](#), and thus, is omitted here. The proof is complete.  $\square$

Note that both [\(3.6\)](#) and [\(3.11\)](#) neglect the signs of the entries of the connection matrices, and thus, differences between excitatory and inhibitory effects are ignored. In the case of  $a_i(x) = 1$ ,  $b_i(x) = b_i x$ , van den Driessche and Zou [\[13\]](#) made an attempt to recognize such a difference. Next we will use the Lyapunov–Razumikhin technique to generalize [Theorem 2.4](#) of [\[13\]](#) to system [\(3.4\)](#).

**Theorem 3.3.** *Suppose that  $(H_1)$ – $(H_4)$  and [\(3.5\)](#) hold. If*

$$\delta \stackrel{\text{def}}{=} L \sum_{k=0}^K \|T_k\|_2 \eta < 1, \tag{3.13}$$

where

$$L = \max_{1 \leq i \leq n} L_i, \quad \eta = \frac{\max_{1 \leq i \leq n} \{\bar{\alpha}_i\}}{\min_{1 \leq i \leq n} \{\alpha_i \gamma_i\}},$$

then, for any input  $J$ , system (1.6) has a unique equilibrium  $x^*$ , which is globally asymptotically stable, independent of the delays.

**Proof.** Let  $V(t) = V(u)(t) = (1/2)\|u\|_2^2$ . Using

$$\|g(u(t - \tau_k))\|_2^2 = \sum_{j=1}^n g_j^2(u(t - \tau_k)) \leq \sum_{j=1}^n L_j^2 u_j^2(t - \tau_k) \leq L^2 \|u(t - \tau_k)\|_2^2,$$

we can estimate the up-right derivative of  $V$  along solutions of (3.4) as follows:

$$\begin{aligned} D^+V(t) &= -u^T A(u)B(u) + u^T A(u) \sum_{k=0}^K T_k g(u(t - \tau_k)) \\ &\leq -\sum_{i=1}^n \alpha_i(u_i)\beta_i(u_i)u_i + \|u\|_2 \|A(u)\|_2 \sum_{k=0}^K \|T_k\|_2 \|g(u(t - \tau_k))\|_2 \\ &\leq -\sum_{i=1}^n \underline{\alpha}_i \gamma_i u_i^2 + \|u\|_2 \max_{1 \leq i \leq n} \{\bar{\alpha}_i\} \sum_{k=0}^K \|T_k\|_2 \|g(u(t - \tau_k))\|_2 \\ &\leq -\min_{1 \leq i \leq n} \{\underline{\alpha}_i \gamma_i\} \sum_{i=1}^n u_i^2 + \max_{1 \leq i \leq n} \{\bar{\alpha}_i\} \sum_{k=0}^K \|T_k\|_2 L \|u(t - \tau_k)\|_2 \|u\|_2. \end{aligned}$$

Let  $\tau = \max\{\tau_k; k = 1, \dots, K\}$ . Then for those  $t$  satisfying

$$u(t) \neq 0 \quad \text{and} \quad \max_{s \in [-\tau_k, 0]} \|u(t + s)\|_2 = \|u(t)\|_2, \tag{3.14}$$

we have

$$D^+V(t) \leq -\left( \min_{1 \leq i \leq n} \{\underline{\alpha}_i \gamma_i\} - \max_{1 \leq i \leq n} \{\bar{\alpha}_i\} L \sum_{k=0}^K \|T_k\|_2 \right) \|u\|_2^2 = -(1 - \delta) \left( \min_{1 \leq i \leq n} \{\underline{\alpha}_i \gamma_i\} \right) \sum_{i=1}^n u_i^2(t) < 0.$$

From Theorem 2.3 in [13], it follows that the trivial solution of system (3.4) is globally asymptotically stable, and therefore,  $x^*$  is globally asymptotically stable for system (1.6). This completes the proof.  $\square$

### 4. Examples and simulations

In this section, we give some examples to demonstrate our criteria. Examples 4.2–4.4 also show that the three criteria do not include one another.

**Example 4.1.** Consider the all excitatory doubly stochastic connection matrix studied in [8,15], i.e.,  $a_i(u) = 1$ ,  $b_i(u) = u$ ,  $k = 0$ ,  $t_{ii} = 0$ ,  $t_{ij} = 1/(n - 1)$  for  $i \neq j$ ,  $s_j(u) = s(u)$ ,  $j = 1, \dots, n$ , is increasingly sigmoid with neuron gain  $s'(0) = \sup_{x \in \mathbb{R}} s'(x) > 0$ . Then,  $\|T_0\| = 1$ ,  $L = s'(0)$ , and all (3.6), (3.11) and (3.13) reduce to

$$s'(0) < 1. \tag{4.1}$$

Note that (4.1) has been proved by Wu [15] to be a sufficient and necessary condition for such a network to have a unique equilibrium that is a global attractor, independent of delay. However, Theorem 1 in [17] cannot be applied to this case if  $\tau \geq s'(0)$ .



**Example 4.2.** Consider

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = - \begin{pmatrix} 2 + \sin x_1 & 0 \\ 0 & 2 + \cos x_2 \end{pmatrix} \times \left[ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \begin{pmatrix} \frac{1}{16} & \frac{1}{12} \\ \frac{7}{48} & \frac{1}{24} \end{pmatrix} \begin{pmatrix} s_1(x_1(t-\tau_1)) \\ s_2(x_2(t-\tau_1)) \end{pmatrix} - \begin{pmatrix} \frac{1}{24} & \frac{1}{48} \\ \frac{1}{16} & \frac{1}{6} \end{pmatrix} \begin{pmatrix} s_1(x_1(t-\tau_2)) \\ s_2(x_2(t-\tau_2)) \end{pmatrix} + \begin{pmatrix} J_1 \\ J_2 \end{pmatrix} \right], \tag{4.2}$$

where  $s_1(u) = \sin(u)$  and  $s_2 = \cos(u)$  satisfy (H<sub>3</sub>) and (H<sub>4</sub>) with  $L = 1$ . Then  $\theta = 1/48 > 0$ ,  $\mu = -1/16 < 0$  and  $\delta = 1.07 > 1$ . **Theorem 3.1** is applicable and shows that the equilibrium  $x^*$  of (4.2) is globally asymptotically stable. However, both **Theorems 3.2 and 3.3** are not applicable here.

**Example 4.3.** Consider

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = - \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} x(t) + \begin{pmatrix} 1 & \frac{1}{4} \\ 1 & \frac{1}{4} \end{pmatrix} \begin{pmatrix} \sin(x_1(t-\tau)) \\ \cos(2x_2(t-\tau)) \end{pmatrix} + \begin{pmatrix} J_1 \\ J_2 \end{pmatrix}. \tag{4.3}$$

In this example,  $\gamma_1 = \gamma_2 = 2$ ,  $L_1 = 1$ ,  $L_2 = 2$  and hence  $L = 2$ ,  $\eta = 1/2$ . By simple calculations, we have  $\theta = -2 < 0$ ,  $\mu = 3/4$ ,  $\delta = 1.458$ , which shows that **Theorem 3.2** is applicable but both **Theorems 3.1 and 3.3** are not.

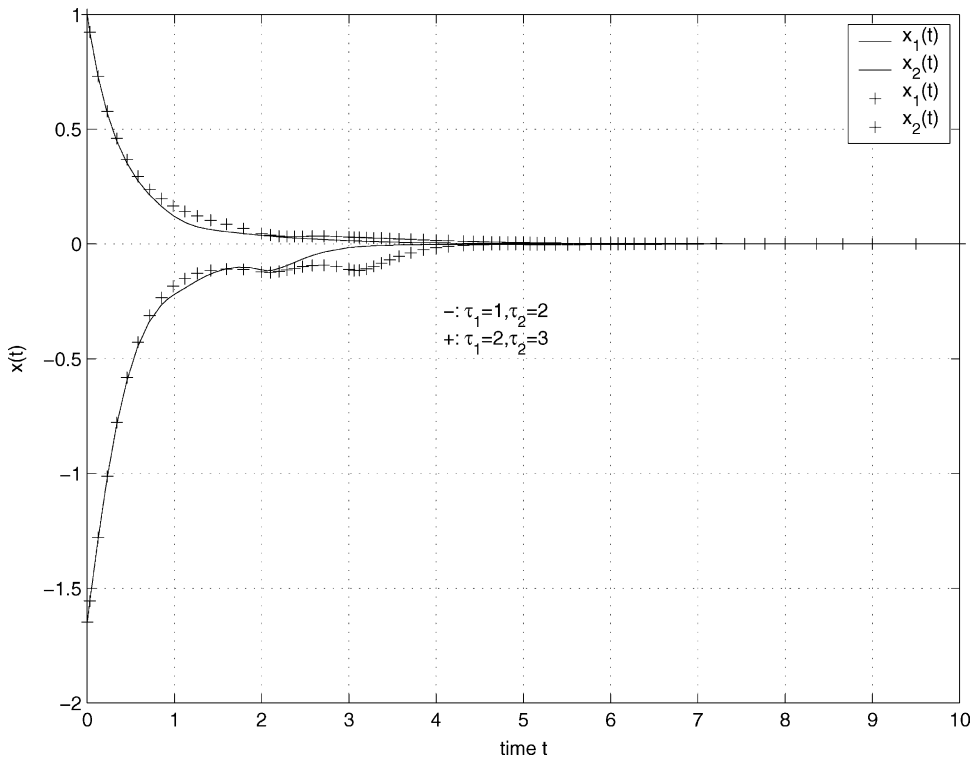


Fig. 1. Numerical solution for (4.2). Here we choose  $J_1 = 5/48$ ,  $J_2 = 5/24$  and two sets of data as: (i)  $\tau_1 = 1$ ,  $\tau_2 = 2$ , and the initial data are:  $x_1(s) = 1$ ,  $x_2(s) = -e^{s+0.5}$ , for  $s \in [-2, 0]$ ; (ii)  $\tau_1 = 2$ ,  $\tau_2 = 3$  and the initial data are:  $x_1(s) = 1$ ,  $x_2(s) = -e^{s+0.5}$ , for  $s \in [-3, 0]$ .

**Example 4.4.** Consider

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} x(t) + \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \sin \left( \frac{2}{\sqrt{3}} x_1(t - \tau) \right) \\ \sin \left( \frac{2}{\sqrt{3}} x_2(t - \tau) \right) \end{pmatrix} + \begin{pmatrix} J_1 \\ J_2 \end{pmatrix}. \tag{4.4}$$

Here,

$$a_i(x_i(t)) = 1, \quad b_i(x_i(t)) = x_i(t), \quad s_i(x_i) = \sin \left( \frac{2x_i}{\sqrt{3}} \right) \quad \text{for } i = 1, 2, \quad K = 1,$$

$$T_0 = 0, \quad \tau_1 = \tau, \quad T_1 = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix},$$

and

$$\theta = 1 - \frac{2}{\sqrt{3}} < 0, \quad \mu = -\frac{1}{3} \quad \text{and} \quad \delta = \sqrt{\frac{2}{3}} < 1.$$

All conditions of [Theorem 3.3](#) are satisfied, therefore, system (4.4) has an equilibrium, which is globally asymptotically stable. None of [Theorems 3.1](#) and [3.2](#) is applicable to this example. The results in [\[17,18\]](#) cannot be applied to (4.4) since the matrix  $T_0 + T_1$  is not symmetric and  $s_i, i = 1, 2$  are not monotonic.

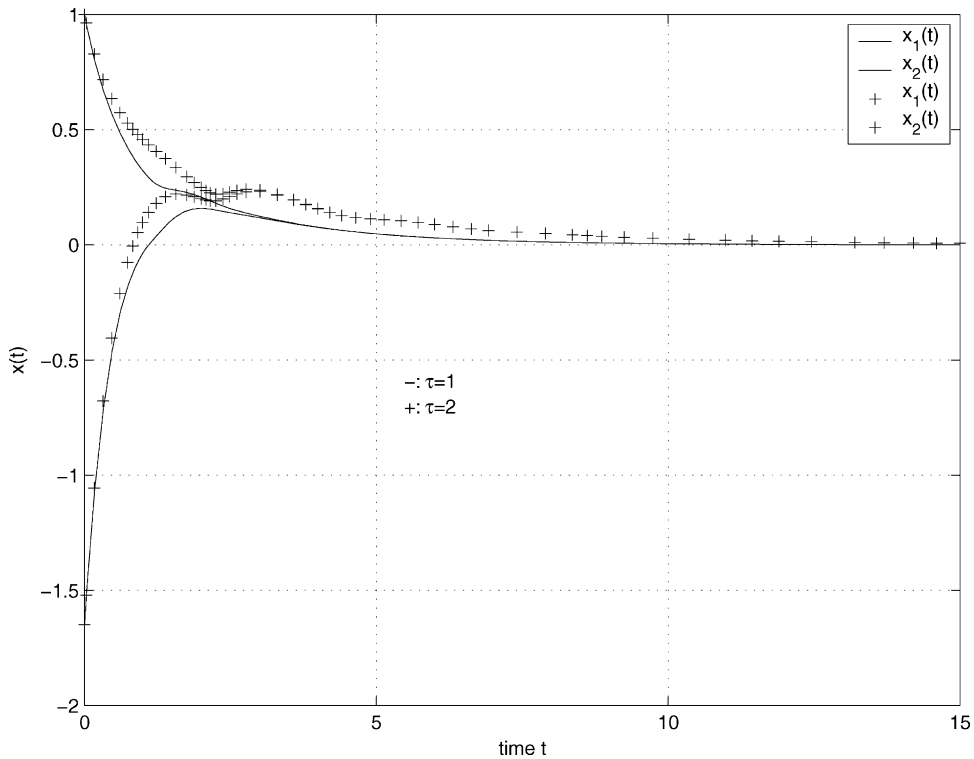


Fig. 2. Numerical solution for (4.3). Here we choose  $J_1 = -1/4 = J_2$  and two sets of data as: (i)  $\tau = 1, \tau = 2, x_1(s) = 1, x_2(s) = -e^{s+0.5}$ , for  $s \in [-1, 0]$ ; (ii)  $\tau = 2, x_1(s) = 1, x_2(s) = -e^{s+0.5}$ , for  $s \in [-2, 0]$ .

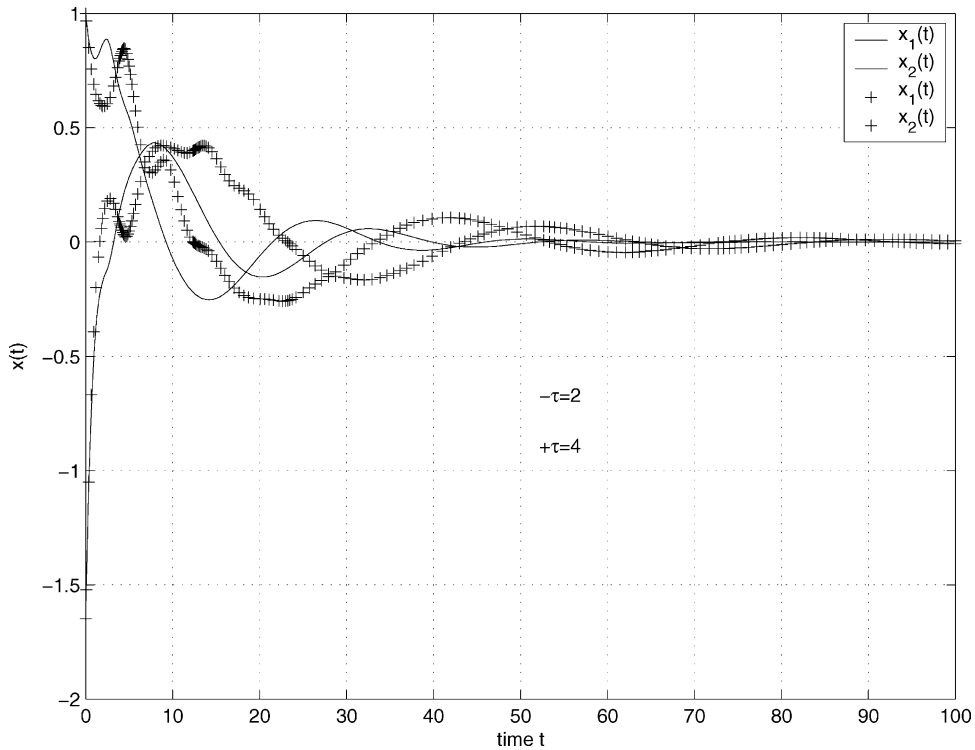


Fig. 3. Numerical solution for (4.4). Here we choose  $J_1 = J_2 = 0$  and two sets of data as: (i)  $\tau = 2$ , the initial data are:  $x_1(s) = 1$ ,  $x_2(s) = -e^{s+0.5}$ , for  $s \in [-2, 0]$ ; (ii)  $\tau = 4$ ,  $x_1(s) = 1$ ,  $x_2(s) = -e^{s+0.5}$ , for  $s \in [-4, 0]$ .

To conclude this section, we present some numerical simulations for Examples 4.2–4.4. The simulations are performed by the DDEs Solver developed by Shampine and Thompson [11], and are shown in Figs. 1–3, respectively.

## 5. Summary

As is widely known, time delays do exist in a neural network, due to the finite speeds of switching and transmission of signals in the network. Although delays do not change the structure of the equilibria of the network, they can destroy the stability of an otherwise stable equilibrium. Even if the delays do not change the stability of an equilibrium, they can affect the basin of the attraction of a stable equilibrium. As far as a unique equilibrium is concerned, seeking conditions under which the unique equilibrium is globally stable is of both theoretical and practical importance for a neural network. In this paper, we discuss the global stability of the Cohen–Grossberg neural network with delays in a very general setting. We first establish the existence of an equilibrium for the network under quite general conditions, using the Brouwer’s fixed point theorem. By constructing appropriate Lyapunov functionals, or Lyapunov functions combined with the Razumikhin technique, we obtain three criteria, each of which guarantees that the equilibrium is globally attractive, which also implies the uniqueness of the equilibrium. All of these criteria are independent of the magnitudes of the delays, and therefore, in the above content, delays are harmless in a network with the structure satisfying one of the criteria. We give some examples to demonstrate our criteria, and also to show that the

three criteria do not include one another. We also present some numerical simulations for these examples, which all support our theoretical conclusions.

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