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Traveling waves of a diffusive SIR epidemic model with general nonlinear incidence and infinitely distributed latency but without demography[☆]

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ABSTRACT

In this paper, we are concerned with existence/non-existence of traveling waves of a diffusive SIR epidemic model with general incidence rate of the form of $f(S)g(I)$ and infinitely distributed latency but without demography. We show that the existence of traveling waves only depends on the basic reproduction number of the corresponding spatial-homogeneous system of delay differential equations, which is determined by the recovery rate, the local properties of f and g and a minimal wave speed c^* that is affected by the distributed delay. The proof of existence of traveling waves is by employing Schauder's fixed point theorem, and the proof of nonexistence is completed with the aid of the bilateral Laplace transform.

1. Introduction

When considering transmission and spread of some infectious diseases that have much short infection duration, one may often neglects the *demographic structure* of the host population and just focus on the *transmission dynamics*. A classic example is the following Kermack–McKendrick ordinary differential

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equation (ODE) model [1]

$$\begin{cases} S'(t) = -\beta S(t)I(t), \\ I'(t) = \beta S(t)I(t) - \gamma I(t), \\ R'(t) = \gamma I(t), \end{cases} \tag{1.1}$$

where $S(t)$, $I(t)$ and $R(t)$ represent the sizes of the susceptible, infected and removed individuals at time t respectively, $\beta > 0$ denotes the transmission coefficient, and $\gamma > 0$ is the recovery/remove rate.

When the spatial spread is concerned, one can incorporate spatial diffusion into (1.1), leading to the following diffusive version:

$$\begin{cases} \dfrac{\partial S(x,t)}{\partial t} = d_1 \dfrac{\partial^2 S(x,t)}{\partial x^2} - \beta S(x,t)I(x,t), \\ \dfrac{\partial I(x,t)}{\partial t} = d_2 \dfrac{\partial^2 I(x,t)}{\partial x^2} + \beta S(x,t)I(x,t) - \gamma I(x,t), \\ \dfrac{\partial R(x,t)}{\partial t} = d_3 \dfrac{\partial^2 R(x,t)}{\partial x^2} + \gamma I(x,t), \end{cases} \tag{1.2}$$

where $S(x,t)$, $I(x,t)$ and $R(x,t)$ represent the populations of the susceptible, infected and removed individuals at location x and time t, respectively. Here, d_1, d_2 and d_3 are positive constants representing the diffusion rates of the susceptible, infective and removed individuals, respectively. For more details, see [2,3] and references cited therein.

In (1.1) and (1.2), mass action infection mechanism is adopted. Since the demographic structure is ignored, the infection mechanism plays a crucial or even a decisive role in such models without demography, and there have been some efforts of exploring various infection mechanisms. For example, Kennedy and Aris [4] replaced the mass action term by an incidence term of the form $f(S)I$ and conducted some linear analysis for the special case with zero diffusion rate of susceptible individuals. Capasso and Serio [5] introduced a saturated incidence rate $\beta SI/(1 + kI)$ into epidemic models to prevent the unboundedness of the contact rate. Liu et al. [6] proposed the incidence rate $kI^l S/(1 + \alpha I^h)$ $(l, h > 0)$ which was also used in many papers, where kI^l measures the infection force of the disease and $1/(1 + \alpha I^h)$ measures the inhibition effect from the behavioral change of the susceptible individuals when their number increases or from the crowding effect of the infective individuals; see also [7]. The effect of nonlinear incidence rate of the general form $f(S)g(I)$ on the dynamics of epidemiological models is also studied in [8]. More examples about nonlinear incidence rates can be found in [9] and the references therein; and it is known that a nonlinear incidence can have important impact on the disease dynamics, see, e.g. [6,7,10] and the references therein. In particular, recently Wang et al. [11] replaced the mass action in (1.2) by the standard incidence function $\beta SI/(S + I)$ and investigated the existence/non-existence of traveling waves. We point out that $\beta SI/(S+I)$ is not in the form of separated functions of S and I respectively. More recently, Shu et al. [12] further considered a more general form $\phi(S, I, R)$ for the incidence function and explored the existence and non-existence of traveling waves of the model system resulted from replacing the mass action in (1.2) by such a general form $\phi(S, I, R)$ satisfying some conditions.

The aforementioned works are for SIR type dynamics of diseases that are transmitted *within a single host population*. When considering transmission of SIR type *vector-borne* (e.g., mosquito-borne) diseases, the number of variables will typically be doubled. However, following the work of Cooke [13], one can actually reduce the number of variables in vector-borne disease models. The main idea is that, because the new infection of mosquitoes is a result of biting infectious hosts, one may assume that the population of infectious vectors is proportional to the populations of infectious host at some previous time due to latency. This idea has been used by some other researchers, see, e.g. [14–18]. To demonstrate this idea, let us denote the populations of susceptible and infectious hosts (e.g., humans) and vectors (e.g. mosquitoes) by $S(t), I(t), V_S(t)$ and $V_I(t)$ respectively. Assume that it takes an infected vector τ time units to become infectious. Since a vector population is usually quite large, it is assumed in [13] that $V_I(t)$ is proportional to $I(t-\tau)$, that is, $V_I(t) = pI(t-\tau)$ for some $p > 0$. If the host's infection rate is of the form $f(S)h(V_I)$, then the force of infection for host at time t is given by

$$f(S(t))h(V_I(t)) = f(S(t))h(pI(t-\tau)) =: f(S(t))g(I(t-\tau)), \tag{1.3}$$

where $g(u) = h(pu)$. Considering the fact that the time it takes an infected vector to become infectious varies from individual to individual, we introduce a function $k(\tau)$ to denote the probability that an infected vector becomes infectious τ time units after infection. With this consideration of variance in latency between individuals, we obtain the infection rate for the host at time t as

$$f(S(t))\int_0^{\infty} k(\xi)g(I(t-\xi))\mathrm{d}\xi = f(S(t))\int_{-\infty}^{t} k(t-s)g(I(s))\mathrm{d}s. \tag{1.4}$$

Adding spatial variable x, this immediately leads to the following model system parallel to (1.2):

$$\begin{cases} \dfrac{\partial S(x,t)}{\partial t} = d_1\dfrac{\partial^2 S(x,t)}{\partial x^2} - f(S(x,t))\displaystyle\int_{-\infty}^{t} k(t-s)g(I(x,s))\mathrm{d}s, \\ \dfrac{\partial I(x,t)}{\partial t} = d_2\dfrac{\partial^2 I(x,t)}{\partial x^2} + f(S(x,t))\displaystyle\int_{-\infty}^{t} k(t-s)g(I(x,s))\mathrm{d}s - \gamma I(x,t), \\ \dfrac{\partial R(x,t)}{\partial t} = d_3\dfrac{\partial^2 R(x,t)}{\partial x^2} + \gamma I(x,t). \end{cases} \tag{1.5}$$

We point out that Bai and Zhang [14] recently considered a spatial case of (1.5) in the sense that $f(S) = \beta S$ and the kernel $k(\xi)$ has support only on a bounded interval $[0,r]$. That is, they studied

$$\begin{cases} \dfrac{\partial S(x,t)}{\partial t} = d_1\dfrac{\partial^2 S(x,t)}{\partial x^2} - \beta S(x,t)\displaystyle\int_{t-r}^{t} k(t-s)g(I(x,s))\mathrm{d}s, \\ \dfrac{\partial I(x,t)}{\partial t} = d_2\dfrac{\partial^2 I(x,t)}{\partial x^2} + \beta S(x,t)\displaystyle\int_{t-r}^{t} k(t-s)g(I(x,s))\mathrm{d}s - \gamma I(x,t). \end{cases} \tag{1.6}$$

Allowing non-compact support for the kernel $k(\xi)$ and allowing a more general $f(S)$ will bring in some challenges. In addition, as we give our assumptions on the function g in the next section and compare with those in [14], we find that the conditions in [14] are strong in some aspects but are still not sufficient for one of its key lemmas. See Remarks 2.1 and 2.2 in Section 2. Obviously, our model system (1.5) can include many (if not all) diffusive SIR models without demography in previous works, and is more difficult to analyze.

For diffusive disease models without demography, typically there is a continuum of disease free equilibria and hence, when considering traveling wave solutions for such model systems, one is naturally confined to, after dropping the R equation since it is decoupled from the other two, those connecting two equilibria $(S_*,0)$ and $(S_\infty,0)$ with $S_\infty < S_*$. Here S_* is related to the initial susceptible population and S_∞ is related to the population size of the susceptible class after the epidemics, which is often referred to as the final size (of the susceptible population). For more details of such traveling waves and the various approaches in exploring the existence of such traveling waves, see [2,3,11,12,19–24].

If there exists such a traveling wave solution, the wave speed accounts for how fast the disease spreads geographically and S_∞ reflects how severe the disease is. Non-existence of such traveling waves implies that the disease cannot invade spatially. Thus, existence/non-existence of traveling wave solutions of the above mentioned type for the model system (1.5) is of practical significance. Mathematically, (1.5) is a diffusive system with predator–prey type interaction, time delay and spatial non-locality, and the research on its existence/non-existence of traveling waves is challenging and is of theoretical importance.

The main goal of this paper is to discuss the existence/non-existence of traveling wave solutions of system (1.5). Our methods in this paper are mainly based on those in [11] and several early studies [24–27]. To prove the existence theorem, we will employ the Schauder's fixed point theorem for a *partially quasi-monotone* mapping, and the challenging and difficult task is to construct an appropriate convex set that is invariant under this mapping. In particular, we give some useful properties of a second-order linear differential operator and its inverse to provide a general scheme for verifying the compact of the mapping. The proof of the nonexistence theorem is based on an idea of applying bilateral Laplace transform, which was first introduced by Carr and Chmaj [28], and further used in [11,14,24]. But our argument in the proof of nonexistence is different from those in the aforementioned works [14,24] and we deduce a contradiction in a *new way*. More precisely, unlike proving the unboundedness of the left-hand of the characteristic equation $\Upsilon(c,\lambda)=0$, we discuss the analyticity of the bilateral Laplace transform of the populations of the infected individuals I at some finite point to derive a contradiction.

The rest of the paper is organized as follows. In Section 2, we give some simple assumptions about the incidence functions and state the two main theorems on the existence and nonexistence of traveling wave solution respectively. Some remarks are also given to compare our results with some previous ones. In Section 3, some preliminary results are given. Similar to [11], notations of differential and integral operators are introduced, and some useful properties of the integral operator are discussed more detailedly. In Section 4, we construct a profile set in which, the task of looking for a traveling wave can be reduced to the existence of fixed point of a mapping. To apply the Schauder's fixed point theorem, we also prove the continuity and compactness of the mapping and the invariance of the profile set under this mapping. In Section 5, to complete the proof of the existence theorem, we further verify that the fixed point satisfies the boundary conditions for the desired traveling wave solution. In Section 6, the nonexistence theorem is proven for the two sub-cases respectively if the existence conditions are violated. In particular, we accomplish it by a way of contradiction different from the existing ones for the first sub-case.

2. Main results

We are interested in positive traveling wave solutions of (1.5). Since $R(x,t)$ does not appear in the equations of $\frac{\partial S}{\partial t}$ and $\frac{\partial I}{\partial t}$, we only need to study the following system:

$$\begin{cases} \dfrac{\partial S(x,t)}{\partial t} = d_1 \dfrac{\partial^2 S(x,t)}{\partial x^2} - f(S(x,t)) \displaystyle\int_{-\infty}^{t} k(t-s) g(I(x,s)) \mathrm{d}s, \\ \dfrac{\partial I(x,t)}{\partial t} = d_2 \dfrac{\partial^2 I(x,t)}{\partial x^2} + f(S(x,t)) \displaystyle\int_{-\infty}^{t} k(t-s) g(I(x,s)) \mathrm{d}s - \gamma I(x,t). \end{cases} \tag{2.1}$$

Letting $\xi = x + ct$, where c is a positive constant corresponding to the wave speed, we will look for special solutions of (2.1) in the form of $(S(\xi), I(\xi))$ where the profile $(S(\xi), I(\xi))$ satisfies the following associated system of differential equations:

$$cS'(\xi) = d_1 S''(\xi) - f(S(\xi)) \int_0^{\infty} k(s) g(I(\xi - cs)) \mathrm{d}s, \tag{2.2a}$$

$$cI'(\xi) = d_2 I''(\xi) + f(S(\xi)) \int_0^{\infty} k(s) g(I(\xi - cs)) \mathrm{d}s - \gamma I(\xi) \tag{2.2b}$$

with the boundary conditions:

$$S(-\infty) = S_*, \quad S(+\infty) = S_\infty \in [0, S_*), \quad I(\pm\infty) = 0. \tag{2.3}$$

Here $S_* > 0$ is a constant reflecting the initial homogeneous population distribution for the susceptible class.

Throughout this paper, we make the following assumptions.

(A1) $k(s)$ is a nonnegative and Lebesgue integrable function on $[0,\infty)$, and $\int_0^\infty k(s)\mathrm{d}s = 1$, $\int_0^\infty sk(s)\mathrm{d}s < \infty$.

(A2) f, g are nonnegative and continuous functions on $[0,\infty)$ satisfying $f(0)=0, f(S_*)>0, g(0)=0$. The function g has a right-hand derivative at origin with $g'_+(0)>0$, and $g(u)\le g'_+(0)u$ for $u\in[0,S_*]$. Further assume that there exist positive constants L_1 and L_2 such that

$$0\le \frac{f(u)-f(v)}{u-v}\le L_1, \tag{2.4}$$

$$0\le \frac{g(u)-g(v)}{u-v}\le L_2 \tag{2.5}$$

hold for all $u,v\in[0,S_*]$ with $u\ne v$.

(A3) There exist $\delta_0>0, \theta>1$ and $\omega>0$ such that $g(u)\ge g'_+(0)u-\omega u^\theta$ for $u\in[0,\delta_0]$.

Remark 2.1. The inequalities (2.4) and (2.5) are equivalent to the conditions that f and g are nondecreasing and Lipschitz continuous on $[0,S_*]$. In this paper, we *do not need* to assume the *monotonicity and Lipschitz continuity* of f and g on $[S_*,\infty)$.

Remark 2.2. The assumption (A3) is used to construct a lower solution (see Lemma 4.3). The condition that $g''_+(0)$ exists can guarantee that this assumption holds. Note that although (A3) is relatively weak, it is not implied by (A2). Indeed, we can find a function g which satisfies (A2), but not (A3). For example,

$$g(u)=\begin{cases}0, & u=0,\\ u+\frac{u}{\ln u}, & u\in(0,\mathrm{e}^{-2}),\\ \frac{1}{4}(u+\mathrm{e}^{-2}), & u\in[\mathrm{e}^{-2},\infty).\end{cases}$$

It is worth noting that the example g above satisfies $g'_+(0)=1$ and the conditions (H1) and (H2) in [14]. However, since this function does not satisfy the assumption (A3), neither can it satisfy the inequality $I-g(I)\le I^2$ for all $0<I<\delta_0$ which was used in the proof of Lemma 2.4 in [14]. This indicates that the proof of Lemma 2.4 in [14] is incorrect.

We are now in a position to state main results on the existence of a traveling wave solution to (2.1).

Theorem 2.1. *Assume that (A1)–(A3) hold. If $R_0:=f(S_*)g'_+(0)/\gamma>1$, there exists a $c^*>0$ such that for each $c>c^*$ system (2.1) admits a non-trivial and positive traveling wave solution (S,I) satisfying the boundary conditions (2.3). Furthermore, S is nonincreasing on $\mathbb{R}$, $0<I(\xi)\le S_*-S_\infty$ for all $\xi\in\mathbb{R}$, and*

$$\gamma\int_{-\infty}^{\infty} I(\xi)\mathrm{d}\xi=\int_{-\infty}^{\infty} f(S(\xi))\left(\int_0^\infty k(s)g(I(\xi-cs))\mathrm{d}s\right)\mathrm{d}\xi=c(S_*-S_\infty). \tag{2.6}$$

Remark 2.3. Suppose that a function g satisfies the associated conditions in the assumption (A2). Let

$$\tilde g(u)=\begin{cases}g(u), & u\in[0,S_*],\\ g(S_*), & u\in(S_*,\infty).\end{cases}$$

It is easily seen that, if for the function $\tilde g$ above there exists a traveling wave solution of system (2.1) satisfying the boundary conditions (2.3) and $0<I(\xi)\le S_*-S_\infty\le S_*$ for all $\xi\in\mathbb{R}$, this traveling wave solution is also the one of system (2.1) with the function g. Therefore, we only need to prove Theorem 2.1 under the following additional condition:

$$g(u)\equiv g(S_*)\quad \text{for all } u>S_*.$$

Together with the assumption (A2), this implies that g is a bounded function and $g(u)\le g'_+(0)u$ for all $u\ge 0$. We can also easily verify that g is nondecreasing and Lipschitz continuous on $[0,\infty)$, that is, (2.5) holds for all $u,v\in[0,\infty)$ with $u\ne v$.

Remark 2.4. In order to prove the existence of traveling wave, the key step of a method via Schauder fixed point theorem is to construct a convex set bounded by some upper and lower solutions. Different from the monotone systems in which comparison principle guarantees the upper and lower solutions are independent, for non-monotone systems, the upper and lower solutions are coupled [12]. Moreover, it is more difficult to construct the upper and lower solutions for nonmonotone systems with time delay [24–27]. In this paper, the incidence infection function we consider is separable. A non-separable infection function incorporated into our model with time delay will make the construction of upper and lower solutions even more challenging. We also need to find some constraints on the non-separable infection function to ensure the invariance of the convex set under the mapping. We leave them as our future work.

On the other hand, we can state the following theorem on the nonexistence of a traveling wave solution to (2.1), which implies that c^* is the minimal wave speed.

Theorem 2.2. *Assume that (A1)–(A3) hold. If $R_0 > 1$ and $c < c^*$, or $R_0 < 1$, there is no nontrivial and nonnegative traveling wave solution (S, I) to system (2.1), satisfying the boundary conditions (2.3).*

3. Preliminaries

Given $\mu > 0$, we first introduce the function spaces

$$\mathscr{L}_\mu = \left\{\phi : \mathbb{R} \to \mathbb{R} \,\middle|\, \mathrm{e}^{-\mu|\cdot|}\phi(\cdot) \in L^\infty(\mathbb{R}, \mathbb{R})\right\},$$

$$\mathscr{B}_\mu = \left\{\phi \in C(\mathbb{R}, \mathbb{R}) \,\middle|\, \sup_{\xi\in\mathbb{R}} \mathrm{e}^{-\mu|\xi|}|\phi(\xi)| < \infty\right\},$$

$$\mathscr{B}_\mu \times \mathscr{B}_\mu = \{\Phi = (\phi_1, \phi_2) \,|\, \phi_i \in \mathscr{B}_\mu, i = 1, 2\}.$$

Moreover, $\mathscr{B}_\mu$ is equipped with the exponential decay norm defined by

$$|\phi|_\mu = \sup_{\xi\in\mathbb{R}} \mathrm{e}^{-\mu|\xi|}|\phi(\xi)|,$$

and $\mathscr{B}_\mu \times \mathscr{B}_\mu$ is equipped with the norm defined by

$$|\Phi|_\mu = \max\{|\phi_1|_\mu, |\phi_2|_\mu\}.$$

It is easy to show that $(\mathscr{B}_\mu, |\cdot|_\mu)$ and $(\mathscr{B}_\mu \times \mathscr{B}_\mu, |\cdot|_\mu)$ are Banach spaces.

Next, similar to [11], we will introduce the second-order linear differential operator Δ_i and its inverse Δ_i^{-1} for $i = 1, 2$. Note that for any positive number $\alpha_i (i = 1, 2)$, the equation

$$-d_i\lambda^2 + c\lambda + \alpha_i = 0$$

has two real roots

$$-\lambda_{1i} = \frac{c - \sqrt{c^2 + 4d_i\alpha_i}}{2d_i} < 0, \quad \lambda_{2i} = \frac{c + \sqrt{c^2 + 4d_i\alpha_i}}{2d_i} > 0. \tag{3.1}$$

Then the second-order linear differential operator Δ_i is defined as

$$\Delta_i h := -d_i h'' + ch' + \alpha_i h. \tag{3.2}$$

We can also define the corresponding integral operator Δ_i^{-1} as

$$(\Delta_i^{-1}h)(\xi) := \frac{1}{d_i(\lambda_{2i} + \lambda_{1i})}\left[\int_{-\infty}^{\xi} \mathrm{e}^{-\lambda_{1i}(\xi-x)}h(x)\mathrm{d}x + \int_{\xi}^{\infty} \mathrm{e}^{\lambda_{2i}(\xi-x)}h(x)\mathrm{d}x\right]. \tag{3.3}$$

The fact that $\lambda_{2i} > \lambda_{1i}$ for $i = 1, 2$ implies that the integral $\Delta_i^{-1}h$ is well-defined for any $h \in \mathscr{L}_{\lambda_{1i}}$. Let μ_0 and μ be two positive numbers such that

$$\mu_0 < \mu < \min\{\lambda_{11}, \lambda_{12}\}. \tag{3.4}$$

Note that

$$\mathscr{B}_{\mu_0} \subset \mathscr{B}_{\mu} \subset \mathscr{L}_{\lambda_{1i}}.$$

Hence, the operator Δ_i^{-1} can be restricted in the Banach space $\mathscr{B}_{\mu_0}$ or $\mathscr{B}_{\mu}$, and we have the following Lemma about the properties of Δ_i^{-1}.

Lemma 3.1. *For any μ_0 and μ satisfying $0 < \mu_0 < \mu < \min\{\lambda_{11}, \lambda_{12}\}$, we have*
(i) *$\Delta_i^{-1} : \mathscr{B}_{\mu} \to \mathscr{B}_{\mu}$ is a bounded linear operator;*
(ii) *$\Delta_i^{-1} : \mathscr{B}_{\mu_0} \to \mathscr{B}_{\mu}$ is a compact operator.*

Proof. Assume that $h \in \mathscr{B}_{\mu}$, then

$$|(\Delta_i^{-1}h)(\xi)|e^{-\mu|\xi|} \le |h|_{\mu} M(\xi)$$

for all $\xi \in \mathbb{R}$, where

$$M(\xi) := \frac{e^{-\mu|\xi|}}{d_i(\lambda_{2i} + \lambda_{1i})}\left[\int_{-\infty}^{\xi} e^{-\lambda_{1i}(\xi - x) + \mu|x|}dx + \int_{\xi}^{\infty} e^{\lambda_{2i}(\xi - x) + \mu|x|}dx\right].$$

$M(\xi)$ is continuous and differentiable with respect to ξ on $\mathbb{R}$, and by L'Hôpital's rule we can show that

$$M(-\infty) = \frac{1}{\lambda_{1i} - \mu} + \frac{1}{\lambda_{2i} + \mu}, \quad M(\infty) = \frac{1}{\lambda_{1i} + \mu} + \frac{1}{\lambda_{2i} - \mu}.$$

Hence, there exists a constant $M_1 > 0$ such $|M(\xi)| < M_1$ for all $\xi \in \mathbb{R}$, and

$$|(\Delta_i^{-1}h)(\xi)|e^{-\mu|\xi|} \le M_1|h|_{\mu}. \tag{3.5}$$

This inequality implies that $\Delta_i^{-1}h \in \mathscr{B}_{\mu}$ for all $h \in \mathscr{B}_{\mu}$, and

$$|\Delta_i^{-1}h|_{\mu} \le M_1|h|_{\mu}.$$

Therefore, $\Delta_i^{-1} : \mathscr{B}_{\mu} \to \mathscr{B}_{\mu}$ is a bounded linear operator.

To prove that $\Delta_i^{-1} : \mathscr{B}_{\mu_0} \to \mathscr{B}_{\mu}$ is compact, we shall employ Arzela–Ascoli theorem and a standard diagonal process. First, similar to the proof above, we have $\Delta_i^{-1} : \mathscr{B}_{\mu_0} \to \mathscr{B}_{\mu_0}$ is also bounded, that is, there exists a constant $M_0 > 0$ such that

$$|\Delta_i^{-1}h|_{\mu_0} \le M_0|h|_{\mu_0}.$$

Assume that $\{h_n\}$ is a bounded sequence in $\mathscr{B}_{\mu_0}$, and denote $u_n = \Delta_i^{-1}h_n$. Thus, $\{u_n\}$ is a bounded sequence in $\mathscr{B}_{\mu_0}$. Note that $|u_n(\xi)| \le e^{\mu_0|\xi|}|u_n|_{\mu_0}$ for all $\xi \in \mathbb{R}$. Hence, for each fixed $k \in \mathbb{N}$, $\{u_n\}$ can be viewed as a bounded sequence in $C([-k, k], \mathbb{R})$ with respect to the maximum norm, i.e., $\{u_n(\xi)\}$ is uniformly bounded on $[-k, k]$. Since

$$\begin{aligned}\left|(\Delta_i^{-1}h)'(\xi)\right| &= \frac{1}{d_i(\lambda_{2i} + \lambda_{1i})}\left|-\lambda_{1i}\int_{-\infty}^{\xi} e^{-\lambda_{1i}(\xi - x)}h(x)dx + \lambda_{2i}\int_{\xi}^{\infty} e^{\lambda_{2i}(\xi - x)}h(x)dx\right| \\ &\le \lambda_{2i}|h|_{\mu_0}\left|[\Delta_i^{-1}(e^{\mu_0|\cdot|})](\xi)\right|\end{aligned}$$

$$\begin{aligned}&\leq \lambda_{2i}|h|_{\mu_0} M_0 \left|e^{\mu_0|\cdot|}\right|_{\mu_0} e^{\mu_0|\xi|}\\&= \lambda_{2i}|h|_{\mu_0} M_0 e^{\mu_0|\xi|},\end{aligned}$$

$\{u_n(\xi)\}$ is also equi-continuous on $[-k,k]$. According to the Arzela–Ascoli theorem and the standard diagonal process, we can extract a subsequence $\{h_{n_k}\}$ such that $u_{n_k} = \Delta_i^{-1} h_{n_k}$ converges in any $C([-k,k],\mathbb{R})$ with respect to the maximum norm. We shall also prove that the subsequence $\{u_{n_k}\}$ converges in $\mathscr{B}_\mu$ with respect to the norm $|\cdot|_\mu$.

Given any $\epsilon > 0$, there exists an integer $N > 0$ independent to u_{n_k} such that

$$\begin{aligned}e^{-\mu|\xi|}|u_{n_k}(\xi) - u_{n_m}(\xi)| &\leq e^{-\mu|\xi|}(|u_{n_k}(\xi)| + |u_{n_m}(\xi)|)\\&\leq e^{-(\mu-\mu_0)|\xi|}(|u_{n_k}|_{\mu_0} + |u_{n_m}|_{\mu_0})\\&< \epsilon\end{aligned}$$

for any $|\xi| > N$ and $k, m \in \mathbb{N}$. Since $\{u_{n_k}\}$ converges uniformly on the interval $I_N = [-N,N]$, we can find $K \in \mathbb{N}$ such that

$$e^{-\mu|\xi|}|u_{n_k}(\xi) - u_{n_m}(\xi)| \leq |u_{n_k}(\xi) - u_{n_m}(\xi)| < \epsilon$$

for any $\xi \in [-N,N]$ and $k, m > K$. The above two inequalities imply that $\{u_{n_k}\}$ is a Cauchy sequence in $\mathscr{B}_\mu$. Since $\mathscr{B}_\mu$ is a Banach space, $\{u_{n_k}\}$ converges in $\mathscr{B}_\mu$. This proves the compactness of the operator $\Delta_i^{-1} : \mathscr{B}_{\mu_0} \to \mathscr{B}_\mu$. The proof of the lemma is complete. □

From Lemma 3.1 in [11], we can obtain that

$$\Delta_i(\Delta_i^{-1} h) = h \tag{3.6}$$

for any $h \in \mathscr{B}_\mu$ (in fact, it also holds for any $h \in \mathscr{L}_{\lambda_{1i}}$), and

$$\Delta_i^{-1}(\Delta_i h) = h \tag{3.7}$$

for any $h \in \mathscr{B}_\mu$ such that $h', h'' \in \mathscr{B}_\mu$. Thus, Δ_i^{-1} is actually the inverse operator of Δ_i in some sense. Furthermore, according to the theory of impulsive systems [29], we have the following more general conclusion, which is an extension to Lemma 2.1 in [30] and is useful for the proof of invariance of the cone in the sequel.

Lemma 3.2. *Let $i = 1$ or 2. Assume that $h \in \mathscr{L}_{\lambda_{1i}}$ satisfies the following conditions: (i) $h', h'' \in \mathscr{L}_{\lambda_{1i}}$;(ii) h'' is continuous on $\mathbb{R}\backslash\{\xi_j\}$, where $\{\xi_j\}$ is a finite increasing sequence; (iii) $h(\xi_j+), h(\xi_j-)$, $h'(\xi_j+)$ and $h'(\xi_j-)$ exist. Then $\Delta_i^{-1}(\Delta_i h) \in C(\mathbb{R},\mathbb{R})$ and*

$$[\Delta_i^{-1}(\Delta_i h)](\xi) = h(\xi) + \frac{1}{\lambda_{2i}+\lambda_{1i}}\left(\sum_{\xi_j>\xi}(b_j + \lambda_{1i}a_j)e^{\lambda_{2i}(\xi-\xi_j)} + \sum_{\xi_j<\xi}(b_j - \lambda_{2i}a_j)e^{-\lambda_{1i}(\xi-\xi_j)}\right) \tag{3.8}$$

for any $\xi \notin \{\xi_j\}$, where $a_j = h(\xi_j+) - h(\xi_j-)$ and $b_j = h'(\xi_j+) - h'(\xi_j-)$. In particular, further assume $h \in C(\mathbb{R},\mathbb{R})$, (3.8) can be reduced to

$$[\Delta_i^{-1}(\Delta_i h)](\xi) = h(\xi) + \frac{1}{\lambda_{2i}+\lambda_{1i}}\left(\sum_{\xi_j\geq\xi} b_j e^{\lambda_{2i}(\xi-\xi_j)} + \sum_{\xi_j<\xi} b_j e^{-\lambda_{1i}(\xi-\xi_j)}\right) \tag{3.9}$$

for all $\xi \in \mathbb{R}$.

Proof. The assumptions of h imply that $\Delta_i h = -d_i h'' + ch' + \alpha_i h \in \mathscr{L}_{\lambda_{1i}}$. Thus, $\Delta_i^{-1}(\Delta_i h)$ is well defined, and the function $[\Delta_i^{-1}(\Delta_i)h](\xi)$ is continuous with respect to ξ on $\mathbb{R}$. By Theorem 87 in [29], (3.8) holds for any $\xi \notin \{\xi_j\}$. Furthermore, the continuity of h is equivalent to $a_j = 0$ for all j. Thus, with this additional assumption, the continuity of $[\Delta_i^{-1}(\Delta_i h)]$ implies that (3.9) holds for all $\xi \in \mathbb{R}$. □

Finally, note that the characteristic equation corresponding to the linearization of Eq. (2.2b) at $(S_*, 0)$ is

$$\Upsilon(c,\lambda) := d_2\lambda^2 - c\lambda + f(S_*)g_+'(0)\int_0^\infty k(s)\mathrm{e}^{-\lambda cs}\mathrm{d}s - \gamma = 0. \tag{3.10}$$

Since

$$\begin{aligned}
&\Upsilon(c,0) = f(S_*)g_+'(0) - \gamma,\\
&\lim_{\lambda\to\infty}\Upsilon(c,\lambda) = \infty \quad \text{for all } c \ge 0, \quad \lim_{c\to\infty}\Upsilon(c,\lambda) = -\infty \quad \text{for all } \lambda > 0,\\
&\frac{\partial\Upsilon(c,\lambda)}{\partial c} = -\lambda[1 + f(S_*)g_+'(0)\int_0^\infty sk(s)\mathrm{e}^{-\lambda cs}\mathrm{d}s] < 0,\\
&\frac{\partial^2\Upsilon(c,\lambda)}{\partial\lambda^2} = 2d_2 + c^2 f(S_*)g_+'(0)\int_0^\infty s^2k(s)\mathrm{e}^{-\lambda cs}\mathrm{d}s > 0,
\end{aligned}$$

then it is easy to obtain the following lemma, which is similar to Lemma 2.5 in [31] and Lemma 2.1 in [32] and Lemma 2.2 in [33].

Lemma 3.3. *Assume that $R_0 = f(S_*)g_+'(0)/\gamma > 1$. Then there exists a positive constant*

$$c^* := \sup\{c > 0 \,|\, \Upsilon(c,\lambda) > 0 \quad \textit{for all}\ \lambda > 0\},$$

and the following statements hold.

(i) *If $c = c^*$, then $\Upsilon(c,\lambda) = 0$ has a unique positive root λ^*.*

(ii) *If $c > c^*$, then $\Upsilon(c,\lambda) = 0$ has two distinct positive roots $\lambda_1 < \lambda_2$, and*

$$\Upsilon(c,\lambda)\begin{cases} < 0, & \lambda \in (\lambda_1,\lambda_2),\\ > 0, & \lambda \in [0,\lambda_1)\cup(\lambda_2,\infty).\end{cases}$$

In what follows, we always suppose $R_0 > 1$ and fix $c > c^*$ except for Section 6.

4. Construction of an invariant cone

In order to construct a profile set, we firstly state three lemmas.

Lemma 4.1. *The function $I_+(\xi) = \mathrm{e}^{\lambda_1\xi}$ satisfies the following differential inequality*

$$cI_+'(\xi) \ge d_2 I_+''(\xi) + f(S_*)\int_0^\infty k(s)g(I_+(\xi - cs))\mathrm{d}s - \gamma I_+(\xi). \tag{4.1}$$

Proof. By the characteristic equation (3.10), it is easy to verify that

$$cI_+'(\xi) = d_2 I_+''(\xi) + f(S_*)g_+'(0)\int_0^\infty k(s)I_+(\xi - cs)\mathrm{d}s - \gamma I_+(\xi).$$

Noting that $g(u) \le g_+'(0)u$ for all $u \ge 0$ (see Remark 2.3), we obtain that $g(I_+(\xi - cs)) \le g_+'(0)I_+(\xi - cs)$ always holds. Therefore, (4.1) is true and the proof is completed □

Lemma 4.2. *There exist* $\beta \in (0, \lambda_1)$ *and* $\varrho \geq S_*$ *such that the function* $S_-(\xi) = \max\{S_* - \varrho e^{\beta\xi}, 0\}$ *satisfies*

$$cS'_-(\xi) \leq d_1 S''_-(\xi) - f(S_-(\xi)) \int_0^\infty k(s) g(I_+(\xi - cs)) ds \tag{4.2}$$

for any $\xi \neq \zeta_1$*, where* $\zeta_1 := \frac{1}{\beta} \ln \frac{S_*}{\varrho}$.

Proof. We first choose a positive number $\beta = \frac{1}{2}\min\{\lambda_1, \frac{c}{d_1}\}$, then $c\beta - d_1\beta^2 > 0$. Let

$$\varrho := \max\left\{ S_*, \frac{f(S_*) g'_+(0) \int_0^\infty k(s) e^{-\lambda_1 cs} ds}{c\beta - d_1\beta^2} \right\},$$

then $\zeta_1 = \frac{1}{\beta} \ln \frac{S_*}{\varrho} \leq 0$ and

$$-(c\beta - d_1\beta^2)\varrho + f(S_*) g'_+(0) \int_0^\infty k(s) e^{-\lambda_1 cs} ds \leq 0.$$

If $\xi > \zeta_1$, then $S_-(\xi) = 0$. Clearly, (4.2) holds for $\xi > \zeta_1$.

If $\xi < \zeta_1$, then $S_-(\xi) = S_* - \varrho e^{\beta\xi}$. Recalling the choice of β and ϱ, and noting that $g(u) \leq g'_+(0)u$ for all $u \geq 0$ (see Remark 2.3), we have

$$\begin{aligned}
& cS'_-(\xi) - d_1 S''_-(\xi) + f(S_-(\xi)) \int_0^\infty k(s) g(I_+(\xi - cs)) ds \\
&= -(c\beta - d_1\beta^2)\varrho e^{\beta\xi} + f(S_* - \varrho e^{\beta\xi}) \int_0^\infty k(s) g(e^{\lambda_1\xi - \lambda_1 cs}) ds \\
&\leq -(c\beta - d_1\beta^2)\varrho e^{\beta\xi} + f(S_*) g'_+(0) e^{\lambda_1\xi} \int_0^\infty k(s) e^{-\lambda_1 cs} ds \\
&\leq \left[-(c\beta - d_1\beta^2)\varrho + f(S_*) g'_+(0) \int_0^\infty k(s) e^{-\lambda_1 cs} ds \right] e^{\beta\xi} \\
&\leq 0.
\end{aligned}$$

Hence, (4.2) also holds for $\xi < \zeta_1$. This completes the proof. ☐

Lemma 4.3. *Let* $\eta = \min\{\beta, (\theta - 1)\lambda_1, \frac{1}{2}(\lambda_2 - \lambda_1)\}$. *Then there exists* $M \geq 1$ *such that* $I_-(\xi) = \max\{e^{\lambda_1\xi}(1 - Me^{\eta\xi}), 0\}$ *satisfies*

$$cI'_-(\xi) \leq d_2 I''_-(\xi) + f(S_-(\xi)) \int_0^\infty k(s) g(I_-(\xi - cs)) ds - \gamma I_-(\xi) \tag{4.3}$$

for any $\xi \neq \zeta_2$*, where* $\zeta_2 := \frac{1}{\eta} \ln \frac{1}{M}$.

Proof. Since $\lambda_1 < \eta + \lambda_1 < \lambda_2$, by Lemma 3.3(ii) we have $\Upsilon(c, \lambda_1 + \eta) < 0$. Denote

$$D := \frac{1}{-\Upsilon(c, \lambda_1 + \eta)} \left(L_1 g'_+(0)\varrho \int_0^\infty k(s) e^{-\lambda_1 cs} ds + \omega f(S_*) \int_0^\infty k(s) e^{-\theta\lambda_1 cs} ds \right), \tag{4.4}$$

which is a positive constant. In fact, this expression is deduced later.

Let

$$M := \max\{e^{-\zeta_1\eta}, \delta_0^{-\frac{\eta}{\lambda_1}}, D\}, \tag{4.5}$$

where ζ_1 is in Lemma 4.2, then we have

$$\zeta_2 = \frac{1}{\eta} \ln \frac{1}{M} \leq \zeta_1 \leq 0, \quad \zeta_2 \leq \frac{1}{\lambda_1} \ln \delta_0.$$

If $\xi > \zeta_2$, then $I_-(\xi) = 0$. Noting that

$$f(S_-(\xi)) \int_0^\infty k(s)g(I_-(\xi - cs))\mathrm{d}s \geq 0,$$

(4.3) holds for $\xi > \zeta_2$.

If $\xi < \zeta_2$, then $I_-(\xi) = \mathrm{e}^{\lambda_1 \xi}(1 - M\mathrm{e}^{\eta\xi})$, $S_-(\xi) = S_* - \varrho \mathrm{e}^{\beta\xi}$. Thus, $I_-(\xi) \leq \mathrm{e}^{\lambda_1\xi} \leq \mathrm{e}^{\lambda_1\zeta_2} \leq \delta_0$. By the assumption (A3), we can obtain that $g(I_-(\xi - cs)) \geq g'_+(0)I_-(\xi - cs) - \omega I_-^\theta(\xi - cs)$ for $s \geq 0$. Therefore,

$$\begin{aligned}
&d_2 I''_-(\xi) - cI'_-(\xi) + f(S_-(\xi)) \int_0^\infty k(s)g(I_-(\xi - cs))\mathrm{d}s - \gamma I_-(\xi) \\
&\geq d_2 I''_-(\xi) - cI'_-(\xi) + f(S_-(\xi)) \int_0^\infty k(s) \left[g'_+(0)I_-(\xi - cs) - \omega I_-^\theta(\xi - cs)\right] \mathrm{d}s - \gamma I_-(\xi) \\
&\geq d_2 \left[\lambda_1^2 - (\lambda_1 + \eta)^2 M\mathrm{e}^{\eta\xi}\right] \mathrm{e}^{\lambda_1\xi} - c\left[\lambda_1 - (\lambda_1 + \eta)M\mathrm{e}^{\eta\xi}\right] \mathrm{e}^{\lambda_1\xi} - \gamma \mathrm{e}^{\lambda_1\xi}(1 - Me^{\eta\xi}) \\
&\quad + g'_+(0) f(S_-(\xi)) \int_0^\infty k(s)\mathrm{e}^{\lambda_1(\xi - cs)} \left[1 - Me^{\eta(\xi - cs)}\right] \mathrm{d}s - \omega f(S_-(\xi)) \int_0^\infty k(s)\mathrm{e}^{\lambda_1\theta(\xi - cs)}\mathrm{d}s \\
&\geq -\Upsilon(c, \lambda_1 + \eta)M\mathrm{e}^{(\lambda_1+\eta)\xi} - g'_+(0)\left[f(S_*) - f(S_* - \varrho\mathrm{e}^{\beta\xi})\right] \int_0^\infty k(s)\mathrm{e}^{\lambda_1(\xi - cs)}\mathrm{d}s \\
&\quad + Mg'_+(0)\left[f(S_*) - f(S_* - \varrho\mathrm{e}^{\beta\xi})\right] \int_0^\infty k(s)\mathrm{e}^{(\lambda_1+\eta)(\xi - cs)}\mathrm{d}s - \omega f(S_* - \varrho\mathrm{e}^{\beta\xi}) \int_0^\infty k(s)\mathrm{e}^{\lambda_1\theta(\xi - cs)}\mathrm{d}s \\
&\geq \left[-\Upsilon(c, \lambda_1 + \eta)M\mathrm{e}^{\eta\xi} - L_1 g'_+(0)\varrho\mathrm{e}^{\beta\xi} \int_0^\infty k(s)\mathrm{e}^{-\lambda_1 cs}\mathrm{d}s - \omega f(S_*)\mathrm{e}^{(\theta-1)\lambda_1\xi} \int_0^\infty k(s)\mathrm{e}^{-\theta\lambda_1 cs}\mathrm{d}s\right] \mathrm{e}^{\lambda_1\xi} \\
&\geq \left[-\Upsilon(c, \lambda_1 + \eta)M - L_1 g'_+(0)\varrho \int_0^\infty k(s)\mathrm{e}^{-\lambda_1 cs}\mathrm{d}s - \omega f(S_*) \int_0^\infty k(s)\mathrm{e}^{-\theta\lambda_1 cs}\mathrm{d}s\right] \mathrm{e}^{(\lambda_1+\eta)\xi}.
\end{aligned}$$

Here, in the last step we have used the fact that $\xi < \zeta_2 \leq 0$, $\eta \leq \beta$ and $\eta \leq (\theta - 1)\lambda_1$. It follows from (4.4) and (4.5) that (4.3) holds for $\xi < \zeta_2$. This completes the proof of the lemma. □

Setting

$$\alpha_1 := \max\{d_1\lambda_1^2 + c\lambda_1 + 1, L_1 g(S_*)\}, \quad \alpha_2 := \max\{d_2\lambda_1^2 + c\lambda_1 + 1, \gamma\},$$

it follows from (3.1) that the inequality

$$\lambda_1 < \min\{\lambda_{11}, \lambda_{12}\}$$

holds. Then we can choose μ_0, μ so that

$$\lambda_1 < \mu_0 < \mu < \min\{\lambda_{11}, \lambda_{12}\}.$$

Let $S_+(\xi) \equiv S_*$. Using these four functions S_+, S_-, I_+ and I_- to specify the boundary of a profile set, we can define this set as

$$\varGamma := \{(S, I) \in \mathscr{B}_\mu \times \mathscr{B}_\mu : S_- \leq S \leq S_+, I_- \leq I \leq I_+\}. \tag{4.6}$$

Clearly, $\varGamma$ is a nonempty, bounded, closed and convex set in $\mathscr{B}_\mu \times \mathscr{B}_\mu$. We will look for traveling wave solutions of system (2.1) in $\varGamma$, that is, $(S, I) \in \varGamma$ satisfying system (2.2) with the boundary conditions (2.3).

We are now ready to define $H_1(S, I), H_2(S, I)$ by

$$H_1(S, I)(\xi) := \alpha_1 S(\xi) - f(S(\xi)) \int_0^\infty k(s)g(I(\xi - cs))\mathrm{d}s,$$

$$H_2(S, I)(\xi) := \alpha_2 I(\xi) + f(S(\xi)) \int_0^\infty k(s)g(I(\xi - cs))\mathrm{d}s - \gamma I(\xi)$$

for any $(S, I) \in \varGamma$. We give the following lemma about the properties of the operators H_1 and H_2.

Lemma 4.4. *The following statements hold.*

(i) *$H_i(\Gamma)$ is a bounded set in $\mathscr{B}_{\mu_0}$ for $i = 1, 2$;*

(ii) *$H_i : \Gamma \to \mathscr{B}_\mu$ is a continuous mapping for $i = 1, 2$.*

Proof. Due to the monotonicity of f and the boundedness of g (see Remark 2.3), for any $(S, I) \in \Gamma$ we have

$$\begin{aligned}|H_1(S,I)(\xi)| &\le \alpha_1 S(\xi) + f(S(\xi))\int_0^\infty k(s)g(I(\xi - cs))\mathrm{d}s\\ &\le \alpha_1 S_* + f(S_*)\int_0^\infty k(s)g(S_*)\mathrm{d}s\\ &\le \alpha_1 S_* + f(S_*)g(S_*),\end{aligned}$$

$$\begin{aligned}|H_2(S,I)(\xi)| &\le (\alpha_2 - \gamma)I(\xi) + f(S(\xi))\int_0^\infty k(s)g(I(\xi - cs))\mathrm{d}s\\ &\le (\alpha_2 - \gamma)I_+(\xi) + f(S_*)\int_0^\infty k(s)g(S_*)\mathrm{d}s\\ &\le (\alpha_2 - \gamma)\mathrm{e}^{\lambda_1\xi} + f(S_*)g(S_*).\end{aligned}$$

Since $\lambda_1 < \mu_0$, $|H_1(S,I)|_{\mu_0} \le \alpha_1 S_* + f(S_*)g(S_*)$ and $|H_2(S,I)|_{\mu_0} \le \alpha_2 - \gamma + f(S_*)g(S_*)$. Hence, the sets $H_1(\Gamma)$ and $H_2(\Gamma)$ are bounded in $\mathscr{B}_{\mu_0}$.

Recalling that $\mathscr{B}_{\mu_0} \subset \mathscr{B}_\mu$, the operator H_1 can also be viewed as a mapping from Γ to $\mathscr{B}_\mu$. Given $\epsilon > 0$, there exists $A > 0$ such that

$$\int_A^\infty k(s)\mathrm{d}s < \epsilon.$$

For any $(S_1, I_1), (S_2, I_2) \in \Gamma$, we have

$$\begin{aligned}&|H_1(S_1,I_1)(\xi) - H_1(S_2,I_2)(\xi)|\mathrm{e}^{-\mu|\xi|}\\ &\le \alpha_1|S_1(\xi) - S_2(\xi)|\mathrm{e}^{-\mu|\xi|} + |f(S_1(\xi)) - f(S_2(\xi))|\mathrm{e}^{-\mu|\xi|}\int_0^\infty k(s)g(I_2(\xi - cs))\mathrm{d}s\\ &\quad + f(S_1(\xi))\mathrm{e}^{-\mu|\xi|}\int_0^\infty k(s)|g(I_1(\xi - cs)) - g(I_2(\xi - cs))|\mathrm{d}s\\ &\le \left(\alpha_1 + L_1\int_0^\infty k(s)g(S_*)\mathrm{d}s\right)|S_1(\xi) - S_2(\xi)|\mathrm{e}^{-\mu|\xi|}\\ &\quad + f(S_*)\mathrm{e}^{-\mu|\xi|}\int_0^\infty k(s)|g(I_1(\xi - cs)) - g(I_2(\xi - cs))|\mathrm{d}s\\ &\le [\alpha_1 + L_1 g(S_*)]|S_1 - S_2|_\mu + f(S_*)\mathrm{e}^{-\mu|\xi|}\int_0^A k(s)|g(I_1(\xi - cs)) - g(I_2(\xi - cs))|\mathrm{d}s\\ &\quad + f(S_*)\mathrm{e}^{-\mu|\xi|}\int_A^\infty k(s)|g(I_1(\xi - cs)) - g(I_2(\xi - cs))|\mathrm{d}s\\ &\le [\alpha_1 + L_1 g(S_*)]|S_1 - S_2|_\mu + L_2 f(S_*)\int_0^A k(s)|I_1(\xi - cs) - I_2(\xi - cs)|\mathrm{e}^{-\mu|\xi - cs|}\mathrm{e}^{\mu(|\xi - cs| - |\xi|)}\mathrm{d}s\\ &\quad + f(S_*)\mathrm{e}^{-\mu|\xi|}\int_A^\infty k(s)(|g(I_1(\xi - cs))| + |g(I_2(\xi - cs))|)\mathrm{d}s\\ &\le [\alpha_1 + L_1 g(S_*)]|S_1 - S_2|_\mu + L_2 f(S_*)|I_1 - I_2|_\mu\int_0^A k(s)\mathrm{e}^{\mu cs}\mathrm{d}s + 2f(S_*)g(S_*)\int_A^\infty k(s)\mathrm{d}s.\end{aligned}$$

Here in the last two steps we have used the Lipschitz continuity of g and the boundedness of g on $[0, \infty)$ respectively; see Remark 2.3. Let

$$\delta := \min\left\{\epsilon, \frac{\epsilon}{\int_0^A k(s)\mathrm{e}^{\mu cs}\mathrm{d}s}\right\},$$

then for any $(S_1, I_1), (S_2, I_2) \in \Gamma$ satisfying $|S_1 - S_2|_\mu < \delta$ and $|I_1 - I_2|_\mu < \delta$, we have

$$\begin{aligned}&|H_1(S_1, I_1) - H_1(S_2, I_2)|_\mu \\ &\le [\alpha_1 + L_1 g(S_*)]\delta + L_2 f(S_*)\delta \int_0^A k(s)\mathrm{e}^{\mu c s}\mathrm{d}s + 2f(S_*)g(S_*)\epsilon \\ &\le C_0\epsilon,\end{aligned}$$

where $C_0 = \alpha_1 + L_1 g(S_*) + L_2 f(S_*) + 2f(S_*)g(S_*)$ is a positive constant. This implies that the operator $H_1 : \Gamma \to \mathscr{B}_\mu$ is continuous. Similarly, we can prove that the operator $H_2 : \Gamma \to \mathscr{B}_\mu$ is also continuous. This completes the proof.

Next, we define an operator $F := (F_1(S, I), F_2(S, I))$ by

$$F_i(S, I) := \Delta_i^{-1} H_i(S, I), \quad i = 1, 2. \tag{4.7}$$

By Lemmas 3.1 and 4.4, the operator F is also well defined on Γ and the following conclusion holds.

Lemma 4.5. *$F = (F_1, F_2)$ is a continuous and compact mapping from Γ to $\mathscr{B}_\mu \times \mathscr{B}_\mu$.*

Proof. It suffices to prove that $F_i : \Gamma \to \mathscr{B}_\mu$ is continuous and compact for $i = 1, 2$. Firstly, $F_i = \Delta_i^{-1} H_i$ can be viewed as the composite of mappings $H_i : \Gamma \to \mathscr{B}_\mu$ and $\Delta_i^{-1} : \mathscr{B}_\mu \to \mathscr{B}_\mu$. Then, Lemmas 3.1(i) and 4.4(ii) imply that $F_i : \Gamma \to \mathscr{B}_\mu$ is continuous. Secondly, $F_i = \Delta_i^{-1} H_i$ can also be viewed as the composite of mappings $H_i : \Gamma \to \mathscr{B}_{\mu_0}$ and $\Delta_i^{-1} : \mathscr{B}_{\mu_0} \to \mathscr{B}_\mu$. Therefore, it follows from Lemmas 3.1(ii) and 4.4(i) that $F_i : \Gamma \to \mathscr{B}_\mu$ is compact. This completes the proof. □

For $(S, I) \in \Gamma$, we can rewrite (2.2) as

$$\begin{cases} \Delta_1 S = H_1(S, I), \\ \Delta_2 I = H_2(S, I). \end{cases} \tag{4.8}$$

Thus, from (3.6) we can obtain that if the mapping F has a fixed point in Γ, i.e., there exists $(S, I) \in \Gamma$ satisfying

$$\begin{cases} S = F_1(S, I), \\ I = F_2(S, I), \end{cases} \tag{4.9}$$

then it must be a solution of system (4.8) or (2.2). If this solution further satisfies the boundary conditions (2.3), then it gives a traveling wave solution of system (2.1), which is our search target.

In order to employ the Schauder's fixed point theorem to prove the existence of a fixed point of the mapping F in Γ, we also need to verify the cone invariance of Γ under F.

Lemma 4.6. *$F = (F_1, F_2)$ maps Γ into Γ.*

Proof. By Lemma 4.5, $F(\Gamma) \subset \mathscr{B}_\mu \times \mathscr{B}_\mu$. Therefore, we only need to verify

$$S_- \le F_1(S, I) \le S_+ \equiv S_*,$$

$$I_- \le F_2(S, I) \le I_+$$

for all $(S, I) \in \Gamma$.

Due to the assumptions (A1)–(A2), the boundedness of g (see Remark 2.3), and $\alpha_1 \geq L_1 g(S_*)$, we have

$$\begin{aligned}
&H_1(S,I)(\xi) - H_1(S_-,I_+)(\xi)\\
&\geq \alpha_1[S(\xi) - S_-(\xi)] - [f(S(\xi)) - f(S_-(\xi))]\int_0^\infty k(s)g(I(\xi - cs))\mathrm{d}s\\
&\geq \alpha_1[S(\xi) - S_-(\xi)] - g(S_*)[f(S(\xi)) - f(S_-(\xi))]\\
&\geq [\alpha_1 - L_1 g(S_*)][S(\xi) - S_-(\xi)]\\
&\geq 0.
\end{aligned}$$

Thus, $F_1(S,I) \geq F_1(S_-,I_+)$. It follows from (4.2) in Lemma 4.2 that

$$F_1(S_-,I_+) = \Delta_1^{-1}H_1(S_-,I_+) \geq \Delta_1^{-1}(\Delta_1 S_-).$$

Noting that S_- is a continuous function on $\mathbb{R}$ satisfying all of the assumptions in Lemma 3.2, $S_-'(\zeta_1-) < 0$ and $S_-'(\zeta_1+) = 0$, in view of (3.9) we obtain that $\Delta_1^{-1}(\Delta_1 S_-) \geq S_-$. Thus, $F_1(S,I) \geq S_-$. Note that $H_1(S,I) \leq \alpha_1 S \leq \alpha_1 S_+ = \Delta_1 S_+$. Since S_+ is a constant function, it follows from (3.7) that

$$F_1(S,I) = \Delta_1^{-1}H_1(S,I) \leq \Delta_1^{-1}(\Delta_1 S_+) = S_+.$$

So far, we have proven that

$$S_- \leq F_1(S,I) \leq S_+.$$

Since $\alpha_2 \geq \gamma$, the monotonicity of f and g implies that the operator $H_2(S,I)$ is monotone with respect to variables S and I. Thus, we have

$$F_2(S_-,I_-) \leq F_2(S,I) \leq F_2(S_+,I_+).$$

It follows from (4.3) in Lemma 4.3 that

$$F_2(S_-,I_-) = \Delta_2^{-1}H_2(S_-,I_-) \geq \Delta_2^{-1}(\Delta_2 I_-).$$

Note that I_- is continuous on $\mathbb{R}$ satisfying all of the assumptions in Lemma 3.2, $I_-'(\zeta_2-) < 0$ and $I_-'(\zeta_2+) = 0$. By (3.9) we obtain that $\Delta_2^{-1}(\Delta_2 I_-) > I_-$. Hence, $F_2(S_-,I_-) > I_-$. Since $\lambda_1 < \mu$, we have $I_+, I_+', I_+'' \in \mathscr{B}_\mu$. It follows from (4.1) in Lemma 4.1 and (3.7) that

$$F_2(S_+,I_+) = \Delta_2^{-1}H_2(S_+,I_+) \leq \Delta_2^{-1}(\Delta_2 I_+) = I_+.$$

Hence, we have also shown that

$$I_- \leq F_2(S,I) \leq I_+.$$

This completes the proof of the lemma. □

5. Proof of the existence of traveling wave

By Lemmas 4.5–4.6, we know that F is a continuous and compact mapping from the bounded closed convex set Γ of $\mathscr{B}_\mu$ into itself. It follows from the Schauder fixed point theorem that the mapping F has a fixed point $(S,I) \in \Gamma$, that is, there exists a point $(S,I) \in \Gamma$ such that (4.9) holds. Consequently, we obtain that this point (S,I) satisfies the system (2.2). In the following, we shall verify this solution (S,I) satisfies the boundary conditions (2.3).

Firstly, since $S_- \leq S \leq S_+$ and $I_- \leq I \leq I_+$, we can get that

$$\lim_{\xi\to-\infty} S(\xi) = S_*, \quad \lim_{\xi\to-\infty} I(\xi) = 0, \tag{5.1}$$

and

$$\lim_{\xi\to-\infty} \mathrm{e}^{-\lambda_1\xi} I(\xi) = 1. \tag{5.2}$$

The equality (5.2) also implies that I is non-trivial. In fact, it can be shown that $I(\xi) > 0$ for any $\xi \in \mathbb{R}$. Assume by contradiction that there is $\xi_0 \in \mathbb{R}$ such that $I(\xi_0) = 0$, then there exist constants $a, b \in \mathbb{R}$ such that $\xi_0, \zeta_2 \in (a, b)$. It follows that I attains its minimum in (a, b) for $\xi \in [a, b]$. From (2.2b), we obtain that

$$-d_2 I''(\xi) + cI'(\xi) + \gamma I(\xi) \geq 0, \qquad \xi \in [a, b].$$

By the elliptic strong maximum principle, it follows that $I(\xi) \equiv 0$ for $\xi \in [a, b]$, which contradicts the fact that $I(\xi) > 0$ for $\xi \in [a, \zeta_2)$ from Lemma 3.2. The monotonicity of g and $g'_+(0) > 0$ imply that $g(u) > 0$ for all $u > 0$; see the assumption (A2) and Remark 2.3. Together with the assumption (A1), we have

$$\int_0^\infty k(s) g(I(\xi - cs)) \mathrm{d}s > 0$$

for all $\xi \in \mathbb{R}$. Therefore, by (2.2a) and (5.1), it is easy to see that S is also non-trivial.

Recalling the definition of $\varDelta_i^{-1}$, we have

$$(\varDelta_i^{-1} h)'(\xi) = \frac{1}{d_i(\lambda_{2i} + \lambda_{1i})} \left(-\lambda_{1i} \int_{-\infty}^{\xi} \mathrm{e}^{-\lambda_{1i}(\xi - x)} h(x) \mathrm{d}x + \lambda_{2i} \int_{\xi}^{\infty} \mathrm{e}^{\lambda_{2i}(\xi - x)} h(x) \mathrm{d}x \right), \tag{5.3}$$

$$(\varDelta_i^{-1} h)''(\xi) = \frac{1}{d_i(\lambda_{2i} + \lambda_{1i})} \left(\lambda_{1i}^2 \int_{-\infty}^{\xi} \mathrm{e}^{-\lambda_{1i}(\xi - x)} h(x) \mathrm{d}x + \lambda_{2i}^2 \int_{\xi}^{\infty} \mathrm{e}^{\lambda_{2i}(\xi - x)} h(x) \mathrm{d}x \right) - \frac{1}{d_i} h(\xi) \tag{5.4}$$

for any $h \in \mathscr{B}_\mu$. Hence, if $\lim_{\xi\to-\infty} h(\xi)$ exists, then by L'Hôpital's rule, we have

$$\lim_{\xi\to-\infty} (\varDelta_i^{-1} h)'(\xi) = \lim_{\xi\to-\infty} (\varDelta_i^{-1} h)''(\xi) = 0.$$

By Lemma 4.4, we know that $H_1(S, I), H_2(S, I) \in \mathscr{B}_\mu$. It is easily seen that $H_1(S, I)(\xi) \to \alpha_1 S_*$ and $H_2(S, I)(\xi) \to 0$ as $\xi \to -\infty$. Consequently, from (4.9) we obtain that

$$\lim_{\xi\to-\infty} S'(\xi) = \lim_{\xi\to-\infty} [\varDelta_1^{-1} H_1(S, I)]'(\xi) = 0, \quad \lim_{\xi\to-\infty} S''(\xi) = \lim_{\xi\to-\infty} [\varDelta_1^{-1} H_1(S, I)]''(\xi) = 0, \tag{5.5}$$

and

$$\lim_{\xi\to-\infty} I'(\xi) = \lim_{\xi\to-\infty} [\varDelta_2^{-1} H_2(S, I)]'(\xi) = 0, \quad \lim_{\xi\to-\infty} I''(\xi) = \lim_{\xi\to-\infty} [\varDelta_2^{-1} H_2(S, I)]''(\xi) = 0. \tag{5.6}$$

Secondly, we intend to study the asymptotic behavior of S and I as $x \to \infty$. For the sake of convenience, we denote

$$\psi(\xi) := f(S(\xi)) \int_0^\infty k(s) g(I(\xi - cs)) \mathrm{d}s. \tag{5.7}$$

We will rewrite (2.2a) as

$$d_1 S''(\xi) - cS'(\xi) = \psi(\xi). \tag{5.8}$$

Integrating (5.8) from $-\infty$ to ξ yields

$$d_1 S'(\xi) = c(S(\xi) - S_*) + \int_{-\infty}^{\xi} \psi(x) \mathrm{d}x. \tag{5.9}$$

Due to the boundedness of S, it can be shown that the integral in the equality above should be uniformly bounded by contradiction. Thus, it follows that

$$\int_{-\infty}^{\infty} \psi(x) \mathrm{d}x < \infty, \tag{5.10}$$

which in turn yields S' is bounded on $\mathbb{R}$. Since

$$|\psi(\xi)| \leq f(S_*)g(S_*) \tag{5.11}$$

for all $\xi \in \mathbb{R}$, it follows from (5.8) that S'' is also bounded on $\mathbb{R}$. Note that (5.8) implies that

$$[\mathrm{e}^{-c\xi/d_1}S'(\xi)]' = \frac{1}{d_1}\mathrm{e}^{-c\xi/d_1}\psi(\xi).$$

Integrating this equality from ξ to ∞ gives

$$\mathrm{e}^{-c\xi/d_1}S'(\xi) = -\frac{1}{d_1}\int_\xi^\infty \mathrm{e}^{-cx/d_1}\psi(x)\mathrm{d}x.$$

Thus, S is nonincreasing. Since we have shown that S is non-trivial, one can obtain that

$$0 \leq S(\infty) < S(-\infty) = S_*.$$

Moreover, since S' is a non-positive and integrable function on $\mathbb{R}$ and we have shown that S'' is bounded on $\mathbb{R}$, it is easily seen that $S'(\xi) \to 0$ as $\xi \to \infty$. Therefore, letting $\xi \to \infty$, it follows from (5.9) that

$$c(S_* - S(\infty)) = \int_{-\infty}^\infty \psi(x)\mathrm{d}x. \tag{5.12}$$

Note that (2.2b) can also be rewritten as

$$-d_2I''(\xi) + cI'(\xi) + \gamma I(\xi) = \psi(\xi). \tag{5.13}$$

By the fundamental theory of second-order linear ordinary differential equations, we obtain that

$$I(\xi) = C_1\mathrm{e}^{-\tilde\lambda_1\xi} + C_2\mathrm{e}^{\tilde\lambda_2\xi} + \frac{1}{d_2(\tilde\lambda_2+\tilde\lambda_1)}\left(\int_{-\infty}^\xi \mathrm{e}^{-\tilde\lambda_1(\xi-x)}\psi(x)\mathrm{d}x + \int_\xi^\infty \mathrm{e}^{\tilde\lambda_2(\xi-x)}\psi(x)\mathrm{d}x\right), \tag{5.14}$$

where C_1, C_2 are constants, and

$$\tilde\lambda_1 = \frac{-c+\sqrt{c^2+4d_2\gamma}}{2d_2}, \quad \tilde\lambda_2 = \frac{c+\sqrt{c^2+4d_2\gamma}}{2d_2}.$$

Note that (5.11) guarantees the integral in (5.14) is well defined, and

$$\left|\frac{1}{d_2(\tilde\lambda_2+\tilde\lambda_1)}\left(\int_{-\infty}^\xi \mathrm{e}^{-\tilde\lambda_1(\xi-x)}\psi(x)\mathrm{d}x + \int_\xi^\infty \mathrm{e}^{\tilde\lambda_2(\xi-x)}\psi(x)\mathrm{d}x\right)\right| \leq \frac{1}{\gamma}f(S_*)g(S_*). \tag{5.15}$$

Since $I(\xi) \to 0$ as $\xi \to -\infty$, then $C_1 = 0$. Note that $I(\xi) \leq I_+(\xi) = \mathrm{e}^{\lambda_1\xi}$. Recalling the definition of λ_1 in Lemma 3.3, together with (3.10), it is easily seen that $\lambda_1 < \tilde\lambda_2$. Hence, we can also get $C_2 = 0$. Consequently,

$$I(\xi) = \frac{1}{d_2(\tilde\lambda_2+\tilde\lambda_1)}\left(\int_{-\infty}^\xi \mathrm{e}^{-\tilde\lambda_1(\xi-x)}\psi(x)\mathrm{d}x + \int_\xi^\infty \mathrm{e}^{\tilde\lambda_2(\xi-x)}\psi(x)\mathrm{d}x\right). \tag{5.16}$$

Since (5.10) holds, it follows from the equality above and Fubini's theorem that I is also integrable on $\mathbb{R}$, and

$$\int_{-\infty}^\infty I(\xi)\mathrm{d}\xi = \frac{1}{\gamma}\int_{-\infty}^\infty \psi(x)\mathrm{d}x < \infty, \tag{5.17}$$

which together with (5.12) implies that (2.6) in Theorem 2.1 holds. Moreover, by (5.16) and (5.15), we obtain that for any $\xi \in \mathbb{R}$,

$$\begin{aligned}|I'(\xi)| &= \left|\frac{1}{d_2(\tilde\lambda_2+\tilde\lambda_1)}\left(-\tilde\lambda_1\int_{-\infty}^\xi \mathrm{e}^{-\tilde\lambda_1(\xi-x)}\psi(x)\mathrm{d}x + \tilde\lambda_2\int_\xi^\infty \mathrm{e}^{\tilde\lambda_2(\xi-x)}\psi(x)\mathrm{d}x\right)\right| \\ &\leq \frac{\tilde\lambda_2}{\gamma}f(S_*)g(S_*).\end{aligned}$$

Hence, a combination of the two conclusions that I' is bounded and I is a positive and integrable function on $\mathbb{R}$ yields $I(\xi) \to 0$ as $\xi \to \infty$. Together with (5.1) and (5.6), an integration of (5.13) from $-\infty$ to ξ yields

$$-d_2 I'(\xi) + cI(\xi) + \gamma \int_{-\infty}^{\xi} I(x)\mathrm{d}x = \int_{-\infty}^{\xi} \psi(x)\mathrm{d}x,$$

which implies that $I'(\xi) \to 0$ as $\xi \to \infty$. Since g is a bounded and continuous function with $g(0) = 0$ and $I(\xi) \to 0$ as $\xi \to \infty$, by Lebesgue's dominated convergence theorem, we have

$$\lim_{\xi\to\infty} \int_0^{\infty} k(s)g(I(\xi - cs))\mathrm{d}s = 0.$$

Therefore, it follows from (2.2) that $S''(\xi) \to 0$ and $I''(\xi) \to 0$ as $\xi \to \infty$. We conclude the asymptotic behavior of S and I as $\xi \to \infty$.

$$\lim_{\xi\to\infty} S(\xi) = S_\infty < S_*, \quad \lim_{\xi\to\infty} S'(\xi) = \lim_{\xi\to\infty} S''(\xi) = 0, \tag{5.18}$$

$$\lim_{\xi\to\infty} I(\xi) = \lim_{\xi\to\infty} I'(\xi) = \lim_{\xi\to\infty} I''(\xi) = 0. \tag{5.19}$$

Finally, we are ready to prove $I(\xi) \le S_* - S_\infty$ for all $\xi \in \mathbb{R}$. Since I is a positive and integrable function on $\mathbb{R}$, we can define

$$G(\xi) := I(\xi) + \frac{\gamma}{c}\int_{-\infty}^{\xi} I(x)\mathrm{d}x + \frac{\gamma}{c}\int_{\xi}^{\infty} \mathrm{e}^{c/d_2(\xi - x)} I(x)\mathrm{d}x. \tag{5.20}$$

It follows from (5.1), (5.19), (5.12), (5.17) and L'Hôpital's rule that

$$\lim_{\xi\to-\infty} G(\xi) = 0, \quad \lim_{\xi\to\infty} G(\xi) = \frac{\gamma}{c}\int_{-\infty}^{\infty} I(x)\mathrm{d}x = S_* - S(\infty).$$

By (5.20), we can obtain that

$$G'(\xi) = I'(\xi) + \frac{\gamma}{d_2}\int_{\xi}^{\infty} \mathrm{e}^{c/d_2(\xi - x)} I(x)\mathrm{d}x,$$

$$G''(\xi) = I''(\xi) - \frac{\gamma}{d_2} I(\xi) + \frac{\gamma c}{d_2^2}\int_{\xi}^{\infty} \mathrm{e}^{c/d_2(\xi - x)} I(x)\mathrm{d}x$$

and

$$\lim_{\xi\to-\infty} G'(\xi) = 0, \quad \lim_{\xi\to\infty} G'(\xi) = 0.$$

It follows from (5.13) that

$$-d_2 G''(\xi) + cG'(\xi) = \psi(\xi).$$

Noting that $G'(\infty) = 0$, integrating the equation above from ξ to ∞ yields

$$G'(\xi) = \frac{1}{d_2}\int_{\xi}^{\infty} \mathrm{e}^{c/d_2(\xi - x)} \psi(x)\mathrm{d}x \ge 0,$$

which implies that G is a nondecreasing function on $\mathbb{R}$. Since $G(\infty) = S_* - S(\infty)$, we obtain that

$$I(\xi) \le G(\xi) \le S_* - S_\infty$$

for all $\xi \in \mathbb{R}$. This completes the proof of Theorem 2.1.

6. Proof of the nonexistence of traveling wave

For $R_0 > 1$ and $c < c^*$, or $R_0 < 1$, assume by contradiction that there exists a non-trivial and nonnegative traveling wave solution $(S(x+ct), I(x+ct))$ of (2.1), that is, $(S(\xi), I(\xi))$ satisfies system (2.2) with the boundary conditions (2.3). Firstly, note that if $I(\xi) \equiv 0$, then $S(\xi) \equiv S_*$. Hence I is non-trivial. Similar to the discussion in Section 5, we can know that $I(\xi) > 0$ for all $\xi \in \mathbb{R}$. Secondly, since (S, I) is a bounded solution of system (2.2), then by the fundamental theory of second-order linear ordinary differential equations, we obtain that

$$S(\xi) = \frac{1}{d_1(\hat{\lambda}_2 + \hat{\lambda}_1)} \left(\int_{-\infty}^{\xi} e^{-\hat{\lambda}_1(\xi - x)} \varphi(x) dx + \int_{\xi}^{\infty} e^{\hat{\lambda}_2(\xi - x)} \varphi(x) dx \right), \tag{6.1}$$

$$I(\xi) = \frac{1}{d_2(\tilde{\lambda}_2 + \tilde{\lambda}_1)} \left(\int_{-\infty}^{\xi} e^{-\tilde{\lambda}_1(\xi - x)} \psi(x) dx + \int_{\xi}^{\infty} e^{\tilde{\lambda}_2(\xi - x)} \psi(x) dx \right), \tag{6.2}$$

where

$$\hat{\lambda}_1 = \frac{-c + \sqrt{c^2 + 4d_1\gamma}}{2d_1}, \quad \hat{\lambda}_2 = \frac{c + \sqrt{c^2 + 4d_1\gamma}}{2d_1},$$
$$\tilde{\lambda}_1 = \frac{-c + \sqrt{c^2 + 4d_2\gamma}}{2d_2}, \quad \tilde{\lambda}_2 = \frac{c + \sqrt{c^2 + 4d_2\gamma}}{2d_2},$$
$$\varphi(x) = \gamma S(x) - f(S(x)) \int_0^\infty k(s) g(I(x - cs)) ds,$$

and

$$\psi(x) = f(S(x)) \int_0^\infty k(s) g(I(x - cs)) ds. \tag{6.3}$$

By Lebesgue's dominated convergence theorem, it is easily seen that $\varphi(\pm\infty)$ and $\psi(\pm\infty)$ exist. Therefore, from (6.1) and (6.2), by L'Hôpital's rule we can show that

$$S'(\pm\infty) = S''(\pm\infty) = I'(\pm\infty) = I''(\pm\infty) = 0. \tag{6.4}$$

Furthermore, it is easily seen from (6.2) that

$$-\tilde{\lambda}_1 I(\xi) \le I'(\xi) \le \tilde{\lambda}_2 I(\xi). \tag{6.5}$$

Thirdly, from (2.2a), we can obtain that

$$[e^{-c\xi/d_1} S'(\xi)]' = \frac{1}{d_1} e^{-c\xi/d_1} \psi(\xi).$$

Since we have shown that $S'(\infty) = 0$, integrating the equality above from ξ to ∞ yields

$$e^{-c\xi/d_1} S'(\xi) = -\frac{1}{d_1} \int_\xi^\infty e^{-cx/d_1} \psi(x) dx.$$

Thus, S is nonincreasing and for all $\xi \in \mathbb{R}$,

$$S_* \ge S(\xi) \ge S_\infty \ge 0. \tag{6.6}$$

Finally, let $J(\xi) := \int_{-\infty}^{\xi} I(x) dx$, which is well defined on $\mathbb{R}$ since

$$J(\xi) = \frac{1}{\gamma}[d_1 S'(\xi) + d_2 I'(\xi) - cS(\xi) + cS_* - cI(\xi)].$$

Moreover,

$$J(\infty) = \int_{-\infty}^{\infty} I(x) dx = c(S_* - S_\infty) < \infty.$$

In the following, we distinguish two cases.

6.1. The case $R_0 > 1$ and $c < c^*$

In this case, we will first show that a claim about the function $J(\cdot)$.

Claim. *(I) There exist two constants $\xi_0 < 0$ and $K_1 > 0$ such that $I(\xi) \leq K_1 J(\xi)$ for any $\xi \leq \xi_0$; (II) there is $\rho > 0$ such that $J(\xi) = O(\mathrm{e}^{\rho\xi})$ as $\xi \to -\infty$.*

Since $I(-\infty) = 0$, there exists $\xi_1 < 0$ such that for any $\xi \leq \xi_1$, $I(\xi) < \min\{\delta_0, S_*\}$. According the assumptions (A2)–(A3), we have for any $\xi \leq \xi_1$,

$$g'_+(0) - \frac{\omega \int_0^\infty k(s) I^\theta(\xi - cs)\mathrm{d}s}{\int_0^\infty k(s) I(\xi - cs)\mathrm{d}s} \leq \frac{\int_0^\infty k(s) g(I(\xi - cs))\mathrm{d}s}{\int_0^\infty k(s) I(\xi - cs)\mathrm{d}s} \leq g'_+(0).$$

Noting that $I(-\infty) = 0$ and $\theta > 1$, it is easy to show that

$$\lim_{\xi \to -\infty} \frac{\int_0^\infty k(s) I^\theta(\xi - cs)\mathrm{d}s}{\int_0^\infty k(s) I(\xi - cs)\mathrm{d}s} = 0.$$

Hence, together with $R_0 > 1$, we have

$$\lim_{\xi \to -\infty} \frac{f(S(\xi)) \int_0^\infty k(s) g(I(\xi - cs))\mathrm{d}s}{\int_0^\infty k(s) I(\xi - cs)\mathrm{d}s} = f(S_*) g'_+(0) > \frac{\gamma(R_0 + 1)}{2}.$$

Thus, there exists $\xi_0 < 0$ such that for any $\xi \leq \xi_0$,

$$f(S(\xi)) \int_0^\infty k(s) g(I(\xi - cs))\mathrm{d}s \geq \frac{\gamma(R_0 + 1)}{2} \int_0^\infty k(s) I(\xi - cs)\mathrm{d}s.$$

Therefore, by (2.2b) we have for any $\xi \leq \xi_0$,

$$cI'(\xi) \geq d_2 I''(\xi) + \frac{\gamma(R_0 + 1)}{2} \left(\int_0^\infty k(s) I(\xi - cs)\mathrm{d}s - I(\xi) \right) + \frac{\gamma(R_0 - 1)}{2} I(\xi). \tag{6.7}$$

Integrating both sides of (6.7) from $-\infty$ to ξ yields

$$\frac{\gamma(R_0 - 1)}{2} J(\xi) \leq cI(\xi) - d_2 I'(\xi) + \frac{\gamma(R_0 + 1)}{2} \left(J(\xi) - \int_0^\infty k(s) J(\xi - cs)\mathrm{d}s \right) \tag{6.8}$$

for any $\xi \leq \xi_0$. Integrating both sides of (6.8) from $-\infty$ to ξ gives

$$\begin{aligned}
&\frac{\gamma(R_0 - 1)}{2} \int_{-\infty}^\xi J(u)\mathrm{d}u + d_2 I(\xi) \\
&\leq cJ(\xi) + \frac{\gamma(R_0 + 1)}{2} \int_{-\infty}^\xi \int_0^\infty k(s)[J(x) - J(x - cs)]\mathrm{d}s\mathrm{d}x \\
&= cJ(\xi) + \frac{\gamma(R_0 + 1)}{2} \int_{-\infty}^\xi \int_0^\infty \int_0^1 csk(s) I(x - mcs)\mathrm{d}m\mathrm{d}s\mathrm{d}x \\
&= cJ(\xi) + \frac{\gamma(R_0 + 1)}{2} \int_0^\infty csk(s) \int_0^1 J(\xi - mcs)\mathrm{d}m\mathrm{d}s \\
&\leq \left(1 + \frac{\gamma(R_0 + 1)}{2} \int_0^\infty sk(s)\mathrm{d}s \right) cJ(\xi)
\end{aligned} \tag{6.9}$$

for any $\xi \leq \xi_0$. Let

$$K_0 := c\left(1 + \frac{\gamma(R_0 + 1)}{2} \int_0^\infty sk(s)\mathrm{d}s \right), \quad K_1 := \frac{K_0}{d_2}.$$

From (6.9), we have for any $\xi \le \xi_0$,

$$I(\xi) \le K_1 J(\xi),$$

and

$$\frac{\gamma(R_0-1)}{2}\int_{-\infty}^{\xi} J(u)\mathrm{d}u \le K_0 J(\xi).$$

Since J is a nondecreasing and positive function, then for any $\xi \le \xi_0$ and any $\tau > 0$ we have

$$\frac{\gamma(R_0-1)}{2}\tau J(\xi-\tau) \le K_0 J(\xi).$$

We can choose $\tau = \frac{2\mathrm{e}K_0}{\gamma(R_0-1)}$, then for any $\xi \le \xi_0$,

$$J(\xi-\tau) \le \frac{1}{\mathrm{e}} J(\xi). \tag{6.10}$$

Let

$$\rho = \frac{1}{\tau} = \frac{\gamma(R_0-1)}{2\mathrm{e}K_0},$$

then $\rho > 0$, and by (6.10) we have

$$J(\xi-\tau)\mathrm{e}^{-\rho(\xi-\tau)} \le \frac{1}{\mathrm{e}} J(\xi)\mathrm{e}^{-\rho(\xi-\tau)} = J(\xi)\mathrm{e}^{-\rho\xi}$$

for any $\xi \le \xi_0$. This implies that

$$\sup_{\xi\in(-\infty,\xi_0]} J(\xi)\mathrm{e}^{-\rho\xi} = \max_{\xi\in[\xi_0-\tau,\xi_0]} J(\xi)\mathrm{e}^{-\rho\xi}.$$

Therefore, $J(\xi) = O(\mathrm{e}^{\rho\xi})$ as $\xi \to -\infty$. This completes the proof of the claim.

For $\lambda \in \mathbb{C}$, we define a bilateral Laplace transform of I by

$$\mathcal{L}(\lambda) := \int_{-\infty}^{\infty} \mathrm{e}^{-\lambda\xi} I(\xi)\mathrm{d}\xi = \int_{-\infty}^{\infty} \mathrm{e}^{-\lambda\xi}\mathrm{d}J(\xi). \tag{6.11}$$

Let a, $b \in \mathbb{R}\cup\{\pm\infty\}$ with $a \le b$ denote the abscissas of convergence for $\mathcal{L}(\lambda)$, that is, the integral (6.11) converges in the strip $a < \Re\lambda < b$ and diverges for $\Re\lambda > b$ and for $\Re\lambda < a$. Since I is bounded, $\int_0^{\infty} \mathrm{e}^{-\lambda\xi} I(\xi)\mathrm{d}\xi$ converges for any λ with $\Re\lambda > 0$. Hence, together with the claim (II) above, we know that $(0,\rho) \subset (a,b)$. Furthermore, by (6.5) and the claim (I), it is easily seen that

$$J(\xi) \ge \frac{1}{K_1} I(\xi) \ge K_2 \mathrm{e}^{\tilde{\lambda}_2\xi} \tag{6.12}$$

for any $\xi < \xi_0$, where $K_2 = \frac{I(\xi_0)}{K_1}\mathrm{e}^{-\tilde{\lambda}_2\xi_0} > 0$. The inequality (6.12) implies that b is a finite real number not greater than $\tilde{\lambda}_2$. Since $J(\xi)$ is nondecreasing, by properties of Laplace transforms (Theorem 2.5a and 2.5b, [34]), one can obtain that $\mathcal{L}(\lambda)$ is analytic in the strip $a < \Re\lambda < b$ and $\lambda = b$ is a singular point of $\mathcal{L}(\lambda)$. In the following, we will prove that $\lambda = b$ is an analytic point of $\mathcal{L}(\lambda)$, which causes a contradiction.

From Theorem 2.2.2a in [34], we obtain that for any sufficiently small $\epsilon > 0$, $J(\xi) = o(\mathrm{e}^{(b-\epsilon)\xi})$ as $\xi \to -\infty$. Therefore, it follows from the claim (I) and (6.5) that for any sufficiently small $\epsilon > 0$, as $\xi \to -\infty$,

$$I(\xi) = o(\mathrm{e}^{(b-\epsilon)\xi}) \tag{6.13}$$

and

$$I'(\xi) = o(\mathrm{e}^{(b-\epsilon)\xi}).$$

Hence, together with the boundedness of I and I', we know that the integrals $\int_{-\infty}^{\infty} \mathrm{e}^{-\lambda\xi} I'(\xi)\mathrm{d}\xi$ and $\int_{-\infty}^{\infty} \mathrm{e}^{-\lambda\xi} I''(\xi)\mathrm{d}\xi$ converge at least in the trip $0 < \Re\lambda < b$ (see Theorem 2.2.1 in [34]), and

$$\int_{-\infty}^{\infty} \mathrm{e}^{-\lambda\xi} I'(\xi)\mathrm{d}\xi = \lambda\mathcal{L}(\lambda), \tag{6.14}$$

$$\int_{-\infty}^{\infty} \mathrm{e}^{-\lambda\xi} I''(\xi)\mathrm{d}\xi = \lambda^2\mathcal{L}(\lambda). \tag{6.15}$$

Integrating both sides of (2.2b) from $-\infty$ to ξ gives

$$\int_{-\infty}^{\xi} \psi(x)\mathrm{d}x = cI(\xi) - d_2 I'(\xi) + \gamma J(\xi),$$

where $\psi(x)$ is defined as (6.3). Therefore, from the discussion above we obtain that for any sufficiently small $\epsilon > 0$, as $\xi \to -\infty$,

$$\int_{-\infty}^{\xi} \psi(x)\mathrm{d}x = o(\mathrm{e}^{(b-\epsilon)\xi}). \tag{6.16}$$

By Fubini's theorem, we have

$$\int_{-\infty}^{\infty} \mathrm{e}^{-\lambda\xi} \int_0^{\infty} k(s)I(\xi - cs)\mathrm{d}s\mathrm{d}\xi = \mathcal{L}(\lambda) \int_0^{\infty} k(s)\mathrm{e}^{-\lambda cs}\mathrm{d}s, \tag{6.17}$$

which also implies that the integral on left-hand side converges in the trip $a < \Re\lambda < b$. Eq. (2.2b) can be rewritten as

$$d_2 I''(\xi) - cI'(\xi) + f(S_*)g'_+(0) \int_0^{\infty} k(s)I(\xi - cs)\mathrm{d}s - \gamma I(\xi) = P(\xi), \tag{6.18}$$

where

$$P(\xi) := f(S_*)g'_+(0) \int_0^{\infty} k(s)I(\xi - cs)\mathrm{d}s - f(S(\xi)) \int_0^{\infty} k(s)g(I(\xi - cs))\mathrm{d}s.$$

Taking the bilateral Laplace transforms of both sides of (6.18) and using (6.14), (6.15) and (6.17), we get

$$\Upsilon(c,\lambda)\mathcal{L}(\lambda) = \int_{-\infty}^{\infty} \mathrm{e}^{-\lambda\xi} P(\xi)\mathrm{d}\xi. \tag{6.19}$$

By the boundary conditions (2.3) and the assumptions (A2)–(A3), there exist $M_1 > 0$ and $\xi_2 < 0$ such that for any $\xi < \xi_2$,

$$f(S(\xi)) < M_1$$

and

$$0 \le g'_+(0)I(\xi) - g(I(\xi)) \le \omega I^\theta(\xi).$$

Therefore, together with (6.6), we have for any $\xi < \xi_2$,

$$\begin{aligned} |P(\xi)| \le\ & g'_+(0)|f(S_*) - f(S(\xi))| \int_0^{\infty} k(s)I(\xi - cs)\mathrm{d}s \\ & + f(S(\xi)) \int_0^{\infty} k(s)|g(I(\xi - cs)) - g'_+(0)I(\xi - cs)|\mathrm{d}s. \\ \le\ & g'_+(0)L_1|S_* - S(\xi)| \int_0^{\infty} k(s)I(\xi - cs)\mathrm{d}s \\ & + \omega M_1 \int_0^{\infty} k(s)I^\theta(\xi - cs)\mathrm{d}s. \end{aligned} \tag{6.20}$$

Integrating both sides of (2.2a) from $-\infty$ to ξ gives

$$d_1 S'(\xi) = c(S(\xi) - S_*) + \int_{-\infty}^{\xi} \psi(x)\mathrm{d}x.$$

Then

$$S_* - S(\xi) = [S_* - S(0)]e^{\frac{c}{d_1}\xi} + \frac{1}{d_1}\int_{\xi}^{0} e^{\frac{c}{d_1}(\xi - s)} \int_{-\infty}^{s} \psi(x)\mathrm{d}x\mathrm{d}s.$$

Noting (6.16), it follows from L'Hôpital's rule that for any σ with $0 < \sigma < \min\{b, c/d_1\}$,

$$S_* - S(\xi) = o(e^{\sigma\xi}) \tag{6.21}$$

as $\xi \to -\infty$. In addition, (6.13) implies that for any sufficiently small $\epsilon > 0$, as $\xi \to -\infty$,

$$\int_0^\infty k(s)I(\xi - cs)\mathrm{d}s = o(e^{(b-\epsilon)\xi}), \tag{6.22}$$

and

$$\int_0^\infty k(s)I^\theta(\xi - cs)\mathrm{d}s = o(e^{(b-\epsilon)\theta\xi}). \tag{6.23}$$

Let

$$\sigma_0 := \min\{b, \frac{c}{d_1}, (\theta - 1)b\}.$$

From (6.20), together with (6.21), (6.22) and (6.23), we have for any σ with $0 < \sigma < \sigma_0$,

$$|P(\xi)| = o(e^{(b+\sigma)\xi}) \tag{6.24}$$

as $\xi \to -\infty$. This implies that $\int_{-\infty}^0 e^{-\lambda\xi}P(\xi)\mathrm{d}\xi$ converges for any λ with $\Re\lambda < b + \sigma_0$. On the other hand, noting that $P(\xi)$ is bounded, $\int_0^\infty e^{-\lambda\xi}P(\xi)\mathrm{d}\xi$ converges for any λ with $\Re\lambda > 0$. Therefore, the integral $\int_{-\infty}^\infty e^{-\lambda\xi}P(\xi)\mathrm{d}\xi$ converges at least in the strip $0 < \Re\lambda < b + \sigma_0$ and is analytic with respect to λ in this strip.

By Lemma 3.3, we know that when $0 < c < c^*$, $\Upsilon(c, \lambda) > 0$ for all $\lambda > 0$. This implies that $\Upsilon(c, b) \neq 0$. Hence, in a neighborhood of $\lambda = b$, (6.19) can be rewritten as

$$\mathcal{L}(\lambda) = \frac{1}{\Upsilon(c, \lambda)} \int_{-\infty}^\infty e^{-\lambda\xi}P(\xi)\mathrm{d}\xi. \tag{6.25}$$

Consequently, $\lambda = b$ is an analytic point of $\mathcal{L}(\lambda)$, which is a contradiction.

6.2. The case $R_0 < 1$

Recalling $J(\infty) < \infty$, together with (6.4), an integration of (2.2b) on the real line yields

$$J(\infty) = \int_{-\infty}^\infty I(\xi)\mathrm{d}\xi = \frac{1}{\gamma}\int_{-\infty}^\infty f(S(\xi)) \int_0^\infty k(s)g(I(\xi - cs))\mathrm{d}s\mathrm{d}\xi. \tag{6.26}$$

Similar to the discussion in the last part of Section 5, it can also be shown that for all $\xi \in \mathbb{R}$,

$$I(\xi) \le S_* - S_\infty \le S_*.$$

Hence, by the assumption (A2), we have for all $\xi \in \mathbb{R}$,

$$g(I(\xi - cs)) \le g'_+(0)I(\xi - cs). \tag{6.27}$$

Therefore, together with (6.6) and (6.27), it follows from (6.26) that

$$\begin{aligned} J(\infty) &\le \frac{f(S_*)g'_+(0)}{\gamma} \int_{-\infty}^\infty \int_0^\infty k(s)I(\xi - cs)\mathrm{d}s\mathrm{d}\xi \\ &= R_0 \int_0^\infty k(s) \int_{-\infty}^\infty I(\xi)\mathrm{d}\xi\mathrm{d}s \\ &= R_0 \int_{-\infty}^\infty I(\xi)\mathrm{d}\xi \\ &< J(\infty). \end{aligned}$$

This is a contradiction, which completes the proof of Theorem 2.2. □

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