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Global asymptotic stability of a class of nonautonomous integro-differential systems and applications

Meng Fan[a,1], Xingfu Zou[b,*,2]

[a] *School of Mathematics and Statistics, Key Laboratory for Vegetation Ecology of the Education Ministry, Northeast Normal University, 5268 Renmin Street, Changchun, Jilin 130024, The People's Republic of China*

[b] *Department of Applied Mathematics, University of Western Ontario, London, Ont., Canada N6A 5B7*

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Abstract

In this paper, we study the global asymptotic stability of a class of nonautonomous integro-differential systems. By constructing suitable Lyapunov functionals, we establish new and explicit criteria for the global asymptotic stability in the sense of Definition 2.1. In the autonomous case, we discuss the global asymptotic stability of a unique *equilibrium* of the system, and in the case of periodic system, we establish sufficient criteria for existence, uniqueness and global asymptotic stability of a *periodic solution*. Also explored are applications of our main results to some biological and neural network models. The examples show that our criteria are more general and easily applicable, and improve and generalize some existing results.

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* Corresponding author.
E-mail addresses: mfan@nenu.edu.cn (M. Fan), xzou@uwo.ca (X. Zou).

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1. Introduction

Consider the n-dimensional nonautonomous integro-differential system of the form

$$\dot{x}_i(t) = -b_i(t)x_i(t) + f_i(t, x_1(t), \ldots, x_n(t); x_1(t-\tau_{i1}(t)), \ldots, x_n(t-\tau_{in}(t));$$
$$\int_{-\infty}^{t} k_{i1}(t-s)x_1(s)\,\mathrm{d}s, \ldots,$$
$$\int_{-\infty}^{t} k_{in}(t-s)x_n(s)\,\mathrm{d}s), \quad t \geqslant 0, \;\; i = 1, \ldots, n \tag{1.1}$$

together with the following assumptions:

(A_{11}) For each $i \in \{1, 2, \ldots, n\}$, $b_i(t)$ is bounded and continuous on $\mathbf{R}^+$ with $b_i(t) > 0$ for $t \geqslant 0$;

(A_{12}) For each $i, j \in \{1, 2, \ldots, n\}$, $\tau_{ij}(t)$ is continuously differentiable with $\tau_{ij}(t) \geqslant 0$ and $1 - \dot{\tau}_{ij}(t) > 0$ for $t \geqslant 0$;

(A_{13}) For each $i, j \in \{1, 2, \ldots, n\}$, $k_{ij} : \mathbf{R}^+ \to \mathbf{R}^+$ satisfies

$$\int_0^\infty k_{ij}(s)\,\mathrm{d}s < +\infty, \qquad \int_0^\infty s k_{ij}(s)\,\mathrm{d}s < +\infty;$$

(A_{14}) For each $i \in \{1, 2, \ldots, n\}$, $f_i : \mathbf{R}^+ \times \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ is bounded, continuous and there exist nonnegative, bounded and continuous functions $\alpha_{ij}(t), \beta_{ij}(t), \gamma_{ij}(t)$ defined on $\mathbf{R}^+$ such that

$$|f_i(t, u_1, \ldots, u_n; v_1, \ldots, v_n; w_1, \ldots, w_n) - f_i(t, \bar{u}_1, \ldots, \bar{u}_n; \bar{v}_1, \ldots, \bar{v}_n; \bar{w}_1, \ldots, \bar{w}_n)|$$
$$\leqslant \sum_{j=1}^{n} [\alpha_{ij}(t)|u_j - \bar{u}_j| + \beta_{ij}(t)|v_j - \bar{v}_j| + \gamma_{ij}(t)|w_j - \bar{w}_j|], \quad i = 1, \ldots, n$$

for any $(u_1, \ldots, u_n), (\bar{u}_1, \ldots, \bar{u}_n), (v_1, \ldots, v_n), (\bar{v}_1, \ldots, \bar{v}_n), (w_1, \ldots, w_n), (\bar{w}_1, \ldots, \bar{w}_n) \in \mathbf{R}^n$.

We consider initial conditions of the form

$$x_i(t) = \varphi_i(t), \quad t \in (-\infty, 0], \;\; i = 1, \ldots, n, \tag{1.2}$$

where $\varphi_i(t)$ is bounded and continuous on $(-\infty, 0]$.

The interest in studying system (1.1) is justified by the fact that it includes many important models arising from mathematical biology and neural networks, which have been intensively and extensively studied in the literature. The term $-b_i(t)x_i(t)$ in (1.1) could represent the death rate in a population growth model in mathematical biology, and could account for the resistance of a neuron amplifier in an artificial neural network. For example, (1.1) includes the scalar delay differential equation

$$\dot{x}(t) = -x(t) + f(x(t-\tau)), \tag{1.3}$$

which has been used to model many biological, physical and physiological phenomena by choosing appropriate nonlinear function f (see, e.g., [1,7,12,13,17,20,26,27,36,37] and the references therein). Systems (1.1) also includes as a special case the Hopfield neural network

$$C_i \frac{\mathrm{d}u_i(t)}{\mathrm{d}t} = -\frac{u_i(t)}{R_i} + \sum_{j=1}^{n} T_{ij}(t) g_j(u_j(t)) + I_i, \quad i = 1, 2, \ldots, n \tag{1.4}$$

and its various modified versions with delays (see, e.g., [4–6,8–10,14–16,18,19,21,24 25,28–32,34,35,38,39,41]). A general modification of (1.4) will be discussed in Section 5 while seeking application of our main results. Another special case of (1.1) is the bidirectional associate memory (BAM) neural network

$$\frac{\mathrm{d}u_i(t)}{\mathrm{d}t} = -b_i u_i(t) + \sum_{j=n+1}^{n+m} T_{ij} g_j(u_j(t)) + I_i, \quad i = 1, \ldots, n,$$

$$\frac{\mathrm{d}u_i(t)}{\mathrm{d}t} = -b_i u_i(t) + \sum_{j=1}^{n} T_{ij} g_j(u_j(t)) + I_i, \quad i = n+1, \ldots, n+m, \tag{1.5}$$

which was first proposed by Kosko [22] and has since then been modified and studied by others (e.g., [15,22,23,40]). Our main results will also be applied to a generalization of (1.5) in Section 5.

Most of the works mentioned above for (1.3)–(1.5) focus on autonomous cases of (1.1) in which existence and stability of equilibria of the systems and Hopf bifurcations have been the main concerns. However, as far as non-autonomous cases are concerned, there seems to be very little work in the literature on the asymptotic behaviour of the solutions. Of particular importance is the property of global asymptotic stability which is a needed property whenever a neural circuit is designed for solving various optimization problems, for parallel computations, or for signal processing in real time. The main purpose of this work is to present sufficient conditions for the global asymptotic stability of (1.1)–(1.2). The criteria obtained in this paper are explicit, and thus, are easily applicable.

The paper is organized as follows. In Section 2, we study system (1.1) in the general form, and establish some fundamental criteria for global asymptotic stability in the sense that the distance of any two solutions of (1.1) approaches zero. In Section 3, we consider the autonomous case of (1.1), and obtain existence and global asymptotic stability of an unique equilibrium. Section 4 is dedicated to the periodic case of (1.1), where existence and global asymptotic stability of a unique periodic solution is explored. In order to illustrate some features of our main results, we discuss in Section 5, some particular models arising from mathematical biology and neural networks.

2. General case

For convenience, we first introduce the following definition and lemmas.

Definition 2.1. System (1.1) is said to be global asymptotically stable if for any two solutions $x_i(t)$ and $\bar{x}_i(t)$ of (1.1)–(1.2), one has

$$\lim_{t\to+\infty}\sum_{i=1}^{n}|x_i(t)-\bar{x}_i(t)|=0.$$

Lemma 2.1. *All the solutions of* (1.1)–(1.2) *are bounded on* $\mathbf{R}^+$.

Proof. By the boundedness of f_i, there exist positive constants $M_i > 0$ such that any solution of (1.1)–(1.2) satisfy

$$-b_i(t)x_i(t)-M_i \leqslant \dot{x}_i(t) \leqslant -b_i(t)x_i(t)+M_i.$$

By the above inequality and the boundedness of $b_i(t)$, the conclusion follows. □

Lemma 2.2 (see Barbălat [3]). *Let f be a nonnegative function defined on* $[0,+\infty)$ *such that f is integrable on* $[0,+\infty)$ *and is uniformly continuous on* $[0,+\infty)$. *Then*

$$\lim_{t\to+\infty} f(t)=0.$$

We now take some suitable Lyapunov functionals and by some analysis and estimates, obtain some sufficient conditions for the global asymptotic stability of system (1.1), as stated below.

Theorem 2.1. *Assume that there exist positive constants $\mu_i > 0$, $i = 1,2,\ldots,n$, such that*

$$\inf_{t\in\mathbf{R}^+}\left\{\mu_i b_i(t)-\sum_{j=1}^{n}\mu_j\left[\alpha_{ji}(t)+\frac{\beta_{ji}(\zeta_{ji}^{-1}(t))}{1-\dot{\tau}_{ji}(\zeta_{ji}^{-1}(t))}\right.\right.$$
$$\left.\left.+\int_0^{\infty}\gamma_{ji}(t+s)k_{ji}(s)\,\mathrm{d}s\right]\right\}>0 \tag{2.1}$$

or

$$\inf_{t\in\mathbf{R}^+}\left\{2\mu_i b_i(t)-\sum_{j=1}^{n}\left[\mu_i\left(\alpha_{ij}(t)+\beta_{ij}(t)+\gamma_{ij}(t)\int_0^{+\infty}k_{ij}(s)\,\mathrm{d}s\right)\right.\right.$$
$$\left.\left.+\mu_j\left(\alpha_{ji}(t)+\frac{\beta_{ji}(\zeta_{ji}^{-1}(t))}{1-\dot{\tau}_{ji}(\zeta_{ji}^{-1}(t))}+\int_0^{\infty}\gamma_{ji}(t+s)k_{ji}(s)\,\mathrm{d}s\right)\right]\right\}>0, \tag{2.2}$$

where $\zeta_{ij}^{-1}(t)$ is the inverse function of $\zeta_{ij}(t)=t-\tau_{ij}(t)$. Then system (1.1) *is global asymptotically stable.*

Proof. Let $x_i(t), \bar{x}_i(t)$ be any two solutions of (1.1)–(1.2).

If (2.1) is satisfied, consider the Lyapunov functional $V(t)$ defined by

$$V(t)=\sum_{i=1}^{n}\mu_i\left\{|x_i(t)-\bar{x}_i(t)|+\sum_{j=1}^{n}\int_{t-\tau_{ij}(t)}^{t}\frac{\beta_{ij}(\zeta_{ij}^{-1}(s))}{1-\dot{\tau}_{ij}(\zeta_{ij}^{-1}(s))}|x_j(s)-\bar{x}_j(s)|\,\mathrm{d}s\right.$$

$$\left.+\sum_{j=1}^{n}\int_{0}^{\infty}\int_{t-s}^{t}\gamma_{ij}(\theta+s)k_{ij}(s)|x_j(\theta)-\bar{x}_j(\theta)|\,\mathrm{d}\theta\,\mathrm{d}s\right\}, \qquad (2.3)$$

for $t\geqslant 0$. One can easily show that

$$V(t)\geqslant\sum_{i=1}^{n}\mu_i|x_i(t)-\bar{x}_i(t)|, \quad t\geqslant 0 \text{ and } V(0)<+\infty.$$

Calculating the upper right derivative $V'(t)$ of $V(t)$, we get

$$V'(t)$$

$$\leqslant\sum_{i=1}^{n}\mu_i\left\{-b_i(t)|x_i(t)-\bar{x}_i(t)|+\sum_{j=1}^{n}\alpha_{ij}(t)|x_j(t)-\bar{x}_j(t)|\right.$$

$$+\sum_{j=1}^{n}\beta_{ij}(t)|x_j(t-\tau_{ij}(t))-\bar{x}_j(t-\tau_{ij}(t))|$$

$$+\sum_{j=1}^{n}\gamma_{ij}(t)\left|\int_{-\infty}^{t}k_{ij}(t-s)(x_j(s)-\bar{x}_j(s))\,\mathrm{d}s\right|$$

$$+\sum_{j=1}^{n}\left[\frac{\beta_{ij}(\zeta_{ij}^{-1}(t))}{1-\dot{\tau}_{ij}(\zeta_{ij}^{-1}(t))}|x_j(t)-\bar{x}_j(t)|-\beta_{ij}(t)|x_j(t-\tau_{ij}(t))-\bar{x}_j(t-\tau_{ij}(t))|\right]$$

$$+\sum_{j=1}^{n}\left[\int_{0}^{\infty}\gamma_{ij}(t+s)k_{ij}(s)|x_j(t)-\bar{x}_j(t)|\,\mathrm{d}s\right.$$

$$\left.\left.-\int_{0}^{\infty}\gamma_{ij}(t)k_{ij}(s)|x_j(t-s)-\bar{x}_j(t-s)|\,\mathrm{d}s\right]\right\}$$

$$\leqslant\sum_{i=1}^{n}\mu_i\left\{-b_i(t)|x_i(t)-\bar{x}_i(t)|+\sum_{j=1}^{n}\alpha_{ij}(t)|x_j(t)-\bar{x}_j(t)|\right.$$

$$+\sum_{j=1}^{n}\frac{\beta_{ij}(\zeta_{ij}^{-1}(t))}{1-\dot{\tau}_{ij}(\zeta_{ij}^{-1}(t))}|x_j(t)-\bar{x}_j(t)|$$

$$+\sum_{j=1}^{n}\int_0^{\infty}\gamma_{ij}(t+s)k_{ij}(s)|x_j(t)-\bar{x}_j(t)|\,ds\Bigg\}$$

$$\leqslant \sum_{i=1}^{n}\Bigg\{-\mu_i b_i(t)|x_i(t)-\bar{x}_i(t)|+\sum_{j=1}^{n}\mu_j\alpha_{ji}(t)|x_i(t)-\bar{x}_i(t)|$$

$$+\sum_{j=1}^{n}\frac{\mu_j\beta_{ji}(\zeta_{ji}^{-1}(t))}{1-\dot{\tau}_{ji}(\zeta_{ji}^{-1}(t))}|x_i(t)-\bar{x}_i(t)|$$

$$+\sum_{j=1}^{n}\mu_j\int_0^{\infty}\gamma_{ji}(t+s)k_{ji}(s)\,ds|x_i(t)-\bar{x}_i(t)|\Bigg\}$$

$$\leqslant \sum_{i=1}^{n}\Bigg\{-\mu_i b_i(t)+\sum_{j=1}^{n}\mu_j\Bigg[\alpha_{ji}(t)+\frac{\beta_{ji}(\zeta_{ji}^{-1}(t))}{1-\dot{\tau}_{ji}(\zeta_{ji}^{-1}(t))}$$

$$+\int_0^{\infty}\gamma_{ji}(t+s)k_{ji}(s)\,ds\Bigg]\Bigg\}|x_i(t)-\bar{x}_i(t)|.$$

Now let

$$c_1 := \min_{1\leqslant i\leqslant n}\inf_{t\in\mathbf{R}^+}\Bigg\{\mu_i b_i(t)-\sum_{j=1}^{n}\mu_j\Bigg[\alpha_{ji}(t)+\frac{\beta_{ji}(\zeta_{ji}^{-1}(t))}{1-\dot{\tau}_{ji}(\zeta_{ji}^{-1}(t))}$$

$$+\int_0^{\infty}\gamma_{ji}(t+s)k_{ji}(s)\,ds\Bigg]\Bigg\}>0.$$

Then, from (2.1) we know $c_1>0$, and the above estimate leads to

$$V'(t)\leqslant -c_1\sum_{i=1}^{n}|x_i(t)-\bar{x}_i(t)|. \tag{2.4}$$

Integrating both sides of (2.4) with respect to t gives

$$V(t)+c_1\int_0^t\sum_{i=1}^{n}|x_i(s)-\bar{x}_i(s)|\,ds\leqslant V(0)<+\infty,\quad t\geqslant 0.$$

Therefore, $V(t)$ is bounded on $0\leqslant t<\infty$, and also

$$\int_0^t\sum_{i=1}^{n}|x_i(s)-\bar{x}_i(s)|\,ds\leqslant\frac{V(0)}{c_1}<+\infty,\quad t\geqslant 0,$$

which implies

$$\sum_{i=1}^{n}|x_i(t)-\bar{x}_i(t)|\in L^1[0,+\infty).$$

On the other hand, by the boundedness of $b_i(t), f_i(t,\cdot,\cdot,\cdot), x_i(t)$ and $\bar{x}_i(t)$ and (1.1), we known that $\dot{x}_i(t), \dot{\bar{x}}_i(t)$ are also bounded and therefore $\sum_{i=1}^{n} |x_i(t)-\bar{x}_i(t)|$ is uniformly continuous on $[0,+\infty)$. By Lemma 2.2 we can conclude that

$$\lim_{t\to\infty}\sum_{i=1}^{n}|x_i(t)-\bar{x}_i(t)|=0,$$

which proves that system (1.1) is global asymptotically stable.

If (2.2) is satisfied, consider the Lyapunov functional defined by

$$V(t)=\sum_{i=1}^{n}\mu_i\left\{\frac{1}{2}(x_i(t)-\bar{x}_i(t))^2+\frac{1}{2}\sum_{j=1}^{n}\int_{t-\tau_{ij}(t)}^{t}\frac{\beta_{ij}(\zeta_{ij}^{-1}(s))}{1-\dot{\tau}_{ij}(\zeta_{ij}^{-1}(s))}(x_j(s)-\bar{x}_j(s))^2\,\mathrm{d}s\right.$$
$$\left.+\frac{1}{2}\sum_{j=1}^{n}\int_0^{\infty}\int_{t-s}^{t}\gamma_{ij}(\theta+s)k_{ij}(s)(x_j(\theta)-\bar{x}_j(\theta))^2\,\mathrm{d}\theta\,\mathrm{d}s\right\}.$$

Calculating the upper right derivative of $V(t)$ and using the inequality $2ab\leqslant a^2+b^2$, one has

$$V'(t)$$
$$\leqslant\sum_{i=1}^{n}\mu_i\left\{-b_i(t)(x_i(t)-\bar{x}_i(t))^2+\sum_{j=1}^{n}\alpha_{ij}(t)|x_i(t)-\bar{x}_i(t)||x_j(t)-\bar{x}_j(t)|\right.$$
$$+\sum_{j=1}^{n}\beta_{ij}(t)|x_i(t)-\bar{x}_i(t)||x_j(t-\tau_{ij}(t))-\bar{x}_j(t-\tau_{ij}(t))|$$
$$+\sum_{j=1}^{n}\gamma_{ij}(t)\int_{-\infty}^{t}|k_{ij}(t-s)||x_i(t)-\bar{x}_i(t)||x_j(s)-\bar{x}_j(s)|\,\mathrm{d}s$$
$$+\frac{1}{2}\sum_{j=1}^{n}\frac{\beta_{ij}(\zeta_{ij}^{-1}(t))}{1-\dot{\tau}_{ij}(\zeta_{ij}^{-1}(t))}(x_j(t)-\bar{x}_j(t))^2$$
$$-\frac{1}{2}\sum_{j=1}^{n}\beta_{ij}(t)(x_j(t-\tau_{ij}(t))-\bar{x}_j(t-\tau_{ij}(t)))^2$$
$$+\frac{1}{2}\sum_{j=1}^{n}\int_0^{\infty}\gamma_{ij}(t+s)k_{ij}(s)(x_j(t)-\bar{x}_j(t))^2\,\mathrm{d}s$$
$$\left.-\frac{1}{2}\sum_{j=1}^{n}\int_0^{\infty}\gamma_{ij}(t)k_{ij}(s)(x_j(t-s)-\bar{x}_j(t-s))^2\,\mathrm{d}s\right\}$$

$$
\begin{aligned}
&\leqslant \sum_{i=1}^{n}\mu_i \left\{-b_i(t)(x_i(t)-\bar{x}_i(t))^2+\frac{1}{2}\sum_{j=1}^{n}\alpha_{ij}(t)[(x_i(t)-\bar{x}_i(t))^2+(x_j(t)-\bar{x}_j(t))^2]\right.\\
&+\frac{1}{2}\sum_{j=1}^{n}\beta_{ij}(t)[(x_i(t)-\bar{x}_i(t))^2+(x_j(t-\tau_{ij}(t))-\bar{x}_j(t-\tau_{ij}(t)))^2\\
&+\frac{1}{2}\sum_{j=1}^{n}\gamma_{ij}(t)\left[\int_{-\infty}^{t}k_{ij}(t-s)(x_i(t)-\bar{x}_i(t))^2\,\mathrm{d}s\right.\\
&\left.+\int_{-\infty}^{t}k_{ij}(t-s)(x_j(s)-\bar{x}_j(s))^2\,\mathrm{d}s\right]\\
&+\frac{1}{2}\sum_{j=1}^{n}\frac{\beta_{ij}(\zeta_{ij}^{-1}(t))}{1-\dot{\tau}_{ij}(\zeta_{ij}^{-1}(t))}(x_j(t)-\bar{x}_j(t))^2\\
&-\frac{1}{2}\sum_{j=1}^{n}\beta_{ij}(t)(x_j(t-\tau_{ij}(t))-\bar{x}_j(t-\tau_{ij}(t)))^2\\
&+\frac{1}{2}\sum_{j=1}^{n}\int_{0}^{\infty}\gamma_{ij}(t+s)k_{ij}(s)(x_j(t)-\bar{x}_j(t))^2\,\mathrm{d}s\\
&\left.-\frac{1}{2}\sum_{j=1}^{n}\int_{0}^{\infty}\gamma_{ij}(t)k_{ij}(s)(x_j(t-s)-\bar{x}_j(t-s))^2\,\mathrm{d}s\right\}\\
&\leqslant \sum_{i=1}^{n}\mu_i \left\{-b_i(t)(x_i(t)-\bar{x}_i(t))^2+\frac{1}{2}\sum_{j=1}^{n}\alpha_{ij}(t)(x_i(t)-\bar{x}_i(t))^2\right.\\
&+\frac{1}{2}\sum_{j=1}^{n}\alpha_{ij}(t)(x_j(t)-\bar{x}_j(t))^2\\
&+\frac{1}{2}\sum_{j=1}^{n}\beta_{ij}(t)(x_i(t)-\bar{x}_i(t))^2+\frac{1}{2}\sum_{j=1}^{n}\frac{\beta_{ij}(\zeta_{ij}^{-1}(t))}{1-\dot{\tau}_{ij}(\zeta_{ij}^{-1}(t))}(x_j(t)-\bar{x}_j(t))^2\\
&+\frac{1}{2}\sum_{j=1}^{n}\gamma_{ij}(t)\int_{-\infty}^{t}k_{ij}(t-s)(x_i(t)-\bar{x}_i(t))^2\,\mathrm{d}s\\
&\left.+\frac{1}{2}\sum_{j=1}^{n}\int_{0}^{\infty}\gamma_{ji}(t+s)k_{ji}(s)(x_j(t)-\bar{x}_j(t))^2\,\mathrm{d}s\right\}
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant \sum_{i=1}^{n}\Bigg\{-\mu_i b_i(t)(x_i(t)-\bar{x}_i(t))^2+\frac{1}{2}\,\mu_i\sum_{j=1}^{n}\alpha_{ij}(t)(x_i(t)-\bar{x}_i(t))^2\\
&+\frac{1}{2}\sum_{j=1}^{n}\mu_j\alpha_{ji}(t)(x_i(t)-\bar{x}_i(t))^2\\
&+\frac{1}{2}\sum_{j=1}^{n}\mu_i\beta_{ij}(t)(x_i(t)-\bar{x}_i(t))^2+\frac{1}{2}\sum_{j=1}^{n}\mu_j\,\frac{\beta_{ji}(\zeta_{ji}^{-1}(t))}{1-\dot{\tau}_{ji}(\zeta_{ji}^{-1}(t))}\,(x_i(t)-\bar{x}_i(t))^2\\
&+\frac{1}{2}\sum_{j=1}^{n}\mu_i\gamma_{ij}(t)\int_{-\infty}^{t}k_{ij}(t-s)\,\mathrm{d}s(x_i(t)-\bar{x}_i(t))^2\\
&+\frac{1}{2}\sum_{j=1}^{n}\mu_j\int_{0}^{\infty}\gamma_{ji}(t+s)k_{ji}(s)\,\mathrm{d}s(x_i(t)-\bar{x}_i(t))^2\Bigg\}\\
&\leqslant -\,c_2\sum_{j=1}^{n}(x_i(t)-\bar{x}_i(t))^2,
\end{aligned}
$$

where $c_2>0$ is defined by

$$
\begin{aligned}
c_2=\min_{1\leqslant i\leqslant n}\inf_{t\in\mathbf{R}^+}\Bigg\{&\mu_i b_i(t)-\frac{1}{2}\sum_{j=1}^{n}\mu_i\alpha_{ij}(t)-\frac{1}{2}\sum_{j=1}^{n}\mu_j\alpha_{ji}(t)-\frac{1}{2}\sum_{j=1}^{n}\mu_i\beta_{ij}(t)\\
&-\frac{1}{2}\sum_{j=1}^{n}\frac{\mu_j\beta_{ji}(\zeta_{ji}^{-1}(t))}{1-\dot{\tau}_{ji}(\zeta_{ji}^{-1}(t))}\\
&-\frac{1}{2}\sum_{j=1}^{n}\mu_i\gamma_{ij}(t)\int_{-\infty}^{t}k_{ij}(t-s)\,\mathrm{d}s-\frac{1}{2}\sum_{j=1}^{n}\mu_j\int_{0}^{\infty}\gamma_{ji}(t+s)k_{ji}(s)\,\mathrm{d}s\Bigg\}>0.
\end{aligned}
$$

Proceeding as in the proof of the first part, we also obtain the global asymptotic stability of (1.1). □

Remark 2.1. For the purpose of interpreting the results biologically, one may prefer explicit conditions in terms of $b_i(t)$, $\alpha_{ij}(t)$, $\beta_i(t)$ and γ_{ij}. Various sets of such convenient conditions can be easily derived by choosing different values for $\mu_i>0$. Since the paper is already lengthy, we do not list such specifications here, and will have to leave it for readers. This remark also applies to Theorems 3.1, 4.2 and 5.2 for the corresponding special cases.

3. Autonomous case

In this section, we consider an autonomous case of (1.1), that is,

$$\dot{x}_i(t) = -b_i x_i(t) + f_i\Bigg(x_1(t),\ldots,x_n(t); x_1(t-\tau_{i1}),\ldots,x_n(t-\tau_{in});$$
$$\int_{-\infty}^{t} k_{i1}(t-s)x_1(s)\,\mathrm{d}s,\ldots,\int_{-\infty}^{t} k_{in}(t-s)x_n(s)\,\mathrm{d}s\Bigg), \quad t \geqslant 0. \tag{3.1}$$

The corresponding assumptions become

(A$_{31}$) For each $i,j\in\{1,2,\ldots,n\}$, $b_i>0$, $\tau_{ij}\geqslant 0$ and $k_{ij}:\mathbf{R}^+\to\mathbf{R}^+$ satisfies
$$\int_0^{\infty} k_{ij}(s)\,\mathrm{d}s < +\infty, \qquad \int_0^{\infty} s k_{ij}(s)\,\mathrm{d}s < +\infty;$$

(A$_{32}$) $f_i:\mathbf{R}^n\times\mathbf{R}^n\times\mathbf{R}^n\to\mathbf{R}$ is bounded, continuous and there exist nonnegative constants $\alpha_{ij}\geqslant 0$, $\beta_{ij}\geqslant 0$ and $\gamma_{ij}\geqslant 0$ such that for any $(u_1,\ldots,u_n)$, $(\bar{u}_1,\ldots,\bar{u}_n)$, $(v_1,\ldots,v_n),(\bar{v}_1,\ldots,\bar{v}_n)$, $(w_1,\ldots,w_n),(\bar{w}_1,\ldots,\bar{w}_n)\in\mathbf{R}^n$, we have
$$|f_i(u_1,\ldots,u_n;v_1,\ldots,v_n;w_1,\ldots,w_n) - f_i(\bar{u}_1,\ldots,\bar{u}_n;\bar{v}_1,\ldots,\bar{v}_n;\bar{w}_1,\ldots,\bar{w}_n)|$$
$$\leqslant \sum_{j=1}^{n}[\alpha_{ij}|u_j-\bar{u}_j| + \beta_{ij}|v_j-\bar{v}_j| + \gamma_{ij}|w_j-\bar{w}_j|], \quad i=1,\ldots,n.$$

Lemma 3.1. *The set of equilibria of* (3.1) *is nonempty in* $\mathbf{R}^n$.

Proof. Consider the mapping $F:\mathbf{R}^n\to\mathbf{R}^n$ defined by

$$\begin{aligned}(F(x))_i &:= \frac{1}{b_i} f_i\left(x_1,\ldots,x_n;x_1,\ldots,x_n;\int_{-\infty}^{t} k_{i1}(t-s)x_1\,\mathrm{d}s,\ldots,\int_{-\infty}^{t} k_{in}(t-s)x_n\,\mathrm{d}s\right)\\ &= \frac{1}{b_i} f_i(x_1,\ldots,x_n;x_1,\ldots,x_n;a_{i1}x_1,\ldots,a_{in}x_n)\\ &= \frac{1}{b_i} f_i(x;x;A_i x),\end{aligned}$$

where $a_{ij}=\int_{-\infty}^{t} k_{ij}(t-s)\,\mathrm{d}s$, $A_i=diag(a_{i1},\ldots,a_{in})$, $x=(x_1,\ldots,x_n)^{\mathrm{T}}\in\mathbf{R}^n$, $i,j=1,\ldots,n$. It is obvious that F is continuous. By the boundedness of f_i, there exists positive constant $M>0$ such that for any $x=(x_1,\ldots,x_n)^{\mathrm{T}}\in\mathbf{R}^n$, we have

$$\|F(x)\|_\infty \leqslant M,$$

where $\|x\| := \max_{1\leqslant i\leqslant n}|x_i|$, $x\in\mathbf{R}^n$.

Let $\Omega := \{x\in\mathbf{R}^n : \|x\|_\infty \leqslant M\}$, it is easy to show that $\Omega\subset\mathbf{R}^n$ is convex and compact. By the Brouwer fixed point theorem, there exists $x^*\in\mathbf{R}^n$ such that $x^*=F(x^*)$, that is

$$x_i^* := \frac{1}{b_i} f_i\left(x_1^*,\ldots,x_n^*;x_1^*,\ldots,x_n^*;\int_{-\infty}^{t} k_{i1}(t-s)x_1^*\,\mathrm{d}s,\ldots,\int_{-\infty}^{t} k_{in}(t-s)x_n^*\,\mathrm{d}s\right),$$

which implies x^* is an equilibrium of (3.1). □

Applying Theorem 2.1 to system (3.1), we can claim

Theorem 3.1. *If there exist positive constants* $\mu_i > 0$, $i = 1, 2, \ldots, n$, *such that*

$$\mu_i b_i - \sum_{j=1}^{n} \mu_j(\alpha_{ji} + \beta_{ji} + \gamma_{ji}) > 0, \quad i = 1, \ldots, n$$

or

$$2\mu_i b_i - \sum_{j=1}^{n} [\mu_i(\alpha_{ij} + \beta_{ij} + \gamma_{ij}) + \mu_j(\alpha_{ji} + \beta_{ji} + \gamma_{ji})] > 0, \quad i = 1, \ldots, n.$$

Then system (3.1) *has a unique equilibrium that is global asymptotically stable.*

4. Periodic case

The main purpose of this section is to establish criteria for existence and global asymptotic stability of a unique periodic solution of (1.1) when it is a periodic system. So, we consider the system in the same form of (1.1)

$$\dot{x}_i(t) = -b_i(t)x_i(t) + f_i\Bigg(t, x_1(t), \ldots, x_n(t); x_1(t-\tau_{i1}(t)), \ldots, x_n(t-\tau_{in}(t)); \int_{-\infty}^{t} k_{i1}(t-s)x_1(s)\,ds, \ldots, \int_{-\infty}^{t} k_{in}(t-s)x_n(s)\,ds\Bigg), \quad t \geqslant 0,\ i = 1, \ldots, n, \tag{4.1}$$

but with different assumptions corresponding to the periodic case:

(A_{41}) For each $i \in \{1, 2, \ldots, n\}$, $b_i(t)$ is continuous with $b_i(t) > 0$ and $b_i(t+\omega) = b_i(t)$ for $t \in \mathbf{R}^+$;

(A_{42}) For each $i, j \in \{1, 2, \ldots, n\}$, τ_{ij} is continuous with $\tau_{ij}(t+\omega) = \tau_{ij}(t)$ on $\mathbf{R}^+$;

(A_{43}) For each $i \in \{1, 2, \ldots, n\}$, $f_i : \mathbf{R}^+ \times \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ is bounded, continuous and ω-periodic in the first variable;

(A_{44}) For each $i, j \in \{1, 2, \ldots, n\}$, $k_{ij} : \mathbf{R}^+ \to \mathbf{R}^+$ satisfies

$$\int_0^\infty k_{ij}(s)\,ds < +\infty, \qquad \int_0^\infty s k_{ij}(s)\,ds < +\infty;$$

(A_{45}) For each $i, j \in \{1, 2, \ldots, n\}$, τ_{ij} is continuously differentiable with $\tau_{ij}(t) \geqslant 0$ and $1 - \dot{\tau}_{ij}(t) > 0$ on $\mathbf{R}^+$;

(A_{46}) There exist nonnegative, bounded and continuous functions $\alpha_{ij}(t), \beta_{ij}(t), \gamma_{ij}(t)$ defined on $\mathbf{R}^+$ such that

$$|f_i(t, u_1, \ldots, u_n; v_1, \ldots, v_n; w_1, \ldots, w_n) - f_i(t, \bar{u}_1, \ldots, \bar{u}_n; \bar{v}_1, \ldots, \bar{v}_n; \bar{w}_1, \ldots, \bar{w}_n)| \leqslant \sum_{j=1}^{n} [\alpha_{ij}(t)|u_j - \bar{u}_j| + \beta_{ij}(t)|v_j - \bar{v}_j| + \gamma_{ij}(t)|w_j - \bar{w}_j|], \quad i = 1, \ldots, n,$$

for any $(u_1, \ldots, u_n), (\bar{u}_1, \ldots, \bar{u}_n), (v_1, \ldots, v_n), (\bar{v}_1, \ldots, \bar{v}_n), (w_1, \ldots, w_n), (\bar{w}_1, \ldots, \bar{w}_n) \in \mathbf{R}^n$.

The method used here to prove the existence of periodic solutions, will be the continuation theorem developed from coincidence degree theory. For the reader's convenience, we shall first summarize below a few concepts and results from [11] that will be used in this section.

Let X, Z be normed vector spaces, $L: Dom\,L \subset X \to Z$ a linear mapping, and $N: X \to Z$ a continuous mapping. The mapping L will be called a Fredholm mapping of index zero if $dim(Ker\,L) = codim(Im\,L) < +\infty$ and $Im\,L$ is closed in Z. If L is a Fredholm mapping of index zero, then there exist continuous projectors $P: X \to X$ and $Q: Z \to Z$ such that $Im\,P = Ker\,L$, $Im\,L = Ker\,Q = Im(I - Q)$. It follows that $L|_{Dom\,L \cap Ker\,P}: (I - P)X \to Im\,L$ is invertible. We denote the inverse of that map by K_P. If Ω is an open bounded subset of X, the mapping N will be called L-compact on $\bar{\Omega}$ if $QN(\bar{\Omega})$ is bounded and $K_P(I - Q)N: \bar{\Omega} \to X$ is compact. Since $Im\,Q$ is isomorphic to $Ker\,L$, there exists isomorphisms $J: Im\,Q \to Ker\,L$.

In the proof of our existence theorem below, we will use the continuation theorem from Gaines and Mawhin [11, p. 40].

Lemma 4.1 (continuation theorem). *Let L be a Fredholm mapping of index zero and let N be L-compact on $\bar{\Omega}$. Suppose*

(a) *For each $\lambda \in (0, 1)$, every solution x of $Lx = \lambda Nx$ is such that $x \notin \partial\Omega$;*
(b) *$QNx \neq 0$ for each $x \in \partial\Omega \cap Ker\,L$ and*

$$\deg\{JQN, \Omega \cap Ker\,L, 0\} \neq 0.$$

Then the equation $Lx = Nx$ has at least one solution lying in $Dom\,L \cap \bar{\Omega}$.

Now we state our main result on the existence of periodic solutions of (4.1)

Theorem 4.1. *If* (A_{41})–(A_{44}) *hold, then system* (4.1) *has at least one periodic solution with period ω.*

Proof. Let

$$X = Z = \{x(t) = (x_1(t), \ldots, x_n(t))^{\mathrm{T}} \in C(\mathbf{R}, \mathbf{R}^n) | x(t + \omega) = x(t),\ t \in \mathbf{R}\}$$

and

$$\|x\| = \left(\sum_{i=1}^{n} \left(\max_{t \in [0,\omega]} |x_i(t)| \right)^2 \right)^{1/2} \qquad \text{for any } x \in X(Z).$$

Then X and Z are both Banach spaces when they are endowed with the norm $\|\cdot\|$. Let

$$Nx = \Bigg(-b_i(t)x_i(t) + f_i(t, x_1(t), \ldots, x_n(t); x_1(t - \tau_{i1}(t)), \ldots, x_n(t - \tau_{in}(t));$$

$$\int_{-\infty}^{t} k_{i1}(t - s)x_1(s)\,\mathrm{d}s, \ldots, \int_{-\infty}^{t} k_{in}(t - s)x_n(s)\,\mathrm{d}s \Bigg)_{n \times 1},$$

for any $x \in X$ and

$$Lx = \dot{x}, \quad Px = \frac{1}{\omega}\int_0^{\omega} x(t)\,\mathrm{d}t, \quad x \in X; \qquad Qz = \frac{1}{\omega}\int_0^{\omega} z(t)\,\mathrm{d}t, \quad z \in Z.$$

Then, it follows that

$$Ker\,L = \{x | x \in X, x = h, h \in R^n\}, \quad Im\,L = \left\{z | z \in Z, \int_0^{\omega} z(t)\,\mathrm{d}t = 0\right\}$$

and

$$dim(Ker\,L) = n = codim(Im\,L).$$

Obviously, $Im\,L$ is closed in Z, therefore, L is a Fredholm mapping of index zero. One can easily show that P and Q are continuous projectors such that

$$Im\,P = Ker\,L, \quad Ker\,Q = Im\,L = Im(I - Q).$$

Furthermore, the generalized inverse (to L) $K_P : Im\,L \to Ker\,P \cap Dom\,L$ exists and is given by

$$K_P(z) = \int_0^t z(s)\,\mathrm{d}s - \frac{1}{\omega}\int_0^{\omega}\int_0^t z(s)\,\mathrm{d}s\,\mathrm{d}t.$$

Thus

$$QNx = \left(\frac{1}{\omega}\int_0^{\omega}\left[-b_i(t)x_i(t) + f_i(t, x_1(t), \ldots, x_n(t); x_1(t - \tau_{i1}(t)), \ldots, x_n(t - \tau_{in}(t)); \int_{-\infty}^{t} k_{i1}(t-s)x_1(s)\,\mathrm{d}s, \ldots, \int_{-\infty}^{t} k_{in}(t-s)x_n(s)\,\mathrm{d}s)\right]\,\mathrm{d}t\right)_{n\times 1}$$

and

$$\begin{aligned} &K_P(I - Q)Nx \\ &= \left(\int_0^t \left[-b_i(s)x_i(s) + f_i\left(s, x_1(s), \ldots, x_n(s); x_1(s - \tau_{i1}(s)), \ldots, x_n(s - \tau_{in}(s)); \int_{-\infty}^{s} k_{i1}(s-u)x_1(u)\,\mathrm{d}u, \ldots, \int_{-\infty}^{t} k_{in}(s-u)x_n(u)\,\mathrm{d}u\right)\right]\,\mathrm{d}s\right)_{n\times 1} \\ &- \left(\frac{1}{\omega}\int_0^{\omega}\int_0^t \left[-b_i(s)x_i(s) + f_i(s, x_1(s), \ldots, x_n(s); x_1(s - \tau_{i1}(s)), \ldots, x_n(s - \tau_{in}(s)); \int_{-\infty}^{s} k_{i1}(s-u)x_1(u)\,\mathrm{d}u, \ldots, \int_{-\infty}^{t} k_{in}(s-u)x_n(u)\,\mathrm{d}u)\right]\,\mathrm{d}s\,\mathrm{d}t\right)_{n\times 1} \end{aligned}$$

$$-\left(\left(\frac{t}{\omega}-\frac{1}{2}\right)\int_0^\omega\left[-b_i(s)x_i(s)+f_i\left(s,x_1(s),\ldots,x_n(s);x_1(s-\tau_{i1}(s)),\ldots,x_n(s-\tau_{in}(s));\right.\right.\right.$$

$$\left.\left.\left.\int_{-\infty}^{s}k_{i1}(s-u)x_1(u)\,\mathrm{d}u,\ldots,\int_{-\infty}^{t}k_{in}(s-u)x_n(u)\,\mathrm{d}u\right)\right]\,\mathrm{d}s\right)_{n\times 1}.$$

Clearly, QN and $K_P(I-Q)N$ are continuous. Using the Arzela–Ascoli theorem, it is not difficult to show that $\overline{K_P(I-Q)N(\bar{\Omega})}$ is compact for any open bounded set $\Omega\subset X$. Moreover, $QN(\bar{\Omega})$ is bounded. Thus, N is L-compact on $\bar{\Omega}$ with any open bounded set $\Omega\subset X$. The isomorphism J of $Im\,Q$ onto $Ker\,L$ can be taken to be identity mapping, since $Im\,Q=Ker\,L$.

Now we are in the position to search for an appropriate open, bounded subset Ω for the application of the continuation theorem. Corresponding to the operator equation $Lx=\lambda Nx$, $\lambda\in(0,1)$, we have

$$\dot{x}_i(t)=\lambda\left[-b_i(t)x_i(t)+f_i\left(t,x_1(t),\ldots,x_n(t);x_1(t-\tau_{i1}(t)),\ldots,x_n(t-\tau_{in}(t));\right.\right.$$

$$\left.\left.\int_{-\infty}^{t}k_{i1}(t-s)x_1(s)\,\mathrm{d}s,\ldots,\int_{-\infty}^{t}k_{in}(t-s)x_n(s)\,\mathrm{d}s\right)\right],$$

$$t\geqslant 0,\ \ i=1,\ldots,n. \tag{4.2}$$

Assume that $x=x(t)\in X$ is a solution of (4.2) for a certain $\lambda\in(0,1)$, we shall first show that $x(t)$ is uniformly bounded with respect to λ. Multiplying both sides of (4.2) by $x_i(t)$ and integrating over the interval $[0,\omega]$, we have

$$\begin{aligned}0&=\int_0^\omega x_i(t)\dot{x}_i(t)\,\mathrm{d}t\\&=\lambda\left[-\int_0^\omega b_i(t)(x_i(t))^2\,\mathrm{d}t+\int_0^\omega x_i(t)f_i\left(t,x_1(t),\ldots,x_n(t);\right.\right.\\&\quad x_1(t-\tau_{i1}(t)),\ldots,x_n(t-\tau_{in}(t));\\&\quad\left.\left.\int_{-\infty}^{t}k_{i1}(t-s)x_1(s)\,\mathrm{d}s,\ldots,\int_{-\infty}^{t}k_{in}(t-s)x_n(s)\,\mathrm{d}s\right)\,\mathrm{d}t\right],\quad i=1,\ldots,n,\end{aligned}\tag{4.3}$$

then

$$\int_0^\omega b_i(t)|x_i(t)|^2\mathrm{d}t\leqslant\int_0^\omega|x_i(t)|\left|f_i\left(t,x_1(t),\ldots,x_n(t);x_1(t-\tau_{i1}(t)),\ldots,x_n(t-\tau_{in}(t));\right.\right.$$

$$\left.\left.\int_{-\infty}^{t}k_{i1}(t-s)x_1(s)\,\mathrm{d}s,\ldots,\int_{-\infty}^{t}k_{in}(t-s)x_n(s)\,\mathrm{d}s\right)\right|\,\mathrm{d}t$$

$$\leqslant M_i \int_0^{\omega} |x_i(t)|\,\mathrm{d}t$$

$$\leqslant M_i\sqrt{\omega}\left(\int_0^{\omega} |x_i(t)|^2\,\mathrm{d}t\right)^{1/2}, \quad i = 1,\ldots,n,$$

where M_i is the bound of f_i, and hence

$$\int_0^{\omega} |x_i(t)|^2\,\mathrm{d}t \leqslant \left(\frac{M_i}{b_i^l}\right)^2 \omega, \quad i = 1,\ldots,n. \tag{4.4}$$

From (4.4), it follows that there exists a $t_0 \in [0,\omega]$ such that

$$|x_i(t_0)| \leqslant \frac{M_i}{b_i^l}, \quad i = 1,\ldots,n.$$

Therefore

$$|x_i(t)| \leqslant |x_i(t_0)| + \left|\int_{t_0}^{t} |\dot{x}_i(s)|\,\mathrm{d}s\right| \leqslant \frac{M_i}{b_i^l} + \int_0^{\omega} |\dot{x}_i(s)|\,\mathrm{d}s, \tag{4.5}$$

for any $t \in [0,\omega]$. From (4.2), it follows that

$$\begin{aligned}
\int_0^{\omega} |\dot{x}_i(t)|\,\mathrm{d}t \leqslant{}& \int_0^{\omega} b_i(t)|x_i(t)|\,\mathrm{d}t \\
&+ \int_0^{\omega} \Bigg| f_i \Bigg(t, x_1(t),\ldots,x_n(t); x_1(t-\tau_{i1}(t)),\ldots,x_n(t-\tau_{in}(t)); \\
&\int_{-\infty}^{t} k_{i1}(t-s)x_1(s)\,\mathrm{d}s,\ldots,\int_{-\infty}^{t} k_{in}(t-s)x_n(s)\,\mathrm{d}s \Bigg) \Bigg|\,\mathrm{d}t \\
\leqslant{}& b_i^u \int_0^{\omega} |x_i(t)|\,\mathrm{d}t + M_i\omega \\
\leqslant{}& b_i^u \sqrt{\omega}\left(\int_0^{\omega} |x_i(t)|^2\,\mathrm{d}t\right)^{1/2} + M_i\omega \\
\leqslant{}& b_i^u \frac{M_i}{b_i^l}\,\omega + M_i\omega = \frac{M_i}{b_i^l}\,(b_i^u + b_i^l)\omega,
\end{aligned}$$

which, together with (4.5), implies

$$|x_i(t)| \leqslant \frac{M_i}{b_i^l} + \frac{M_i}{b_i^l}\,(b_i^u + b_i^l)\omega := B_i, \quad i = 1,\ldots,n.$$

Let

$$\Omega = \{x(t) \in X \mid \|x\| < B\}, \tag{4.6}$$

where

$$B := \max\left\{ \left(\sum_{i=1}^{n} B_i^2\right)^{1/2}, \frac{\sum_{i=1}^{n} M_i}{\min\{b_1^l, \ldots, b_n^l\}} \right\}.$$

Then for any $x \in \partial\Omega \cap Ker\, L$, we have $Lx \neq \lambda Nx$, for $\lambda \in (0,1)$, which implies that Ω satisfies the requirement (a) in Lemma 4.1. In addition, when $x \in \partial\Omega \cap Ker\, L = \partial\Omega \cap \mathbf{R}^n$, x is a constant vector in $\mathbf{R}^n$ with $\|x\| = B$, and

$$\begin{aligned}
x^{\mathrm{T}}(QNx) &= \sum_{i=1}^{n} x_i \left[-\frac{1}{\omega}\int_0^{\omega} b_i(t) x_i \,\mathrm{d}t + \frac{1}{\omega}\int_0^{\omega} f_i\Big(t, x_1(t), \ldots, x_n(t); x_1(t-\tau_{i1}(t)), \ldots, x_n \right. \\
&\quad \left. (t-\tau_{in}(t)); \int_{-\infty}^{t} k_{i1}(t-s)x_1(s)\,\mathrm{d}s, \ldots, \int_{-\infty}^{t} k_{in}(t-s)x_n(s)\,\mathrm{d}s\Big)\,\mathrm{d}t \right] \\
&< \sum_{i=1}^{n} \left[-\frac{1}{\omega}\int_0^{\omega} b_i(t)|x_i|^2 \,\mathrm{d}t + |x_i|\,\frac{1}{\omega}\int_0^{\omega} \Big| f_i\Big(t, x_1(t), \ldots, x_n(t); x_1(t-\tau_{i1}(t)), \ldots, x_n \right. \\
&\quad \left. (t-\tau_{in}(t)); \int_{-\infty}^{t} k_{i1}(t-s)x_1(s)\,\mathrm{d}s, \ldots, \int_{-\infty}^{t} k_{in}(t-s)x_n(s)\,\mathrm{d}s\Big)\Big|\,\mathrm{d}t \right] \\
&< \sum_{i=1}^{n} [-b_i^l |x_i|^2 + |x_i| M_i] \\
&< -\min\{b_1^l, \ldots, b_n^l\}\|x\|^2 + \sum_{i=1}^{n} M_i \|x\| < 0.
\end{aligned}$$

Therefore, $QNx \neq 0$ for $x \in \partial\Omega \cap Ker\, L$. Let

$$F(\mu, x) = -\mu x + (1-\mu) QNx, \quad \mu \in (0,1),$$

then $x^{\mathrm{T}} F(\mu, x) < 0$ for $x \in \partial\Omega \cap Ker\, L$. From the homotopy invariance of Brouwer degree, it follows that

$$\deg\{JQN, \Omega \cap Ker\, L, 0\} = \deg\{QN, \Omega \cap Ker\, L, 0\} = \deg\{-I, \Omega \cap Ker\, L, 0\} \neq 0,$$

where J and I are the identity operator. By now we have proved that Ω verifies all the requirements in Lemma 4.1. Hence, the system (1.1) has at least one ω-periodic solution $x^*(t)$ in $\bar{\Omega}$ and it is clear that $x^*(t)$ is bounded by B. □

Note that if a ω-periodic solution $x^*(t)$ of (4.1) is global asymptotically stable then it is in fact unique. Now combining Theorems 2.1 and 4.1, we immediately have

Theorem 4.2. *Assume* (A_{41})–(A_{46}) *hold. If either* (2.1) *or* (2.2) *is also satisfied, then system* (4.1) *has a unique periodic solution of period* ω, *which is global asymptotically stable*.

Remark 4.1. In (A_{42}), $\tau_{ij} \geqslant 0$ is not required. This means Theorem 4.1 is valid for both the retarded case (with all $\tau_{ij} \geqslant 0$) and the mixed case (with some $\tau_{ij} < 0$) of (1.1).

5. Applications and discussion

In this section, we illustrate our results established in the previous sections by considering some particular mathematical models arising from mathematical biology and neural networks.

Example 5.1. Consider the following delayed red blood cells model

$$\dot{N}(t) = -a(t)N(t) + b(t)e^{-N(t-\tau(t))}, \tag{5.1}$$

where $a(t)$ and $b(t)$ are positive, bounded and continuous functions. $\tau(t)$ is continuously differentiable for $t \geqslant 0$, $\tau(t) \geqslant 0$ and $1-\dot{\tau}(t) > 0$ on $[0, +\infty)$. $N(t)$ denotes the number of red blood cells at time t, $a(t)$ is the probability (in some sense) of death of a red blood cell at time t, $b(t)$ is related to the production of red blood cells per unit time, $\tau(t)$ is the time required to produce a red blood cell.

System (5.1) has been investigated by many authors (see, e.g., [7,20,37]). Chow [7] studied the existence of periodic solution of (5.1) under the assumption that $a(t)$, $b(t)$ and $\tau(t)$ are all positive constants; Jiang and Wei [20] also studied the existence of positive periodic solution. But none of [7,20] discussed stability of periodic solutions. In contrast, we will examine all the three aspects corresponding to Sections 2–4.

It is not difficult to show that (5.1) satisfies the assumptions on (1.1). The initial conditions of (5.1) is of the form

$$x(t) = \varphi(t), \quad t \in (-\infty, 0], \ \ \varphi(0) > 0,$$

where φ is bounded and continuous on $(-\infty, 0]$. A standard argument will result in the following positive invariant result for (5.1).

Lemma 5.1. *For any solution* $N(t, \varphi)$ *of* (5.1), φ *as above, we have* $N(t, \varphi) > 0$ *for* $t > 0$.

Applying Theorems 2.1, 3.1 and 4.1 to (5.1), we obtain the following theorem.

Theorem 5.1. *If*

$$\inf_{t \in \mathbf{R}^+} \left\{ a(t) - \frac{b(\zeta^{-1}(t))}{1 - \dot{\tau}(\zeta^{-1}(t))} \right\} > 0,$$

where $\zeta^{-1}(t)$ is the inverse function of $\zeta(t) = t - \tau(t)$. Then system (5.1) is global asymptotically stable. In particular,

(i) *if $a(t)$, $b(t)$ and $\tau(t)$ are periodic of period ω, then system (5.1) has a unique positive ω-periodic solution, which is global asymptotically stable;*
(ii) *if $a(t) = a, b(t) = b$ and $\tau(t) = \tau$ are all constants with $a > b$, then the unique equilibrium determined by $aN = be^{-N}$ is global asymptotically stable.*

Remark 5.1. One can easily show that Lemma 5.1 and Theorem 5.1 also hold for the following equation:

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} = -a(t)y(t) + \frac{b(t)}{1 + y(t - \tau(t))},$$

where the assumptions on $a(t)$, $b(t)$ and $\tau(t)$ and the initial conditions are the same as those in (5.1), This equation has also been used to model the blood cells and existence of periodic solutions of the equation was discussed by Mallet-Paret and Nussbaum [25].

Example 5.2. Consider the system of delay differential equations

$$\begin{cases} \dfrac{\mathrm{d}x_1(t)}{\mathrm{d}t} = -b_1(t)x_1(t) + \alpha_1(t)\arctan x_2(t - \tau_2(t)), \\ \dfrac{\mathrm{d}x_2(t)}{\mathrm{d}t} = -b_2(t)x_2(t) + \alpha_2(t)\arctan x_1(t - \tau_1(t)), \end{cases} \tag{5.2}$$

where $b_i(t), \alpha_i(t)$ are nonnegative, bounded and continuous and $\tau_i(t)$ are nonnegative, continuously differentiable with $1 - \dot{\tau}_i(t) > 0$, $i = 1, 2$.

When $b_i(t) \equiv 1$, $\tau_i(t) \equiv 1$ and $\alpha_1(t) = \alpha_2(t) \equiv$ *constant*, then (5.2) reduces to the example used in [2,33], where existence of periodic solutions was established via bifurcation.

Theorem 5.2. *If there exist positive constants $\mu_1 > 0$, $\mu_2 > 0$ such that*

$$\inf_{t\in\mathbf{R}^+}\left\{\mu_1 b_1(t) - \mu_2\frac{\alpha_2(\zeta_1^{-1}(t))}{1 - \dot{\tau}_1(\zeta_1^{-1}(t))}\right\} > 0$$

and

$$\inf_{t\in\mathbf{R}^+}\left\{\mu_2 b_2(t) - \mu_1\frac{\alpha_1(\zeta_2^{-1}(t))}{1 - \dot{\tau}_2(\zeta_2^{-1}(t))}\right\} > 0,$$

or

$$\begin{cases} \inf\limits_{t\in\mathbf{R}^+}\left\{2\mu_1 b_1(t) - \left(\mu_1\alpha_1(t) + \mu_2\dfrac{\alpha_2(\zeta_1^{-1}(t))}{1 - \dot{\tau}_1(\zeta_1^{-1}(t))}\right)\right\} > 0 \\ \inf\limits_{t\in\mathbf{R}^+}\left\{2\mu_2 b_2(t) - \left(\mu_2\alpha_2(t) + \mu_1\dfrac{\alpha_1(\zeta_2^{-1}(t))}{1 - \dot{\tau}_2(\zeta_2^{-1}(t))}\right)\right\} > 0, \end{cases}$$

where $\zeta^{-1}(t)$ is the inverse function of $\zeta_i(t)=t-\tau_i(t)$, $i=1,2$. Then system (5.2) is global asymptotically stable. In particular,

(i) *if $\alpha_i(t)$, $b_i(t)$ and $\tau_i(t)$ $(i=1,2)$ are periodic of period ω, then system (5.2) has a unique ω-periodic solution, which is global asymptotically stable;*
(ii) *if $b_i(t)=b_i$, $\tau_i(t)=\tau_i$, $\alpha_i=\alpha_i$ are all constants, and there exist $\mu_i>0$, $i=1,2$, such that*

$$\mu_1 b_1 > \mu_2\alpha_2, \quad \mu_2 b_2 > \mu_1\alpha_1$$

or

$$\min\{2\mu_1 b_1, 2\mu_2 b_2\} > \mu_1\alpha_1 + \mu_2\alpha_2.$$

then system (5.2) has a unique equilibrium, which is global asymptotically stable.

Example 5.3. Consider the neural network model without self-connection.

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = -x(t) + a\tanh[c_1 y(t-\tau)],$$
$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} = -y(t) + a\tanh[c_2 x(t-\tau)], \tag{5.3}$$

where a, c_1, c_2, τ are all nonnegative constants. This is the system studied in [16], where it is proved that if $ac_1<1$, $ac_2<1$, then the zero solution of (5.3) is global asymptotically stable. An immediate application of Theorem 3.1 to (5.3) gives an improvement to the above result, as stated below.

Theorem 5.3. *If there exist $\mu_1>0$ and $\mu_2>0$ such that*

$$\mu_1 > \mu_2 ac_2, \quad \mu_2 > \mu_1 ac_1$$

or

$$\min\{2\mu_1, 2\mu_2\} > \mu_1 ac_1 + \mu_2 ac_2.$$

Then the zero solution of (5.3) is global asymptotically stable.

As promised in the introduction, we will apply our main results to general modifications of both Hopfield neural network and the BAM neural network. We start with the former.

Example 5.4. Consider the following nonautonomous Hopfield-type neural network with infinite delay

$$C_i(t)\frac{\mathrm{d}u_i(t)}{\mathrm{d}t} = -\frac{u_i(t)}{R_i(t)} + \sum_{j=1}^{n} T_{ij}^{(1)}(t)g_j(u_j(t)) + \sum_{j=1}^{n} T_{ij}^{(2)}(t)g_j(u_j(t-\tau_{ij}(t)))$$
$$+\sum_{j=1}^{n} T_{ij}^{(3)}(t)\int_{-\infty}^{t} k_{ij}(t-s)g_j(u_j(s))\,\mathrm{d}s + I_i(t), \quad t\geqslant 0, \tag{5.4}$$

where $C_i(t)>0$, $R_i(t)>0$, $T_{ij}^{(1)}(t)$, $T_{ij}^{(2)}(t)$, $T_{ij}^{(3)}(t)$, $I_i(t)$ are bounded and continuous on $\mathbf{R}^+$; $\tau_{ij}(t)$ is nonnegative, continuously differentiable such that $1-\dot{\tau}_{ij}(t)>0$; $k_{ij}:\mathbf{R}^+\to\mathbf{R}$ is integrable and is normalized such that

$$\int_0^{+\infty} k_{ij}(s)\,\mathrm{d}s = 1, \qquad \int_0^{+\infty} s k_{ij}(s)\,\mathrm{d}s < +\infty;$$

$g_i:\mathbf{R}\to\mathbf{R}$ is bounded and continuous and there exist positive constants $G_i\geqslant 0$ such that for any $u_1,u_2\in\mathbf{R}$, we have

$$|g_i(u_1)-g_i(u_2)|\leqslant G_i|u_1-u_2|.$$

System (5.4) describes the dynamics of a large-scale system composed of n elementary units, also called neurons, which are massively interconnected. More precisely, u_i corresponds to the ith neuron activation state, while $g_i(u_i(\cdot))$, represent the input –output activation functions of the neurons and are typically assumed to have a sigmoidal shape (i.e. they are bounded and increasing functions of u_i and have bounded derivatives). Similar to [34], here this assumption is weakened to a Lipschitz-type condition. I_i represent varying external input signals to the neural network and $T_{ij}^{(k)}$ define the interconnections (or synaptic strength) between the n neurons. System (5.4) represents a natural generalization of Hopfield's original model by taking into account the processing time of individual neurons and the finite switching speed of the neuron amplifiers by incorporating time delay of various forms into the system. The presence of the term involving $T_{ij}^{(1)}$ assumes, in addition to the delayed propagation of signals, a set of local interactions in the network whose propagation time is instantaneous.

System (5.4) has many applications in the fields of neurobiological modelling and analogue computing and is general enough to include many Hopfield-type neural network models as special cases. For instance, let us assume that all the parameters in (5.4) are constants. In addition, if $T_{ij}^{(2)}\equiv T_{ij}^{(3)}\equiv 0$, then (5.4) represents Hopfield's original neural network model [19]; if $T_{ij}^{(2)}\equiv 0$, then (5.4) is the systems considered by Cao [6] and [41]; if $T_{ij}^{(3)}\equiv 0$, then (5.4) is the systems investigated by Gopalsamy and He [14] and Joy [21]; if $T_{ij}^{(1)}\equiv T_{ij}^{(3)}\equiv 0$, then (5.4) becomes the system considered by Bélair [4], Campbell and Bélair [5], Gopalsamy and He [14], Liao and Xiao [25], and Roska et al. [31]. It has been argued (see [21,25] and references therein) that it is reasonable to study neural network models with varying time delays and time-dependent coefficients.

The next theorem gives sufficient conditions for the global asymptotic stability of the generalized Hopfield-type neural network model (5.4).

Theorem 5.4. *If there exist positive constants* $\mu_i\geqslant 0$, $i=1,2,\ldots,n$, *such that*

$$\inf_{t\in\mathbf{R}^+}\left\{\frac{\mu_i}{C_i(t)R_i(t)}-\sum_{j=1}^n\mu_jG_j\left[\frac{|T_{ji}^{(1)}(t)|}{C_j(t)}+\frac{|T_{ji}^{(2)}(\zeta_{ji}^{-1}(t))|}{C_j(\zeta_{ji}^{-1}(t))(1-\dot{\tau}_{ji}(\zeta_{ji}^{-1}(t)))}\right.\right.$$

$$\left.\left.+\int_0^{+\infty}\frac{|T_{ji}^{(3)}(t+s)|}{C_j(t+s)}k_{ji}(s)\,\mathrm{d}s\right]\right\}>0$$

or

$$\inf_{t\in\mathbf{R}^+}\left\{\frac{2\mu_i}{C_i(t)R_i(t)}-\sum_{j=1}^{n}\left[\mu_i G_i\left(\frac{|T_{ij}^{(1)}(t)|}{C_i(t)}+\frac{|T_{ij}^{(2)}(t)|}{C_i(t)}+\frac{|T_{ij}^{(3)}(t)|}{C_i(t)}\right)\right.\right.$$
$$+\mu_j G_j\left(\frac{|T_{ji}^{(1)}(t)|}{C_j(t)}+\frac{|T_{ji}^{(2)}(\zeta_{ji}^{-1}(t))|}{C_j(\zeta_{ji}^{-1}(t))(1-\dot{\tau}_{ji}(\zeta_{ji}^{-1}(t)))}\right.$$
$$\left.\left.\left.+\int_0^{+\infty}\frac{|T_{ji}^{(3)}(t+s)|}{C_j(t+s)}k_{ji}(s)\,\mathrm{d}s\right)\right]\right\}>0,$$

where $\zeta_{ij}^{-1}(t)$ *is the inverse function of* $\zeta_{ij}(t)=t-\tau_{ij}(t)$. *Then system* (5.4) *is global asymptotically stable. In particular*,

(i) *if further assume that* $C_i(t), R_i(t), T_{ij}^1(t), T_{ij}^2(t), T_{ij}^3(t), I_i(t)$ *and* $\tau_{ij}(t)$ *are periodic of period* ω, *then system* (5.4) *has a unique* ω*-periodic solution, which is global asymptotically stable*;

(ii) *if* $C_i(t)=C_i$, $R_i(t)=R_i$, $T_{ij}^1(t)=T_{ij}^1$, $T_{ij}^2(t)=T_{ij}^2$, $T_{ij}^3(t)=T_{ij}^3(t)$, $I_i(t)=I_i$ *and* $\tau_{ij}(t)=\tau_{ij}$ *are all constants and there exist positive constants* $\mu_i>0$, $i=1,2,\dots,n$, *such that*

$$\frac{\mu_i}{C_iR_i}-\sum_{j=1}^{n}\frac{\mu_jG_j}{C_j}\left[|T_{ij}^{(1)}|+|T_{ji}^{(2)}|+|T_{ji}^{(3)}|\right]>0$$

or

$$\frac{2\mu_i}{C_iR_i}-\sum_{j=1}^{n}\left[\frac{\mu_iG_i}{C_i}(|T_{ij}^{(1)}|+|T_{ij}^{(2)}|+|T_{ij}^{(3)}|)+\frac{\mu_jG_j}{C_j}(|T_{ji}^{(1)}|+|T_{ji}^{(2)}|+|T_{ji}^{(3)}|)\right]>0.$$

then system (5.4) *has a unique equilibrium, which is global asymptotically stable*.

Remark 5.2. Theorem 5.4 improves and generalizes the main results of [6,8,14,39]. For example, if the coefficients of (5.4) are constants and $T_{ij}^{(2)}\equiv T_{ij}^{(3)}\equiv 0$, then (5.4) is the system investigated by Hirsch [18], and the second sufficient criterion in Theorem 5.4(ii) agrees with Theorem 3 in [18]. Note, however, that the conditions in [18] are more restrictive, since it has been assumed that g_i is C^1 and that there exist positive constants G_i, for $i=1,\dots,n$, such that $0\leqslant g_i'(u_i)\leqslant G_i$, for all u_i. When $T_{ij}^{(1)}\equiv T_{ij}^{(3)}\equiv 0$, (5.4) becomes the system investigated in [14,34], and the first sufficient criterion in Theorem 5.4(ii) with $\mu_i=1, T_{ij}^{(1)}\equiv T_{ij}^{(3)}\equiv 0$ improves the main theorem in [14] and Theorem 2.1 in [34]. As for the periodic case, Theorem 5.4(i) with $T_{ij}^2(t)\equiv T_{ij}^3(t)\equiv 0$, $C_i(t)=C_i>0$, $R_i(t)=R_i>0$, $T_{ij}^1=T_{ij}$ being constants, and $I_i(t)$ being periodic improves the main results in [20].

Example 5.5. Consider a general modification of the BAM neural network (1.5) to the following

$$\frac{\mathrm{d}u_i(t)}{\mathrm{d}t} = -b_i(t)u_i(t) + \sum_{j=n+1}^{n+m} T_{ij}^{(1)}(t)g_j(u_j(t)) + \sum_{j=n+1}^{n+m} T_{ij}^{(2)}(t)g_j(u_j(t-\tau_{ij}(t)))$$

$$+ \sum_{j=n+1}^{n+m} T_{ij}^{(3)}(t) \int_{-\infty}^{t} k_{ij}(t-s)g_j(u_j(s))\,\mathrm{d}s + I_i(t), \quad t \geqslant 0,\ i = 1,\dots,n,$$

$$\frac{\mathrm{d}u_i(t)}{\mathrm{d}t} = -b_i(t)u_i(t) + \sum_{j=1}^{n} T_{ij}^{(1)}(t)g_j(u_j(t)) + \sum_{j=1}^{n} T_{ij}^{(2)}(t)g_j(u_j(t-\tau_{ij}(t)))$$

$$+ \sum_{j=1}^{n} T_{ij}^{(3)}(t) \int_{-\infty}^{t} k_{ij}(t-s)g_j(u_j(s))\,\mathrm{d}s + I_i(t),$$

$$t \geqslant 0,\ i = n+1,\dots,n+m, \tag{5.5}$$

where $b_i(t) > 0$ is bounded and continuous on $\mathbf{R}^+$, and $T_{ij}^{(1)}(t), T_{ij}^{(2)}(t), T_{ij}^{(3)}(t), I_i(t), \tau_{ij}(t), k_{ij}, g_i$ are all as in Example 5.4. Such a model, initially proposed by Kosko [22] in the original form of (1.5), demonstrates a two-layer architecture for hetero associate memories. Various special cases of (5.5) have been discussed by many authors (see, e.g., [15,22,23,40]). System (5.5) can be formally simplified to a system of Hopfield-type (5.4) by suitably choosing nonlinear terms, but such a simplification will alter the bidirectional interplay of the input–output nature of the two layers, and thus may result in ignorance of the two-layer structure.

An immediate consequence of Theorems 2.1, 3.1 and 4.1 is the following.

Theorem 5.5. *Assume there exist positive constants $\mu_i > 0$, $i = 1,2,\dots,n+m$, such that*

$$\inf_{t\in\mathbf{R}^+} \left\{ \mu_i b_i(t) - \sum_{j=n+1}^{n+m} \mu_j G_j \left[|T_{ji}^{(1)}(t)| + \frac{|T_{ji}^{(2)}(\zeta_{ji}^{-1}(t))|}{1-\dot{\tau}_{ji}(\zeta_{ji}^{-1}(t))} + \int_0^{+\infty} |T_{ji}^{(3)}(t+s)| k_{ji}(s)\,\mathrm{d}s \right] \right\} > 0, \quad i = 1,\dots,n,$$

$$\inf_{t\in\mathbf{R}^+} \left\{ \mu_i b_i(t) - \sum_{j=1}^{n} \mu_j G_j \left[|T_{ji}^{(1)}(t)| + \frac{|T_{ji}^{(2)}(\zeta_{ji}^{-1}(t))|}{1-\dot{\tau}_{ji}(\zeta_{ji}^{-1}(t))} + \int_0^{+\infty} |T_{ji}^{(3)}(t+s)| k_{ji}(s)\,\mathrm{d}s \right] \right\} > 0, \quad i = n+1,\dots,n+m$$

or

$$\inf_{t\in\mathbf{R}^+}\left\{2\mu_i b_i(t)-\sum_{j=n+1}^{n+m}\left[\mu_i G_i(|T_{ij}^{(1)}(t)|+|T_{ij}^{(2)}(t)|+|T_{ij}^{(3)}(t)|)\right.\right.$$
$$\left.\left.+\mu_j G_j\left(|T_{ji}^{(1)}(t)|+\frac{|T_{ji}^{(2)}(\zeta_{ji}^{-1}(t))|}{1-\dot{\tau}_{ji}(\zeta_{ji}^{-1}(t))}+\int_0^{+\infty}|T_{ji}^{(3)}(t+s)|k_{ji}(s)\,\mathrm{d}s\right)\right]\right\}>0,$$
$$i=1,\ldots,n,$$

$$\inf_{t\in\mathbf{R}^+}\left\{2\mu_i b_i(t)-\sum_{j=1}^{n}\left[\mu_i G_i(|T_{ij}^{(1)}(t)|+|T_{ij}^{(2)}(t)|+|T_{ij}^{(3)}(t)|)\right.\right.$$
$$\left.\left.+\mu_j G_j\left(|T_{ji}^{(1)}(t)|+\frac{|T_{ji}^{(2)}(\zeta_{ji}^{-1}(t))|}{1-\dot{\tau}_{ji}(\zeta_{ji}^{-1}(t))}+\int_0^{+\infty}|T_{ji}^{(3)}(t+s)|k_{ji}(s)\,\mathrm{d}s\right)\right]\right\}>0,$$
$$i=n+1,\ldots,n+m,$$

where $\zeta_{ij}^{-1}(t)$ *is the inverse function of* $\zeta_{ij}(t)=t-\tau_{ij}(t)$. *Then system* (5.5) *is global asymptotically stable. In particular*

(i) *if further assume that* $b_i(t), T_{ij}^1(t), T_{ij}^2(t), T_{ij}^3(t), I_i(t)$ *and* $\tau_{ij}(t)$, $i,j=1,\ldots,n$ *are periodic of period* ω, *then system* (5.5) *has a unique* ω*-periodic solution*, *which is global asymptotically stable*;

(ii) *if* $b_i(t)=b_i, T_{ij}^{(1)}(t)=T_{ij}^{(1)}, T_{ij}^{(2)}(t)=T_{ij}^{(2)}, T_{ij}^{(3)}(t)=T_{ij}^{(3)}, I_i(t)=I_i$ *and* $\tau_{ij}(t)=\tau_{ij}$ *are all constants*, *and there exist positive constants* $\mu_i>0$, $i=1,2,\ldots,n+m$, *such that*

$$\mu_i b_i-\sum_{j=n+1}^{n+m}\mu_j G_j(|T_{ji}^{(1)}|+|T_{ji}^{(2)}|+|T_{ji}^{(3)}|)>0,\quad i=1,\ldots,n,$$
$$\mu_i b_i-\sum_{j=1}^{n}\mu_j G_j(|T_{ji}^{(1)}|+|T_{ji}^{(2)}|+|T_{ji}^{(3)}|)>0,\quad i=n+1,\ldots,n+m$$

or

$$2\mu_i b_i-\sum_{j=n+1}^{n+m}[\mu_i G_i(|T_{ij}^{(1)}|+|T_{ij}^{(2)}|+|T_{ij}^{(3)}|)$$
$$+\mu_j G_j(|T_{ji}^{(1)}|+|T_{ji}^{(2)}|+|T_{ji}^{(3)}|)]>0,\quad i=1,\ldots,n,$$

$$2\mu_i b_i-\sum_{j=1}^{n}[\mu_i G_i(|T_{ij}^{(1)}|+|T_{ij}^{(2)}|+|T_{ij}^{(3)}|)$$
$$+\mu_j G_j(|T_{ji}^{(1)}|+|T_{ji}^{(2)}|+|T_{ji}^{(3)}|)]>0,\quad i=n+1,\ldots,n+m,$$

then system (5.5) *has a unique equilibrium*, *which is global asymptotically stable*.

Remark 5.3. Theorem 5.7 improves and generalizes the main results obtained in [15,22,40].

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