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Exponential stability of Cohen–Grossberg neural networks ☆

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Abstract

Exponential stabilities of the Cohen–Grossberg neural network with and without delays are analyzed. By Liapunov functions/functionals, sufficient conditions are obtained for general exponential stability, while by using a comparison result from the theory of monotone dynamical systems, componentwise exponential stability is also discussed. All results are established without assuming any symmetry of the connection matrix, and the differentiability and monotonicity of the activation functions. © 2002 Published by Elsevier Science Ltd.

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1. Introduction

Consider the Cohen–Grossberg neural network model which is described by the system of ordinary differential equations

$$\dot{x}_i = -a_i(x_i)\left(b_i(x_i) - \sum_{j=1}^{n} t_{ij}s_j(x_j)\right), \qquad i = 1, \ldots, n. \tag{1.1}$$

Here $n \geq 2$ is the number of neurons in the network, x_i denotes the state variable associated with the ith neuron, a_i represents an amplification function, and b_i is an appropriately behaved function. The $n \times n$ connection matrix $T = (t_{ij})$ tells how the neurons are connected in the network, and the activation functions s_js show how neurons respond to each other. Functions a_is, b_is and s_is are subject to certain conditions to be specified later. This model was initially proposed and studied by Cohen and Grossberg, and as pointed in Cohen and Grossberg (1983), it includes a number of models from neurobiology, population biology, evolutionary theory. System (1.1) also includes the Hopfield neural network as a special case, which is an artificial neural network and is of the form

$$C_i\dot{x}_i = -\frac{x_i}{R_i} + \sum_{j=1}^{n} t_{ij}s_j(x_j) + I_i, \qquad i = 1, 2, \ldots, n, \tag{1.2}$$

where the positive constants C_is and R_is are the neuron amplifier input capacitances and resistances, respectively, and I_is are the constant inputs from outside of the network and x_is, s_js and $T = (t_{ij})$ are the same as in Eq. (1.1).

Established in the pioneering work of Cohen and Grossberg (1983) and Hopfield (1984) was the 'global limit property' of systems (1.1) and (1.2), respectively, meaning that given any initial conditions, the solution of the system (1.1) (or (1.2)) will converge to some equilibrium of the system. Such a global limit property in Cohen and Grossberg (1983) and Hopfield (1984) was obtained by considering some potential functions under the assumption that the connection matrix T is *symmetric*. This global limit property confirms the ability of *global pattern formation* which is crucial for a network. However, the symmetry assumption is not plausible in many network applications because it lays a restriction on the connection topology of the networks. Moreover, the global limit property does not give a description or even an estimate of the region of attraction for each equilibrium. In other words, given a set of initial conditions, one knows that the solution will converge to some equilibrium but does not know exactly to which one it will converge. In terms of associative memories, one does not know what initial conditions are needed in order to retrieve a particular pattern stored in the network.

On the other hand, in applications of neural networks to parallel computation, signal processing and other problems involving the solutions of optimization

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problems, it is required that there be a well-defined computable solution for all possible initial states. In other words, it is required that the network have a *unique* equilibrium that is globally attractive. In fact, earlier applications of neural networks to optimization problems have suffered from the existence of a complicated set of equilibria (see Tank and Hopfield (1986)). Thus, the global attractivity of a unique equilibrium for the system (1.1) or (1.2) is of great importance for both practical and theoretical purposes, and has been the major concern of many authors. See, e.g. Forti (1994), Forti, Manetti, and Marini (1992), Hirsch (1989) and Matsuoka (1992).

Due to the finite speeds of switching and transmission of signals, time delays unavoidably exist in a working network and thus should be incorporated into the model equations of the network. For Eq. (1.2), Marcus and Westervelt (1989) first introduced a single delay into the model and observed, both numerically and experimentally, sustained oscillations even still with symmetric connections. Such delay-induced oscillations have stimulated further studies of global attractivity for Eq. (1.2) with various delays incorporated (Bélair, 1993; Cao & Wu, 1996; van den Driessche & Zou, 1998; Gopalsamy & He, 1994; Guan, Chen, & Qin, 2000; Joy, 2000; Lu, 2000; Wu, 1999). For Eq. (1.1), Ye, Michel, and Wang (1995) introduced delays into it by considering the following system:

$$\dot{x}_i(t) = -a_i(x_i)\left[b_i(x_i) - \sum_{k=0}^{K}\sum_{j=1}^{n} t_{ij}^{(k)} s_j(x_j(t-\tau_k))\right], \quad i = 1,2,\ldots,n, \tag{1.3}$$

where $n \times n$ matrix $T_k = (t_{ij}^{(k)})$ represents the interconnections which are associated with delay τ_k and the delays τ_k, $k = 0,1,\ldots,K$, are arranged such that $0 = \tau_0 < \tau_1 < \cdots < \tau_K$. They confirmed that the global limit property remains for Eq. (1.3) provided: (i) the connection possesses some kind of *symmetry;* and (ii) the delays are sufficiently *small*. As for the global attractivity of Eq. (1.3), Wang and Zou (2000) established some delay independent criteria, without assuming monotonicity and differentiability of the activation functions and any symmetry of the connections.

In designing and implementing a network, it is preferable and desirable that the neural network not only converge, but also converge as fast as possible. It is well known that exponential stability gives a fast convergence rate to the equilibrium. The purpose of this paper is to obtain some criteria for the exponential stability of a unique equilibrium for the following system:

$$\dot{x}_i(t) = -a_i(x_i)\left[b_i(x_i) - \sum_{j=1}^{n} t_{ij} s_j(x_j(t)) + J_i\right], \quad i = 1,2,\ldots,n, \tag{1.4}$$

and

$$\dot{x}_i(t) = -a_i(x_i)\left[b_i(x_i) - \sum_{j=1}^{n} t_{ij} s_j(x_j(t-\tau_{ij})) + J_i\right], \quad i = 1,2,\ldots,n, \tag{1.5}$$

where J_is denote the constant inputs from outside of the system, $\tau_{ij} \geq 0$ are delays caused from switching and transmission processes. We do not confine ourselves to the symmetric connections and differentiable activation functions and thus allow much broader connection topologies for the network.

This paper is organized as follows. In Section 2, we introduce some necessary notations and assumptions. In Section 3, we establish our main results on the exponential stability of Eqs. (1.4) and (1.5). We employ Liapunov functions/functionals to obtain the general exponential stability, and use a comparison result from the theory of monotone dynamical systems to derive a criterion for componentwise exponential stability. Some examples are also given in Section 3 to demonstrate the main results.

2. Preliminaries

Let R denote the set of real numbers and $R^n = \underbrace{R \times R \times \cdots \times R}_{n}$. If $x \in R^n$, then $x^{\mathrm{T}} = (x_1,\ldots,x_n)$ denotes the transpose of x, $\|x\|_2 = (x^{\mathrm{T}}x)^{1/2}$. Let $R^{n\times n}$ denote the set of $n \times n$ real matrices. For $Z \in R^{n\times n}$, the spectral norm of Z is defined as

$$\|Z\|_2 = (\max\{|\lambda| : \lambda \text{ is an eigenvalue of } Z^{\mathrm{T}}Z\})^{1/2}.$$

The initial conditions associated with Eq. (1.4) are of the form

$$x_i(s) = x_i(0), \qquad i = 1,2,\ldots,n, \tag{2.1}$$

and the initial conditions associated with Eq. (1.5) are of the form

$$x_i(s) = \phi_i(s) \in C([-\tau,0],R), \qquad s \in [-\tau,0], \quad i = 1,2,\ldots,n, \tag{2.2}$$

where $\tau = \max\{\tau_{ij}, 1 \leq i,j \leq n\}$.

We list some assumptions which will be used in the main results.

(H_1) For each $i \in \{1,2,\ldots,n\}$, a_i is bounded, positive and locally Lipschitz continuous, furthermore we assume $0 < \underline{\alpha}_i \leq a_i(u) \leq \bar{\alpha}_i$.

(H_2) For each $i \in \{1,2,\ldots,n\}$, b_i and b_i^{-1} are locally Lipschitz continuous.

For the activation functions $s_i(x)$, $i = 1,2,\ldots,n$, they are typically assumed to be sigmoid which implies that they are monotone and smooth, that is, they are required to satisfy the following:

(A_1) $s_i \in C^1(R)$, $s_i{}'(x) > 0$ for $x \in R$ and $s_i'(0) = \sup_{x\in R} s_i'(x) > 0$, $i = 1, 2, \ldots, n$.
(A_2) $s_i(0) = 0$ and $\lim_{x\to\pm\infty} s_i(x) = \pm 1$.

We will give a modification of (A_1) and (A_2) to the following

(S_1) For each $i \in \{1, 2, \ldots, n\}$, $s_i : R \to R$ is Lipschitz continuous with a Lipschitz constant L_i.
(S_2) For each $i \in \{1, 2, \ldots, n\}$, $|s_i(x)| \le M_i$, $x \in R$ for some constant $M_i > 0$.

3. Main results

We first give an existence result for the equilibrium of Eq. (1.4) (or (1.5))

Theorem 3.1. *If* (H_1)–(H_2) *and* (S_1)–(S_2) *hold, then for every input J, there exists an equilibrium for system* (1.4) ((1.5)).

Proof. Note that Eqs. (1.4) and (1.5) have the same equilibrium set. By (H_1), we know that x^* is an equilibrium of Eqs. (1.4) and (1.5) if and only if $x^* = (x_1^*, \ldots, x_n^*)^T$ is a solution of equations

$$b_i(x_i^*) - \sum_{j=1}^n t_{ij} s_j\left(x_j^*\right) + J_i = 0, \qquad i = 1, \ldots, n. \tag{3.1}$$

For every fixed J

$$\left|\sum_{j=1}^n t_{ij} s_j(x_j) + J_i\right| \le \sum_{j=1}^n |t_{ij}| M_j + |J_i| := P_i.$$

Now consider

$$x_i = h_i(x_1, x_2, \ldots, x_n) = b_i^{-1}\left(\sum_{j=1}^n t_{ij} s_j(x_j) + J_i\right),$$

for $i = 1, 2, \ldots, n$.

Then we have

$$|h_i(x_1, x_2, \ldots, x_n)| \le \max\{|b_i^{-1}(s)\ |, s \in [-P_i, P_i]\} := D_i,$$

for $i = 1, 2, \ldots, n$.

It follows that $(h_1, h_2, \ldots, h_n)^T$ maps $D = [-D_1, D_1] \times [-D_2, D_2] \times \cdots \times [-D_n, D_n]$ into itself. Thus, by Brouwer's fixed point theorem, we get the existence of an equilibrium located in D. The proof is complete. □

3.1. Exponential stability for Eq. (1.4)

Let x^* be an equilibrium of Eq. (1.4) and $u(t) = x(t) - x^*$. Substituting $x(t) = u(t) + x^*$ into Eq. (1.4) leads to

$$\dot{u}_i(t) = -a_i\left(u_i + x_i^*\right) \times \left[b_i\left(u_i + x_i^*\right) - b_i\left(x_i^*\right) - \sum_{j=1}^n t_{ij}\left(s_j(u_j) + x_j^*\right) - s_j\left(x_j^*\right)\right]. \tag{3.2}$$

Let $\alpha_i(u_i(t)) = a_i(u_i(t) + x_i^*)$, $\beta_i(u_i(t)) = b_i(u_i(t) + x_i^*) - b_i(x_i^*)$, $g_j(u_j(t)) = s_j(u_j(t) + x_j^*) - s_j(s_j^*)$. Then Eq. (3.2) further reduces to

$$\dot{u}_i(t) = -\alpha_i(u_i(t))\left[\beta_i(u_i(t)) - \sum_{j=1}^n t_{ij} g_j(u_j)\right], \tag{3.3}$$

$i = 1, 2, \ldots, n$.

If we denote $A(u) = \mathrm{diag}\{\alpha_1(u_1), \ldots, \alpha_n(u_n)\}$, $u = (u_1, \ldots, u_n)^T \in R^n$, $B(u) = (\beta_1(u_1), \ldots, \beta_n(u_n)^T \in R^n$, $T = [t_{ij}]_{n\times n}$, $g(u) = (g_1(u_1), \ldots, g_n(u_n))^T$, then system (3.3) can be rewritten as

$$\dot{u} = -A(u)(B(u) - Tg(u(t))). \tag{3.4}$$

Theorem 3.2. *Assume that* (H_1)–(H_2) *and* (S_1)–(S_2) *hold. Suppose there exists* $\gamma_i > 0$ *such that*

$$u\beta_i(u) \ge \gamma_i u^2, \qquad \text{for } u \in R,\ i = 1, 2, \ldots, n, \tag{3.5}$$

and

$$\delta \overset{\text{def}}{=} L\|T\|_2 \eta < 1, \tag{3.6}$$

where $L = \max_{1\le i\le n} L_i$ *and* $\eta = \max_{1\le i\le n}\{\bar{\alpha}_i\} / \min_{1\le i\le n}\{\underline{\alpha}_i\gamma_i\}$. *Then, for any fixed input J,* Eq. (1.4) *has a unique equilibrium* x^* *which is exponentially stable in the sense that every solution* $x(t)$ *of* Eq. (1.4) *satisfies*

$$\sum_{i=1}^n \left(x_i(t) - x_i^*\right)^2 \le e^{-2\sigma t} \sum_{i=1}^n \left(x_i(0) - x_i^*\right)^2, \tag{3.7}$$

where $\sigma = (1/2)(1 - \delta)(\min_{1\le i\le n}\{\underline{\alpha}_i\gamma_i\})$.

Proof. Existence of an equilibrium x^* is guaranteed by Theorem 3.1, and the uniqueness will be implied by the estimate (3.7). So, we only need to prove Eq. (3.7), which is equivalent to

$$\|u(t)\|_2 \le e^{-\sigma t}\|u(0)\|_2, \tag{3.8}$$

where $u(t) = x(t) - x^*$. Let $V(u) = (1/2)\|u\|^2$. Then

$$\begin{aligned}\frac{dV(u(t))}{dt} &= -u^{T}A(u)B(u) + u^{T}A(u)Tg(u)\\ &\le -\sum_{i=1}^{n}\alpha_i(u_i)\beta_i(u_i)u_i + \|u\|_2\|A(u)\|_2\|T\|_2\|g(u)\|_2\\ &\le -\sum_{i=1}^{n}\underline{\alpha}_i\gamma_i u_i^2 + \|u\|_2 \max_{1\le i\le n}\{\bar{\alpha}_i\}\|T\|_2\|g(u)\|_2\\ &\le -\min_{1\le i\le n}\{\underline{\alpha}_i\gamma_i\}\sum_{i=1}^{n}u_i^2 + \max_{1\le i\le n}\{\bar{\alpha}_i\}\|T\|_2 L\|u\|_2^2\\ &\le -\big(\min_{1\le i\le n}\{\underline{\alpha}_i\gamma_i\} - \max_{1\le i\le n}\{\bar{\alpha}_i\}L\|T\|_2\big)\|u\|_2^2\\ &= -(1-\delta)\big(\min_{1\le i\le n}\{\underline{\alpha}_i\gamma_i\}\big)\|u\|_2^2(t) = -2\sigma V(u(t),\end{aligned}$$

which leads to

$$V(u(t) \le e^{-2\sigma t}V(u(0), \tag{3.9}$$

i.e.

$$\|u(t)\|_2 \le e^{-\sigma t}\|u(0)\|_2.$$

This completes the proof. □

Motivated by Hirsch's work (Hirsch, 1989) for the global asymptotic stability of differentiable dynamic systems such as the Hopfield neural networks, in which the negative diagonal connection weights did contribution to the stability, and by constructing a suitable Liapunov function, we establish the following theorem.

Theorem 3.3. *Suppose* (H_1)–(H_2) *and* (S_1)–(S_2) *hold and Eq.* (3.5) *is satisfied. In addition, if for each* $i = 1, 2, \dots, n$

$$ug_i(u) > 0, \qquad u \ne 0, \tag{3.10}$$

and

$$\Big[t_{ii} + \sum_{j\ne i}|t_{ji}|\Big]^{+} < \frac{\gamma_i}{L_i} \tag{3.11}$$

are satisfied, where $[y]^+ = \max\{y,0\}$. *Then, for every input* J, system (1.4) *has a unique equilibrium* x^* *which is exponentially stable in the sense that every solution* $x(t)$ *of Eq.* (1.4) *satisfies*

$$\sum_{i=1}^{n}|x_i(t) - x_i^*| \le c\, e^{-\sigma_1 t}\sum_{i=1}^{n}|x_i(0) - x_i^*|, \qquad t > 0, \tag{3.12}$$

where $c = \bar{\alpha}/\underline{\alpha}$ and $\sigma_1 = \underline{\alpha}\min_{1\le i\le n}\{\gamma_i - [t_{ii} - \sum_{j\ne i}^{n}|t_{ji}|]^{+}L_i\}$.

Proof. Similar to the proof of Theorem 3.2, we only need to prove that every solution $u(t)$ of Eq. (3.4) satisfies

$$\sum_{i=1}^{n}|u_i(t)| \le c\, e^{-\sigma_1 t}\sum_{i=1}^{n}|u_i(0)|, t > 0. \tag{3.13}$$

Let

$$V(u(t)) = \sum_{i=1}^{n}\int_0^{u_i}\frac{\operatorname{sgn}(s)}{\alpha_i(s)}ds. \tag{3.14}$$

Then

$$\begin{aligned}\frac{dV(u)}{dt} &= \sum_{i=1}^{n} -\operatorname{sgn}(u_i)\Big(\beta_i(u_i) - \sum_{j=1}^{n}t_{ij}g_j(u_j)\Big)\\ &\le \sum_{i=1}^{n}\Big(-\gamma_i|u_i| + t_{ii}\operatorname{sgn}(u_i)g_i(u_i) + \sum_{j\ne i}^{n}t_{ij}g_j(u_j)\Big)\\ &\le \sum_{i=1}^{n}\Big(-\gamma_i|u_i| + t_{ii}|g_i(u_i)| + \sum_{j\ne i}^{n}|t_{ij}||g_j(u_j)|\Big)\\ &= \sum_{i=1}^{n}\Big(\gamma_i|u_i| + \Big(t_{ii} + \sum_{j\ne i}^{n}|t_{ji}|\Big)|g_i(u_i)|\Big)\\ &\le -\sum_{i=1}^{n}\gamma_i|u_i| + \sum_{i=1}^{n}\Big[t_{ii} - \sum_{j\ne i}^{n}|t_{ji}|\Big]^{+}|g_i(u_i)|\\ &\le -\sum_{i=1}^{n}\Big(\gamma_i - \Big[t_{ii} + \sum_{j\ne i}^{n}t_{ji}\Big]^{+}L_i\Big)|u_i|.\end{aligned}$$

Let $\nu := \min_{1\le i\le n}\{\gamma_i - [t_{ii} + \sum_{j\ne i}^{n}|t_{ji}|]^{+}L_i\}$, then we have

$$\frac{dV(u)}{dt} \le -\nu\sum_{i=1}^{n}|u_i|. \tag{3.15}$$

From Eq. (3.14) we get

$$\frac{1}{\bar{\alpha}}\sum_{i=1}^{n}|u_i| \le V(u) \le \frac{1}{\underline{\alpha}}\sum_{i=1}^{n}|u_i|, \tag{3.16}$$

where $\bar{\alpha} = \max_{1\le i\le n}\{\bar{\alpha}_i\}$ and $\underline{\alpha} = \min_{1\le i\le n}\{\underline{\alpha}_i\}$. Combining Eqs. (3.15) and (3.16), we have

$$\frac{dV(u)}{dt} \le -\nu\underline{\alpha}V(u), \tag{3.17}$$

which leads to

$$V(u(t)) \le e^{-\nu\underline{\alpha}t}V(u(0)). \tag{3.18}$$

Hence we can conclude from Eq. (3.16) that

$$\sum_{i=1}^{n}|u_i(t)| \le c\, e^{-\sigma_1 t}\sum_{i=1}^{n}|u_i(0)|.$$

This completes the proof. □

Remark 3.1. By Gerschgorin's circle theorem, we note that the condition (3.11) implies that all eigenvalues of the connection matrix T have real parts which are less than $\max\{\gamma_i/L_i,\ i = 1, 2, \dots, n\}$. A similar result was also observed in Hirsch (1989).

Theorem 3.4. *Suppose that* (H_1)–(H_2) *and* (S_1)–(S_2) *and Eq.* (3.5) *hold. In addition, assume*

$$\gamma_i > L_i \sum_{j=1}^{n} |t_{ji}|, \qquad i = 1, 2, \dots, n. \tag{3.19}$$

Then, for every input J, Eq. (1.4) *has a unique equilibrium* x^* *which is exponentially stable is the sense that every solution* $x(t)$ *of Eq.* (1.4) *satisfies*

$$\sum_{i=1}^{n} |x_i(t) - x_i^*| \le c\,\mathrm{e}^{-\sigma_2 t} \sum_{i=1}^{n} |x_i(0) - x_i^*|, \tag{3.20}$$

where $c = \bar{\alpha}/\underline{\alpha}$ *and* $\sigma_2 = \min_{1\le i\le n}\{\gamma_i - L_i \sum_{j=1}^{n} |t_{ji}|\}$.

Proof. We only need to prove that every solution $u(t)$ of Eq. (3.4) satisfies

$$\sum_{i=1}^{n} |u_i(t)| \le c\,\mathrm{e}^{-\sigma_2 t} \sum_{i=1}^{n} |u_i(0)|, \qquad t > 0. \tag{3.21}$$

Let

$$V(u(t)) = \sum_{i=1}^{n} \int_0^{u_i} \frac{\mathrm{sgn}(s)}{\alpha_i(s)}\,\mathrm{d}s. \tag{3.22}$$

Then

$$\begin{aligned}\frac{\mathrm{d}V(u)}{\mathrm{d}t} &= \sum_{i=1}^{n} -\mathrm{sgn}(u_i)\left[\beta_i(u_i) - \sum_{j=1}^{n} t_{ij} g_j(u_j)\right] \\ &\le \sum_{i=1}^{n}\left(-\gamma_i|u_i| + \sum_{j=1}^{n} |t_{ij}||g_j(u_j)|\right) \\ &\le \sum_{i=1}^{n}\left(-\gamma_i|u_i| + \sum_{j=1}^{n} |t_{ij}|L_j|u_j|\right) \\ &= -\sum_{i=1}^{n}\left(\gamma_i - L_i\sum_{j=1}^{n}|t_{ji}|\right)|u_i|.\end{aligned}$$

Let

$$\nu := \min_{1\le i\le n}\left\{\gamma_i - L_i \sum_{j=1}^{n} |t_{ji}|\right\}, \tag{3.23}$$

then we have

$$\frac{\mathrm{d}V(u)}{\mathrm{d}t} \le -\nu \sum_{i=1}^{n} |u_i|. \tag{3.24}$$

Similar to the proof of the above theorem, we can obtain

$$\sum_{i=1}^{n} |u_i(t)| \le c\,\mathrm{e}^{-\sigma_2 t} \sum_{i=1}^{n} |x_i(0) - x_i^*|. \tag{3.25}$$

□

Consequently, we have

Corollary 3.1. *Suppose that* (H_1)–(H_2), (S_1)–(S_2) *and Eq.* (3.5) *hold. In addition, assume*

$$\underline{\alpha}_i \gamma_i > L_i \sum_{j=1}^{n} \bar{\alpha}_j |t_{ji}|, \qquad i = 1, 2, \dots, n. \tag{3.26}$$

Then for any input J, Eq. (1.4) *has a unique equilibrium* x^* *which is exponentially stable in the sense that every solution* $x(t)$ *of Eq.* (1.4) *satisfies*

$$\sum_{i=1}^{n} |x_i(t) - x_i^*| \le \mathrm{e}^{-\sigma_3 t} \sum_{i=1}^{n} |x_i(0) - x_i^*|, \tag{3.27}$$

where $\sigma_3 = \min_{1\le i\le n}\{\underline{\alpha}_i\gamma_i - \bar{\alpha} L_i \sum_{j=1}^{n} |t_{ji}|\}$.

3.2. Exponential stability for Eq. (1.5)

Note that Eqs. (1.4) and (1.5) have the same equilibrium set. Thus, if all the conditions in Corollary 3.1 are satisfied, then Eq. (1.5) also has an equilibrium x^*. With respect to the exponential stability of x^* for the delay system (1.5), we have the following result which is independent of the delays.

Theorem 3.5. *Suppose all conditions of Corollary* 3.1 *hold. Then there exist constants* $c_2 \ge 1$ *and* $\sigma_4 > 0$ *such that every solution* $x(t)$ *of Eqs.* (1.5) *and* (2.2) *satisfies*

$$\sum_{i=1}^{n} |x_i(t) - x_i^*| \le c_2\,\mathrm{e}^{-\sigma_4 t} \sum_{i=1}^{n} \sup_{s\in[-\tau,0]} |\phi_i(s) - x_i^*|, \qquad t > 0. \tag{3.28}$$

Proof. From Eq. (3.26), we know that for any fixed values of $\tau_{ij} \ge 0$, $i, j = 1, 2, \dots, n$, we can find a positive constant σ_4 such that

$$\underline{\alpha}_i\gamma_i - \sigma_4 - L_i \sum_{j=1}^{n} \bar{\alpha}_j |t_{ji}| \mathrm{e}^{\sigma_4 \tau_{ji}} > 0, \qquad \text{for } i = 1, 2, \dots, n. \tag{3.29}$$

Combining Eqs. (1.5) and (3.5), we can get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}|x_i(t) - x_i^*| &\le -\underline{\alpha}_i\gamma_i|x_i(t) - x_i^*| \\ &\quad + \bar{\alpha}_i \sum_{j=1}^{n} |t_{ij}|L_j\left|x_j(t-\tau_{ij}) - x_j^*\right|,\end{aligned} \tag{3.30}$$

$t > 0,$

for $i = 1, \dots, n$. Let

$$y_i(t) = \mathrm{e}^{\sigma_4 t}|x_i(t) - x_i^*|, \qquad i = 1, \dots, n, \qquad t \in [0, \infty). \tag{3.31}$$

Direct calculation shows

$$\frac{\mathrm{d}y_i(t)}{\mathrm{d}t} \le -(\underline{\alpha}_i\gamma_i - \sigma_4)y_i + \bar{\alpha}_i \sum_{j=1}^{n} |t_{ij}| L_j \,\mathrm{e}^{\sigma_4\tau_{ij}} y_j(t-\tau_{ij}),$$

$$t > 0, \tag{3.32}$$

for $i = 1, \ldots, n$. Now let

$$V(t) = \sum_{i=1}^{n}\left(y_i(t) + \bar{\alpha}_i \sum_{j=1}^{n} |t_{ij}| L_j \,\mathrm{e}^{\sigma_4\tau_{ij}} \int_{t-\tau_{ij}}^{t} y_j(s)\mathrm{d}s \right), \tag{3.33}$$

$t > 0.$

It follows from Eq. (3.32) that

$$\frac{\mathrm{d}V(t)}{\mathrm{d}t} \le -\sum_{i=1}^{n}\left(\underline{\alpha}_i\gamma_i - \sigma_4 - \bar{\alpha} L_i \sum_{j=1}^{n} |t_{ji}| \mathrm{e}^{\sigma_4\tau_{ji}} \right) y_i(t), \tag{3.34}$$

$t > 0.$

This implies $V(t) \le V(0)$ for $t > 0$, and hence

$$\begin{aligned}
\sum_{i=1}^{n} y_i(t) &\le V(t) \le V(0) \\
&= \sum_{i=1}^{n}\left(y_i(0) + L_i \sum_{j=1}^{n} |t_{ji}| \mathrm{e}^{\sigma_4\tau_{ji}} \int_{-\tau_{ji}}^{0} y_i(s)\mathrm{d}s \right) \\
&\le \sum_{i=1}^{n}\left(1 + \tau L_i \sum_{j=1}^{n} |t_{ji}| \mathrm{e}^{\sigma_4\tau_{ji}} \right) \sup_{s\in[-\tau,0]} |\phi_i(s) - x_i^*| \\
&\le c_2 \sum_{i=1}^{n} \sup_{s\in[-\tau,0]} |\phi_i(s) - x_i^*|,
\end{aligned}$$

$t > 0,$

where $c_2 = \max_{1\le i\le n}(1 + \tau L_i \sum_{j=1}^{n} |t_{ji}| \mathrm{e}^{\tau_{ji}\sigma_4})$. Now by Eq. (3.31), we finally obtain

$$\sum_{i=1}^{n} |x_i(t) - x_i^*| \le c_2 \,\mathrm{e}^{-\sigma_4 t} \sum_{i=1}^{n} \sup_{s\in[-\tau,0]} |\phi_i(s) - x_i^*|, \qquad t > 0.$$

This completes the proof. □

3.3. Componentwise exponential stability

Under certain circumstances, one may want to estimate the rate of convergence of each or some of the neurons in the network. In this section, we will establish a criterion for the componentwise exponential stability. The approach is by a comparison result from the theory of monotone dynamical system, which will be stated below. First we introduce some notations. For $x, y \in R^n$, we write $x \le y$ if $x_i \le y_i$ for $1 \le i \le n$. Let $C = \{\varphi : \varphi = (\varphi_1, \ldots, \varphi_n)^{\mathrm{T}}, \varphi_i \in C([-\tau, 0], R)\}$. For $\varphi, \psi \in C$, we write $\varphi \le \psi$ if for each $i \in \{1, 2, \ldots, n\}$, $\varphi_i(s) \le \psi_i(s)$ for $s \in [-\tau, 0]$. Let Ω be an open subset of $R \times C$ and $f : \Omega \to R^n$ be continuous. f is said to be quasi-monotone in Ω if it satisfies the following.

(QM) For any $(t, \varphi), (t, \psi) \in \Omega$, if $\varphi \le \psi$ and $\varphi_i(0) = \psi_i(0)$ for some i, then $f_i(t, \varphi) \le f_i(t, \psi)$.

Consider the functional differential equation

$$x'(t) = f(t, x_t) \tag{3.35}$$

with initial conditions

$$x_i(s) = \varphi_i(s) \in C([-\tau, 0], R), \qquad i = 1, 2, \ldots, n. \tag{3.36}$$

To emphasize the dependence of the solution on the right hand side functional, we denote the solution of (3.35) and (3.36) by $x(t, \varphi, f)$. The following comparison theorem is from Smith (1987).

Theorem 3.6. *Let $g, h : \Omega \to R^n$ and assume either g or h satisfies (QM). Suppose that $g(t, \varphi) \le h(t, \varphi)$ for all $(t, \varphi) \subset \Omega$. If $\varphi, \psi \in C$ with $\varphi \le \psi$ then*

$$x(t, \varphi, g) \le x(t, \psi, h)$$

for all $t \ge 0$.

By applying Theorem 3.6, we can prove the following result.

Theorem 3.7. *Suppose that* (H_1)–(H_2), (S_1)–(S_2) *and Eq.* (3.5) *hold. If*

$$\underline{\alpha}_i\gamma_i > \bar{\alpha}_i \sum_{j=1}^{n} |t_{ij}| L_j, \qquad i = i, \ldots, n, \tag{3.37}$$

then Eq. (1.5) *has a unique equilibrium x^*, and there exists a constant $\sigma_5 > 0$ such that every solution $x(t)$ of Eqs.* (1.5) *and* (2.2) *satisfies*

$$|x_i(t) - x_i^*| \le \mathrm{e}^{-\sigma_5 t} \max_{1\le i\le n} \sup_{s\in[-\tau,0]} |\phi_i(s) - x_i^*|, \tag{3.38}$$

$i = 1, \ldots, n.$

Proof. From Eq. (1.5) we know that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}|x_i(t) - x_i^*| \le{}& -\underline{\alpha}_i\gamma_i |x_i(t) - x_i^*| \\
&+ \bar{\alpha}_i \sum_{j=1}^{n} |t_{ij}| L_j \left| x_j(t-\tau_{ij}) - x_j^* \right|,
\end{aligned} \tag{3.39}$$

$t > 0,$

for $i = 1, \dots, n$, or

$$\frac{d}{dt}|u_i(t)| \le -\underline{\alpha}_i \gamma_i |u_i(t)| + \bar{\alpha}_i \sum_{j=1}^{n} |t_{ij}| L_j |u_j(t-\tau_{ij})|,$$

$$t > 0, \qquad i = 1, \dots, n, \tag{3.40}$$

where $u_i(t) = x_i(t) - x_i^*$, $i = 1, 2, \dots, n$. Let $h = (h_1, h_2, \dots, h_n)^{\mathrm{T}}$ be defined by $h_i(\varphi) = -\underline{\alpha}_i \gamma_i \varphi_i(0) + \bar{\alpha}_i \sum_{j=1}^{n} |t_{ij}| L_j \varphi_j(-\tau_{ij})$, $i = 1, \dots, n$. Then, h satisfies (QM). Denote the solutions of

$$\begin{cases} w' = h(w_t), \\ w_{i_0} = \max\limits_{1 \le i \le n} \sup\limits_{s \in [-\tau, 0]} |\phi_i(s) - x_i^*|, i = 1, 2, \dots, n \end{cases} \tag{3.41}$$

by w and let v be any solution of

$$\begin{cases} v' \ge h(v_t), \\ v_{i_0} = \max\limits_{1 \le i \le n} \sup\limits_{s \in [-\tau, 0]} |\phi_i(s) - x_i^*|, \qquad i = 1, 2, \dots, n. \end{cases} \tag{3.42}$$

Then from Theorem 3.6, we have

$$|u_i(t)| \le w_i(t) \le v_i(t) \tag{3.43}$$

for each $i \in \{1, 2, \dots, n\}$. Now, by Eq. (3.37), we know that for fixed $\tau_{ij} \ge 0, i, j = 1, \dots, n$, there exists $\sigma_5 > 0$ such that

$$-\sigma_5 \ge -\underline{\alpha}_i \gamma_i + \bar{\alpha}_i \sum_{j=1}^{n} |t_{ij}| L_j \, e^{\tau_{ij} \sigma_5}. \tag{3.44}$$

For such $\sigma_5 > 0$, a direct verification shows that $v^* = (v_1^*, \dots, v_n^*)$ with

$$v_i^*(t) = e^{-\sigma_5 t} \max_{1 \le i \le n} \sup_{s \in [-\tau, 0]} |\phi_i(s) - x_i^*|, \qquad i = 1, 2, \dots, n, \tag{3.45}$$

is a solution of Eq. (3.42). Therefore, we should have

$$|u_i(t)| \le e^{-\sigma_5 t} \max_{1 \le i \le n} \sup_{s \in [-\tau, 0]} |\phi_i(s) - x_i^*|, \qquad i = 1, 2, \dots, n, \tag{3.46}$$

which proves Eq. (3.38). □

Remark 3.2. Although the assertions of exponential stability in Theorems (3.5) and (3.7) are independent of the delays, the convergence rate σ_4 and σ_5 do depend on the delays τ_{ij}.

3.4. Examples

In this section, we give some examples to demonstrate our results. As we will see, once a_i, b_i and s_i are given, we can adjust the connection weights matrix T so that our criteria are applicable.

Example 3.1. Consider

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = -\begin{pmatrix} 2 + \sin x_1 & 0 \\ 0 & 2 + \cos x_2 \end{pmatrix} \times \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \begin{pmatrix} -\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} \\ -\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} \end{pmatrix} \begin{pmatrix} s_1(x_1(t)) \\ s_2(x_2(t)) \end{pmatrix} \right], \tag{3.47}$$

where s_1 and s_2 satisfy (S_1) and (S_2) with $L = 1$.

In this example, $\gamma_1 = \gamma_2 = 1$, $\bar{\alpha}_i = 3$, $\underline{\alpha}_i = 1$, $i = 1, 2$ and hence $\eta = 3$, $\|T\|_2 = 1/4$. Thus we have $\delta = 3/4 < 1$. It follows from Theorem 3.2 that Eq. (3.47) has a unique equilibrium, which is exponentially stable in the sense of Eq. (3.7) with $\sigma = 1/8$.

Example 3.2. Consider

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = -\begin{pmatrix} 2 + \sin x_1 & 0 \\ 0 & 2 + \cos x_2 \end{pmatrix} \times \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \begin{pmatrix} \frac{1}{4} & \frac{1}{8} \\ \frac{1}{16} & \frac{1}{8} \end{pmatrix} \begin{pmatrix} s_1(x_1(t)) \\ s_2(x_2(t)) \end{pmatrix} \right], \tag{3.48}$$

where s_1 and s_2 satisfy (S_1) and (S_2) with $L = 1$.

Since $t_{11} + t_{21} = 5/16$, $t_{12} + t_{22} = 1/4$, then the conditions of Corollary 3.1 are satisfied. Therefore, Eq. (3.48) has a unique exponentially stable equilibrium and Eq. (3.27) holds with $\sigma_3 = 1/16$.

Example 3.3. Consider

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x} \end{pmatrix} = \begin{pmatrix} -x_1(t) \\ -x_2(t) \\ -x_3(t) \end{pmatrix} + \begin{pmatrix} \frac{1}{9} & \frac{1}{9} & \frac{1}{10} \\ \frac{1}{10} & \frac{1}{10} & \frac{1}{9} \\ \frac{1}{9} & \frac{1}{10} & \frac{1}{9} \end{pmatrix} \times \begin{pmatrix} \tanh(x_1(t-\tau_1)) \\ \tanh(x_2(t-\tau_2)) \\ \tanh(x_3(t-\tau_3)). \end{pmatrix} \tag{3.49}$$

In this example, $a_i(u) = 1$, $\Gamma_i = 1$, $L_i = 1$, $i = 1, 2, 3$, τ_1, $\tau_2 = 1.4$, $\tau_3 = 2$, and $t_{11} = 1/9$, $t_{12} = 1/9$, $t_{13} = 1/10$; $t_{21} = 1/10$, $t_{22} = 1/10$, $t_{23} = 1/9$; $t_{31} = 1/9$, $t_{32} = 1/10$, $t_{33} = 1/9$; $R_i = t_{i1} + t_{i2} t_{i3}$, $i = 1, 2, 3$. Then, $R_1 = R_3 = 29/90$ and

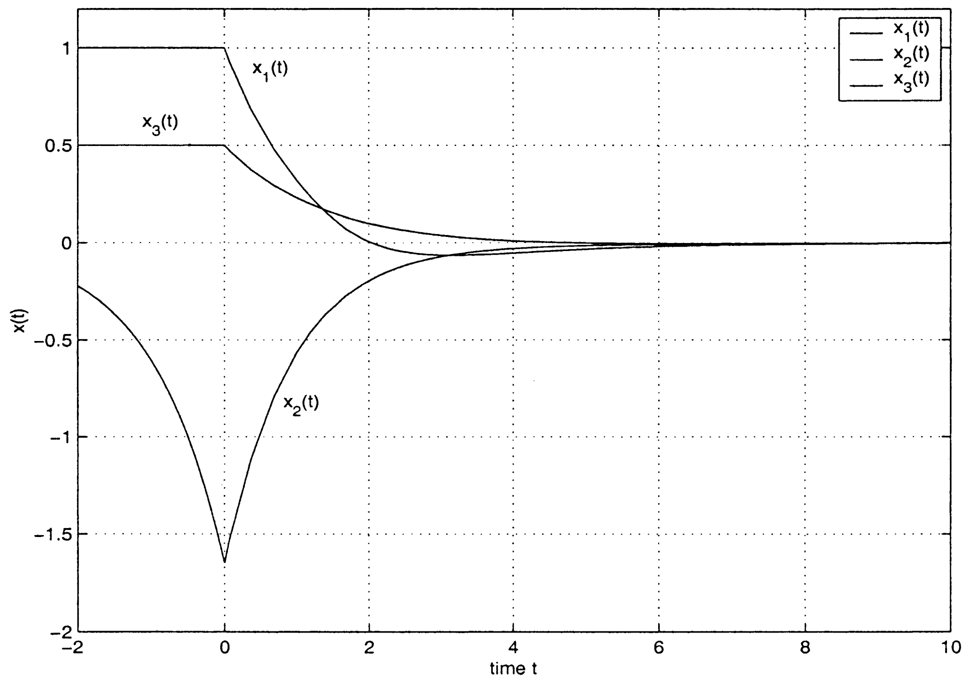

Fig. 1. Numeric simulation for componentwise exponential stability of system (3.49). Here we choose $\tau_1 = 1$, $\tau_2 = 1.4$, $\tau_3 = 2$ and the initial data are: $x_1(s) = 1$, $x_2(s) = -e^{s+0.5}$, and $x_3(s) = 0.5$ for $s \in [-2, 0]$.

$R_2 = 28/90$, and thus all conditions in Theorem 3.7 hold. Therefore zero is the unique equilibrium of Eq. (3.49), which is componentwise exponentially stable. An estimation for the exponential decay rate σ_5 can be obtained from Eq. (3.44). A numeric simulation for Example 3.3 is shown in Fig. 1.

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