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Asymptotic speed of propagation and traveling wavefronts in a non-local delayed lattice differential equation

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Abstract

In this paper, we study a very general non-local lattice differential equation with delay. We obtain the existence of the asymptotic speed of propagation, the existence and uniqueness of the traveling wavefront and the minimal speed of the traveling wavefront for the system. We also confirm that the asymptotic speed of propagation and the minimal speed of the traveling wavefront coincide.
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1. Introduction

Lattice differential equations are infinite systems of ordinary differential equations indexed by points on spatial lattices. Such systems arise, on one hand, from practical backgrounds, such as modeling population growth over patchy environments [5,17,32] and modeling the phase transitions (see, e.g., [3,4]). On the other hand, they are also natural results of discretizing the corresponding models of partial differential equations in which continuous spatial variables are used.

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For nonlinear reaction–diffusion equations models describing a variety of physical and biological phenomena, traveling wave solutions are an important class of solutions since in many situations they (i) determine the long term behavior of other solutions, and (ii) account for phase transitions between different states of physical systems, propagation of patterns, and domain invasion of species in population biology.

A simple but typical lattice differential system is

$$u_n'(t) = D[u_{n+1}(t) + u_{n-1}(t) - 2u_n(t)] + f(u_n(t)), \qquad n \in \mathbb{Z}, \quad t > 0, \tag{1.1}$$

which was initially used in Bell and Cosner [5] and Keener [17] to model myelinated axons in nerve systems. For such a system and its various generalizations, when the nonlinear term $f(u)$ is of *bistable type*, the study on traveling wavefronts of such lattice differential equations has been extensive and intensive, and has led to many interesting and significant results, some of which, have revealed some essential difference between a discrete model and its continuous version. For details, see, for example, [6–8,12,3–5,17,22,24,26,27,34,35], and the references therein. However, when the nonlinear term $f(u)$ is *monostable*, that is, $f(u)$ satisfies

(A) $f(0) = f(k) = 0$ for some $k > 0$; and $f(w) > 0$ for $w \in (0, k)$,

results are still very limited. Zinner et al. [36] addressed the existence and minimal speed of traveling wavefront for the discrete Fisher equation. Recently, Chen and Guo [10,11] discussed a more general class of system

$$u_n'(t) = g(u_{n+1}(t)) + g(u_{n-1}(t)) - 2g(u_n(t)) + f(u_n(t)), \qquad n \in \mathbb{Z}, \quad t > 0. \tag{1.2}$$

where $g(u)$ is increasing and $f(u)$ is monostable. Established in [10,11] are such results as existence, uniqueness and stability (in some sense) as well as minimal wave speed for (1.3). Also in a very recent paper, Carr and Chmaj [9] established uniqueness of traveling wavefronts for the nonlocal *monostable* ODE system

$$u_n' = \sum_{\imath \in \mathbb{Z}\setminus\{0\}} J(i)u_{n-i} - u_n + f(u_n), \qquad n \in \mathbb{Z}, \tag{1.3}$$

which reduces to the discrete reaction–diffusion system (1.1) when taking $J(1) = J(-1) = 1/2$ and $J(i) = 0$ elsewhere. System (1.3) was derived in [4] for an l_2 gradient flow for a Helmholtz free energy functional with general long range linear coupling.

On the other hand, in modeling population growth and transmission of signals in the nerve systems, temporal delays seem to be inevitable accounting for the maturation time of the species under consideration and the time needed for the signals to travel along axons and to cross synapses. The existence of traveling wave solutions of delayed lattice differential equations with monostable nonlinearities was initially studied by Wu and Zou [33] and Zou [37], later by Hsu and Lin [15] and Ma et al. [21], and recently by Huang and Zou [16]. We point out that not addressed in [33,37,15,21,16] were problems of minimal wave speed, uniqueness and stability of traveling wave solutions to delayed lattice differential equations. It is well known that the presence of delay in an ODE changes a finite dimensional system to an infinite dimensional one, and this increases the difficulty level in addressing the above problems. Encouraged by the work of Chen and Guo [10,11], and Carr and Chmaj [9], more recently, Ma and Zou [23] obtained the minimal wave speed for the following delayed lattice differential system

$$u_n'(t) = D[u_{n+1}(t) + u_{n-1}(t) - 2u_n(t)] - du_n(t) + b(u_n(t-r)), \quad n \in \mathbb{Z}, \quad t > 0, \tag{1.4}$$

and established some results on uniqueness and asymptotic stability of its traveling wavefronts, under the *monostable* assumption.

Note that the coupling in system (1.4) is only through linear diffusion, meaning that each unit in the lattice $\mathbb{Z}$ only interacts with its nearest (adjacent) neighbors in the form of linear diffusion. This may not be true in some situations. Indeed, in their recent work, Weng et al. [32] derived a discrete *nonlocal* model parallel to the continuous nonlocal model in [28], which takes the form

$$u_n'(t) = D[u_{n+1}(t) + u_{n-1}(t) - 2u_n(t)] - du_n(t) + \sum_{j=-\infty}^{\infty} \Gamma(n, j) b(u_j(t-r)), \quad n \in \mathbb{Z}, \quad t > 0, \tag{1.5}$$

and includes (1.4) as a special case. In addition to the isotropic property of solutions and the asymptotic speed of propagation, [32] also addressed the existence of traveling wavefronts, and the existence and uniqueness of the associated initial value problem to (1.5) under *monostable* and some *quasi-monotonic* conditions on $b(u)$. However, they did not consider the uniqueness of the traveling wavefronts, and the existence of the minimal wave speed, let alone the relation of the two speeds.

We may say that (1.3) has *non-local diffusion* and *local interaction*, while (1.5) has *local diffusion* and *non-local interaction*. In this paper, instead of addressing the above remaining problems for any of the systems (1.1)–(1.5), we will consider the following more general lattice differential system

$$u_n'(t) = D \sum_{i \in \mathbb{Z} \setminus \{0\}} J(i)[u_{n-i}(t) - u_n(t)] - du_n(t) + \sum_{i \in \mathbb{Z}} K(i) b(u_{n-i}(t-r)), \tag{1.6}$$

where $x \in \mathbb{R}$, $t > 0$, $D, d > 0$, $r \geq 0$, $b(\cdot)$ is a Lipschitz continuous function on any compact interval and $b(0) = dK - b(K) = 0$ for some $K > 0$. Obviously, (1.6) contains both *non-local diffusion* and *non-local interaction*, and includes (1.1)–(1.5) as special cases. Our main concern is the existence of the asymptotic speed of propagation, the existence and uniqueness of traveling wavefronts, and the minimal wave speed and its relation with the asymptotic speed of propagation.

We point out that the asymptotic speed of propagation is an important notion in population biology and for a quite general reaction–diffusion equation or an integral equation, the asymptotic speed of propagation coincides with the minimal wave speed of the equation (see, e.g., [1,2,13, 14,18,19,25,29–31]). One naturally would like to know if this is also true for lattice differential equations.

Throughout this paper, we always assume that the kernel functions J and K satisfy $J(i) = J(-i) \geq 0$ and $K(i) = K(-i) \geq 0$ for all $i \in \mathbb{Z} \setminus \{0\}$, and

$$\sum_{i \in \mathbb{Z} \setminus \{0\}} J(i) = 1, \qquad \sum_{i \in \mathbb{Z} \setminus \{0\}} J(i) \mathrm{e}^{-\lambda i} < +\infty,$$
$$\sum_{i \in \mathbb{Z}} K(i) = 1, \qquad \sum_{i \in \mathbb{Z}} K(i) \mathrm{e}^{-\lambda i} < +\infty,$$

for any $\lambda > 0$. We also assume that the support of J contains either $i = 1$ or two relatively prime integers, $i = p$ and $i = q$.

We also need the following assumptions:

(H1) $b'(0) > d$;
(H2) $\min\{b'(0)u, dK^*\} \geq b(u) > 0$ for some $K^* \geq K$ and all $u \in (0, K^*]$;
(H3) $b(u) > du$ for all $u \in (0, K)$;
(H4) $b'(u) \geq 0$ for all $u \in (0, K)$;
(H5) $b'(0)u - b(u) \leq Mu^{1+\nu}$ for all $u \in (0, K)$, some $M > 0$ and some $\nu \in (0, 1]$;
(H6) $b'(K) < d$.

In the above assumptions, by $b'(0) > d$, we mean that $b(u)$ is differentiable at $u = 0$ and $b'(0) > d$, and the others can be treated similarly. It is easily seen that if $b \in C^2([0, K])$, then (H5) holds spontaneously. A prototype of such functions which has been widely used in the mathematical biology literature is $b(u) = pu\mathrm{e}^{-\alpha u}$ for a wide range of parameters $p > 0$ and $\alpha > 0$.

In the present paper, in addition to the asymptotic speed of propagation, we are also interested in finding monotonic traveling waves $u_n(t) = U(n + ct)$ of (1.6), with U saturating at 0 and K. To this end, we need to find an increasing function $U(\xi)$, where $\xi = n + ct$, for the following associated wave equation

$$-cU'(\xi) + D\sum_{i\neq 0} J(i)[U(\xi - i) - U(\xi)] - dU(\xi) + \sum_{i} K(i)b(U(\xi - i - cr)) = 0, \tag{1.7}$$

subject to the boundary conditions

$$U(-\infty) := \lim_{\xi\to-\infty} U(\xi) = 0, \qquad U(+\infty) := \lim_{\xi\to+\infty} U(\xi) = K. \tag{1.8}$$

We now summarize our main results in the following two theorems.

Theorem 1.1. *Assume that* (H1) *and* (H2) *hold. Then there exists $c_* > 0$ such that c_* is the asymptotic speed of propagation for* (1.6) *in the sense that for any initial data $\varphi = \{\varphi_n\}_{n\in\mathbb{Z}}$ with $\varphi_n \in C([-r, 0], [0, K^*])$, the following statements hold true:*

(i) *if $\limsup_{n\to-\infty} \max_{s\in[-r,0]} \varphi_n(s)\mathrm{e}^{-\lambda n} < +\infty$ for some $\lambda > \Lambda_1(c)$ with $c > c_*$, then*

$$\lim_{t\to+\infty} \sup_n \{u_n(t, \varphi) |\ n \leq -ct\} = 0,$$

(ii) *if $\limsup_{n\to+\infty} \max_{s\in[-r,0]} \varphi_n(s)\mathrm{e}^{\lambda n} < +\infty$ for some $\lambda > \Lambda_1(c)$ with $c > c_*$, then*

$$\lim_{t\to+\infty} \sup_n \{u_n(t, \varphi) |\ n \geq ct\} = 0,$$

(iii) *if $\varphi_{n_0}(0) > 0$ for some $n_0 \in \mathbb{Z}$, then for any $c \in (0, c_*)$,*

$$\liminf_{t\to+\infty} \min_n \{u_n(t, \varphi) |\ |n| \leq ct\} \geq K_*,$$

where $K_* = \frac{1}{d} \inf_{u\in(0,K^*]}\{b(u) |\ b(u) \leq du\} > 0$ and $\lambda = \Lambda_1(c)$ is the smallest solution to the equation $c\lambda - D\sum_{i\neq 0} J(i)\mathrm{e}^{-\lambda i} + D + d - b'(0)\sum_i K(i)\mathrm{e}^{-\lambda(i+cr)} = 0$.

Theorem 1.2. *Assume that* (H1)–(H6) *hold and let $c_* > 0$ be as in Theorem 1.1. Then c_* is also the minimal wave speed for (1.6) in the sense that for $c \in (0, c_*)$, (1.6) has no non-constant traveling wave $U(n + ct)$ with $U(\xi) \in [0, K]$ for all $\xi \in \mathbb{R}$, and for $c \geq c_*$, the equation admits a strictly increasing traveling wavefront $U(n + ct)$ with U saturating at 0 and K. Furthermore, for each $c > c_*$, the traveling wavefront $U(n+ct)$ is unique (up to a translation) under the additional condition $\limsup_{\xi\to-\infty} U(\xi)e^{-\Lambda_1(c)\xi} < +\infty$, where $\Lambda_1(c)$ is defined as in Theorem 1.1.*

Remark 1.1. In Weng et al. [32], in addition to the existence of traveling wavefronts, the authors also obtained some results on the asymptotic speed of propagation for (1.5). Our Theorem 1.1 is a sharp extension of the corresponding results in [32]. In particular, among the other assumptions, Weng et al. [32] assume that the birth function $b(\cdot)$ is non-decreasing. In contrast to the existing literature [1,2,13,14,29,30,32], in our Theorem 1.1, we do not assume that the birth function $b(\cdot)$ is non-decreasing.

Remark 1.2. In our Theorem 1.2, the assumption (H4) is a crucial one by which, the delayed term $b(u)$ is increasing on the interval $[0, K]$. Thus, we can apply the upper–lower solutions and monotonic iteration technique established in [33] or use an argument similar to that in [20] and the Schauder's fixed point theorem to establish the existence of monotonic traveling wavefronts. When K is such that $b(u)$ is not increasing on $[0, K]$, the problem becomes much harder due to lack of quasi-monotonicity. For such delayed equations without quasi-monotonicity, some existence results for traveling waves have been obtained in [33] by using the idea of the so called exponential ordering for delayed differential equations. Application of these results to particular model equations is not trivial as it requires construction of very demanding upper–lower solutions. Uniqueness and stability of traveling waves of such systems seem to be very interesting and challenging problems.

The rest of this paper is organized as follows. In Section 2, by using the squeezing technique [1,2,13,14,29,32], we show that there exists an asymptotic speed of propagation for (1.6). In Section 3, we establish the existence of a traveling wavefront by using an argument as used in [20] and the Schauder's fixed point theorem, and prove that the traveling wavefront is unique up to a translation in some sense. The results in Sections 2 and 3 also confirm the coincidence of the asymptotic speed of propagation and the minimal speed of traveling wavefronts for (1.6).

2. Asymptotic speed of propagation

In this section, we shall show that there exists a constant $c^* > 0$ so that c_* is the asymptotic speed of propagation.

Assume that $b'(0) > d$. Define a new function as follows

$$b_*(u) = \begin{cases} \inf\limits_{\eta\in[u,K^*]} b(\eta), & \text{for } u \leq K^*, \\ b(u), & \text{for } u > K^*. \end{cases}$$

Then $b(u) \geq b_*(u)$ for all $u \in \mathbb{R}$ and $b_*(\cdot)$ is non-decreasing in $(-\infty, K^*]$. Furthermore, $b_*(0) = dK_* - b_*(K_*) = 0$ and $b_*(u) > du$ for all $u \in (0, K_*)$, here and in what follows, $K_* = \frac{1}{d} \inf_{u\in(0,K^*]}\{b(u)|\, b(u) \leq du\} > 0$.

Consider the following initial value problems

$$\begin{cases} u_n'(t) = -(D+d)u_n(t) + H[u_n](t), & n \in \mathbb{Z}, \quad t > 0, \\ u_n(s) = \varphi_n(s), \quad n \in \mathbb{Z}, \quad s \in [-r, 0], \end{cases} \tag{2.1}$$

and

$$\begin{cases} v_n'(t) = -(D+d)v_n(t) + H_*[v_n](t), & n \in \mathbb{Z},\ t > 0, \\ v_n(s) = \psi_n(s), \quad n \in \mathbb{Z},\ s \in [-r, 0], \end{cases} \tag{2.2}$$

where

$$H[u_n](t) = D \sum_{i \neq 0} J(i) u_{n-i}(t) + \sum_i K(i) b(u_{n-i}(t-r)),$$

and

$$H_*[v_n](t) = D \sum_{i \neq 0} J(i) v_{n-i}(t) + \sum_i K(i) b_*(v_{n-i}(t-r)).$$

For the initial value problems (2.1) and (2.2), we have the following existence and comparison result.

Lemma 2.1. *Assume that* (H1) *and* (H2) *hold. Then for all initial data* $\varphi = \{\varphi_n\}_{n\in\mathbb{Z}}, \varphi_n \in C([-r, 0], [0, K^*])$, (2.1) *admits a unique solution* $u = \{u_n\}_{n\in\mathbb{Z}}$ *with* $u_n \in C([0, +\infty), [0, K^*])$. *Moreover, the same conclusion holds for* (2.2) *and if* $\varphi_n(s) \geq \psi_n(s)$ *for all* $n \in \mathbb{Z}$ *and* $s \in [-r, 0]$, *then*

$$u_n(t) \geq v_n(t) \geq \sum_{n_1, n_2 \in \mathbb{Z}, n_1 p + n_2 q = n-k} \frac{(Dt)^{|n_1|+|n_2|}}{(|n_1| + |n_2|)!} [J(p)]^{|n_1|} [J(q)]^{|n_2|} \psi_k(0) \mathrm{e}^{-(D+d)t}, \tag{2.3}$$

for every $n, k \in \mathbb{Z}$ *and* $t > 0$, *here and in what follows,* $p = q = 1$ *or* p *and* q *are two relatively prime integers.*

Proof. Clearly, (2.1) is equivalent to

$$u_n(t) = \varphi_n(0)\mathrm{e}^{-(D+d)t} + \int_0^t \mathrm{e}^{(D+d)(\tau - t)} H[u_n](\tau)\, \mathrm{d}\tau.$$

For $u = \{u_n\}$ with $u_n \in C([-r, +\infty), [0, K^*])$ and $u_n(t) = \varphi_n(t)$ for $t \in [-r, 0]$, define

$$G_n[u](t) = \begin{cases} \varphi_n(0)\mathrm{e}^{-(D+d)t} + \displaystyle\int_0^t \mathrm{e}^{(D+d)(\tau-t)} H[u_n](\tau)\, \mathrm{d}\tau, & \text{for } n \in \mathbb{Z} \text{ and } t > 0. \\ \varphi_n(t), & \text{for } n \in \mathbb{Z} \text{ and } t \in [-r, 0]. \end{cases}$$

Then for $t > 0$, we have

$$0 \leq G_n[u](t) \leq K^* \mathrm{e}^{-(D+d)t} + K^*(D+d) \int_0^t \mathrm{e}^{(D+d)(\tau-t)}\, \mathrm{d}\tau = K^*,$$

and hence, $G = \{G_n\}_{n\in\mathbb{Z}} : S \to S$ is well-defined, where

$$S := \{u = \{u_n\}_{n\in\mathbb{Z}} \mid u_n \in C([-r, +\infty), [0, K^*]), \quad u_n(t) = \varphi_n(t) \quad \text{for } t \in [-r, 0]\}.$$

For $\lambda > 0$, let

$$X_\lambda := \{u = \{u_n\}_{n\in\mathbb{Z}} \mid u_n \in C([-r, +\infty), \mathbb{R}), \quad \sup_{t\geq -r, n\in\mathbb{Z}} |u_n(t)|e^{-\lambda t} < +\infty\},$$

$$\|u\|_\lambda := \sup_{t\geq -r, n\in\mathbb{Z}} |u_n(t)|e^{-\lambda t} < +\infty.$$

Then $(X_\lambda, \|\cdot\|_\lambda)$ is a Banach space and $S \subset X_\lambda$ is a closed subset of X_λ.

For any $u, \bar{u} \in S$, let $w = \{w_n\}_{n\in\mathbb{Z}}$, $w_n(t) = u_n(t) - \bar{u}_n(t)$ for $n \in \mathbb{Z}$, then for $t > 0$, we have

$$\begin{aligned}
&|G_n[u](t) - G_n[\bar{u}](t)|e^{-\lambda t} \\
&\quad\leq e^{-(D+d+\lambda)t} \int_0^t e^{(D+d)\tau} |H[u_n](\tau) - H[\bar{u}_n](\tau)| \,d\tau \\
&\quad\leq \int_0^t e^{(D+d+\lambda)(\tau-t)} \left\{ D \sum_{i\neq 0} J(i)|w_{n-i}(\tau)|e^{-\lambda\tau} \right. \\
&\qquad \left. + L_{K^*}e^{-\lambda r} \sum_i K(i)|w_{n-i}(\tau - r)|e^{-\lambda(\tau-r)} \right\} d\tau \\
&\quad\leq \|w\|_\lambda (D + L_{K^*}e^{-\lambda r}) \int_0^t e^{(D+d+\lambda)(\tau-t)} \,d\tau \\
&\quad\leq \frac{D + L_{K^*}e^{-\lambda r}}{D + d + \lambda} \|w\|_\lambda,
\end{aligned}$$

where and in what follows, L_{K^*} is the Lipschitz constant of $b(\cdot)$ on $[0, K^*]$. Therefore, we can choose $\lambda > 0$ large enough so that $G : S \to S$ is a contracting map. Clearly, the unique fixed point $u \in S$ is a solution of (2.1) on $[0, +\infty)$.

Assume that $\psi_n(s) \leq \varphi_n(s)$ for $n \in \mathbb{Z}$ and $s \in [-r, 0]$. Put $w_n(t) := v_n(t) - u_n(t)$ for $n \in \mathbb{Z}$ and $t \geq -r$. Then $w_n(t)$ is continuous and bounded. Therefore, $\omega(t) := \sup_{n\in\mathbb{Z}} w_n(t)$ is continuous on $[-r, +\infty)$. Let $M_0 > 0$ be such that $M_0 > d + L_{K^*}e^{-M_0 r}$. Suppose that there exists $t_0 > 0$ such that $\omega(t_0) > 0$ and

$$\omega(t_0)e^{-M_0 t_0} = \sup_{t\geq -r} \{\omega(t)e^{-M_0 t}\} > \omega(\tau)e^{-M_0\tau}, \qquad \text{for all } \tau \in [0, t_0). \tag{2.4}$$

Let $\{n_j\}_{j=1}^\infty$ be a sequence such that $w_{n_j}(t_0) > 0$ for all $j \geq 1$ and $\lim_{j\to+\infty} w_{n_j}(t_0) = \omega(t_0)$. Let $\{t_j\}_{j=1}^\infty$ be a sequence in $(0, t_0]$ such that

$$e^{-M_0 t_j} w_{n_j}(t_j) = \max_{t\in[0,t_0]} \{e^{-M_0 t} w_{n_j}(t)\}. \tag{2.5}$$

It follows from (2.4) that $\lim_{j\to+\infty} t_j = t_0$. Since

$$e^{-M_0 t_0} w_{n_j}(t_0) \leq e^{-M_0 t_j} w_{n_j}(t_j) \leq e^{-M_0 t_j} \omega(t_j) \leq e^{-M_0 t_0} \omega(t_0),$$

we have

$$e^{-M_0(t_0-t_j)} w_{n_j}(t_0) \leq w_{n_j}(t_j) \leq e^{-M_0(t_0-t_j)} \omega(t_0),$$

which yields $\lim_{j\to+\infty} w_{n_j}(t_j) = \omega(t_0)$.

In view of (2.5), for each $j \geq 1$, we obtain

$$0 \leq \frac{d}{dt}\{e^{-M_0 t} w_{n_j}(t)\}|_{t=t_j-} = e^{-M_0 t_j}[w'_{n_j}(t_j) - M_0 w_{n_j}(t_j)],$$

and hence

$$\begin{aligned} M_0 w_{n_j}(t_j) &\le w'_{n_j}(t_j) \\ &= -(D+d)w_{n_j}(t_j) + D\sum_{i\neq 0} J(i)w_{n_j-i}(t_j) \\ &\quad + \sum_i K(i)[b_*(v_{n_j-i}(t_j-r)) - b(u_{n_j-i}(t_j-r))] \\ &\le -(D+d)w_{n_j}(t_j) + D\omega(t_j) \\ &\quad + \sum_i K(i)[b_*(v_{n_j-i}(t_j-r)) - b_*(u_{n_j-i}(t_j-r))] \\ &\le -(D+d)w_{n_j}(t_j) + D\omega(t_j) + L_{K^*}\max\{0,\omega(t_j-r)\}. \end{aligned}$$

Sending $j \to +\infty$ we get

$$M_0\omega(t_0) \le d\omega(t_0) + L_{K^*}\mathrm{e}^{-M_0 r}\omega(t_0),$$

which together with $\omega(t_0) > 0$ implies that $M_0 \le d + L_{K^*}\mathrm{e}^{-M_0 r}$, which contradicts $M_0 > d + L_{K^*}\mathrm{e}^{-M_0 r}$. This contradiction shows that $w_n(t) = v_n(t) - u_n(t) \le 0$ for $n \in \mathbb{Z}$ and $t > 0$.

Sine (2.2) is equivalent to

$$v_n(t) = \psi_n(0)\mathrm{e}^{-(D+d)t} + \int_0^t \mathrm{e}^{(D+d)(\tau-t)} H_*[v_n](\tau)\,\mathrm{d}\tau,$$

it follows that

$$v_n(t) \ge \psi_n(0)\mathrm{e}^{-(D+d)t} + D\sum_{i\neq 0} J(i)\int_0^t \mathrm{e}^{(D+d)(\tau-t)} v_{n-i}(\tau)\,\mathrm{d}\tau. \tag{2.6}$$

Therefore, we have

$$v_n(t) \ge \psi_n(0)\mathrm{e}^{-(D+d)t}, \qquad t \ge 0,$$

which together with (2.6) yields

$$v_n(t) \ge \mathrm{e}^{-(D+d)t}\left\{\psi_n(0) + Dt\sum_{i_1\neq 0} J(i_1)\psi_{n-i_1}(0)\right\}.$$

An induction argument shows that

$$v_n(t) \ge \mathrm{e}^{-(D+d)t}\left\{\psi_n(0) + \sum_{m=1}^{\infty}\frac{(Dt)^m}{m!}\sum_{i_1 i_2\cdots i_m\neq 0} J(i_1)J(i_2)\cdots J(i_m)\psi_{n-i_1-i_2-\cdots-i_m}(0)\right\},$$

from which (2.3) follows, and the proof is complete. □

We set

$$\Delta(c,\lambda) := c\lambda - D\sum_{i\neq 0} J(i)\mathrm{e}^{-\lambda i} + D + d - b'(0)\sum_i K(i)\mathrm{e}^{-\lambda(i+cr)}. \tag{2.7}$$

If $b'(0) > d$, we have $\Delta(c,0) = d - b'(0) < 0$ for all $c \ge 0$ and $\lim_{\lambda\to+\infty}\Delta(c,\lambda) = -\infty$. For fixed $c \ge 0$, and any $\lambda_1, \lambda_2 \ge 0$ with $\lambda_1 \neq \lambda_2$, we have

$$\begin{aligned}
&\frac{1}{2}[\Delta(c,\lambda_1)+\Delta(c,\lambda_2)] \\
&\quad = c\frac{\lambda_1+\lambda_2}{2} - D\sum_{i\neq 0} J(i)\left[\frac{e^{-\lambda_1 i}+e^{-\lambda_2 i}}{2}-1\right] \\
&\qquad + d - b'(0)\sum_i K(i)\frac{e^{-\lambda_1(i+cr)}+e^{-\lambda_2(i+cr)}}{2} \\
&\quad < c\frac{\lambda_1+\lambda_2}{2} - D\sum_{i\neq 0} J(i)\left[e^{-(\lambda_1+\lambda_2)i/2}-1\right] + d - b'(0)\sum_i K(i)e^{-(\lambda_1+\lambda_2)(i+cr)/2} \\
&\quad = \Delta\left(c,\frac{\lambda_1+\lambda_2}{2}\right).
\end{aligned}$$

Differentiating $\Delta(c,\lambda)$ with respect to c, we get

$$\frac{\partial}{\partial c}\Delta(c,\lambda) = \lambda + \lambda r b'(0)\sum_i K(i)e^{-\lambda(i+cr)} > 0, \qquad \text{for all } \lambda > 0.$$

Furthermore, for each fixed $\lambda > 0$, we have $\lim_{c\to+\infty}\Delta(c,\lambda) = +\infty$ and

$$\Delta(0,\lambda) = -D\sum_{i\neq 0} J(i)e^{-\lambda i} - b'(0)\sum_i K(i)e^{-\lambda i} < 0.$$

Therefore, we have the following observations:

Lemma 2.2. *Assume that $b'(0) > d$. Then there exists a unique $c_* > 0$ such that*

(i) *if $c \geq c_*$, then there exist two positive numbers $\Lambda_1(c)$ and $\Lambda_2(c)$ with $\Lambda_1(c) \leq \Lambda_2(c)$ such that*

$$\Delta(c,\Lambda_1(c)) = \Delta(c,\Lambda_2(c)) = 0;$$

(ii) *if $c < c_*$, then $\Delta(c,\lambda) < 0$ for all $\lambda \geq 0$;*

(iii) *if $c = c_*$, then $\Lambda_1(c) = \Lambda_2(c) := \Lambda_*$, and if $c > c_*$, then $\Lambda_1(c) < \Lambda_* < \Lambda_2(c)$ and*

$$\Delta(c,\cdot) > 0 \quad \textit{in } (\Lambda_1(c),\Lambda_2(c)), \qquad \Delta(c,\cdot) < 0 \quad \textit{in } \mathbb{R}\setminus[\Lambda_1(c),\Lambda_2(c)],$$

(iv) *$\Lambda_1(c)$ is strictly decreasing and $\Lambda_2(c)$ is strictly increasing in $(c_*,+\infty)$.*

In what follows, we also write $\Delta(c,\lambda) = 0$ as

$$1 = \frac{1}{D+d+c\lambda}\left[D\sum_{i\neq 0} J(i)e^{-\lambda i} + b'(0)\sum_i K(i)e^{-\lambda(i+cr)}\right].$$

Let

$$L_c(\lambda) := \frac{1}{D+d+c\lambda}\left[D\sum_{i\neq 0} J(i)e^{-\lambda i} + b'(0)\sum_i K(i)e^{-\lambda(i+cr)}\right]. \tag{2.8}$$

Then the constant $c_* > 0$ defined in Lemma 2.2 can also be written as

$$c_* = \inf\{c > 0 \mid L_c(\lambda) \leq 1 \text{ for some } \lambda \geq 0\}.$$

Theorem 2.1. *Assume that* (H1) *and* (H2) *hold, and let* $c > c_*$ *and* $\varphi_n \in C([-r, 0], [0, K^*])$. *Then the following statements hold true:*

(i) *If*

$$\limsup_{n\to-\infty} \max_{s\in[-r,0]} \varphi_n(s)\mathrm{e}^{-\lambda n} < +\infty \tag{2.9}$$

for some $\lambda > \Lambda_1(c)$, *then*

$$\lim_{t\to+\infty} \sup_n\{u_n(t,\varphi) \mid n \le -ct\} = 0.$$

(ii) *If*

$$\limsup_{n\to+\infty} \max_{s\in[-r,0]} \varphi_n(s)\mathrm{e}^{\lambda n} < +\infty \tag{2.10}$$

for some $\lambda > \Lambda_1(c)$, *then*

$$\lim_{t\to+\infty} \sup_n\{u_n(t,\varphi) \mid n \ge ct\} = 0.$$

Proof. Define a sequence as follows

$$u^{(j)} = \{u_n^{(j)}\}_{n\in\mathbb{Z}}, \qquad u_n^{(j)}(t) = G_n[u^{(j-1)}](t), \quad t \ge -r,$$
$$u^{(0)} = \{u_n^{(0)}\}_{n\in\mathbb{Z}}, \qquad u_n^{(0)}(t) = \begin{cases} \varphi_n(0), & t > 0, \\ \varphi_n(t), & t \in [-r, 0]. \end{cases}$$

Then an argument similar to that of Lemma 2.1 shows that $u_n^{(j)}(t) \in [0, K^*]$ for all j, and $u = \{u_n\}_{n\in\mathbb{Z}}$ with

$$u_n(t) = \lim_{j\to+\infty} u_n^{(j)}(t), \qquad n \in \mathbb{Z}, \quad t \ge -r$$

is a solution of (2.1).

For any $c > c_*$, take $c_1 \in (c_*, c)$. If (2.9) holds, then by the definition of $u_n^{(0)}(t)$, we can choose $M > 0$ so that

$$u_n^{(0)}(t)\mathrm{e}^{-\lambda(n+c_1 t)} \le M, \qquad \text{for all } n \in \mathbb{Z} \text{ and } t \ge -r. \tag{2.11}$$

Without loss of generality, we may assume that $\lambda \in (\Delta_1(c), \Delta_*)$ and choose $c_1 \in (c_*, c)$ in such a way that $\Delta(c_1, \lambda) = 0$. Then $L_{c_1}(\lambda) = 1$, and for $t > 0$, by (2.4) and (2.11) and the fact that $b(w) \le b'(0)w$ for $w \in [0, K^*]$, we have

$$\begin{aligned}
&u_n^{(1)}(t)\mathrm{e}^{-\lambda(n+c_1 t)} \\
&\quad = \mathrm{e}^{-\lambda(n+c_1 t)} \Bigg\{\varphi_n(0)\mathrm{e}^{-(D+d)t} + \int_0^t \mathrm{e}^{(D+d)(\tau-t)} \Bigg[D\sum_{i\ne 0} J(i)u_{n-i}^{(0)}(\tau) \\
&\qquad + \sum_i K(i)b(u_{n-i}^{(0)}(\tau-r))\Bigg]\,\mathrm{d}\tau\Bigg\} \\
&\quad \le \mathrm{e}^{-(D+d+\lambda c_1)t} \Bigg\{\varphi_n(0)\mathrm{e}^{-\lambda n} + D\int_0^t \mathrm{e}^{(D+d+\lambda c_1)\tau} \sum_{i\ne 0} J(i)u_{n-i}^{(0)}(\tau) \\
&\qquad \times \mathrm{e}^{-\lambda(n-i+c_1\tau)} \cdot \mathrm{e}^{-\lambda i}\,\mathrm{d}\tau
\end{aligned}$$

$$
\begin{aligned}
&\quad + b'(0)\mathrm{e}^{-\lambda c_1 r} \int_0^t \mathrm{e}^{(D+d+\lambda c_1)\tau} \sum_j K(i) u_{n-i}^{(0)}(\tau - r) \mathrm{e}^{-\lambda(n-i+c_1(\tau-r))} \cdot \mathrm{e}^{-\lambda i}\, \mathrm{d}\tau \Bigg\} \\
&\le \frac{M}{D+d+\lambda c_1} \Bigg\{ D \sum_{i\neq 0} J(i)\mathrm{e}^{-\lambda i} + b'(0)\mathrm{e}^{-\lambda c_1 r} \sum_i K(i)\mathrm{e}^{-\lambda i} \\
&\quad + \left[D + d + \lambda c_1 - D \sum_{i\neq 0} J(i)\mathrm{e}^{-\lambda i} - b'(0)\mathrm{e}^{-\lambda c_1 r} \sum_i K(i)\mathrm{e}^{-\lambda i} \right] \mathrm{e}^{-(D+d+\lambda c_1)t} \Bigg\} \\
&= \frac{M}{D+d+\lambda c_1} \\
&\quad \times \left\{ D \sum_{i\neq 0} J(i)\mathrm{e}^{-\lambda i} + b'(0)\mathrm{e}^{-\lambda c_1 r} \sum_i K(i)\mathrm{e}^{-\lambda i} + \Delta(c_1, \lambda)\mathrm{e}^{-(D+d+\lambda c_1)t} \right\} \\
&\le M L_{c_1}(\lambda) \\
&= M,
\end{aligned}
$$

and for $t \in [-r, 0]$, we also have

$$
u_n^{(1)}(t)\mathrm{e}^{-\lambda(n+c_1 t)} = u_n^{(0)}(t)\mathrm{e}^{-\lambda(n+c_1 t)} \le M.
$$

By using an induction argument, we may obtain

$$
u_n^{(j)}(t)\mathrm{e}^{-\lambda(n+c_1 t)} \le M, \qquad \text{for all } j \in \mathbb{N} \text{ and } t \ge -r. \tag{2.12}
$$

Therefore, for $n \le -ct$, we have

$$
0 \le u_n(t) \le M\mathrm{e}^{\lambda(n+c_1 t)} \le M\mathrm{e}^{-\lambda(c-c_1)t} \to 0,
$$

as $t \to +\infty$, from which (i) follows. The statement (ii) can be proved in a similar way and the proof is complete.

As a direct consequence of Theorem 2.1, we have the following

Corollary 2.1. *Assume that* (H1) *and* (H2) *hold. Then for any $c > c_*$,* (1.6) *has no nonconstant traveling wave solution $U(n+ct)$ satisfying $U(\xi) \in [0, K^*]$ for all $\xi \in \mathbb{R}$, and*

$$
\limsup_{\xi\to-\infty} U(\xi)\mathrm{e}^{-\lambda\xi} < +\infty
$$

for some $\lambda > \Lambda_1(c)$.

Remark 2.1. Instead of (H2), we assume that $0 < b(u) \le \min\{Lu, dK^*\}$ for all $u \in (0, K^*]$, some $L \ge b'(0)$ and $K^* \ge K$. Define

$$
\begin{aligned}
c^* = \inf \Bigg\{ c > 0 \,\Bigg|\, &\frac{1}{D+d+c\lambda} \left[D \sum_{i\neq 0} J(i)\mathrm{e}^{-\lambda i} + L \sum_i K(i)\mathrm{e}^{-\lambda(i+cr)} \right] \\
&\le 1 \qquad \text{for some } \lambda \ge 0 \Bigg\}.
\end{aligned}
$$

Then $c^* \ge c_*$, and for $c > c^*$, the same conclusion of Theorem 2.1 holds.

For any $T > 0$ and $\phi = \{\phi_n\}_{n\in\mathbb{Z}}$ with $\phi_n \in C([-r, +\infty), [0, K_*])$, define

$$E_n^T[\phi](t) = \int_0^T e^{-(D+d)\tau} \left\{ D \sum_{i\neq 0} J(i)\phi_{n-i}(t-\tau) + \sum_i K(i) b_*(\phi_{n-i}(t-\tau-r)) \right\} d\tau, \tag{2.13}$$

where $n \in \mathbb{Z}$ and $t \geq T$. Then we have the following comparison principle.

Lemma 2.3. *Let $\phi = \{\phi_n\}_{n\in\mathbb{Z}}$ with $\phi_n \in C([-r, +\infty), [0, K_*])$ be such that for any $\bar{t} \geq T$, the set $\{n \in \mathbb{Z} | \ \phi_n(t) \neq 0$ for some $t \in [T, \bar{t}]\}$ is bounded and*

$$E_n^T[\phi](t) \geq \phi_n(t), \qquad \text{for all } n \in \mathbb{Z} \text{ and } t \geq T. \tag{2.14}$$

If there exists $t_0 \geq 0$ such that the solution $v_n(t)$ of (2.2) satisfies

$$v_n(t_0) > 0 \qquad \text{for all } n \in \mathbb{Z},$$

and

$$v_n(t_0 + t) \geq \phi_n(t) \qquad \text{for } t \in [-r, T].$$

Then

$$v_n(t_0 + t) \geq \phi_n(t) \qquad \text{for all } t \geq -r.$$

Proof. Let

$$t' = \sup\{t \geq T | \ v_n(t_0 + t) \geq \phi_n(t), \ \text{for all } n \in \mathbb{Z}\}.$$

If $t' < +\infty$, then there exists $\{(n_j, t_j)\}_{j=1}^{\infty}$ such that $t_j \searrow t'$ and $0 \leq v_{n_j}(t_0 + t_j) < \phi_{n_j}(t_j)$. Therefore, $\{n_j\}_{j=1}^{\infty}$ is bounded, and hence $\{n_j\}_{j=1}^{\infty}$ is composed of finite integers and contains a constant sub-sequence $\{n'\}$. Thus, we have

$$v_{n'}(t_0 + t') \leq \phi_{n'}(t'). \tag{2.15}$$

Notice that $t' \geq T$, $t_0 \geq 0$ and $v_n(t_0) > 0$ for all $n \in \mathbb{Z}$, it follows from the definition of t' and (2.14) that

$$\begin{aligned} v_{n'}(t_0 + t') &= v_{n'}(t_0)e^{-(D+d)t'} + \int_{t_0}^{t_0+t'} e^{(D+d)(\tau-t')} \left\{ D \sum_{i\neq 0} J(i) v_{n'-i}(\tau) + \sum_i K(i) b_*(v_{n'-i}(\tau - r)) \right\} d\tau \\ &> \int_0^{t'} e^{(D+d)(\tau+t_0-t')} \left\{ D \sum_{i\neq 0} J(i) v_{n'-i}(\tau + t_0) + \sum_i K(i) b_*(v_{n'-i}(\tau + t_0 - r)) \right\} d\tau \\ &\geq \int_0^{t'} e^{(D+d)(\tau-t')} \left\{ D \sum_{i\neq 0} J(i) v_{n'-i}(\tau + t_0) + \sum_i K(i) b_*(v_{n'-i}(\tau + t_0 - r)) \right\} d\tau \end{aligned}$$

$$
\begin{aligned}
&= \int_0^{t'} \mathrm{e}^{-(D+d)\tau} \left\{ D \sum_{i \neq 0} J(i) v_{n'-i}(t_0 + t' - \tau) \right. \\
&\quad \left. + \sum_i K(i) b_*(v_{n'-i}(t_0 + t' - \tau - r)) \right\} \mathrm{d}\tau \\
&\geq \int_0^{T} \mathrm{e}^{-(D+d)\tau} \left\{ D \sum_{i \neq 0} J(i) v_{n'-i}(t_0 + t' - \tau) \right. \\
&\quad \left. + \sum_i K(i) b_*(v_{n'-i}(t_0 + t' - \tau - r)) \right\} \mathrm{d}\tau \\
&\geq \int_0^{T} \mathrm{e}^{-(D+d)\tau} \left\{ D \sum_{i \neq 0} J(i) \phi_{n'-i}(t' - \tau) + \sum_i K(i) b_*(\phi_{n'-i}(t' - \tau - r)) \right\} \mathrm{d}\tau \\
&= E_{n'}^T[\phi](t') \geq \phi_{n'}(t'),
\end{aligned}
$$

which contradicts (2.15). This contradiction shows that $t' = +\infty$ and the proof is complete.

Define a function with two parameters $\omega \in \mathbb{R}$ and $\beta > 0$ as follows

$$
f(y; \omega, \beta) = \begin{cases} \mathrm{e}^{-\omega y} \sin(\beta y), & \text{for } y \in \left[0, \dfrac{\pi}{\beta}\right], \\ 0, & \text{for } y \in \mathbb{R} \setminus \left[0, \dfrac{\pi}{\beta}\right]. \end{cases} \tag{2.16}
$$

Then we have the following lemma.

Lemma 2.4. *Let $c \in (0, c_*)$, then there exist $T > 0$, $h \in (d, b'(0))$, $N > 0$, $\beta_0 > 0$ and a continuous function $\tilde{\omega} = \tilde{\omega}(\beta)$ defined on $[0, \beta_0]$ such that*

$$
\begin{aligned}
&\int_0^T \mathrm{e}^{-(D+d)\tau} \left\{ D \sum_{0<|i|\leq N} J(i) f(y + c\tau - i) + h \sum_{|i|\leq N} K(i) f(y + c\tau + cr - i) \right\} \mathrm{d}\tau \\
&\quad \geq f(y),
\end{aligned} \tag{2.17}
$$

for all $y \in \mathbb{R}$, where $f(y) = f(y; \tilde{\omega}(\beta), \beta)$.

Proof. Define

$$
\begin{aligned}
L(\lambda) = L(\lambda, T, N, h) &:= \int_0^T \mathrm{e}^{-(D+d)\tau} \left\{ D \sum_{0<|i|\leq N} J(i) \mathrm{e}^{-\lambda(c\tau - i)} \right. \\
&\quad \left. + h \sum_{|i|\leq N} K(i) \mathrm{e}^{-\lambda(c\tau + cr - i)} \right\} \mathrm{d}\tau \\
&= \left\{ D \sum_{0<|i|\leq N} J(i) \mathrm{e}^{\lambda i} + h \sum_{|i|\leq N} K(i) \mathrm{e}^{\lambda(i - cr)} \right\} \\
&\quad \times \int_0^T \mathrm{e}^{-(D+d+\lambda c)\tau} \, \mathrm{d}\tau.
\end{aligned}
$$

Firstly, we assert that there exist $h \in (d, b'(0))$, $T > 0$ and $N \in \mathbb{N}$ with $b'(0) - h > 0$ sufficiently small, $T > 0$ and $N > 0$ sufficiently large, such that

$$L(\lambda) = L(\lambda, T, N, h) > 1, \qquad \text{for all } \lambda \in \mathbb{R}. \tag{2.18}$$

Since $L(-\lambda) \geq L(\lambda)$ for $\lambda \geq 0$, we only need to show that $L(\lambda) > 1$ for $\lambda \geq 0$. We observe that for any $T > 0$, $h \in (d, b'(0))$ and any $N \in \mathbb{N}$ with $J(i_0) > 0$ for some $i_0 \in \{1, 2, \dots, N\}$,

$$L(\lambda) = L(\lambda, T, N, h) \geq \frac{D \sum_{0<i\leq N} J(i)\mathrm{e}^{\lambda i}}{D + d + \lambda c_*}[1 - \mathrm{e}^{-(D+d)T}] \to +\infty, \qquad \text{as } \lambda \to +\infty.$$

So we can choose $T_0 > 0$, $N_0 > 0$ and $\lambda_0 > 0$ so that $L(\lambda) = L(\lambda, T, N, h) > 1$ for all $\lambda \geq \lambda_0$, $T \geq T_0$, $N \geq N_0$ and $h \in (d, b'(0))$.

If the assertion is not true, then there exist $\{h_j\}_{j=1}^{\infty}$, $\{T_j\}_{j=1}^{\infty}$, $\{N_j\}_{j=1}^{\infty}$ and $\{\lambda_j\}_{j=1}^{\infty}$ satisfying $h_j \nearrow b'(0)$, $T_j \nearrow +\infty$, $N_j \nearrow +\infty$, $\lambda_j \in [0, \lambda_0]$ such that

$$L(\lambda_j, T_j, N_j, h_j) \leq 1, \qquad \text{for all } j \in \mathbb{N}. \tag{2.19}$$

Without loss of generality, we assume $\lambda_j \to \bar{\lambda} \in [0, \lambda_0]$. Passing to the limit as $j \to \infty$ in (2.19) gives

$$1 < L_c(\bar{\lambda}) = \lim_{j\to\infty} L(\lambda_j, T_j, N_j, h_j) \leq 1,$$

which leads to a contradiction and establishes the assertion.

Let $\lambda = \omega + i\beta$, then

$$L(\omega + i\beta) = \Re[L(\omega + i\beta)] + i\Im[L(\omega + i\beta)],$$

where

$$\Re[L(\omega + i\beta)] = \int_0^T \mathrm{e}^{-(D+d)\tau} \left\{ D \sum_{0<|i|\leq N} J(i)\mathrm{e}^{-\omega(c\tau - i)} \cos\beta(c\tau - i) \right.$$
$$\left. + h \sum_{|i|\leq N} K(i)\mathrm{e}^{-\omega(c\tau+cr-i)} \cos\beta(c\tau + cr - i) \right\} \mathrm{d}\tau, \tag{2.20}$$

and

$$\Im[L(\omega + i\beta)] = -\int_0^T \mathrm{e}^{-(D+d)\tau} \left\{ D \sum_{0<|i|\leq N} J(i)\mathrm{e}^{-\omega(c\tau - i)} \sin\beta(c\tau - i) \right.$$
$$\left. + h \sum_{|i|\leq N} K(i)\mathrm{e}^{-\omega(c\tau+cr-i)} \sin\beta(c\tau + cr - i) \right\} \mathrm{d}\tau. \tag{2.21}$$

Since $L''(\lambda) > 0$ for all $\lambda \in \mathbb{R}$ and $\lim_{|\lambda|\to+\infty} L(\lambda) = +\infty$, it follows that $L(\lambda)$ can achieve its minimum, say at $\lambda = \lambda_0$. Therefore, we have

$$L'(\omega_0) = -\int_0^T \mathrm{e}^{-(D+d)\tau} \left\{ D \sum_{0<|i|\leq N} J(i)(c\tau - i)\mathrm{e}^{-\omega_0(c\tau - i)} \right.$$
$$\left. + h \sum_{|i|\leq N} K(i)(c\tau + cr - i)\mathrm{e}^{-\omega_0(c\tau+cr-i)} \right\} \mathrm{d}\tau = 0.$$

We now define a function $H = H(\omega, \beta)$ by

$$\begin{cases} H(\omega, \beta) = \dfrac{1}{\beta}\Im[L(\omega + i\beta)], & \text{for } \beta \neq 0, \\ H(\omega, 0) = \lim\limits_{\beta\to 0} H(\omega, \beta) = L'(\omega). \end{cases}$$

Then $H(\omega_0, 0) = 0$ and

$$\frac{\partial H}{\partial \omega}(\omega_0, 0) = L''(\omega_0) > 0.$$

The Implicit Function Theorem then implies that there exist $\beta_1 > 0$ and a continuous function $\tilde{\omega} = \tilde{\omega}(\beta)$ defined on $[0, \beta_1]$ with $\tilde{\omega}(0) = \omega_0$ such that $H(\tilde{\omega}(\beta), \beta) = 0$ for $\beta \in [0, \beta_1]$. Hence, we have

$$\Im[L(\tilde{\omega}(\beta) + i\beta)] = 0, \qquad \text{for } \beta \in [0, \beta_1]. \tag{2.22}$$

Since $L(\omega_0) > 1$, we can choose $\beta_2 > 0$ sufficiently small so that

$$\Re[L(\tilde{\omega}(\beta) + i\beta)] > 1, \qquad \text{for } \beta \in [0, \beta_2]. \tag{2.23}$$

Let

$$0 < \beta \leq \beta_0 := \min\left\{\beta_1, \beta_2, \frac{\pi}{N + c_*(T + r)}\right\}. \tag{2.24}$$

Then for $y \in [0, \frac{\pi}{\beta}]$, $|i| \leq N$ and $\tau \in [0, T]$, we have

$$-\frac{\pi}{\beta} < -N \leq y + c\tau - i \leq y + c\tau + cr - i \leq \frac{\pi}{\beta} + c_*(T + r) + N \leq \frac{2\pi}{\beta}.$$

Since $\sin \beta z \leq 0$ for $z \in [-\frac{\pi}{\beta}, 0] \cup [\frac{\pi}{\beta}, \frac{2\pi}{\beta}]$, it follows from (2.20)–(2.24) that for $y \in [0, \frac{\pi}{\beta}]$,

$$\begin{aligned} &\int_0^T e^{-(D+d)\tau} \left\{D \sum_{0<|i|\leq N} J(i) f(y + c\tau - i) + h \sum_{|i|\leq N} K(i) f(y + c\tau + cr - i)\right\} d\tau \\ &\geq \int_0^T e^{-(D+d)\tau} \left\{D \sum_{0<|i|\leq N} J(i) e^{-\tilde{\omega}(\beta)(y+c\tau-i)} \sin \beta(y + c\tau - i)\right. \\ &\qquad \left. + h \sum_{|i|\leq N} K(i) e^{-\tilde{\omega}(\beta)(y+c\tau+cr-i)} \sin \beta(y + c\tau + cr - i)\right\} d\tau \\ &= e^{-\tilde{\omega}(\beta)y} \sin \beta y \cdot \Re[L(\tilde{\omega}(\beta) + i\beta)] - e^{-\tilde{\omega}(\beta)y} \cos \beta y \cdot \Im[L(\tilde{\omega}(\beta) + i\beta)] \\ &\geq e^{-\tilde{\omega}(\beta)y} \sin \beta y = f(y). \end{aligned}$$

This completes the proof. □

Define

$$R(y; \omega, \beta, \chi) := \max_{\eta\geq -\chi} f(y + \eta; \omega, \beta)$$
$$= \begin{cases} \varpi, & \text{for } y \leq \chi + \varrho, \\ f(y - \chi; \omega, \beta), & \text{for } \chi + \varrho \leq y \leq \chi + \dfrac{\pi}{\beta}, \\ 0, & \text{for } y \geq \chi + \dfrac{\pi}{\beta}, \end{cases} \tag{2.25}$$

where

$$\varpi = \varpi(\omega, \beta) := \max\left\{ f(y; \omega, \beta) \,\middle|\, 0 \le y \le \frac{\pi}{\beta} \right\}, \tag{2.26}$$

and $\varrho = \varrho(\omega, \beta)$ is the point where the above maximum ϖ is achieved.

Lemma 2.5. *Let $c \in (0, c_*)$ be given, then there exist $T > 0, \beta > 0, \omega \in \mathbb{R}, B > 0$ and $\sigma_0 > 0$ such that for any $\sigma \in (0, \sigma_0)$ and for any $t \ge T$,*

$$E_n^T[\sigma\phi](t) \ge \sigma\phi_n(t), \tag{2.27}$$

where $\phi_n(t) = R(|n|; \omega, \beta, B + ct), n \in \mathbb{Z}, t \ge -r$.

Proof. By Lemma 2.4, we can choose $T > 0$, $h \in (d, b'(0))$, $N > 0$, $\beta > 0$ and $\omega = \tilde{\omega}(\beta)$ such that (2.17) holds.

Take $B = 2N + c_*r + 1$. Let σ_h be the smallest positive root of the equation $b_*(w) = hw$. Then $b_*(w) > hw$ for $w \in (0, \omega_h)$. Choose $\sigma_0 \in (0, \frac{\sigma_h}{\varpi})$. Let $\sigma \in (0, \sigma_0)$ and $t \ge T$, then we have

$$\begin{aligned} &E_n^T[\sigma\phi](t) \\ &\quad = \int_0^T e^{-(D+d)\tau} \left\{ \sigma D \sum_{i \ne 0} J(i)\phi_{n-i}(t-\tau) + \sum_i K(i) b_*(\sigma\phi_{n-i}(t-\tau-r)) \right\} d\tau \\ &\quad \ge \sigma \int_0^T e^{-(D+d)\tau} \left\{ D \sum_{0<|i|\le N} J(i)\phi_{n-i}(t-\tau) + h \sum_{|i|\le N} K(i)\phi_{n-i}(t-\tau-r) \right\} d\tau. \end{aligned} \tag{2.28}$$

We now distinguish between two cases:

Case (i). $|n| \le B + \varrho + c(t - T - r) - N$. In this case, we have $|n - i| \le B + c(t - \tau - r) + \varrho \le B + c(t - \tau) + \varrho$ for $\tau \in [0, T]$ and $|i| \le N$, and hence, it follows from (2.28) and the definition of $\phi_n(t)$ that

$$\begin{aligned} E_n^T[\sigma\phi](t) &\ge \sigma\varpi \left\{ D \sum_{0<|i|\le N} J(i) + h \sum_{|i|\le N} K(i) \right\} \int_0^T e^{-(D+d)\tau} d\tau \\ &= \sigma\varpi \left\{ D \sum_{0<|i|\le N} J(i) + h \sum_{|i|\le N} K(i) \right\} \cdot \frac{1}{D+d}[1 - e^{-(D+d)T}] \\ &\ge \sigma\varpi = \sigma\phi_n(t), \end{aligned}$$

provided that $T > 0$ and $N > 0$ are large enough.

Case (ii). $B + \varrho + c(t - T - r) - N \le |n| \le B + ct + \frac{\pi}{\beta}$. In this case, $|n| \ge N + 1$. Therefore, for $|i| \le N$, we have $|n - i| = n - i = |n| - i$ if $n > 0$ and $|n - i| = -n + i = |n| + i$ if $n < 0$. Hence, it follows from (2.17) and (2.28), the definition of $\phi_n(t)$ and the evenness of $J(i)$ and $K(i)$ that

$$
\begin{aligned}
E_n^T[\sigma\phi](t) &\geq \sigma \int_0^T \mathrm{e}^{-(D+d)\tau} \left\{ D \sum_{0<|i|\leq N} J(i) \max_{\eta\geq -B-c(t-\tau)} f(|n|-i+\eta) \right. \\
&\qquad \left. + h \sum_{|i|\leq N} K(i) \max_{\eta\geq -B-c(t-\tau-r)} f(|n|-i+\eta) \right\} \mathrm{d}\tau \\
&= \sigma \int_0^T \mathrm{e}^{-(D+d)\tau} \left\{ D \sum_{0<|i|\leq N} J(i) \max_{\eta\geq -B-ct} f(|n|-i+c\tau+\eta) \right. \\
&\qquad \left. + h \sum_{|i|\leq N} K(i) \max_{\eta\geq -B-ct} f(|n|-i+c\tau+cr+\eta) \right\} \mathrm{d}\tau \\
&\geq \sigma \max_{\eta\geq -B-ct} f(|n|+\eta) = \sigma\phi_n(t).
\end{aligned}
$$

Combining (i) and (ii), we obtain (2.27) and complete the proof. □

Theorem 2.2. *Assume that* (H1) *and* (H2) *hold. Assume that* $\varphi = \{\varphi_n\}_{n\in\mathbb{Z}}$ *with* $\varphi_n \in C([-r,0],[0,K^*])$ *satisfies* $\varphi_{n_0}(0) > 0$ *for some* $n_0 \in \mathbb{Z}$. *Then for any* $c \in (0, c_*)$, *there holds*

$$\liminf_{t\to+\infty} \min_n\{u_n(t,\varphi) \mid |n| \leq ct\} \geq K_*, \tag{2.29}$$

where $K_* = \frac{1}{d} \inf_{u\in(0,K^*]}\{b(u) \mid b(u) \leq du\} > 0$.

Proof. Take $\psi = \{\psi_n\}_{n\in\mathbb{Z}}$ where $\psi_n \in C([-r,0],[0,K_*])$ satisfies $\varphi_n(s) \geq \psi_n(s)$ for all $n\in\mathbb{Z}$, $s\in[-r,0]$ and $\psi_{n_0}(0) > 0$. Then by virtue of Lemma 2.1, we have $u_n(t,\varphi) \geq v_n(t,\psi)$ for all $n \in \mathbb{Z}$ and $t > 0$. So it suffices to show that

$$\liminf_{t\to+\infty} \min_n\{v_n(t,\psi) \mid |n| \leq ct\} \geq K_*, \tag{2.30}$$

where $v(t) := v(t,\psi) = \{v_n(t,\psi)\}_{n\in\mathbb{Z}}$ is the unique solution of (2.2).

For any $c \in (0,c_*)$, choose $c_1 \in (c, c_*)$. By Lemma 2.5, there exist constants $T > 0$, $\beta > 0$, $\omega \in \mathbb{R}$, $B > 0$ and $\sigma_0 > 0$ such that for any $\sigma \in (0,\sigma_0)$ and any $t \geq T$,

$$E_n^T[\sigma\phi](t) \geq \sigma\phi_n(t), \tag{2.31}$$

where $\phi_n(t) = R(|n|;\omega,\beta,B+c_1t)$, $n\in\mathbb{Z}$, $t\geq -r$.

By Lemma 2.1, we see that $v_n(t) = v_n(t,\psi) > 0$ for all $n \in \mathbb{Z}$ and $t > 0$. Choose $t_0 > r$ and denote $\phi(n,t) = \phi_n(t)$ for $n \in \mathbb{Z}$ and $t \geq -r$. Since for any $t \in [-r,T]$, $\operatorname{supp}\phi(\cdot,t) \subset \operatorname{supp}\phi(\cdot,T)$ are bounded sets, we can choose $\varsigma \in (0,\sigma_0)$ such that

$$\varsigma\varpi < K_* \tag{2.32}$$

and

$$v_n(t_0+t) \geq \varsigma\phi_n(t), \qquad \text{for } n \in \operatorname{supp}\phi(\cdot,T) \text{ and } t\in[-r,T]. \tag{2.33}$$

It then follows from Lemma 2.3 that

$$v_n(t_0+t) \geq \varsigma\phi_n(t), \qquad \text{for all } n \in \operatorname{supp}\phi(\cdot,T) \text{ and } t \geq -r,$$

from which and the definition of $\phi_n(t)$, we obtain

$$v_n(t_0+t) \geq \varsigma\varpi \qquad \text{for } t\geq -r \text{ and } |n| \leq B + c_1t + \varrho.$$

By (2.2), we find

$$v_n(t_0+t) \geq \int_0^t e^{-(D+d)\tau} \left\{ D \sum_{i\neq 0} J(i) v_{n-i}(t_0+t-\tau) + \sum_i K(i) b_*(v_{n-i}(t_0+t-\tau-r)) \right\} d\tau. \tag{2.34}$$

Let $a = \varsigma\varpi = Q_0(t, N)$, $t \geq -r$ and for $j \in \mathbb{N}$, let

$$Q_j(t, N) = \int_0^t e^{-(D+d)\tau} \left\{ D \sum_{0<|i|\leq N} J(i) Q_{j-1}(t-\tau, N) + \sum_{|i|\leq N} K(i) b_*(Q_{j-1}(t-\tau-r, N)) \right\}, \qquad \text{for } t > 0,$$

and

$$Q_j(t, N) = 0, \qquad \text{for } t \in [-r, 0].$$

Then for $t \geq 0$ and $|n| \leq B + c_1 t + \varrho - N$, we have

$$v_n(t_0+t) \geq \int_0^t e^{-(D+d)\tau} \left\{ D \sum_{0<|i|\leq N} J(i) Q_0(t-\tau, N) + \sum_{|i|\leq N} K(i) b_*(Q_0(t-\tau-r, N)) \right\} d\tau = Q_1(t, N).$$

By an induction argument, it is easily seen that

$$v_n(t_0+t) \geq Q_j(t, N), \qquad \text{for } t \geq -r \text{ and } |n| \leq B + c_1 t + \varrho - jN. \tag{2.35}$$

We claim that for any $\epsilon > 0$, there exist $\bar{t}(\epsilon) > 0$, $\bar{N}(\epsilon) \in \mathbb{N}$ and $\bar{J}(\epsilon) \in \mathbb{N}$ such that

$$Q_j(t, N) \geq K_* - \epsilon, \qquad \text{for } N \geq \bar{N}(\epsilon),\ j \geq \bar{J}(\epsilon) \text{ and } t \geq j(\bar{t}(\epsilon) + r). \tag{2.36}$$

To see this, we firstly observe that

$$0 < a = Q_0(t, N) < K_* \qquad \text{and} \qquad 0 < 1 - e^{-(D+d)t} < 1,\ \text{for } t > 0.$$

and an induction argument shows that

$$0 < Q_j(t, N) < K_*, \qquad \text{for all } t > 0,\ \ j, N \in \mathbb{N} \text{ with } N \text{ large enough.}$$

For small $\epsilon > 0$. Since $b_*'(0) = b'(0) > d$ and $Dw + b_*(w) > (D+d)w$ for $w \in (0, K_*)$, we have

$$\Lambda(\epsilon) = \inf \left\{ \frac{Dw + b_*(w)}{(D+d)w} \,\middle|\, 0 < w \leq K_* - \epsilon \right\} > 1.$$

Choose $\alpha(\epsilon) \in (\frac{1}{\Lambda(\epsilon)}, 1)$. Then

$$\frac{\alpha(\epsilon)}{D+d}[Dw + b_*(w)] > \frac{1}{\Lambda(\epsilon)(D+d)}[Dw + b_*(w)] \geq w, \qquad \text{for } w \in (0, K_* - \epsilon]. \tag{2.37}$$

Define a sequence as follows

$$M_0 = a, \qquad M_j = \frac{\alpha(\epsilon)}{D+d}[DM_{j-1} + b_*(M_{j-1})], \quad j \geq 1.$$

Then we have the following observations:

(i) if $0 < M_j \leq K_* - \epsilon$, then $M_{j+1} \geq M_j$;
(ii) if $M_j > K_* - \epsilon$, then

$$M_{j+1} > \frac{\alpha(\epsilon)}{D+d}[D(K_* - \epsilon) + b_*(K_* - \epsilon)] \geq K_* - \epsilon.$$

If $M_j \leq K_* - \epsilon$ for all $j \in \mathbb{N}$, then by (i) $\lim_{j\to\infty} M_j = \bar{M}$ exists and satisfies

$$0 < \bar{M} = \frac{\alpha(\epsilon)}{D+d}[D\bar{M} + b_*(\bar{M})] \leq K_* - \epsilon,$$

which contradicts (2.37). Therefore, there exists $\bar{J}(\epsilon) \in \mathbb{N}$ such that $M_{\bar{J}(\epsilon)} > K_* - \epsilon$, and hence, it follows from (ii) that $M_j > K_* - \epsilon$ for all $j \geq \bar{J}(\epsilon)$.

Choose $\bar{N}(\epsilon) > 0$ and $\bar{t}(\epsilon) > 0$ sufficiently large so that

$$[1 - \mathrm{e}^{-(D+d)\bar{t}(\epsilon)}] \cdot \min\left\{ \sum_{0<|i|\leq \bar{N}(\epsilon)} J(i), \sum_{|i|\leq \bar{N}(\epsilon)} K(i) \right\} \geq \alpha(\epsilon).$$

For $N \geq \bar{N}(\epsilon)$, if $Q_j(t, N) \geq M_j$ for some j and every $t \geq j(\bar{t}(\epsilon) + r)$, then for all $t \geq (j+1)(\bar{t}(\epsilon) + r)$, we have

$$\begin{aligned} Q_{j+1}(t, N) &= \int_0^t \mathrm{e}^{-(D+d)\tau} \left\{ D \sum_{0<|i|\leq N} J(i) Q_j(t-\tau, N) \right. \\ &\quad \left. + \sum_{|i|\leq N} K(i) b_*(Q_j(t-\tau-r, N)) \right\} \mathrm{d}\tau \\ &\geq \int_0^{\bar{t}(\epsilon)} \mathrm{e}^{-(D+d)\tau} \left\{ D \sum_{0<|i|\leq N} J(i) Q_j(t-\tau, N) \right. \\ &\quad \left. + \sum_{|i|\leq N} K(i) b_*(Q_j(t-\tau-r, N)) \right\} \mathrm{d}\tau \\ &\geq \frac{1 - \mathrm{e}^{-(D+d)\bar{t}(\epsilon)}}{D+d} \left[D \sum_{0<|i|\leq \bar{N}(\epsilon)} J(i) M_j + \sum_{|i|\leq \bar{N}(\epsilon)} K(i) b_*(M_j) \right] \\ &\geq \frac{\alpha(\epsilon)}{D+d}[DM_j + b_*(M_j)] = M_{j+1}. \end{aligned}$$

Since $Q_0(t, N) = a \geq M_0, t \geq 0$, by induction, we conclude that $Q_j(t, N) \geq M_j$ for all $j \geq 0$, $N \geq \bar{N}(\epsilon)$ and $t \geq j(\bar{t}(\epsilon) + r)$. Therefore, $Q_j(t, N) > K_* - \epsilon$ for $j \geq \bar{J}(\epsilon)$, $N \geq \bar{N}(\epsilon)$ and $t \geq j(\bar{t}(\epsilon) + r)$. This establishes the assertion.

So it follows from (2.35) and (2.36) that

$$v_n(t) \geq K_* - \epsilon, \text{ for } t \geq t_0 + \bar{J}(\epsilon)(\bar{t}(\epsilon) + r) \text{ and } |n| \leq B + c_1(t - t_0) + \varrho - \bar{J}(\epsilon)\bar{N}(\epsilon). \quad (2.38)$$

Define

$$t_1 = \max\left\{t_0 + \bar{J}(\epsilon)(\bar{t}(\epsilon) + r), \frac{\bar{J}(\epsilon)\bar{N}(\epsilon) + c_1 t_0 - B - \varrho}{c_1 - c}\right\}.$$

Since $c_1 > c$, it follows from (2.38) that

$$v_n(t) \geq K_* - \epsilon, \qquad \text{for } t \geq t_1 \text{ and } |n| \leq ct,$$

from which (2.30) follows and the proof is complete. □

As a direct consequence of Theorem 2.2, we have the following

Corollary 2.2. *Assume that* (H1) *and* (H2) *hold. Then for any* $c \in (0, c_*)$, (1.2) *has no traveling wave solution* $U(n + ct)$ *satisfying* $U(\xi) \in [0, K^*]$ *for all* $\xi \in \mathbb{R}$ *and* $U(\xi_0) \in (0, K_*)$ *for some* $\xi_0 \in \mathbb{R}$.

3. Existence and uniqueness of traveling waves

In this section, we first show the existence of traveling waves of (1.6) by using the sub-supersolution technique and an iteration scheme.

For any absolutely continuous function $\phi : \mathbb{R} \to \mathbb{R}$, we set

$$N_c[\phi](\xi) := c \lim_{h \searrow 0} \frac{\phi(\xi) - \phi(\xi - h)}{h} - D \sum_{i \neq 0} J(i)[\phi(\xi - i) - \phi(\xi)] + d\phi(\xi) - \sum_i K(i) b(\phi(\xi - i - cr)). \quad (3.1)$$

Defintion 3.1. An absolutely continuous function $\phi : \mathbb{R} \to [0, K]$ is called a supersolution (a subsolution, resp.) of (1.7) if for almost every $\xi \in \mathbb{R}$, $N_c[\phi](\xi) \geq 0$ (≤ 0, resp.).

Lemma 3.1. *Assume that* (H1)–(H5) *hold. Let* $c > c_*$ *and* $\Lambda_1(c), \Lambda_2(c)$ *be defined as in* Lemma 2.2. *Then for every* $\beta \in (1, \min\{1 + \nu, \frac{\Lambda_2(c)}{\Lambda_1(c)}\})$, *there exists* $Q(c, \beta) \geq 1$, *such that for any* $q \geq Q(c, \beta)$ *and any* $\xi^\pm \in \mathbb{R}$, *the functions* $\phi^\pm$ *defined by*

$$\phi^+(\xi) := \min\{K, \mathrm{e}^{\Lambda_1(c)(\xi + \xi^+)} + q\mathrm{e}^{\beta\Lambda_1(c)(\xi + \xi^+)}\}, \qquad \xi \in \mathbb{R} \quad (3.2)$$

and

$$\phi^-(\xi) := \max\{0, \mathrm{e}^{\Lambda_1(c)(\xi + \xi^-)} - q\mathrm{e}^{\beta\Lambda_1(c)(\xi + \xi^-)}\}, \qquad \xi \in \mathbb{R} \quad (3.3)$$

are a supersolution and a subsolution to (1.7), *respectively.*

Proof. It is easily seen that there exists $\xi^* \leq -\xi^+ - \frac{1}{\beta\Lambda_1(c)} \ln \frac{q}{K}$, such that $\phi^+(\xi) = K$ for $\xi > \xi^*$ and $\phi^+(\xi) = \mathrm{e}^{\Lambda_1(c)(\xi + \xi^+)} + q\mathrm{e}^{\beta\Lambda_1(c)(\xi + \xi^+)}$ for $\xi \leq \xi^*$.

For $\xi > \xi^*$, we have

$$\begin{aligned} N_c[\phi^+](\xi) &= -D\sum_{i\neq 0} J(i)[\phi^+(\xi-i)-K] + dK - \sum_i K(i)b(\phi^+(\xi-i-cr)) \\ &\geq -D\sum_{i\neq 0} J(i)[\phi^+(\xi-i)-K] - \sum_i K(i)[b(\phi^+(\xi-i-cr)) - b(K)] \\ &\geq 0. \end{aligned}$$

For $\xi \leq \xi^*$, we have

$$\begin{aligned} N_c[\phi^+](\xi) &\geq e^{\Lambda_1(c)(\xi+\xi^+)}\left[c\Lambda_1(c) - D\sum_{i\neq 0} J(i)e^{-\Lambda_1(c)i} + D + d\right] \\ &\quad + qe^{\beta\Lambda_1(c)(\xi+\xi^+)}\left[c\beta\Lambda_1(c) - D\sum_{i\neq 0} J(i)e^{-\beta\Lambda_1(c)i} + D + d\right] \\ &\quad - \sum_i K(i)b(\phi^+(\xi-i-cr)) \\ &\geq qe^{\beta\Lambda_1(c)(\xi+\xi^+)}\Delta(c,\beta\Lambda_1(c)) + b'(0)\sum_i K(i)\phi^+(\xi-i-cr) \\ &\quad - \sum_i K(i)b(\phi^+(\xi-i-cr)) \\ &> 0. \end{aligned}$$

Therefore, ϕ^+ is a supersolution of (1.7).

Let $\xi_* = -\xi^- - \frac{1}{(\beta-1)\Lambda_1(c)}\ln q$. If $q \geq 1$, then $\xi_* \leq -\xi^-$. Clearly, $\phi^-(\xi) = 0$ for $\xi > \xi_*$ and $\phi^-(\xi) = e^{\Lambda_1(c)(\xi+\xi^-)} - qe^{\beta\Lambda_1(c)(\xi+\xi^-)}$ for $\xi \leq \xi_*$.

For $\xi > \xi_*$, we have

$$N_c[\phi^-](\xi) = -D\sum_{i\neq 0} J(i)\phi^-(\xi-i) - \sum_i K(i)b(\phi^-(\xi-i-cr)) \leq 0.$$

For $\xi \leq \xi_*$, we have $\xi + \xi^- \leq -\frac{1}{(\beta-1)\Lambda_1(c)}\ln q$, and hence

$$\begin{aligned} N_c[\phi^-](\xi) &\leq e^{\Lambda_1(c)(\xi+\xi^-)}\left[c\Lambda_1(c) - D\sum_{i\neq 0} J(i)e^{-\Lambda_1(c)i} + D + d\right] \\ &\quad - qe^{\beta\Lambda_1(c)(\xi+\xi^-)}\left[c\beta\Lambda_1(c) - D\sum_{i\neq 0} J(i)e^{-\beta\Lambda_1(c)i} + D + d\right] \\ &\quad - \sum_i K(i)b(\phi^-(\xi-i-cr)) \\ &\leq -qe^{\beta\Lambda_1(c)(\xi+\xi^-)}\Delta(c,\beta\Lambda_1(c)) + b'(0)\sum_i K(i)\phi^-(\xi-i-cr) \\ &\quad - \sum_i K(i)b(\phi^-(\xi-i-cr)) \\ &\leq -qe^{\beta\Lambda_1(c)(\xi+\xi^-)}\Delta(c,\beta\Lambda_1(c)) + M\sum_i K(i)[\phi^-(\xi-i-cr)]^{1+\nu} \end{aligned}$$

$$
\begin{aligned}
&\leq -q e^{\beta \Lambda_1(c)(\xi+\xi^-)} \Delta(c, \beta \Lambda_1(c)) + M \sum_i K(i) e^{(1+\nu)\Lambda_1(c)(\xi+\xi^- - i)} \\
&\leq \left\{ -q \Delta(c, \beta \Lambda_1(c)) + M \sum_i K(i) e^{(1+\nu)\Lambda_1(c) i} e^{(1+\nu-\beta)\Lambda_1(c)(\xi+\xi^-)} \right\} \\
&\quad \times e^{\beta \Lambda_1(c)(\xi+\xi^-)} \\
&\leq \left\{ -q \Delta(c, \beta \Lambda_1(c)) + M \sum_i K(i) e^{(1+\nu)\Lambda_1(c) i} \right\} e^{\beta \Lambda_1(c)(\xi+\xi^-)} \\
&\leq 0,
\end{aligned}
$$

provided that $q \geq Q(c, \beta) := \max\left\{1, \frac{M}{\Delta(c,\beta\Lambda_1(c))} \sum_i K(i) e^{(1+\nu)\Lambda_1(c) i}\right\}$. Therefore, ϕ^- is a subsolution of (1.7). The proof is complete. □

The following theorem is our main result for the existence of traveling waves.

Theorem 3.1. *Assume* (H1)–(H5) *hold. Let $c_* > 0$ be as in Lemma 2.2. Then for each $c \geq c_*$, (1.6) admits a traveling wave solution $u_n(t) = U(n + ct)$ satisfying $U(-\infty) = 0$, $U(+\infty) = K$ and $U' > 0$ on $\mathbb{R}$. Furthermore, for $c > c_*$, U also satisfies*

$$
\lim_{\xi \to -\infty} U(\xi) e^{-\Lambda_1(c)\xi} = 1, \qquad \lim_{\xi \to -\infty} U'(\xi) e^{-\Lambda_1(c)\xi} = \Lambda_1(c), \tag{3.4}
$$

where $\lambda = \Lambda_1(c)$ is the smallest solution to the equation

$$
\Delta(c, \lambda) = c\lambda - D \sum_{i \neq 0} J(i) e^{-\lambda i} + D + d - b'(0) \sum_i K(i) e^{-\lambda(i+cr)} = 0.
$$

Proof. For $c > c_*$, by virtue of Lemma 3.1, ϕ^+ and ϕ^- with $\xi^\pm = 0$ are a supersolution and a subsolution to (1.7), respectively. For any $\lambda \in (0, \Lambda_1(c))$, let

$$
X = \left\{ \phi \in C(\mathbb{R}, \mathbb{R}) \mid \sup_{\xi \in \mathbb{R}} |\phi(\xi)| e^{-\lambda \xi} < +\infty \right\}, \qquad \|\phi\|_\lambda = \sup_{\xi \in \mathbb{R}} |\phi(\xi)| e^{-\lambda \xi}.
$$

Then $(X, \|\cdot\|_\lambda)$ is a Banach space. Since $\phi^-(\xi) \leq \phi^+(\xi)$ for all $\xi \in \mathbb{R}$ and $\phi^+(\xi)$ is nondecreasing on $\mathbb{R}$, by using a argument as used in [20], it is easily known that the set

$$
\Gamma := \left\{ \phi \in C(\mathbb{R}, [0, K]) \;\middle|\; \begin{array}{ll} \text{(i)} & \phi(\xi) \text{ is nondecreasing on } \mathbb{R}; \\ \text{(ii)} & \phi^-(\xi) \leq \phi(\xi) \leq \phi^+(\xi) \text{ for all } \xi \in \mathbb{R}; \\ \text{(iii)} & |\phi(\xi_1) - \phi(\xi_2)| \leq \dfrac{2K(D+d)}{c} |\xi_1 - \xi_2| \text{ for all } \xi_1, \xi_2 \in \mathbb{R}. \end{array} \right\}.
$$

is nonempty, convex and compact in X.

Define $F : \Gamma \to \Gamma$ by

$$
F(\phi)(\xi) = \frac{1}{c} e^{-\frac{D+d}{c}\xi} \int_{-\infty}^{\xi} e^{\frac{D+d}{c}\tau} H(\phi)(\tau)\, d\tau,
$$

where $H(\phi)(\xi) = D \sum_{i \neq 0} J(i)\phi(\xi - i) + \sum_i K(i) b(\phi(\xi - i - cr))$, $\xi \in \mathbb{R}$. It is easily seen that F is well-defined and a fixed point of F is a solution of (1.7) and (1.8).

Since for any $\phi, \psi \in \Gamma$,

$$
\begin{aligned}
&|F(\phi)(\xi) - F(\psi)(\xi)| e^{-\lambda \xi} \\
&\quad \le \frac{1}{c} e^{-(\frac{D+d}{c}+\lambda)\xi} \int_{-\infty}^{\xi} e^{\frac{D+d}{c}\tau} |H(\phi)(\tau) - H(\psi)(\tau)| \, d\tau \\
&\quad \le \frac{1}{c} e^{-(\frac{D+d}{c}+\lambda)\xi} \int_{-\infty}^{\xi} e^{\frac{D+d}{c}\tau} \Bigg\{ D \sum_{i \neq 0} J(i) |\phi(\tau - i) - \psi(\tau - i)| e^{-\lambda(\tau - i)} \cdot e^{\lambda(\tau - i)} \\
&\qquad + L_K \sum_{i} K(i) |\phi(\tau - i - cr) - \psi(\tau - i - cr)| e^{-\lambda(\tau - i - cr)} \cdot e^{\lambda(\tau - i - cr)} \Bigg\} \, d\tau \\
&\quad \le \frac{\|\phi - \psi\|_\lambda}{c} e^{-(\frac{D+d}{c}+\lambda)\xi} \int_{-\infty}^{\xi} e^{(\frac{D+d}{c}+\lambda)\tau} \\
&\qquad \times \Bigg\{ D \sum_{i \neq 0} J(i) e^{-\lambda i} + L_K \sum_{i} K(i) e^{\lambda i} \cdot e^{-\lambda c r} \Bigg\} \, d\tau \\
&\quad = \frac{\|\phi - \psi\|_\lambda}{D + d + c\lambda} \Bigg\{ D \sum_{i \neq 0} J(i) e^{-\lambda i} + L_K \sum_{i} K(i) e^{\lambda i} \cdot e^{-\lambda c r} \Bigg\},
\end{aligned}
$$

it follows that $F : \Gamma \to \Gamma$ is continuous. Therefore, by virtue of Schauder's Fixed Point Theorem, it follows that F has a fixed point U_c in X, which will be denoted by (U_c, c) and satisfies

$$
e^{\Lambda_1(c)\xi} - q e^{\beta \Lambda_1(c)\xi} \le U_c(\xi) \le e^{\Lambda_1(c)\xi} + q e^{\beta \Lambda_1(c)\xi}, \qquad \xi \in \mathbb{R}. \tag{3.5}
$$

Clearly, (U_c, c) is also a weak solution of (1.7), i.e., for any $\phi \in C_0^\infty(\mathbb{R})$, we have

$$
\begin{aligned}
&c \int_{\mathbb{R}} U_c \phi' + D \sum_{i \neq 0} J(i) \int_{\mathbb{R}} U_c(\cdot) \phi(\cdot + i) - (D + d) \int_{\mathbb{R}} U_c \phi \\
&\qquad + \sum_{i} K(i) \int_{\mathbb{R}} b(U_c(\cdot)) \phi(\cdot + i + cr) = 0.
\end{aligned} \tag{3.6}
$$

Take $u^* \in (0, K)$, then for each $c > c_*$, there exists $\xi_c \in \mathbb{R}$ such that $U_c(\xi_c) = u^*$. By Helly's Theorem, there exists a sequence $c_m > c_*$ with $c_m \searrow c_*$ as $m \to +\infty$, such that $\tilde{U}_{c_m}(\cdot) := U_{c_m}(\cdot + \xi_{c_m})$ converges pointwise to a nondecreasing function U_{c_*} as $m \to +\infty$.

Applying the Lebesgue's Dominated Convergence Theorem to (3.6) with c replaced by c_m and U_c replaced by $\tilde{U}_{c_m}$ then gives

$$
\begin{aligned}
&c_* \int_{\mathbb{R}} U_{c_*} \phi' + D \sum_{i \neq 0} J(i) \int_{\mathbb{R}} U_{c_*}(\cdot) \phi(\cdot + i) - (D + d) \int_{\mathbb{R}} U_{c_*} \phi \\
&\qquad + \sum_{i} K(i) \int_{\mathbb{R}} b(U_{c_*}(\cdot)) \phi(\cdot + i + c_* r) = 0,
\end{aligned}
$$

for all $\phi \in C_0^\infty(\mathbb{R})$. Since $c_* > 0$, the last equality implies that $U_{c_*} \in W^{1,\infty}(\mathbb{R})$, and hence, a bootstrap argument shows that U_{c_*} is of class C^1 and thus a solution of (1.7). Since $U_{c_*}(0) = u^* \in (0, K)$ and $b(u) > du$ for $u \in (0, K)$, it follows that $U_{c_*}(-\infty) = 0$ and $U_{c_*}(+\infty) = K$.

Next, we show that for each $c \geq c_*$, $U_c' > 0$ on $\mathbb{R}$. Suppose on the contrary that $U_c'(x_0) = 0$ for some $x_0 \in \mathbb{R}$. Since $U_c' \geq 0$ on $\mathbb{R}$, we have $U_c''(x_0) = 0$, and hence

$$0 = cU_c''(x_0) = D \sum_{i \neq 0} J(i)U_c'(x_0 - i) + \sum_i K(i)b'(U_c(x_0 - i - cr))U_c'(x_0 - i - cr),$$

which together with the fact that $b'(0) > d > 0$ implies that $U_c'(x_0 - i) = U_c'(x_0) = 0$ for $i \neq 0$ with $J(i) > 0$ and $U_c'(x_0 - i_0 - cr) = 0$ for i_0 with $K(i_0) > 0$ if $-x_0 > 0$ is sufficiently large. So by using an induction argument, we conclude that

$$U_c'(x_0 + n - mcr) = 0, \qquad \text{for all } n, m \in \mathbb{Z} \text{ with } m \geq 0.$$

Let $w_{n,m}(t) := U_c'(x_0 + n - mcr + t)$, then $w_{n,m}$ satisfies the initial value problem

$$\begin{aligned} w_{n,m}' &= \frac{D}{c} \sum_{i \neq 0} J(i)[w_{n-i,m} - w_{n,m}] - \frac{d}{c} w_{n,m} \\ &\quad + \frac{1}{c} \sum_i K(i)b'(U_c(x_0 + n - i - (m+1)cr + t))w_{n-i,m+1}, \\ w_{n,m}(0) &= 0, \end{aligned}$$

where $n, m \in \mathbb{Z}$ with $m \geq 0$. By the uniqueness of the initial value problem, we have $w_{n,m}(t) \equiv 0$, and hence $U \equiv$ const., which is a contradiction.

If $c > c_*$, it then follows from (3.5) that

$$\lim_{\xi \to -\infty} |U_c(\xi)\mathrm{e}^{-\Lambda_1(c)\xi} - 1| \leq \lim_{\xi \to -\infty} q\mathrm{e}^{(\beta-1)\Lambda_1(c)\xi} = 0.$$

Since $0 \leq b'(0)u - b(u) \leq Mu^{1+\nu}$ for $u \in (0, K)$, we have

$$\lim_{\xi \to -\infty} |b(U_c(\xi)) - b'(0)U_c(\xi)|\mathrm{e}^{-\Lambda_1(c)\xi} \leq \lim_{\xi \to -\infty} M[U_c(\xi)]^{1+\nu}\mathrm{e}^{-\Lambda_1(c)\xi} = 0.$$

Hence, for $c > c_*$, it follows from the following analog of the Lebesgue's Dominated Convergence Theorem that

$$\begin{aligned} &\lim_{\xi \to -\infty} U_c'(\xi)\mathrm{e}^{-\Lambda_1(c)\xi} \\ &\quad = \frac{1}{c} \lim_{\xi \to -\infty} \left\{ D \sum_{i \neq 0} J(i)[U_c(\xi - i) - U_c(\xi)] - dU_c(\xi) + \sum_i K(i)b(U_c(\xi - i - cr)) \right\} \\ &\quad\quad \times \mathrm{e}^{-\Lambda_1(c)\xi} \\ &\quad = \frac{1}{c} \left\{ D \sum_{i \neq 0} J(i)[\mathrm{e}^{-\Lambda_1(c)i} - 1] - d + b'(0) \sum_i K(i)\mathrm{e}^{-\Lambda_1(c)(i+cr)} \right\} \\ &\quad = \Lambda_1(c). \end{aligned}$$

This completes the proof. □

Lemma 3.2. *Let* $\{f_j(x)\}$, $j \in \mathbb{Z}$, $x \in \mathbb{R}$, *be a sequence of functions such that* $\sum_j f_j(x)$ *exists for any* $x \in \mathbb{R}$ *and* $f_j(x) \to \bar{f}_j$ *as* $x \to x_0 \in \{\mathbb{R}, -\infty, +\infty\}$ *for all* $j \in \mathbb{Z}$. *If there exists a summable sequence* $\{g_j\}$ *such that* $|f_j(x)| \le g_j$ *for all* $j \in \mathbb{Z}$ *and* $x \in \mathbb{R}$, *then*

$$\sum_j f_j(x) \to \sum_j \bar{f}_j, \qquad \text{as } x \to x_0.$$

The proof of Lemma 3.2 is similar to that of Lebesgue's dominated convergence theorem and is omitted.

In what follows, we study the uniqueness of our solutions, and establish the following main result, which shows that for any fixed $c > c_*$ the solution to (1.7) and (1.8) is unique up to a translation.

Theorem 3.2. *Assume* (H1)–(H6) *hold. For each* $c > c_*$, *let* (U, c) *be the solution to* (1.7) *and* (1.8) *as given in Theorem* 3.1. *Let* $(\hat{U}, c)$ *be another solution to* (1.7) *and* (1.8) *satisfying*

$$\limsup_{\xi \to -\infty} \hat{U}(\xi) e^{-\Lambda_1(c)\xi} < +\infty. \tag{3.7}$$

Then there exists $\bar{z} \in \mathbb{R}$ *such that* $\hat{U}(\cdot) = U(\cdot + \bar{z})$.

Proof. Firstly, we observe that if $(\hat{U}, c)$ is a solution to (1.7) and (1.8), then

$$\hat{U} \le K. \tag{3.8}$$

Suppose otherwise that there exists x_0 so that $\hat{U}(x_0) > K$ and $\hat{U}(x) \le \hat{U}(x_0)$ for all $x \in \mathbb{R}$. Then we have $\hat{U}'(x_0) = 0$ and so

$$\begin{aligned} 0 &\ge -c\hat{U}'(x_0) + D \sum_{i \ne 0} J(i)[\hat{U}(x_0 - i) - \hat{U}(x_0)] \\ &= d\hat{U}(x_0) - \sum_i K(i) b(\hat{U}(x_0 - i - cr)) \\ &\ge d\hat{U}(x_0) - b(\hat{U}(x_0)) > 0, \end{aligned}$$

which is a contradiction.

In what follows, we denote by (U, c) the solution of (1.7) and (1.8) given in Theorem 3.1. Since $b'(K) < d < b'(0)$, we can choose $\alpha > 0$ such that

$$d > 2\alpha \max\left\{1, \left[e^{-\Lambda_1(c)cr} \sum_i K(i) e^{-\Lambda_1(c)i}\right]^{-1}\right\} + b'(K). \tag{3.9}$$

Choose $\kappa > 0$ sufficiently small and $N \in \mathbb{N}$ sufficiently large so that

$$b'(\eta) \le b'(K) + \frac{\alpha}{2} \min\left\{1, \left[e^{-\Lambda_1(c)cr} \sum_i K(i) e^{-\Lambda_1(c)i}\right]^{-1}\right\}, \quad \text{for } \eta \in [K - \kappa, K + \kappa], \tag{3.10}$$

and

$$b'_{\max} \max\left\{\sum_{|i|>N} K(i), \mathrm{e}^{-\Lambda_1(c)cr} \sum_{|i|>N} K(i)\mathrm{e}^{-\Lambda_1(c)i}\right\} \le \alpha/2. \tag{3.11}$$

Take $M_1 > N + cr$ sufficiently large so that

$$U(\xi) \ge K - \kappa/2, \qquad \text{for } \xi \ge M_1 - N - cr. \tag{3.12}$$

Since $\lim_{x\to-\infty} U'(x)\mathrm{e}^{-\Lambda_1(c)x} = \Lambda_1(c) > 0$, we can take $M_2 > 0$ sufficiently large that

$$U'(x)\mathrm{e}^{-\Lambda_1(c)x} \ge \frac{1}{2}\Lambda_1(c), \qquad \text{for } x \le -M_2. \tag{3.13}$$

Denote

$$\varrho := \min\{U'(\xi); -M_2 \le \xi \le M_1\} > 0.$$

Let $\mu \in (0, \kappa/2)$ and define

$$B = \max\left\{\frac{2\mu}{\alpha\varrho} b'_{\max} \mathrm{e}^{\Lambda_1(c)M_1} \sum_i K(i)\mathrm{e}^{-\Lambda_1(c)i}, \frac{3\mu}{\alpha\Lambda_1(c)} b'_{\max} \sum_i K(i)\mathrm{e}^{-\Lambda_1(c)i}\right\}. \tag{3.14}$$

We claim that for $\mu \in (0, \kappa/2)$ given above, there exists $z \ge M_1$, such that

$$U(x+z) + \mu \min\{1, \mathrm{e}^{\Lambda_1(c)x}\} > \hat{U}(x), \qquad \text{for all } x \in \mathbb{R}. \tag{3.15}$$

In fact, we can first choose $z_1 \ge M > 0$ such that $\mathrm{e}^{\Lambda_1(c)z_1} > \rho := \limsup_{x\to-\infty} \hat{U}(x)\mathrm{e}^{-\Lambda_1(c)x}$. Since

$$\lim_{x\to-\infty} U(x+z_1)\mathrm{e}^{-\Lambda_1(c)x} = \mathrm{e}^{\Lambda_1(c)z_1} > \rho,$$

there exists $M_3 > 0$ such that

$$U(x+z_1) > \hat{U}(x), \qquad \text{for } x \le -M_3.$$

Take $M_4 > 0$ sufficiently large that

$$U(x) + \mu\mathrm{e}^{-\Lambda_1(c)M_3} > K, \qquad \text{for } x \ge M_4.$$

Let $z = z_1 + M_3 + M_4$, then for $x \le -M_3$, we have

$$U(x+z) + \mu \min\{1, \mathrm{e}^{\Lambda_1(c)x}\} - \hat{U}(x) > U(x+z_1) - \hat{U}(x) > 0,$$

and for $x \ge -M_3$, we have $x + z \ge M_4$, and hence, (3.8) implies that

$$\begin{aligned} U(x+z) + \mu \min\{1, \mathrm{e}^{\Lambda_1(c)x}\} - \hat{U}(x) &\ge U(x+z) + \mu\mathrm{e}^{-\Lambda_1(c)M_3} - \hat{U}(x) \\ &> K - \hat{U}(x) \ge 0. \end{aligned}$$

Define

$$w(x,t) = U(x+z+B(1-\mathrm{e}^{-\alpha t})) + \mu \min\{1, \mathrm{e}^{\Lambda_1(c)x}\}\mathrm{e}^{-\alpha t} - \hat{U}(x), \tag{3.16}$$

then we have

$$w(x,0) = U(x+z) + \mu \min\{1, \mathrm{e}^{\Lambda_1(c)x}\} - \hat{U}(x) > 0.$$

We claim that $w(x,t) > 0$ for all $x \in \mathbb{R}$ and $t \ge 0$. To see this, suppose that there exist $x_0 \in \mathbb{R}$ and $t_0 > 0$ such that

$$w(x_0, t_0) = U(P_0) + \mu \min\{1, e^{\Lambda_1(c)x_0}\}e^{-\alpha t_0} - \hat{U}(x_0) = 0 \le w(x, t), \tag{3.17}$$

for all $x \in \mathbb{R}$ and $t \in [0, t_0]$, where

$$P_0 = x_0 + z + B(1 - e^{-\alpha t_0}).$$

Clearly, if $x_0 = 0$, then

$$w_x(x_0-, t_0) = U'(P_0) - \hat{U}'(x_0) + \mu\Lambda_1(c)e^{\Lambda_1(c)x_0} \cdot e^{-\alpha t_0} \le 0,$$

and

$$w_x(x_0+, t_0) = U'(P_0) - \hat{U}'(x_0) \ge 0,$$

which is impossible. So we have $x_0 \neq 0$, and hence

$$w_x(x_0, t_0) = U'(P_0) - \hat{U}'(x_0) + \mu\Lambda_1(c)e^{\Lambda_1(c)x_0} \cdot e^{-\alpha t_0} = 0, \qquad \text{if } x_0 < 0, \tag{3.18}$$

and

$$w_x(x_0, t_0) = U'(P_0) - \hat{U}'(x_0) = 0, \qquad \text{if } x_0 > 0. \tag{3.19}$$

In the case where $x_0 > 0$, we have

$$\begin{aligned}
0 \ge\ & w_t(x_0, t_0) - D\sum_{i\neq 0} J(i)[w(x_0 - i, t_0) - w(x_0, t_0)] \\
=\ & -\alpha\mu e^{-\alpha t_0} + \alpha BU'(P_0)e^{-\alpha t_0} \\
& - \mu D\sum_{i\neq 0} J(i)[\min\{1, e^{\Lambda_1(c)(x_0-i)}\} - 1]e^{-\alpha t_0} \\
& - D\sum_{i\neq 0} J(i)[U(P_0 - i) - U(P_0)] + D\sum_{i\neq 0} J(i)[\hat{U}(x_0 - i) - \hat{U}(x_0)] \\
\ge\ & [-\alpha\mu + \alpha BU'(P_0)]e^{-\alpha t_0} - cU'(P_0) - dU(P_0) + \sum_i K(i)b(U(P_0 - i - cr)) \\
& + c\hat{U}'(x_0) + d\hat{U}(x_0) - \sum_i K(i)b(\hat{U}(x_0 - i - cr)) \\
=\ & [d\mu - \alpha\mu]e^{-\alpha t_0} + \sum_i K(i)[b(U(P_0 - i - cr)) - b(\hat{U}(x_0 - i - cr))] \\
\ge\ & [d\mu - \alpha\mu]e^{-\alpha t_0} + \sum_i K(i)[b(U(P_0 - i - cr)) - b(U(P_0 - i - cr) + \mu e^{-\alpha t_0})] \\
\ge\ & \left[d - \alpha - b'_{\max}\sum_{|i|>N} K(i) - \sum_{|i|\le N} K(i)b'(\eta_i)\right]\mu e^{-\alpha t_0},
\end{aligned} \tag{3.20}$$

where $\eta_i \in (U(P_0 - i - cr), U(P_0 - i - cr) + \mu)$. Since $P_0 > z \ge M_1$, it follows from (3.12) that $\eta_i \ge U(P_0 - i - cr) \ge K - \kappa/2$ for $|i| \le N$, and hence, by (3.9)–(3.11), the right hand side of (3.20) is positive, which is a contradiction.

In the case where $x_0 < 0$, we have

$$\begin{aligned}
0 &\geq w_t(x_0,t_0) - D\sum_{i\neq 0} J(i)[w(x_0-i,t_0) - w(x_0,t_0)]\\
&= -\alpha\mu e^{\Lambda_1(c)x_0}\cdot e^{-\alpha t_0} + \alpha B U'(P_0)e^{-\alpha t_0}\\
&\quad - \mu D\sum_{i\neq 0} J(i)[\min\{1, e^{\Lambda_1(c)(x_0-i)}\} - e^{\Lambda_1(c)x_0}]e^{-\alpha t_0}\\
&\quad - D\sum_{i\neq 0} J(i)[U(P_0-i) - U(P_0)] + D\sum_i J(i)[\hat{U}(x_0-i) - \hat{U}(x_0)]\\
&\geq [-\alpha\mu e^{\Lambda_1(c)x_0} + \alpha B U'(P_0)]e^{-\alpha t_0} - \mu D e^{\Lambda_1(c)x_0}\left[\sum_{i\neq 0} J(i)e^{-\Lambda_1(c)i} - 1\right]e^{-\alpha t_0}\\
&\quad - cU'(P_0) - dU(P_0) + \sum_i K(i)b(U(P_0-i-cr))\\
&\quad + c\hat{U}'(x_0) + d\hat{U}(x_0) - \sum_i K(i)b(\hat{U}(x_0-i-cr))\\
&\geq [-\alpha\mu e^{\Lambda_1(c)x_0} + \alpha B U'(P_0)]e^{-\alpha t_0} - \mu D e^{\Lambda_1(c)x_0-\alpha t_0}\left[\sum_{i\neq 0} J(i)e^{-\Lambda_1(c)i} - 1\right] \qquad (3.21)\\
&\quad + \mu c\Lambda_1(c)e^{\Lambda_1(c)x_0-\alpha t_0} + d\mu e^{\Lambda_1(c)x_0-\alpha t_0}\\
&\quad + \sum_i K(i)[b(U(P_0-i-cr)) - b(U(P_0-i-cr)\\
&\quad + \mu\min\{1, e^{\Lambda_1(c)(x_0-i-cr)}\}e^{-\alpha t_0})]\\
&\geq \left[-\alpha + \frac{\alpha B}{\mu}U'(P_0)e^{-\Lambda_1(c)x_0}\right]\mu e^{\Lambda_1(c)x_0-\alpha t_0}\\
&\quad + \mu b'(0)\sum_i K(i)e^{-\Lambda_1(c)(i+cr)}e^{\Lambda_1(c)x_0-\alpha t_0}\\
&\quad - \mu\sum_i K(i)b'(\eta_i)e^{-\Lambda_1(c)(i+cr)}e^{\Lambda_1(c)x_0-\alpha t_0}\\
&\geq \mu\Bigg[b'(0)e^{-\Lambda_1(c)cr}\sum_i K(i)e^{-\Lambda_1(c)i} - \alpha + \frac{\alpha B}{\mu}U'(P_0)e^{-\Lambda_1(c)P_0}\\
&\quad - b'_{\max}e^{-\Lambda_1(c)cr}\sum_{|i|>N} K(i)e^{-\Lambda_1(c)i} - \sum_{|i|\leq N} K(i)b'(\eta_i)e^{-\Lambda_1(c)(i+cr)}\Bigg]e^{\Lambda_1(c)x_0-\alpha t_0},
\end{aligned}$$

where $\eta_i \in (U(P_0-i-cr), U(P_0-i-cr)+\mu)$.

In this case, if $P_0 \leq -M_2$, then (3.13) and (3.14) imply that

$$\begin{aligned}
&\frac{\alpha B}{\mu}U'(P_0)e^{-\Lambda_1(c)P_0} - \sum_{|i|\leq N} K(i)b'(\eta_i)e^{-\Lambda_1(c)(i+cr)}\\
&\quad\geq \frac{\alpha B\Lambda_1(c)}{2\mu} - b'_{\max}\sum_i K(i)e^{-\Lambda_1(c)i} > 0,
\end{aligned}$$

and hence, by (3.9) and (3.11), the right hand side of (3.21) is positive, which is a contradiction.

If $P_0 \in [-M_2, M_1]$, then by (3.14), we have

$$\begin{aligned}
&\frac{\alpha B}{\mu} U'(P_0) \mathrm{e}^{-\Lambda_1(c)P_0} - \sum_{|i|\le N} K(i) b'(\eta_i) \mathrm{e}^{-\Lambda_1(c)(i+cr)} \\
&\quad \ge \frac{\alpha B \varrho}{\mu} \mathrm{e}^{-\Lambda_1(c)M_1} - b'_{\max} \sum_i K(i) \mathrm{e}^{-\Lambda_1(c)i} > 0,
\end{aligned}$$

and hence, by (3.11), the right hand side of (3.21) is positive, which is a contradiction.

If $P_0 \ge M_1$, then it follows from (3.10) that $\eta_i \ge U(P_0 - i - cr) \ge K - \kappa/2$ for $|i| \le N$, and hence, by (3.9),

$$\begin{aligned}
&b'(0) \mathrm{e}^{-\Lambda_1(c)cr} \sum_i K(i) \mathrm{e}^{-\Lambda_1(c)i} - \sum_{|i|\le N} K(i) b'(\eta_i) \mathrm{e}^{-\Lambda_1(c)(i+cr)} \\
&\quad \ge d \mathrm{e}^{-\Lambda_1(c)cr} \sum_i K(i) \mathrm{e}^{-\Lambda_1(c)i} - b'(K) \mathrm{e}^{-\Lambda_1(c)cr} \sum_i K(i) \mathrm{e}^{-\Lambda_1(c)i} - \alpha/2 \\
&\quad > 3\alpha/2.
\end{aligned}$$

So, by (3.11), the right hand side of (3.21) is positive, which is also a contradiction.

Taking the limit $t \to +\infty$ in (3.16), we get

$$U(x + z + B) \ge \hat{U}(x), \qquad \text{for all } x \in \mathbb{R}.$$

Thus there exists a minimal $\bar{z}$ such that

$$U(x) \ge \hat{U}(x - z), \qquad \text{for all } x \in \mathbb{R} \text{ and } z \ge \bar{z}. \tag{3.22}$$

We assert that if $U(x) \not\equiv \hat{U}(x - \bar{z})$ for some x, then $U(x) > \hat{U}(x - \bar{z})$ for all $x \in \mathbb{R}$. Suppose otherwise that for some x_0, $U(x_0) = \hat{U}(x_0 - \bar{z})$. Let $w(x) = U(x) - \hat{U}(x - \bar{z})$. Then we have $w'(x_0) = 0$ and $w(x) \ge w(x_0) = 0$ for all $x \in \mathbb{R}$, and hence

$$\begin{aligned}
0 &\le D \sum_{i\ne 0} J(i)[w(x_0 - i) - w(x_0)] \\
&= -cw'(x_0) + D \sum_{i\ne 0} J(i)[w(x_0 - i) - w(x_0)] - dw(x_0) \\
&= -cU'(x_0) + D \sum_{i\ne 0} J(i)[U(x_0 - i) - U(x_0)] - dU(x_0) \\
&\quad + c\hat{U}'(x_0 - \bar{z}) - D \sum_{i\ne 0} J(i)[\hat{U}(x_0 - \bar{z} - i) - \hat{U}(x_0 - \bar{z})] + d\hat{U}(x_0 - \bar{z}) \\
&= -\sum_i K(i) b(U(x_0 - i - cr)) + \sum_i K(i) b(\hat{U}(x_0 - \bar{z} - i - cr)) \\
&= -\sum_i K(i) b'(\eta_i) w(x_0 - i - cr) \le 0,
\end{aligned}$$

where $\eta_i \in (\hat{U}(x_0 - \bar{z} - i - cr), U(x_0 - i - cr))$. Since $b'(0) > d > 0$, it follows that $w(x_0 + i) = w(x_0 - i) = w(x_0) = 0$ for $i \ne 0$ with $J(i) > 0$, and $w(x_0 - i_0 - cr) = U(x_0 - i_0 - cr) - \hat{U}(x_0 - i_0 - \bar{z} - cr) = 0$ for some i_0 with $K(i_0) > 0$ if $-x_0 > 0$ is sufficiently large. From which, by an induction argument, we can show that

$$w(x_0 - mcr + n) = 0, \qquad \text{for all } n, m \in \mathbb{Z} \text{ with } m \ge 0. \tag{3.23}$$

Let $v_{n,m}(t) = w(x_0 - mcr + n + ct), n \in \mathbb{Z}, m \geq 0$, then by the Mean Value Theorem, it is easily seen that $v_{n,m}(t)$ satisfies the initial value problem

$$\begin{aligned} v'_{n,m} &= D\sum_{i\neq 0} J(i)[v_{n-i,m} - v_{n,m}] - dv_{n,m} + \sum_i K(i)P_{n-i,m+1}(t)v_{n-i,m+1},\\ v_{n,m}(0) &= 0, \end{aligned}$$

where $n \in \mathbb{Z}, m \geq 0$ and

$$\begin{aligned} P_{n,m}(t) = \int_0^1 & b'[U(x_0 - mcr + n + ct)\\ &+ \alpha(\hat{U}(x_0 - mcr + n - \bar{z} + ct) - U(x_0 - mcr + n + ct))]\,\mathrm{d}\alpha. \end{aligned}$$

By the uniqueness of solutions to the initial value problem, we conclude that $v_{n,m}(t) \equiv 0$, and hence $w(x) \equiv 0$, which leads to a contradiction and establish the assertion.

In what follows, we suppose that $U(x) > \hat{U}(x - \bar{z})$ for all $x \in \mathbb{R}$. Then it follows that

$$1 \geq \rho \mathrm{e}^{-\Lambda_1(c)\bar{z}}, \tag{3.24}$$

where $\rho = \limsup_{x\to-\infty} \hat{U}(x)\mathrm{e}^{-\Lambda_1(c)x}$.

Let $\varepsilon > 0$ and define

$$w(x,t) = U(x - \varepsilon(1 - \mathrm{e}^{-\alpha t})) - \hat{U}(x - \bar{z}), \qquad x \in \mathbb{R},\ t \in \mathbb{R}.$$

Then $w(x, 0) = U(x) - \hat{U}(x - \bar{z}) > 0$ for all $x \in \mathbb{R}$. Suppose that there exist $t_0 > 0$ and $x_0 \in \mathbb{R}$ such that

$$\begin{aligned} w(x_0, t_0) &= U(x_0 - \varepsilon(1 - \mathrm{e}^{-\alpha t_0})) - \hat{U}(x_0 - \bar{z}) = 0 < w(x,t),\\ &\quad \text{for } x \in \mathbb{R} \text{ and } t \in [0, t_0). \end{aligned}$$

Then

$$w_x(x_0, t_0) = U'(x_0 - \varepsilon(1 - \mathrm{e}^{-\alpha t_0})) - \hat{U}'(x_0 - \bar{z}) = 0.$$

Therefore, we have

$$\begin{aligned} 0 &\leq D\sum_{i\neq 0} J(i)[w(x_0 - i, t_0) - w(x_0, t_0)]\\ &= D\sum_{i\neq 0} J(i)[U(P_1 - i) - U(P_1)] - D\sum_{i\neq 0} J(i)[\hat{U}(P_2 - i) - \hat{U}(P_2)]\\ &= c[U'(P_1) - \hat{U}'(P_2)] + d[U(P_1) - \hat{U}(P_2)]\\ &\quad - \sum_i K(i)b(U(P_1 - i - cr)) + \sum_i K(i)b(\hat{U}(P_2 - i - cr))\\ &= -\sum_i K(i)b'(\eta_i)w(x_0 - i - cr, t_0) \leq 0, \end{aligned}$$

where $P_1 = x_0 - \varepsilon(1 - \mathrm{e}^{-\alpha t_0})$, $P_2 = x_0 - \bar{z}$ and $\eta_i \in (\hat{U}(P_2 - i - cr), U(P_1 - i - cr))$. Since $b'(0) > d > 0$, it follows that $w(x_0 + i, t_0) = w(x_0 - i, t_0) = w(x_0, t_0) = 0$ for $i \neq 0$ with $J(i) > 0$, and $w(x_0 - i_0 - cr, t_0) = U(P_1 - i_0 - cr) - \hat{U}(P_2 - i_0 - cr) = 0$ for some i_0 with $K(i_0) > 0$ if $-x_0 > 0$ is sufficiently large. From which, by an induction argument, we can show that

$$w(x_0 - mcr + n, t_0) = 0, \qquad \text{for all } n, m \in \mathbb{Z} \text{ with } m \geq 0.$$

An argument as used above can be used to show that

$$w(x, t_0) = U(x - \varepsilon(1 - e^{-\alpha t_0})) - \hat{U}(x - \bar{z}) = 0, \qquad \text{for all } x \in \mathbb{R}.$$

Therefore, we have

$$\begin{aligned} e^{-\Lambda_1(c)\varepsilon(1-e^{-\alpha t_0})} &= \lim_{x\to-\infty} U(x - \varepsilon(1 - e^{-\alpha t_0}))e^{-\Lambda_1(c)x} \\ &= \limsup_{x\to-\infty} \hat{U}(x - \bar{z})e^{-\Lambda_1(c)x} \\ &= \rho e^{-\Lambda_1(c)\bar{z}}. \end{aligned} \tag{3.25}$$

If $\rho e^{-\Lambda_1(c)\bar{z}} = 1$, then (3.25) leads to a contradiction. If $\rho e^{-\Lambda_1(c)\bar{z}} < 1$, then we can choose $\varepsilon > 0$ in such a way that

$$e^{-\Lambda_1(c)\varepsilon} > \rho e^{-\Lambda_1(c)\bar{z}},$$

therefore, it follows from (3.25) that $e^{\Lambda_1(c)\varepsilon e^{-\alpha t_0}} < 1$, which is also a contradiction. So we have

$$w(x, t) = U(x - \varepsilon(1 - e^{-\alpha t})) - \hat{U}(x - \bar{z}) > 0, \qquad \text{for all } x \in \mathbb{R} \text{ and } t \geq 0. \tag{3.26}$$

Passing to the limit as $t \to +\infty$ in (3.26) gives

$$U(x) \geq \hat{U}(x - (\bar{z} - \varepsilon)), \qquad \text{for all } x \in \mathbb{R},$$

contradicting the minimality of $\bar{z}$ and proving that $U(x) = \hat{U}(x - \bar{z})$ for all $x \in \mathbb{R}$. The proof is complete. □

As a direct consequence of Theorem 3.2, we have the following

Corollary 3.1. *Assume that* (H1)–(H6) *hold. Then for any* $c > c_*$, *there are no solutions* $\hat{U}(n + ct)$ *of* (1.6) *satisfying*

$$\limsup_{\xi\to-\infty} \hat{U}(\xi)e^{-\Lambda_1(c)\xi} \leq 0.$$

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