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The existence and global exponential stability of a periodic solution of a class of delay differential equations

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Abstract
By employing Schauder's fixed point theorem and a non-Liapunov method (matrix theory, inequality analysis), we obtain some new criteria that ensure existence and global exponential stability of a periodic solution to a class of functional differential equations. Applying these criteria to a cellular neural network with time delays (delayed cellular neural network, DCNN) under a periodic environment leads to some new results that improve and generalize many existing ones we know on this topic. These results are of great significance in designs and applications of globally stable periodic DCNNs.

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1. Introduction

For fixed $\tau > 0$ and $\omega > 0$, let

$$X = C([-\tau, 0], \boldsymbol{R}^n) \qquad \text{and} \qquad C_\omega = \{x \in C(\boldsymbol{R}, \boldsymbol{R}^n) : x(t+\omega) = x(t),\ \forall\, t \in \boldsymbol{R}\}.$$

Then X and C_ω are two Banach spaces with the supremum norms:

(i) for $x = (x_1, \ldots, x_n)^{\mathrm{T}} \in X$,

$$||x|| = \sum_{i=1}^{n} |x_i|_\infty, \qquad |x_i|_\infty = \max_{t\in[-\tau,0]} |x_i(t)|;$$

(ii) for $x = (x_1, \ldots, x_n)^{\mathrm{T}} \in C_\omega$,

$$||x|| = \sum_{i=1}^{n} |x_i|_\infty, \qquad |x_i|_\infty = \max_{t\in[0,\omega]} |x_i(t)|.$$

As is customary, for a function $x : \boldsymbol{R} \to \boldsymbol{R}^n$, let x_t denote the element in X defined by

$$x_t(\theta) = x(t+\theta) \qquad \text{for } \theta \in [-\tau, 0].$$

Consider the following system of functional differential equations:

$$x_i'(t) = -c_i(t)x_i(t) + f_i(t, x_t), \qquad i = 1, 2, \ldots, n, \tag{1.1}$$

where $x(t) = (x_1(t), \ldots, x_n(t))^{\mathrm{T}}$ gives the state variables, $c_i \in C(\boldsymbol{R}, \boldsymbol{R})$, $f_i \in C(\boldsymbol{R} \times X, \boldsymbol{R})$, $c_i(t+\omega) = c_i(t)$ and $f_i(t+\omega, \phi) = f_i(t, \phi)$ for $\phi \in X$ and $i = 1, 2, \ldots, n$. A more general system is the following one in vector form:

$$x'(t) = -A(t)x(t) + f(t, x_t), \tag{1.2}$$

where $A \in C(\boldsymbol{R}, \boldsymbol{R}^{n\times n})$, $f \in C(\boldsymbol{R} \times X, \boldsymbol{R}^n)$, $A(t+\omega) = A(t)$ and $f(t+\omega, \phi) = f(t, \phi)$ for $\phi \in X$. When $A = \mathrm{diag}(c_1(t), \ldots, c_n(t))$, (1.2) reduces to (1.1) if written component-wise. Initial conditions associated with system (1.1) (or (1.2)) are of the form

$$x_i(s) = \phi_i(s), \qquad s \in [-\tau, 0], \quad i = 1, 2, \ldots, n, \tag{1.3}$$

where $\phi = (\phi_1(t), \phi_2(t), \ldots, \phi_n(t))^{\mathrm{T}} \in X$.

The main motivation to consider (1.1) and (1.2) is from the study of cellular neural networks (CNNs). A CNN is formed by many units called cells. A cell may contain linear and nonlinear circuit elements, which typically are linear capacitors, linear resistors, linear and nonlinear controlled sources and independent sources. The circuit diagram and connection pattern modelling a CNN can be found in [2, 3]. Nowadays, CNNs are widely used in signal and image processing, associative memories and pattern classification (see, for instance, [3, 14, 18, 21, 23]. In the last decade or so, dynamic behaviours of CNNs have been intensively studied because of the successful hardware implementations for their applications in many real world problems. See, for example, [5, 7–11, 18–21, 23] for stability and periodicity analysis for CNNs.

As pointed out in [21], processing of moving images requires introduction of delays for signal transmission among the cells. Also, the delays in artificial neural networks are usually time varying and sometimes vary violently with time, due to the finite switching speed of amplifiers and faults in the electrical circuit. These justify a class of delayed cellular neural network (DCNN) model described by the following system

$$x_i'(t) = -c_i(t)x_i(t) + \sum_{j=1}^{n} a_{ij}(t)g_j(x_j(t)) + \sum_{j=1}^{n} b_{ij}(t)g_j(x_j(t-\tau_{ij}(t))) + I_i(t),$$
$$i = 1, \ldots, n. \tag{1.4}$$

Here n corresponds to the number of cells in the neural network; $x_i(t)$ denotes the potential (or voltage) of the ith cell at time t; $g_i(\cdot)$ denotes a nonlinear activation; $I_i(t)$ denotes the ith component of an external input source introduced from outside the network to the ith cell at time t ; $c_i(t)$ denotes the rate with which the ith cell resets its potential to the resting state when isolated from other cells and inputs at time t ; $a_{ij}(t)$ and $b_{ij}(t)$ denote the strengths of connectivity between the ith and jth cells at time t, respectively; $\tau_{ij}(t)$ corresponds to the time delay required in processing and transmitting a signal from the jth cell to the ith cell at time t. Model (1.4) includes many frequently used neural network models studied in the literature (see, for instance, [6, 11, 19] and references therein), but is obviously a special case of the more general system (1.1) and (1.2).

We point out that in most situations, the activation functions in neural networks are taken to be bounded, smooth and monotone functions (usually sigmoidal). However, in some applications, one often needs to use unbounded activation functions. For example, when a neural network is designed for solving optimization problems in the presence of constraints (linear, quadratic or more general programming problems), unbounded activations modelled by diode-like exponential type functions are needed to impose constraints' satisfaction (see, e.g., [8]). Yet, the extension of the aforementioned results to DCNNs with unbounded activation functions is not trivial at all. For example, in an autonomous network, when the activation functions are unbounded, existence of an equilibrium for the network becomes a problem (see, e.g., [8]), in contrast to the case with bounded activation functions where the existence of an equilibrium point is always guaranteed [7]. Another fact we would like to mention is that adoption of non-monotone and non-smooth activation functions may improve the performance of a network (see, e.g., [2, 14, 20] and the references therein).

Keeping in mind the above facts about neural networks, it is desirable *not* to assume the boundedness, smoothness and monotonicity of the functions $g_i(u)$ $(i = 1, 2, \ldots, n)$. Instead, the following weaker conditions seem to be more admirable:

(H1) $a_{ij}(t), b_{ij}(t), c_i(t), \tau_{ij}(t)$ and $I_i(t)(i, j = 1, 2, \ldots, n)$ are continuous ω-periodic functions on $\boldsymbol{R}$ with $\int_0^w c_i(s)\,\mathrm{d}s > 0$ for $i = 1, 2, \ldots, n$.

(H2) There exist constants l_j such that $|g_j(u) - g_j(v)| \leqslant l_j|u - v|$ for $u, v \in \boldsymbol{R}$, $j = 1, 2, \ldots, n$.

Back to (1.1) and/or (1.2), (H1)–(H2) on (1.4) would suggest corresponding weaker conditions for (1.1) and (1.2). A natural and important concern is the existence and stability of an ω-periodic solution. The purpose of this paper is to address this concern. More precisely, in section 2, by applying Schauder's fixed point theorem, we will derive a set of new sufficient conditions for the existence of an ω-periodic solution of (1.1); in section 3, we use a non-Lyapunov method (matrix theory and inequality analysis) to establish some criteria that guarantee the global exponential stability of the periodic solution of (1.1). In section 3, we will also derive some conditions for existence and stability of an ω-periodic solution of the more general system (1.2). Section 4 is dedicated to applications of the main results obtained in sections 2 and 3 to the delayed neural network system (1.4). Our results on (1.1) and (1.2), as well as on the neural networks (1.4), greatly improve and generalize many existing ones, and are of significance in designs and applications of neural networks.

2. Existence of a periodic solution

For the sake of convenience in later sections, we first introduce some notations and definitions needed in this paper. Throughout this paper, we always let E_n denote the identity matrix of size n and will adopt the following notations:

$$w^u = \max_{t\in[0,\omega]} w(t), \qquad w^l = \min_{t\in[0,\omega]} w(t), \qquad \bar{w} = \frac{1}{\omega}\int_0^\omega w(s)\,\mathrm{d}s$$

and

$$\Gamma(a,b) = (1 - \mathrm{e}^{-\bar{b}\omega})^{-1} \max_{t\in[0,\omega]} \int_0^\omega a(s+t)\exp\left(-\int_s^\omega b(u+t)\,\mathrm{d}u\right)\mathrm{d}s$$

for any ω-periodic functions $w(t)$, $a(t)$ and $b(t)$ on $\boldsymbol{R}$.

Let $A = (a_{ij})_{m\times n}$ and $B = (b_{ij})_{m\times n}$ be two matrices. We say $A \geqslant 0$ if $a_{ij} \geqslant 0$, $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$ and $A \geqslant B$ if $A - B \geqslant 0$.

For a matrix $B = (b_{ij})_{n\times n}$, we write $B \geqslant 0$ (>0, $\leqslant 0$, <0) if $b_{ij} \geqslant 0$ (>0, $\leqslant 0$, <0) for all $i, j = 1, 2, \ldots, n$.

Definition 2.1. *Let $x^*(t) = (x_1^*(t), x_2^*(t), \ldots, x_n^*(t))^{\mathrm{T}}$ be an ω-periodic solution of system (1.1) (or (1.2)) and (1.3) with initial value $\phi^*(t) = (\phi_1^*(t), \phi_2^*(t), \ldots, \phi_n^*(t))^{\mathrm{T}} \in X$. If there exist constants $\lambda > 0$ and $M \geqslant 1$ such that, for any solution $x(t)$ of (1.1) (or (1.2)) and (1.3),*

$$|x(t) - x^*(t)| \leqslant M||\phi - \phi^*||\mathrm{e}^{-\lambda t}, \qquad \forall t \geqslant 0,$$

then $x^*(t)$ is said to be globally exponentially stable.

Definition 2.2. *A real invertible $n \times n$ matrix $A = (a_{ij})_{n\times n}$ is said to be an M-matrix if $a_{ij} \leqslant 0, i, j = 1, 2, \ldots, n, i \neq j$ and $A^{-1} \geqslant 0$.*

The following three lemmas will be needed in the proofs of our results.

Lemma 2.1 ([4]). *Assume that S is a convex compact set in a Banach space and that $P : S \to S$ is continuous. Then P has a fixed point in S.*

Lemma 2.2 ([1, 15]). *Let $A = (a_{ij})_{n\times n}$ with $a_{ij} \leqslant 0, i, j = 1, 2, \ldots, n, i \neq j$. Then the following statements are equivalent:*

(I) A is an M-matrix;
(II) There exists a vector $\xi = (\xi_1, \xi_2, \ldots, \xi_n)^{\mathrm{T}} > 0$ such that $A\xi > 0$;
(III) $a_{ii} > 0$, $i = 1, 2, \ldots, n$ and there exists a diagonal matrix $D = \mathrm{diag}(d_1, d_2, \ldots, d_n)$ with $d_i > 0$, $i = 1, 2, \ldots, n$ such that AD is strictly diagonally dominant.

Lemma 2.3 ([1, 15]). *Let $A \geqslant 0$ be an $n \times n$ matrix and $\rho(A)$ be the spectral radius of A. If $\rho(A) < 1$, then $E_n - A$ is an M-matrix.*

Now we are in a position to state and prove the main result on the existence of a ω-periodic solution of (1.1).

Theorem 2.1. *Assume that the following conditions are satisfied:*

(D1) $\bar{c}_i = (1/\omega)\int_0^\omega c_i(s)\,\mathrm{d}s > 0$, $i = 1, 2, \ldots n$;
(D2) *There exist constants $M_j > 0$ and non-negative continuous ω-periodic functions $\alpha_{ij}(t)$, $\beta_j(t)$, $i, j = 1, 2, \ldots, n$ such that*

$$|f_i(t, \phi)| \leqslant \sum_{j=1}^{n} \alpha_{ij}(t) M_j + \beta_i(t) \qquad \text{for } \phi \in X \text{ with } |\phi_j|_\infty \leqslant M_j,\ j = 1, 2, \ldots, n,$$

and

$$(E_n - D)(M_1, M_2, \ldots, M_n)^{\mathrm{T}} > (\hat{\beta}_1, \hat{\beta}_2, \ldots, \hat{\beta}_n)^{\mathrm{T}},$$

where $D = (\Gamma(\alpha_{ij}, c_i))_{n\times n}$, $\hat{\beta}_i = \Gamma(\beta_i, c_i)$, $i = 1, 2, \ldots, n$.

Then system (1.1) has at least one ω-periodic solution.

Proof. Define the operator $P : C_\omega \to C_\omega$ as follows:

$$(Px)_i(t) = (1 - \mathrm{e}^{-\bar{c}_i\omega})^{-1} \int_0^\omega f_i(s + t, x_{s+t}) \exp\left(-\int_s^\omega c_i(u + t)\,\mathrm{d}u\right) \mathrm{d}s, \quad i = 1, 2, \ldots, n. \tag{2.1}$$

Then for all $t \in [0, \omega]$, $x \in C_\omega$ with $|x_i|_\infty \leqslant M_i,\ i = 1, 2, \ldots, n$, we have

$$
\begin{aligned}
|(Px)_i(t)| &\leqslant (1 - \mathrm{e}^{-\bar{c}_i\omega})^{-1} \int_0^\omega |f_i(s+t, x_{s+t})| \exp\left(-\int_s^\omega c_i(u+t)\,\mathrm{d}u\right) \mathrm{d}s \\
&\leqslant (1 - \mathrm{e}^{-\bar{c}_i\omega})^{-1} \left\{ \sum_{j=1}^n M_j \int_0^\omega \alpha_{ij}(s+t) \exp\left(-\int_s^\omega c_i(u+t)\,\mathrm{d}u\right) \mathrm{d}s \right. \\
&\quad \left. + \int_0^\omega \beta_i(s+t) \exp\left(-\int_s^\omega c_i(u+t)\,\mathrm{d}u\right) \mathrm{d}s \right\} \\
&\leqslant \sum_{j=1}^n \Gamma(\alpha_{ij}, c_i) M_j + \hat{\beta}_i \\
&< M_i, \qquad i = 1, 2, \ldots, n. \quad \text{(by D2)}
\end{aligned}
\tag{2.2}
$$

Let

$$
S = \{x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^{\mathrm{T}} \in C_\omega : \ |x_i|_\infty \leqslant M_i,\ i = 1, 2, \ldots, n\}. \tag{2.3}
$$

Then, we have shown that $PS \subseteq S$. Observe that

$$
(Px)_i(t) = (1 - \mathrm{e}^{-\bar{c}_i\omega})^{-1} \int_t^{t+\omega} f_i(s, x_s) \exp\left(-\int_s^{t+\omega} c_i(u)\,\mathrm{d}u\right) \mathrm{d}s, \qquad i = 1, 2, \ldots, n.
$$

Differentiating the above and making use of the ω-periodicity of $c_i(t)$, x_t and $f(t, \phi)$ in t, we obtain

$$
(Px)_i'(t) = -c_i(t)(Px)_i(t) + f_i(t, x_t), \qquad i = 1, 2, \ldots, n. \tag{2.4}
$$

From (2.2) and (2.4), we know that for any $(t, x) \in \boldsymbol{R} \times S$,

$$
\begin{aligned}
|(Px)_i'(t)| &\leqslant |c_i(t)||(Px)_i(t)| + |f_i(t, x_t)| \\
&\leqslant |c_i(t)|M_i + \sum_{j=1}^n \alpha_{ij}(t) M_j + \beta_i(t) \\
&\leqslant N_i, \qquad i = 1, 2, \ldots, n,
\end{aligned}
$$

where

$$
N_i = \max_{t \in [0,\omega]} \left\{ |c_i(t)|M_i + \sum_{j=1}^n \alpha_{ij}(t) M_j + \beta_i(t) \right\}, \qquad i = 1, 2, \ldots, n.
$$

Thus

$$
|(Px)_i'(t)| \leqslant N_i \qquad \text{for any } (t, x) \in \boldsymbol{R} \times S, \quad i = 1, 2, \ldots, n. \tag{2.5}
$$

Set

$$
\Omega = \{x(t) \in S : \ |x_i(t_1) - x_i(t_2)| \leqslant N_i |t_1 - t_2|,\ t_1, t_2 \in \boldsymbol{R},\ i = 1, 2, \ldots, n\}. \tag{2.6}
$$

It is easy to verify that Ω is a convex and compact set. Moreover, by (2.5) and the fact that $PS \subseteq S$, we have $P\Omega \subseteq \Omega$. In what follows, we show that $P : \Omega \to \Omega$ is continuous. Let $x^{(k)}, \hat{x} \in \Omega, k = 1, 2, \ldots$ with $||x^{(k)} - \hat{x}|| \to 0$ as $k \to \infty$. Set

$$
y^{(k)}(t) = (Px^{(k)})(t) - (P\hat{x})(t), \qquad k = 1, 2, \ldots.
$$

Then by (2.4), we have

$$
\frac{\mathrm{d}}{\mathrm{d}t} y_i^{(k)}(t) = -c_i(t) y_i^{(k)}(t) + f_i(t, x_t^{(k)}) - f_i(t, \hat{x}_t), \qquad i = 1, 2, \ldots, n. \tag{2.7}
$$

It follows that

$$
\begin{aligned}
|y_i^{(k)}(t)| &= \left|(1-\mathrm{e}^{-\bar{c}_i\omega})^{-1}\int_0^{\omega}[f_i(s+t,x_{s+t}^{(k)})-f_i(s+t,\hat{x}_{s+t})]\right. \\
&\qquad \left.\times \exp\left(-\int_s^{\omega} c_i(u+t)\,\mathrm{d}u\right)\mathrm{d}s\right| \\
&\leqslant (1-\mathrm{e}^{-\bar{c}_i\omega})^{-1}\int_0^{\omega}|f_i(s+t,x_{s+t}^{(k)})-f_i(s+t,\hat{x}_{s+t})| \\
&\qquad \times \exp\left(-\int_s^{\omega} c_i(u+t)\,\mathrm{d}u\right)\mathrm{d}s \\
&\leqslant (1-\mathrm{e}^{-\bar{c}_i\omega})^{-1}\max_{s\in[0,\omega]}|f_i(s,x_s^{(k)})-f_i(s,\hat{x}_s)|\int_0^{\omega}\exp\left(-\int_s^{\omega} c_i(u+t)\,\mathrm{d}u\right)\mathrm{d}s \\
&\leqslant \omega(1-\mathrm{e}^{-\bar{c}_i\omega})^{-1}\mathrm{e}^{\overline{|c_i|}\omega}\max_{s\in[0,\omega]}|f_i(s,x_s^{(k)})-f_i(s,\hat{x}_s)|, \qquad i=1,2,\ldots,n. \qquad (2.8)
\end{aligned}
$$

Note that $[0,\omega]\times\Omega\subset \boldsymbol{R}\times C_\omega$ is a compact set. It follows that $f_i(t,\phi)$ is uniformly continuous in $[0,\omega]\times\Omega$, and so

$$
\max_{t\in[0,\omega]}|f_i(t,x_t^{(k)})-f_i(t,\hat{x}_t)|\to 0 \qquad \text{as } k\to\infty, \quad i=1,2,\ldots,n. \qquad (2.9)
$$

Combining (2.8) and (2.9), we have

$$
\max_{t\in[0,\omega]}|y_i^{(k)}(t)|\to 0, \quad \text{as } k\to\infty,\ i=1,2,\ldots,n
$$

or

$$
\|Px^{(k)}-P\hat{x}\|\to 0, \qquad \text{as } k\to\infty.
$$

Hence, $P:\Omega\to\Omega$ is continuous.

The above verifies the conditions of lemma 2.1 for P, concluding that P has a fixed point $x^*=x^*(t)\in\Omega$. It is easy to show that $x^*(t)$ is a periodic solution of equation (1.1), completing the proof of the theorem.

Corollary 2.1. *Assume that (D1) and the following conditions are satisfied:*

(D3) *There exist non-negative continuous ω- periodic functions $\alpha_{ij}(t)$ and $\beta_j(t)$, $i,j=1,2,\ldots,n$ such that for any $\phi=(\phi_1,\phi_2,\ldots,\phi_n)^{\mathrm{T}}\in X$,*

$$
|f_i(t,\phi)|\leqslant\sum_{j=1}^{n}\alpha_{ij}(t)|\phi_j|_\infty+\beta_i(t), \qquad i=1,2,\ldots,n;
$$

(D4) *E_n-D is an M-matrix, where $D=(\Gamma(\alpha_{ij},c_i))_{n\times n}$.*

Then system (1.1) has at least one ω-periodic solution.

Proof. By (D4) and lemma 2.2, there exists a vector $\xi=(\xi_1,\xi_2,\ldots,\xi_n)^{\mathrm{T}}>0$ such that

$$
\eta=(\eta_1,\eta_2,\ldots,\eta_n)^{\mathrm{T}}=(E_n-D)\xi>0.
$$

Let $\hat{\beta}_i=\Gamma(\beta_i,c_i),\ i=1,2,\ldots,n$. Choose $\gamma>0$ such that $\gamma\eta_i>\hat{\beta}_i,\ i=1,2,\ldots,n$ and set $M_i=\gamma\xi_i,\ i=1,2,\ldots,n$. Then

$$
(E_n-D)(M_1,M_2,\ldots,M_n)^{\mathrm{T}}>(\hat{\beta}_1,\hat{\beta}_2,\ldots,\hat{\beta}_n)^{\mathrm{T}},
$$

and it follows from (D3) that

$$|f_i(t,\phi)| \leqslant \sum_{j=1}^{n} \alpha_{ij}(t)M_j + \beta_i(t) \qquad \text{for } \phi \in X \text{ with } |\phi_j|_\infty \leqslant M_j, \ \ j = 1, 2, \ldots, n.$$

In view of theorem 2.1, system (1.1) has at least one ω-periodic solution. The proof is complete.

Remark 2.1. Corollary 2.1 shows that conditions (D3) and (D4) imply (D2). But (D2) and (D4) cannot lead to (D3). For example, let $\alpha_{ij}(t)$ and $\beta_j(t), i, j = 1, 2, \ldots, n$ be non-negative continuous ω-periodic functions. Set $D = (\Gamma(\alpha_{ij}, c_i))_{n\times n}$ and $\hat{\beta}_i = \Gamma(\beta_i, c_i),\ i = 1, 2, \ldots, n$. By (D4) and lemma 2.2, there exists a vector $\xi = (\xi_1, \xi_2, \ldots, \xi_n)^{\mathrm{T}} > 0$ such that

$$\eta = (\eta_1, \eta_2, \ldots, \eta_n)^{\mathrm{T}} = (E_n - D)\xi > 0.$$

Choose $\gamma > 0$ such that $\gamma\eta_j > \hat{\beta}_j,\ j = 1, 2, \ldots, n$, and set $M_j = \gamma\xi_j,\ j = 1, 2, \ldots, n$ and

$$f_i(t,\phi) = \sum_{j=1}^{n} \alpha_{ij}(t)[\phi_j(-\tau)]^2/M_j + \beta_i(t), \qquad i = 1, 2, \ldots, n,$$

where $\tau > 0$. Then

$$|f_i(t,\phi)| \leqslant \sum_{j=1}^{n} \alpha_{ij}(t)M_j + \beta_i(t) \qquad \text{for } \phi \in X \text{ with } |\phi_j|_\infty \leqslant M_j, \ \ j = 1, 2, \ldots, n$$

and

$$(E_n - D)(M_1, M_2, \ldots, M_n)^{\mathrm{T}} > (\hat{\beta}_1, \hat{\beta}_2, \ldots, \hat{\beta}_n)^{\mathrm{T}}.$$

This shows condition (D2) is satisfied, but condition (D3) does not hold.

3. Uniqueness and exponential stability

In this section, we explore the uniqueness and stability of the ω-periodic solution of (1.1). For this purpose, we need the following condition related to (D3):

(D3') *There exist non-negative continuous ω-periodic functions $\alpha_{ij}(t),\ i, j = 1, 2, \ldots, n$ such that for any $\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_n)^{\mathrm{T}}, \phi = (\phi_1, \phi_2, \ldots, \phi_n)^{\mathrm{T}} \in X$*

$$|f_i(t,\varphi) - f_i(t,\phi)| \leqslant \sum_{j=1}^{n} \alpha_{ij}(t)|\varphi_j - \phi_j|_\infty, \qquad i = 1, 2, \ldots, n.$$

Theorem 3.1. *Assume that (D1), (D3') and (D4) are satisfied. Then system (1.1) has exactly one ω-periodic solution. Moreover, it is globally exponentially stable.*

Proof. By (D3'), we have for any $\phi = (\phi_1, \phi_2, \ldots, \phi_n)^{\mathrm{T}} \in X$

$$|f_i(t,\phi)| \leqslant \sum_{j=1}^{n} \alpha_{ij}(t)|\phi_j|_\infty + |f_i(t,0)|, \qquad i = 1, 2, \ldots, n.$$

This shows that (D3) holds. In view of corollary 2.1, system (1.1) has at least one ω-periodic solution, say $x^*(t) = (x_1^*(t), x_2^*(t), \ldots, x_n^*(t))^{\mathrm{T}}$ with initial value $\phi^*(t) = (\phi_1^*(t), \phi_2^*(t), \ldots, \phi_n^*(t))^{\mathrm{T}} \in X$. Let $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^{\mathrm{T}}$ be an arbitrary solution of system (1.1) and (1.3) with initial value $\phi(t) = (\phi_1(t), \phi_2(t), \ldots, \phi_n(t))^{\mathrm{T}} \in X$. Set

$$y_i(t) = |x_i(t) - x_i^*(t)|, \qquad i = 1, 2, \ldots, n. \tag{3.1}$$

Then, from (1.1) and (D3'), we have

$$\begin{aligned}
D^- y_i(t) &= \limsup_{h\to 0^-} \frac{y_i(t+h) - y_i(t)}{h} \\
&= \limsup_{h\to 0^-} \frac{|x_i(t+h) - x_i^*(t+h)| - |x_i(t) - x_i^*(t)|}{h} \\
&\leqslant \operatorname{sign}(x_i(t) - x_i^*(t))(x_i(t) - x_i^*(t))' \\
&\leqslant -c_i(t)|x_i(t) - x_i^*(t)| + |f_i(t, x_t) - f_i(t, x_t^*)| \\
&\leqslant -c_i(t)|x_i(t) - x_i^*(t)| + \sum_{j=1}^{n} \alpha_{ij}(t) \max_{t-\tau\leqslant s\leqslant t} |x_j(s) - x_j^*(s)| \\
&= -c_i(t) y_i(t) + \sum_{j=1}^{n} \alpha_{ij}(t) \max_{t-\tau\leqslant s\leqslant t} y_j(s), \qquad i = 1, 2, \ldots, n.
\end{aligned}$$

That is

$$D^- y_i(t) \leqslant -c_i(t) y_i(t) + \sum_{j=1}^{n} \alpha_{ij}(t) \max_{t-\tau\leqslant s\leqslant t} y_j(s), \qquad i = 1, 2, \ldots, n. \tag{3.2}$$

It follows that

$$\begin{aligned}
y_i(t) \leqslant y_i(0) \exp\left(-\int_0^{\mathrm{t}} c_i(u)\,\mathrm{d}u\right) \mathrm{d}s \\
+ \sum_{j=1}^{n} \int_0^{\mathrm{t}} \alpha_{ij}(s) \max_{s-\tau\leqslant v\leqslant s} y_j(v) \exp\left(-\int_s^{\mathrm{t}} c_i(u)\,\mathrm{d}u\right) \mathrm{d}s, \\
t \geqslant 0, \quad i = 1, 2, \ldots, n.
\end{aligned} \tag{3.3}$$

Let $t_i^* \in [0, \omega]$ such that

$$y_i(t_i^*) = \max_{t\in[0,\omega]} y_i(t), \qquad i = 1, 2, \ldots, n.$$

Then it follows from (3.3) that

$$\begin{aligned}
y_i(t_i^*) &\leqslant y_i(0) \exp\left(-\int_0^{t_i^*} c_i(u)\,\mathrm{d}u\right) \mathrm{d}s \\
&\quad + \sum_{j=1}^{n} \int_0^{t_i^*} \alpha_{ij}(s) \max_{s-\tau\leqslant v\leqslant s} y_j(v) \exp\left(-\int_s^{t_i^*} c_i(u)\,\mathrm{d}u\right) \mathrm{d}s \\
&\leqslant y_i(0) \mathrm{e}^{\overline{|c_i|}\omega} + \sum_{j=1}^{n} \max_{-\tau\leqslant s\leqslant \omega} y_j(s) \int_0^{t_i^*} \alpha_{ij}(s) \exp\left(-\int_s^{t_i^*} c_i(u)\,\mathrm{d}u\right) \mathrm{d}s \\
&\leqslant y_i(0) \mathrm{e}^{\overline{|c_i|}\omega} + \sum_{j=1}^{n} \max_{-\tau\leqslant s\leqslant \omega} y_j(s) \int_{t_i^*-\omega}^{t_i^*} \alpha_{ij}(s) \exp\left(-\int_s^{t_i^*} c_i(u)\,\mathrm{d}u\right) \mathrm{d}s \\
&\leqslant \kappa|\phi_i - \phi_i^*|_\infty + \sum_{j=1}^{n} \max_{-\tau\leqslant s\leqslant \omega} y_j(s) \int_0^{\omega} \alpha_{ij}(s + t_i^*) \exp\left(-\int_s^{\omega} c_i(u + t_i^*)\,\mathrm{d}u\right) \mathrm{d}s \\
&\leqslant \kappa|\phi_i - \phi_i^*|_\infty + \sum_{j=1}^{n} \Gamma(\alpha_{ij}, c_i) \max_{-\tau\leqslant s\leqslant \omega} y_j(s) \\
&\leqslant \kappa|\phi_i - \phi_i^*|_\infty + \sum_{j=1}^{n} \Gamma(\alpha_{ij}, c_i)[|\phi_j - \phi_j^*|_\infty + y_j(t_j^*)], \qquad i = 1, 2, \ldots, n,
\end{aligned}$$

where

$$\kappa = \max\{\mathrm{e}^{\overline{|c_1|}\omega}, \mathrm{e}^{\overline{|c_2|}\omega}, \ldots, \mathrm{e}^{\overline{|c_n|}\omega}\}.$$

Thus,

$$(E_n - D)(y_1(t_1^*), \ldots, y_n(t_n^*))^{\mathrm{T}} \leqslant (\kappa E_n + D)(|\phi_1 - \phi_1^*|_\infty, \ldots, |\phi_n - \phi_n^*|_\infty)^{\mathrm{T}},$$

and so

$$(y_1(t_1^*), \ldots, y_n(t_n^*))^{\mathrm{T}} \leqslant (E_n - D)^{-1}(\kappa E_n + D)(|\phi_1 - \phi_1^*|_\infty, \ldots, |\phi_n - \phi_n^*|_\infty)^{\mathrm{T}}. \tag{3.4}$$

(3.4) shows that there exists a constant $A > 1$ independent of ϕ and ϕ^* such that

$$\max_{-\tau \leqslant t \leqslant \omega} y_i(t) \leqslant A\|\phi - \phi^*\|, \qquad i = 1, 2, \ldots, n. \tag{3.5}$$

Since $E_n - D$ is an M-matrix, it follows from lemma 2.2 (III) that there exist $m_i > 0$, $i = 1, 2, \ldots, n$ such that

$$m_i > \sum_{j=1}^{n} \Gamma(\alpha_{ij}, c_i) m_j, \qquad i = 1, 2, \ldots, n. \tag{3.6}$$

Then, there exist constants $\sigma > 0$ and λ $(0 < \lambda < \min\{\bar{c}_i : i = 1, 2, \ldots, n\})$ such that

$$-m_i + \mathrm{e}^{\lambda(\tau+\omega)}(1 - \mathrm{e}^{-\omega\bar{c}_i})(1 - \mathrm{e}^{-\omega(\bar{c}_i - \lambda)})^{-1} \sum_{j=1}^{n} \Gamma(\alpha_{ij}, c_i) m_j < -\sigma, \qquad i = 1, 2, \ldots, n. \tag{3.7}$$

Set $z_i(t) = y_i(t)\mathrm{e}^{\lambda t}$, $i = 1, 2, \ldots, n$. Then from (3.3), we have

$$\begin{aligned} z_i(t) \leqslant{}& z_i(0) \exp\left(-\int_0^t (c_i(u) - \lambda)\, \mathrm{d}u\right) \\ &+ \mathrm{e}^{\lambda\tau} \sum_{j=1}^{n} \int_0^t \alpha_{ij}(s) \max_{s-\tau \leqslant v \leqslant s} z_j(v) \exp\left(-\int_s^t (c_i(u) - \lambda)\, \mathrm{d}u\right) \mathrm{d}s, \\ &t \geqslant 0,\ i = 1, 2, \ldots, n. \end{aligned} \tag{3.8}$$

Set

$$w(t) = \max\{m_i^{-1} z_i(t) : i = 1, 2, \ldots, n\}. \tag{3.9}$$

Choose $t_k \in [-\tau, k]$ such that

$$w(t_k) = \max\{w(t) : -\tau \leqslant t \leqslant k\}. \tag{3.10}$$

Then $t_1 \leqslant t_2 \leqslant t_3 \leqslant \cdots$. For $k = 1, 2, \ldots$, let r_k be the integer such that $r_k\omega \leqslant t_k < (r_k + 1)\omega$. Then by (3.5), (3.7)–(3.10), we have

$$\begin{aligned} z_i(t_k) \leqslant{}& z_i(0) \exp\left(-\int_0^{t_k} (c_i(u) - \lambda)\, \mathrm{d}u\right) \\ &+ \mathrm{e}^{\lambda\tau} \sum_{j=1}^{n} \int_0^{t_k} \alpha_{ij}(s) \max_{s-\tau \leqslant v \leqslant s} z_j(v) \exp\left(-\int_s^{t_k} (c_i(u) - \lambda)\, \mathrm{d}u\right) \mathrm{d}s \\ ={}& z_i(0)\mathrm{e}^{\lambda t_k} \exp\left(-\int_0^{t_k} c_i(u)\, \mathrm{d}u\right) \\ &+ \mathrm{e}^{\lambda\tau} \sum_{j=1}^{n} \int_0^{t_k - r_k\omega} \alpha_{ij}(s) \max_{s-\tau \leqslant v \leqslant s} z_j(v) \exp\left(-\int_s^{t_k} (c_i(u) - \lambda)\, \mathrm{d}u\right) \mathrm{d}s \end{aligned}$$

$$
\begin{aligned}
&+\ \mathrm{e}^{\lambda\tau}\sum_{j=1}^{n}\int_{t_k-r_k\omega}^{t_k}\alpha_{ij}(s)\max_{s-\tau\leqslant v\leqslant s}z_j(v)\exp\left(-\int_s^{t_k}(c_i(u)-\lambda)\,\mathrm{d}u\right)\mathrm{d}s\\
&\leqslant z_i(0)\mathrm{e}^{\lambda t_k}\exp\left(-\int_{t_k-r_k\omega}^{t_k}c_i(u)\,\mathrm{d}u+\int_0^{\omega}|c_i(u)|\,\mathrm{d}u\right)\\
&+\ \mathrm{e}^{\lambda(\tau+t_k)}\exp\left(-\int_{t_k-r_k\omega}^{t_k}c_i(u)\,\mathrm{d}u+\int_0^{\omega}|c_i(u)|\,\mathrm{d}u\right)\sum_{j=1}^{n}\max_{-\tau\leqslant s\leqslant\omega}z_j(s)\int_0^{\omega}\alpha_{ij}(s)\,\mathrm{d}s\\
&+\ \mathrm{e}^{\lambda\tau}w(t_k)\sum_{j=1}^{n}m_j\int_{t_k-r_k\omega}^{t_k}\alpha_{ij}(s)\exp\left(-\int_s^{t_k}(c_i(u)-\lambda)\,\mathrm{d}u\right)\mathrm{d}s\\
&=\mathrm{e}^{\lambda(t_k-r_k\omega)-r_k\omega(\bar{c}_i-\lambda)+\omega\overline{|c_i|}}\left[z_i(0)+\mathrm{e}^{\lambda\tau}\omega\sum_{j=1}^{n}\overline{\alpha}_{ij}\max_{-\tau\leqslant s\leqslant\omega}z_j(s)\right]\\
&+\ \mathrm{e}^{\lambda\tau}w(t_k)\sum_{j=1}^{n}m_j\int_{t_k-r_k\omega}^{t_k}\alpha_{ij}(s)\exp\left(-\int_s^{t_k}(c_i(u)-\lambda)\,\mathrm{d}u\right)\mathrm{d}s\\
&\leqslant \mathrm{e}^{(\lambda+\overline{|c_i|})\omega}\left[1+A\mathrm{e}^{\lambda(\tau+\omega)}\omega\sum_{j=1}^{n}\bar{\alpha}_{ij}\right]\|\phi-\phi^*\|\\
&+\ \mathrm{e}^{\lambda\tau}w(t_k)\sum_{j=1}^{n}m_j\sum_{v=1}^{r_k}\int_{t_k-v\omega}^{t_k-(v-1)\omega}\alpha_{ij}(s)\exp\left(-\int_s^{t_k}(c_i(u)-\lambda)\,\mathrm{d}u\right)\mathrm{d}s\\
&=\mathrm{e}^{(\lambda+\overline{|c_i|})\omega}\left[1+A\mathrm{e}^{\lambda(\tau+\omega)}\omega\sum_{j=1}^{n}\bar{\alpha}_{ij}\right]\|\phi-\phi^*\|\\
&+\ \mathrm{e}^{\lambda\tau}w(t_k)\sum_{j=1}^{n}m_j\sum_{v=1}^{r_k}\exp\left(-\int_{t_k-(v-1)\omega}^{t_k}(c_i(u)-\lambda)\,\mathrm{d}u\right)\\
&\times\int_{t_k-v\omega}^{t_k-(v-1)\omega}\alpha_{ij}(s)\exp\left(-\int_s^{t_k-(v-1)\omega}(c_i(u)-\lambda)\,\mathrm{d}u\right)\mathrm{d}s\\
&=R_i\|\phi-\phi^*\|+\mathrm{e}^{\lambda\tau}w(t_k)\sum_{j=1}^{n}m_j\sum_{v=1}^{r_k}\mathrm{e}^{-(v-1)\omega(\bar{c}_i-\lambda)}\\
&\times\int_0^{\omega}\alpha_{ij}(s+t_k)\exp\left(-\int_s^{\omega}(c_i(u+t_k)-\lambda)\,\mathrm{d}u\right)\mathrm{d}s,
\end{aligned}
$$

where

$$
R_i=\left[1+A\mathrm{e}^{\lambda(\tau+\omega)}\omega\sum_{j=1}^{n}\bar{\alpha}_{ij}\right]\mathrm{e}^{(\lambda+\overline{|c_i|})\omega},\qquad i=1,2,\ldots,n.
$$

The second term in the above estimate can be further estimated as

$$
\begin{aligned}
&\mathrm{e}^{\lambda\tau} w(t_k) \sum_{j=1}^{n} m_j \sum_{\nu=1}^{r_k} \mathrm{e}^{-(\nu-1)\omega(\bar{c}_i-\lambda)} \int_0^{\omega} \alpha_{ij}(s+t_k) \exp\left(-\int_s^{\omega} (c_i(u+t_k)-\lambda)\,\mathrm{d}u\right) \mathrm{d}s \\
&\qquad \leqslant \mathrm{e}^{\lambda(\tau+\omega)} w(t_k) \sum_{j=1}^{n} m_j \sum_{\nu=1}^{r_k} \mathrm{e}^{-(\nu-1)\omega(\bar{c}_i-\lambda)} \int_0^{\omega} \alpha_{ij}(s+t_k) \exp\left(-\int_s^{\omega} c_i(u+t_k)\,\mathrm{d}u\right) \mathrm{d}s \\
&\qquad \leqslant \mathrm{e}^{\lambda(\tau+\omega)} (1-\mathrm{e}^{-\omega\bar{c}_i}) w(t_k) \sum_{j=1}^{n} m_j \Gamma(\alpha_{ij}, c_i) \sum_{\nu=1}^{r_k} \mathrm{e}^{-(\nu-1)\omega(\bar{c}_i-\lambda)} \\
&\qquad \leqslant \mathrm{e}^{\lambda(\tau+\omega)} (1-\mathrm{e}^{-\omega\bar{c}_i})(1-\mathrm{e}^{-\omega(\bar{c}_i-\lambda)})^{-1} w(t_k) \sum_{j=1}^{n} \Gamma(\alpha_{ij}, c_i) m_j \\
&\qquad \leqslant (m_i-\sigma) w(t_k), \qquad i = 1, 2, \ldots, n,
\end{aligned}
$$

Here, we have used (3.7) and the fact that

$$\sum_{\nu=1}^{r_k} \mathrm{e}^{-(\nu-1)\omega(\bar{c}_i-\lambda)} = \sum_{\nu=0}^{r_k-1} [\mathrm{e}^{-\omega(\bar{c}_i-\lambda)}]^{\nu} \leqslant \sum_{\nu=0}^{\infty} [\mathrm{e}^{-\omega(\bar{c}_i-\lambda)}]^{\nu} = (1-\mathrm{e}^{-\omega(\bar{c}_i-\lambda)})^{-1}.$$

It follows that

$$m_i^{-1} z_i(t_k) \leqslant m_i^{-1} R_i \|\phi-\phi^*\| + (1-m_i^{-1}\sigma) w(t_k), \qquad i = 1, 2, \ldots, n. \tag{3.11}$$

Set

$$m = \max\{m_i : i = 1, 2, \ldots, n\}, \qquad R = \max\{m_i^{-1} R_i : i = 1, 2, \ldots, n\}.$$

From (3.9), (3.10) and (3.11), we have

$$w(t_k) \leqslant (1-m^{-1}\sigma) w(t_k) + R\|\phi-\phi^*\|, \qquad k = 1, 2, \ldots,$$

and so

$$w(t_k) \leqslant m\sigma^{-1} R \|\phi-\phi^*\|, \qquad k = 1, 2, \ldots. \tag{3.12}$$

Set $M = m^2\sigma^{-1}R$. From (3.9), (3.10) and (3.12), we have

$$z_i(t) \leqslant M\|\phi-\phi^*\|, \qquad t \geqslant 0. \tag{3.13}$$

It follows that

$$|x_i(t) - x_i^*(t)| = |y_i(t)| \leqslant M\|\phi-\phi^*\| \mathrm{e}^{-\lambda t}, \qquad t \geqslant 0,\ i = 1, 2, \ldots, n. \tag{3.14}$$

Thus, if $x(t)$ is also ω-periodic, then we must have $x(t) \equiv x^*(t)$. Furthermore, (3.14) also implies that $x^*(t)$ is globally exponentially stable. This completes the proof.

Combining lemma 2.3 and theorem 3.1, we have the following corollaries.

Corollary 3.1. *Assume that (D1) and (D3') hold, and that* $\rho(D) < 1$, *where* $D = (\Gamma(\alpha_{ij}, c_i))_{n\times n}$. *Then system (1.1) has exactly one* ω*-periodic solution. Moreover, it is globally exponentially stable.*

Note that if $c_i^l \geqslant 0,\ i = 1, 2, \ldots, n$, then

$$\begin{aligned}
\Gamma(\alpha_{ij}, c_i) &= (1 - \mathrm{e}^{-\bar{c}_i\omega})^{-1} \max_{t\in[0,\omega]} \int_0^\omega \alpha_{ij}(s+t) \exp\left(-\int_s^\omega c_i(u+t)\,\mathrm{d}u\right) \mathrm{d}s \\
&\leqslant (1 - \mathrm{e}^{-\bar{c}_i\omega})^{-1} \max_{t\in[0,\omega]} \int_0^\omega \alpha_{ij}(s+t)\,\mathrm{d}s \\
&= (1 - \mathrm{e}^{-\bar{c}_i\omega})^{-1} \int_0^\omega \alpha_{ij}(s)\,\mathrm{d}s \\
&= (1 - \mathrm{e}^{-\bar{c}_i\omega})^{-1} \omega\bar{\alpha}_{ij}, \qquad i, j = 1, 2, \ldots, n.
\end{aligned} \tag{3.15}$$

Also if $c_i^l > 0,\ i = 1, 2, \ldots, n$, then by the ω-periodicity of $c_i(t)$, we have

$$\begin{aligned}
\Gamma(\alpha_{ij}, c_i) &= (1 - \mathrm{e}^{-\bar{c}_i\omega})^{-1} \max_{t\in[0,\omega]} \int_0^\omega \alpha_{ij}(s+t) \exp\left(-\int_s^\omega c_i(u+t)\,\mathrm{d}u\right) \mathrm{d}s \\
&\leqslant (1 - \mathrm{e}^{-\bar{c}_i\omega})^{-1} \left(\frac{\alpha_{ij}}{c_i}\right)^u \max_{t\in[0,\omega]} \int_0^\omega c_i(s+t) \exp\left(-\int_s^\omega c_i(u+t)\,\mathrm{d}u\right) \mathrm{d}s \\
&= (1 - \mathrm{e}^{-\bar{c}_i\omega})^{-1} \left(\frac{\alpha_{ij}}{c_i}\right)^u \max_{t\in[0,\omega]} \left[1 - \exp\left(-\int_0^\omega c_i(u+t)\,\mathrm{d}u\right)\right] \\
&= (1 - \mathrm{e}^{-\bar{c}_i\omega})^{-1} \left(\frac{\alpha_{ij}}{c_i}\right)^u \max_{t\in[0,\omega]} \left[1 - \exp\left(-\int_0^\omega c_i(u)\,\mathrm{d}u\right)\right] \\
&= (1 - \mathrm{e}^{-\bar{c}_i\omega})^{-1} \left(\frac{\alpha_{ij}}{c_i}\right)^u (1 - \mathrm{e}^{-\bar{c}_i\omega}) \\
&= \left(\frac{\alpha_{ij}}{c_i}\right)^u, \qquad i, j = 1, 2, \ldots, n.
\end{aligned} \tag{3.16}$$

Let

$$U = ((1 - \mathrm{e}^{-\bar{c}_i\omega})^{-1}\omega\bar{\alpha}_{ij})_{n\times n}, \qquad V = \left(\left(\frac{\alpha_{ij}}{c_i}\right)^u\right)_{n\times n}. \tag{3.17}$$

Then $0 \leqslant D \leqslant U$ and $0 \leqslant D \leqslant V$. In view of Ky Fan theorem in [4], we have

$$\rho(D) \leqslant \rho(U), \qquad \rho(D) \leqslant \rho(V). \tag{3.18}$$

This and corollary 3.1 lead to the following two corollaries.

Corollary 3.2. *Assume that (D1) and (D3') hold and $c_i(t) \geqslant 0, i = 1, 2, \ldots, n$, and that $\rho(U) < 1$. Then system (1.1) has exactly one ω-periodic solution. Moreover, it is globally exponentially stable.*

Corollary 3.3. *Assume that (D1) and (D3') hold and $c_i(t) > 0,\ i = 1, 2, \ldots, n$, and that $\rho(V) < 1$. Then system (1.1) has exactly one ω-periodic solution. Moreover, it is globally exponentially stable.*

In the rest of this section, we deal with the existence and global exponential stability of periodic solutions for equation (1.2), and obtain some results of the same nature as those established for (1.1). To this end, we need to use the notion of matrix measure for an $n \times n$ real matrix A, denoted by $\mu(A)$, which is defined by

$$\mu(A) = \lim_{\theta\to 0} \frac{|E_n + \theta A| - 1}{\theta}.$$

Here for an $n \times n$ real matrix, $|A|$ denotes the matrix norm induced by a vector norm $|x| = \sum_{i=1}^n |x_i|$. Thus, both $|A|$ and $\mu(A)$ indeed depend on which vector norm is adopted.

We first quote an existence result from theorem 2.1 in [22].

Lemma 3.1 ([22]). *Assume that there exists $M > 0$ such that for any $t \in [0, \omega]$*

$$\frac{1}{M}\int_0^{\omega} \exp\left(-\int_s^{\omega} \mu(A(t+u))\,\mathrm{d}u\right) \max_{x\in C_{\omega}(M)} |f(t+s, x_{t+s})|\,\mathrm{d}s \leqslant 1 - \exp\left(-\int_0^{\omega} \mu(A(s))\,\mathrm{d}s\right), \tag{3.19}$$

where $C_{\omega}(M) = \{x \in C_{\omega} : \max_{s\in[0,\omega]} |x(s)| \leqslant M\}$. Then system (1.2) has at least one ω-periodic solution.

Theorem 3.2. *Assume that there exist a non-negative continuous ω-periodic function $L(t)$ such that for any $\phi, \psi \in X$*

$$|f(t, \phi) - f(t, \psi)| \leqslant L(t) \max_{s\in[-\tau,0]} |\phi(s) - \psi(s)| \tag{3.20}$$

and for any $t \in [0, \omega]$

$$\int_0^{\omega} L(s+t) \exp\left(-\int_s^{\omega} \mu(A(u+t))\,\mathrm{d}u\right) \mathrm{d}s < 1 - \exp\left(-\int_0^{\omega} \mu(A(s))\,\mathrm{d}s\right). \tag{3.21}$$

Then system (1.2) has exactly one ω-periodic solution. Moreover, it is globally exponentially stable.

Proof. It follows from (3.21) that there exists $\gamma > 1$ such that

$$\int_0^{\omega} L(s+t) \exp\left(-\int_s^{\omega} \mu(A(u+t))\,\mathrm{d}u\right) \mathrm{d}s < \frac{1}{\gamma} - \exp\left(-\int_0^{\omega} \mu(A(s))\,\mathrm{d}s\right) \quad \text{for } t \in [0, \omega]. \tag{3.22}$$

Choose $M > 0$ sufficiently large such that

$$\frac{1}{M}\int_0^{\omega} |f(t+s, 0)| \exp\left(-\int_s^{\omega} \mu(A(t+u))\,\mathrm{d}u\right) \mathrm{d}s < \frac{\gamma - 1}{\gamma}. \tag{3.23}$$

By (3.20), for any $x \in C_{\omega}(M)$, we have

$$|f(t, x_t)| \leqslant |f(t, 0)| + L(t) \max_{s\in[0,\tau]} |x(t-s)| \leqslant ML(t) + |f(t, 0)| \qquad \text{for } t \in [0, \omega]. \tag{3.24}$$

From (3.22), (3.23) and (3.24), we have

$$\begin{aligned}
&\frac{1}{M}\int_0^{\omega} \exp\left(-\int_s^{\omega} \mu(A(t+u))\,\mathrm{d}u\right) \max_{x\in C_{\omega}(M)} |f(t+s, x_{t+s})|\,\mathrm{d}s \\
&\quad\leqslant \frac{1}{M}\int_0^{\omega} (ML(t+s) + |f(t+s, 0)|) \exp\left(-\int_s^{\omega} \mu(A(t+u))\,\mathrm{d}u\right) \mathrm{d}s \\
&\quad= \int_0^{\omega} L(t+s) \exp\left(-\int_s^{\omega} \mu(A(t+u))\,\mathrm{d}u\right) \mathrm{d}s \\
&\qquad + \frac{1}{M}\int_0^{\omega} |f(t+s, 0)| \exp\left(-\int_s^{\omega} \mu(A(t+u))\,\mathrm{d}u\right) \mathrm{d}s \\
&\quad\leqslant 1 - \exp\left(-\int_0^{\omega} \mu(A(s))\,\mathrm{d}s\right).
\end{aligned}$$

This shows that (3.19) holds, in view of lemma 3.1, system (1.2) has at least one ω-periodic solution, say $x^*(t)$ with initial value $\phi^* \in X$. Let $x(t)$ be an arbitrary solution of system (1.2) and (1.3) with initial value $\phi \in X$. Set $y(t) = x(t) - x^*(t)$. Then by (1.2), we have

$$y'(t) = -A(t)y(t) + f(t, x_t) - f(t, x_t^*). \tag{3.25}$$

Let $X(t, t_0)$ be the fundamental matrix solution of the following system:

$$x'(t) = -A(t)x(t) \tag{3.26}$$

satisfying that $X(t_0, t_0) = E_n$. Then by [22, lemma 2.3], we have

$$|X(t, t_0)| \leqslant \exp\left(-\int_{t_0}^{t} \mu(A(s))\, \mathrm{d}s\right), \qquad t \geqslant t_0. \tag{3.27}$$

It follows from (3.25) that

$$y(t) = X(t, 0)y(0) + \int_0^t X(t, s)[f(s, x_s) - f(s, x_s^*)]\, \mathrm{d}s, \qquad t \geqslant 0. \tag{3.28}$$

From (3.20), (3.27) and (3.28), we have

$$\begin{aligned} |y(t)| &\leqslant |X(t, 0)||y(0)| + \int_0^t |X(t, s)||f(s, x_s) - f(s, x_s^*)|\, \mathrm{d}s \\ &\leqslant |y(0)| \exp\left(-\int_0^t \mu(A(s))\, \mathrm{d}s\right) \\ &\quad + \int_0^t L(s) \max_{\theta\in[-\tau,0]} |y(s+\theta)| \exp\left(-\int_s^t \mu(A(u))\, \mathrm{d}u\right) \mathrm{d}s, \qquad t \geqslant 0. \end{aligned} \tag{3.29}$$

Let $t^* \in [0, \omega]$ such that

$$|y(t^*)| = \max_{t\in[0,\omega]} |y(t)|, \qquad i = 1, 2, \ldots, n.$$

It follows from (3.22) and (3.29) that

$$\begin{aligned} |y(t^*)| &\leqslant |y(0)| \exp\left(-\int_0^{t^*} \mu(A(u))\, \mathrm{d}u\right) \\ &\quad + \int_0^{t^*} L(s) \max_{\theta\in[-\tau,0]} |y(s+\theta)| \exp\left(-\int_s^{t^*} \mu(A(u))\, \mathrm{d}u\right) \mathrm{d}s \\ &\leqslant \kappa|y(0)| + \max_{-\tau\leqslant s\leqslant\omega} |y(s)| \int_0^{t^*} L(s) \exp\left(-\int_s^{t^*} \mu(A(u))\, \mathrm{d}u\right) \mathrm{d}s \\ &\leqslant \kappa|y(0)| + \max_{-\tau\leqslant s\leqslant\omega} |y(s)| \int_{t^*-\omega}^{t^*} L(s) \exp\left(-\int_s^{t^*} \mu(A(u))\, \mathrm{d}u\right) \mathrm{d}s \\ &\leqslant \kappa\|\phi - \phi^*\| + \max_{-\tau\leqslant s\leqslant\omega} |y(s)| \int_0^{\omega} L(s + t^*) \exp\left(-\int_s^{\omega} \mu(A(u + t^*))\, \mathrm{d}u\right) \mathrm{d}s \\ &\leqslant \kappa\|\phi - \phi^*\| + \frac{1}{\gamma} \max_{-\tau\leqslant s\leqslant\omega} |y(s)| \\ &\leqslant \kappa\|\phi - \phi^*\| + \frac{1}{\gamma} \left(\|\phi - \phi^*\| + |y(t^*)|\right), \end{aligned}$$

where $\kappa = \exp(\int_0^{\omega} |\mu(A(s))|\, \mathrm{d}s)$. It follows that

$$|y(t^*)| \leqslant \frac{\kappa\gamma + 1}{\gamma - 1}\|\phi - \phi^*\|,$$

which implies that

$$\max_{-\tau\leqslant t\leqslant\omega}|y(t)| \leqslant \frac{\kappa\gamma+1}{\gamma-1}\|\phi-\phi^*\|. \tag{3.30}$$

Since $\gamma > 1$ and $\int_0^\omega \mu(A(s))\,\mathrm{d}s > 0$, so we can choose $\lambda \in (0, \frac{1}{\omega}\int_0^\omega \mu(A(s))\,\mathrm{d}s)$ such that

$$\eta = \mathrm{e}^{\lambda\tau}\left[\frac{1}{\gamma} - \exp\left(-\int_0^\omega \mu(A(s))\,\mathrm{d}s\right)\right]\left[\mathrm{e}^{-\lambda\omega} - \exp\left(-\int_0^\omega \mu(A(s))\,\mathrm{d}s\right)\right]^{-1} < 1. \tag{3.31}$$

Set $z(t) = y(t)\mathrm{e}^{\lambda t}$. Then it follows from (3.29) that

$$\begin{aligned}|z(t)| &\leqslant |z(0)|\exp\left(-\int_0^t [\mu(A(s))-\lambda]\,\mathrm{d}s\right)\\ &\quad + \mathrm{e}^{\lambda\tau}\int_0^t L(s)\max_{\theta\in[-\tau,0]}|z(s+\theta)|\exp\left(-\int_s^t [\mu(A(u))-\lambda]\,\mathrm{d}u\right)\mathrm{d}s,\\ &\quad t\geqslant 0.\end{aligned} \tag{3.32}$$

Choose $t_k \in [-\tau, k]$ such that

$$|z(t_k)| = \max\{|z(t)| : -\tau \leqslant t \leqslant k\}. \tag{3.33}$$

Then $t_1 \leqslant t_2 \leqslant t_3 \leqslant \cdots$. Let r_k be an integer such that $r_k\omega \leqslant t_k < (r_k+1)\omega,\ k = 1, 2, \ldots$. Then by (3.22), (3.30), (3.31), (3.32) and (3.33), we have

$$\begin{aligned}|z(t_k)| &\leqslant |z(0)|\exp\left(-\int_0^{t_k}[\mu(A(s))-\lambda]\,\mathrm{d}s\right)\\ &\quad + \mathrm{e}^{\lambda\tau}\int_0^{t_k} L(s)\max_{\theta\in[-\tau,0]}|z(s+\theta)|\exp\left(-\int_s^{t_k}[\mu(A(u))-\lambda]\,\mathrm{d}u\right)\mathrm{d}s\\ &= |z(0)|\exp\left(-\int_0^{t_k}[\mu(A(s))-\lambda]\,\mathrm{d}s\right)\\ &\quad + \mathrm{e}^{\lambda\tau}\int_0^{t_k-r_k\omega} L(s)\max_{\theta\in[-\tau,0]}|z(s+\theta)|\exp\left(-\int_s^{t_k}[\mu(A(u))-\lambda]\,\mathrm{d}u\right)\mathrm{d}s\\ &\quad + \mathrm{e}^{\lambda\tau}\int_{t_k-r_k\omega}^{t_k} L(s)\max_{\theta\in[-\tau,0]}|z(s+\theta)|\exp\left(-\int_s^{t_k}[\mu(A(u))-\lambda]\,\mathrm{d}u\right)\mathrm{d}s\\ &\leqslant |z(0)|\mathrm{e}^{\lambda t_k}\exp\left(-\int_0^{t_k}\mu(A(s))\,\mathrm{d}s\right) + \mathrm{e}^{\lambda(\tau+t_k)}\exp\left(-\int_{t_k-r_k\omega}^{t_k}\mu(A(u))\,\mathrm{d}u\right)\\ &\quad \times\int_0^{t_k-r_k\omega} L(s)\max_{\theta\in[-\tau,0]}|z(s+\theta)|\exp\left(-\int_s^{t_k-r_k\omega}\mu(A(u))\,\mathrm{d}u\right)\mathrm{d}s\\ &\quad + \mathrm{e}^{\lambda\tau}|z(t_k)|\sum_{i=1}^{r_k}\exp\left(-\int_{t_k-(i-1)\omega}^{t_k}[\mu(A(u))-\lambda]\,\mathrm{d}u\right)\\ &\quad \times\int_{t_k-i\omega}^{t_k-(i-1)\omega} L(s)\exp\left(-\int_s^{t_k-(i-1)\omega}[\mu(A(u))-\lambda]\,\mathrm{d}u\right)\mathrm{d}s\\ &\leqslant \left[|z(0)| + \omega\bar{L}\mathrm{e}^{\lambda\tau}\max_{s\in[-\tau,\omega]}|z(s)|\right]\mathrm{e}^{\lambda t_k}\exp\left(-r_k\int_0^\omega \mu(A(s))\,\mathrm{d}s + \int_0^\omega |\mu(A(s))|\,\mathrm{d}s\right)\end{aligned}$$

$$
\begin{aligned}
&+ \mathrm{e}^{\lambda\tau}|z(t_k)| \sum_{i=1}^{r_k} \exp\left(-\int_{t_k-(i-1)\omega}^{t_k} [\mu(A(u)) - \lambda]\,\mathrm{d}u\right) \\
&\times \int_{t_k-i\omega}^{t_k-(i-1)\omega} L(s) \exp\left(-\int_{s}^{t_k-(i-1)\omega} [\mu(A(u)) - \lambda]\,\mathrm{d}u\right) \mathrm{d}s \\
\leqslant &\left[1 + \frac{\kappa\gamma+1}{\gamma-1}\omega\bar{L}\mathrm{e}^{\lambda(\tau+\omega)}\right] \|\phi - \phi^*\| \exp\left(\lambda\omega + \int_0^{\omega} |\mu(A(s))|\,\mathrm{d}s\right) \\
&+ \mathrm{e}^{\lambda\tau}|z(t_k)| \sum_{i=1}^{r_k} \exp\left(-\int_{t_k-(i-1)\omega}^{t_k} [\mu(A(u)) - \lambda]\,\mathrm{d}u\right) \\
&\times \int_{t_k-i\omega}^{t_k-(i-1)\omega} L(s) \exp\left(-\int_{s}^{t_k-(i-1)\omega} [\mu(A(u)) - \lambda]\,\mathrm{d}u\right) \mathrm{d}s \\
= &R\|\phi - \phi^*\| + \mathrm{e}^{\lambda\tau}|z(t_k)| \sum_{i=1}^{r_k} \exp\left(-(i-1)\int_0^{\omega} [\mu(A(u)) - \lambda]\,\mathrm{d}u\right) \\
&\times \int_0^{\omega} L(s+t_k) \exp\left(-\int_s^{\omega} [\mu(A(u+t_k)) - \lambda]\,\mathrm{d}u\right) \mathrm{d}s \\
\leqslant &R\|\phi - \phi^*\| + \mathrm{e}^{\lambda(\tau+\omega)}|z(t_k)| \left[\frac{1}{\gamma} - \exp\left(-\int_0^{\omega} \mu(A(s))\,\mathrm{d}s\right)\right] \\
&\times \sum_{i=1}^{r_k} \exp\left(-(i-1)\int_0^{\omega} [\mu(A(u)) - \lambda]\,\mathrm{d}u\right) \\
\leqslant &R\|\phi - \phi^*\| + \mathrm{e}^{\lambda\tau}|z(t_k)| \left[\frac{1}{\gamma} - \exp\left(-\int_0^{\omega} \mu(A(s))\,\mathrm{d}s\right)\right] \\
&\times \left[\mathrm{e}^{-\lambda\omega} - \exp\left(-\int_0^{\omega} \mu(A(s))\,\mathrm{d}s\right)\right]^{-1} \\
= &R\|\phi - \phi^*\| + \eta|z(t_k)|,
\end{aligned}
\tag{3.34}
$$

where

$$
R = \left[1 + \frac{\kappa\gamma+1}{\gamma-1}\omega\bar{L}\mathrm{e}^{\lambda(\tau+\omega)}\right] \exp\left(\lambda\omega + \int_0^{\omega} |\mu(A(s))|\,\mathrm{d}s\right).
$$

It follows from (3.31) and (3.34) that

$$
|z(t_k)| \leqslant (1-\eta)^{-1} R\|\phi - \phi^*\|, \qquad k = 1, 2, \ldots. \tag{3.35}
$$

It follows that

$$
|x(t) - x^*(t)| = |y(t)| \leqslant (1-\eta)^{-1} R\|\phi - \phi^*\|\mathrm{e}^{-\lambda t}, \qquad t \geqslant 0. \tag{3.36}
$$

Thus, if $x(t)$ is also ω-periodic, then we must have $x(t) \equiv x^*(t)$. Furthermore, (3.36) also implies that $x^*(t)$ is globally exponentially stable. This completes the proof.

If

$$
L(t) < \mu(A(t)), \qquad \forall\, t \in [0, \omega], \tag{3.37}
$$

then there exists a $\theta \in (0, 1)$ by the periodicity of $L(t)$ and $\mu(A(t))$, such that

$$
L(t) \leqslant \theta\mu(A(t)), \qquad \forall\, t \in [0, \omega].
$$

It follows that for any $t \in [0, \omega]$

$$\int_0^{\omega} L(s+t)\exp\left(-\int_s^{\omega}\mu(A(u+t))\,\mathrm{d}u\right)\mathrm{d}s \leqslant \theta\left[1-\exp\left(-\int_0^{\omega}\mu(A(s))\,\mathrm{d}s\right)\right]$$
$$< 1-\exp\left(-\int_0^{\omega}\mu(A(s))\,\mathrm{d}s\right).$$

This shows that (3.21) is implied by the simpler yet stronger condition (3.37). Thus, we have

Corollary 3.4. *In theorem 3.2, if condition (3.21) is replaced by (3.37), then the conclusion still holds.*

Remark 3.1. In theorem 3.1, we only require the conditions $\bar{c}_i > 0$, $i = 1, 2, \ldots, n$, which may allow $c_i(t) \leqslant 0$ for some $t \in [0, \omega]$, not the usually used condition $c_i(t) > 0$, $i = 1, 2, \ldots, n$.

Remark 3.2. Corollary 3.4 reproduces one of the main results, theorem 3.1, in [22].

4. Applications in CNNs

In this section, we apply the general results obtained in sections 2 and 3 to the DCNNs system (1.4) given in the introduction. Firstly, we note that very recently, Huang *et al* [13] studied the existence and exponential stability of the periodic solutions for system (1.4) under (H1) and (H2), and derived some sufficient conditions among which is the following *harsh* condition:

(H0) $c_i^l - \sum_{j=1}^{n}(|\bar{a}_{ij}|+|\bar{b}_{ij}|)l_j \mathrm{e}^{c_j^u\tau} > 0$, $i = 1, \ldots, n$, where $\tau = \max\{\tau_{ij}^u : i, j = 1, 2, \ldots, n\}$.

Applying theorem 3.1 to system (1.4), we have the following theorem:

Theorem 4.1. *Assume that (H1), (H2) and the following condition are satisfied:*

(H3) *$E_n - D$ is an M-matrix, where $D = (\Gamma((|a_{ij}|+|b_{ij}|)l_j, c_i))_{n\times n}$.*

Then system (1.4) has exactly one ω-periodic solution. Moreover, it is globally exponentially stable.

Proof. Set

$$\tau = \max\{\tau_{ij}^u\ : i, j = 1, 2, \ldots, n\}$$

and

$$f_i(t,\phi) = \sum_{j=1}^{n} a_{ij}(t)g_j(\phi_j(0)) + \sum_{j=1}^{n} b_{ij}(t)g_j(\phi_j(-\tau_{ij}(t))) + I_i(t), \qquad i = 1, 2, \ldots, n. \tag{4.1}$$

Then by (H2) and (4.1), for any $\phi = (\phi_1, \phi_2, \ldots, \phi_n)^{\mathrm{T}}$, $\psi = (\psi_1, \psi_2, \ldots, \psi_n)^{\mathrm{T}} \in X$

$$|f_i(t,\phi) - f_i(t,\psi)| \leqslant \sum_{j=1}^{n} l_j(|a_{ij}(t)|+|b_{ij}(t)|)|\phi_j-\psi_j|_\infty, \qquad i = 1, 2, \ldots, n. \tag{4.2}$$

Let

$$\alpha_{ij}(t) = (|a_{ij}(t)|+|b_{ij}(t)|)l_j, \qquad i, j = 1, 2, \ldots, n. \tag{4.3}$$

Then (4.2) and (H3) imply (D3') and (D4), respectively. In view of theorem 3.1, system (1.4) has exactly one ω-periodic solution which is globally exponentially stable. The proof is complete.

Similarly, by corollaries 3.1–3.3, we have the following corollaries.

Corollary 4.1. *Assume that (H1) and (H2) hold, and that $\rho(D) < 1$, where $D = (\Gamma((|a_{ij}| + |b_{ij}|)l_j, c_i))_{n\times n}$. Then system (1.4) has exactly one ω-periodic solution. Moreover, it is globally exponentially stable.*

Corollary 4.2. *Assume that (H1) and (H2) hold and $c_i(t) \geqslant 0$, $i = 1, 2, \ldots, n$. Let*

$$U = ((1 - \mathrm{e}^{-\bar{c}_i\omega})^{-1}\omega(\overline{|a_{ij}|} + \overline{|b_{ij}|})l_j)_{n\times n}.$$

If $\rho(U) < 1$, then system (1.4) has exactly one ω-periodic solution which is globally exponentially stable.

Corollary 4.3. *Assume that (H1) and (H2) hold and $c_i(t) > 0$, $i = 1, 2, \ldots, n$. Let*

$$V = \left(\left(\frac{(|a_{ij}| + |b_{ij}|)l_j}{c_i}\right)^u\right)_{n\times n}.$$

If $\rho(V) < 1$, then system (1.4) has exactly one ω-periodic solution. Moreover, it is globally exponentially stable.

Remark 4.1. When $c_i(t) \equiv c_i > 0$, $i = 1, 2, \ldots, n$, corollary 4.3 reproduces the main result of theorem 4.1 in [17].

Remark 4.2. References [12, 13] also deal with the network system (1.4), which is a special case of (1.1). The general forms of (1.1) and (1.2) are not covered by [12, 13]. In methods, both [12] and [13] use coincidence degree theory to derive their main results, while here in this paper, we use matrix theory and inequality analysis. In the results, [12] requires a smoothness condition on the delay which is not feasible in many applications. Most significantly, we have removed the conditions (H0) in theorem 3.3 in [13], and hence have shown that it is unnecessary. Finally, we point out that the condition $\rho(U) < 1$ in corollary 4.2 is weaker than the condition $\rho(K) < 1$ in theorem 3.3 in [13], where

$$K = \left(\left(\frac{1}{\bar{c}_i} + \omega\right)(\overline{|a_{ij}|} + \overline{|b_{ij}|})l_j\right)_{n\times n}.$$

To see this, we first note that

$$\mathrm{e}^x > 1 + x + \frac{x^2}{2}, \qquad x > 0.$$

It follows that

$$(1 - \mathrm{e}^{-x})^{-1} < \frac{1 + x + \frac{x^2}{2}}{x\left(1 + \frac{x}{2}\right)} = 1 + \frac{2}{x(2 + x)}, \qquad x > 0.$$

Thus, we have

$$(1 - \mathrm{e}^{-\bar{c}_i\omega})^{-1}\omega(\overline{|a_{ij}|} + \overline{|b_{ij}|}) < \left(\frac{2}{\bar{c}_i(2 + \bar{c}_i\omega)} + \omega\right)(\overline{|a_{ij}|} + \overline{|b_{ij}|}), \qquad i, j = 1, 2, \ldots, n.$$

Set

$$W = \left(\left(\frac{2}{\bar{c}_i(2 + \bar{c}_i\omega)} + \omega\right)(\overline{|a_{ij}|} + \overline{|b_{ij}|})l_j\right)_{n\times n}.$$

Then $0 \leqslant U \leqslant W \leqslant K$ and therefore $\rho(U) \leqslant \rho(W) \leqslant \rho(K)$.

In the following, we give two more specific examples to illustrate our results.

Example 4.1. Consider the following scalar delay differential equation:

$$x'(t) = -ax(t) + b(a - \sin t)g(x(t-\tau)) + p(t), \tag{4.4}$$

where $a \in [1, +\infty)$, $\tau \in (0, +\infty)$, $b \in \boldsymbol{R}$, $p \in C_{2\pi}$ and $g \in C^1(\boldsymbol{R}, \boldsymbol{R})$. Let $\omega = 2\pi$, $A(t) = a$ and $f(t, \phi) = b(a - \sin t)g(\phi(-\tau)) + p(t)$. If $|g'(x)| \leqslant 1$, then (3.20) holds with $L(t) = |b|(a - \sin t)$. By a calculation, we have

$$\begin{aligned}
&\int_0^\omega L(s+t)\exp\left(-\int_s^\omega \mu(A(u+t))\,\mathrm{d}u\right)\mathrm{d}s \\
&\quad = |b|\mathrm{e}^{-2a\pi}\int_0^{2\pi}\mathrm{e}^{as}[a - \sin(s+t)]\,\mathrm{d}s \\
&\quad = |b|(1-\mathrm{e}^{-2a\pi})\left(1 - \frac{a\sin t - \cos t}{1+a^2}\right) \\
&\quad \leqslant |b|(1-\mathrm{e}^{-2a\pi})\left(1 + \frac{1}{\sqrt{1+a^2}}\right), \qquad \forall t \in [0, 2\pi].
\end{aligned}$$

In view of theorem 3.2, if

$$|b| < \sqrt{1+a^2}/(1+\sqrt{1+a^2}), \tag{4.5}$$

then (4.5) has exactly one 2π-periodic solution and it is globally exponentially stable. However, the condition in corollary 3.4 corresponding to (4.5) is

$$|b| < a/(1+a). \tag{4.6}$$

Obviously, condition (4.5) is weaker than (4.6).

Example 4.2. Consider the following BAM neural networks

$$\begin{aligned}
x_1'(t) &= -c_1(t)x_1(t) + \sum_{j=1}^{2} a_{1j}(t)g_j(x_j(t)) + \sum_{j=1}^{2} b_{1j}(t)g_j(x_j(t-\tau_{1j}(t))) + I_1(t), \\
x_2'(t) &= -c_2(t)x_2(t) + \sum_{j=1}^{2} a_{2j}(t)f_j(x_j(t)) + \sum_{j=1}^{2} b_{2j}(t)g_j(x_j(t-\tau_{2j}(t))) + I_2(t),
\end{aligned} \tag{4.7}$$

where $c_1(t) = 2 + \sin t$, $c_2(t) = 2 + \cos t$, $I_1(t) = \sin t$, $I_2(t) = \cos t$, $a_{11}(t) = a_{12}(t) = a\sin t$, $a_{21}(t) = a_{22}(t) = a\cos t$, $b_{11}(t) = b_{12}(t) = b\sin t$, $b_{21}(t) = b_{22}(t) = b\cos t$, $g_1(x) = \frac{x}{3} + \sin\frac{2x}{3}$, $g_2(x) = \frac{x}{2} + \cos\frac{x}{2}$, $\tau_{11}(t) = \tau_{21}(t) = \sin t$, $\tau_{12}(t) = \tau_{22}(t) = \cos t$. Then the functions $c_i(t)$, $a_{ij}(t)$, $b_{ij}(t)$ and $I_i(t)$ are 2π-periodic solutions; the functions $g_j(x)$ satisfy the condition (H2) with $l_1 = l_2 = 1$. By a simple calculation, we have

$$\left(\frac{(|a_{ij}| + |b_{ij}|)l_j}{c_i}\right)^u = |a| + |b|,$$

$$V = \begin{pmatrix} |a|+|b| & |a|+|b| \\ |a|+|b| & |a|+|b| \end{pmatrix},$$

and

$$\rho(V) = 2(|a| + |b|).$$

By corollary 4.3, if $|a| + |b| < 0.5$, then system (4.7) has exactly one 2π-periodic solution and it is globally exponentially stable.

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