Journal of Nonlinear Science (2025) 35:113
https://doi.org/10.1007/s00332-025-10205-z

Multi-periodicity in a Predator–Prey System with a Fear Effect

Yijin Zhao[1] · Weinian Zhang[1] · Xingfu Zou[2]

Received: 5 December 2024 / Accepted: 13 August 2025

Abstract
In this paper, we investigate multi-periodicity in a predator–prey system with a fear effect. Overcoming the difficulties in the calculation of focal values and the irreducible decomposition of the algebraic varieties of focal values under some restrictions of biological sense by using the stratified resultant elimination, we find that the weak focus is of multiplicity at most four. Based on this, we identify conditions for the occurrence of exactly one, two, or three small cycles from Hopf bifurcations by determining the independence of focal values. Moreover, applying the Poincaré–Bendixson theorem, we also explore large cycles that are periodic orbits different from those arising from Hopf bifurcations. Further, we prove the existence of the global attractor and obtain its structure by integrating all results about the system. Our work indicates that there are several ways of coexistence for the predator and prey, characterized by monostability, bistability of node–cycle type, and bistability of cycle–cycle type.

Keywords Predator–prey system · Fear effect · Global attractor · Small cycle · Large cycle

Mathematics Subject Classification 34C05 · 34D45 · 92D25

1 Introduction

It is known that there are three major types of interactions between two interacting species—cooperation, competition, and predator–prey (P–P). Among the three, the

Communicated by Alexander Lohse.

Supported by National Key R&D Program of China (2022YFA1005900) and NSFC (#12171336), and by NSERC of Canada (RGPIN-2022-04744).

✉ Xingfu Zou
xzou@uwo.ca

1 Department of Mathematics, Sichuan University, Chengdu 610064, Sichuan, People's Republic of China

2 Department of Applied Mathematics, University of Western Ontario, London, ON N6A 5B7, Canada

P–P type is the most complicated and it allows richer population dynamics. This is because a population dynamics model with the first two types of interactions is a monotone dynamical system for which convergence to equilibria is generic, while a dynamical model with a P–P type interaction generates a non-monotone dynamical system and hence can have richer dynamics. Therefore, P–P type systems are the most interesting not only mathematically but also biologically and have attracted a lot of researchers, and many of them have been focusing on the impact of various nonlinearities in the predation term represented by the functional response function.

On the other hand, more and more field experiments (e.g., Zanette et al. 2011) have shown that many animals can perceive the risks from their predators and respond accordingly to reduce the risk of predation, and the response to the fear can be in various forms. In order to explore the impact of such an anti-predation response of prey due to its fear of the predator, Wang et al. (2016) incorporated a fear factor into a classic P–P model with Holling-II type functional response in such a way that the fear effect can reduce the production, which reflects the scenario of the field study (Zanette et al. 2011). The model reads

$$\begin{cases} \frac{dx}{dt} = (\frac{r}{1+sy} - \delta_1)x - \delta_3 x^2 - \frac{pxy}{1+qx}, \\ \frac{dy}{dt} = -\delta_2 y + \frac{cpxy}{1+qx}, \end{cases} \tag{1.1}$$

where x and y represent the populations of prey and predator, respectively, r is the birth rate of prey, δ_3 represents the death rate of prey due to intra-species competition, δ_1 is the natural death rate of prey, δ_2 is the death rate of predators, c is the conversion efficiency of prey into predators, p is the maximum predation rate, and q is a limitation parameter of the growth of the predator population for increasing prey density. The fear effect in reducing the production is represented by the factor $f(s, y) = 1/(1+sy)$, which satisfies

$$\begin{aligned} f(0, y) = 1, \quad \lim_{s\to\infty} f(s, y) = 0, \quad \frac{\partial f(s,y)}{\partial s} < 0, \\ f(s, 0) = 1, \quad \lim_{y\to\infty} f(s, y) = 0, \quad \frac{\partial f(s,y)}{\partial y} < 0, \end{aligned}$$

where s reflects the level of fear. Here all parameters are positive, and in addition, $r > \delta_1$ is assumed to ensure that in the absence of the predator, the prey population settles at a positive equilibrium $(r - \delta_1)/\delta_3$.

It is shown in Wang et al. (2016) that system (1.1) has two boundary equilibria: $(0, 0)$ which is always unstable, and $((r - \delta_1)/\delta_3, 0)$ which is globally asymptotically stable if $cp - \delta_2 q \le 0$. In the complementary case $cp - \delta_2 q > 0$, they used the rescaling

$$y \mapsto sy, \quad x \mapsto \frac{(cp - \delta_2 q)x}{\delta_2}, \quad dt \mapsto \frac{\delta_2 dt}{(1 + qx)(1 + sy)}, \tag{1.2}$$

to simplify system (1.1) as

$$\begin{cases} \frac{dx}{dt} = x(a_1 + a_2 x - a_3 y - a_4 x^2 - a_5 xy - a_6 y^2 - a_4 x^2 y), \\ \frac{dy}{dt} = y(1+y)(x-1), \end{cases} \tag{1.3}$$

a polynomial system, where $a_1 := (r - \delta_1)/\delta_2$, $a_2 := (rq - \delta_1 q - \delta_3)/(cp - \delta_2 q)$, $a_3 := (\delta_1 s + p)/(s\delta_2)$, $a_4 := \delta_3\delta_2 q/(cp - \delta_2 q)^2$, $a_5 := (\delta_1 q + \delta_3)/(cp - \delta_2 q)$ and $a_6 := p/(s\delta_2)$. It is found that in addition to the two boundary equilibria $E_0 : (0, 0)$, $E_1 : ([(a_2^2 + 4a_1 a_4)^{1/2} + a_2]/(2a_4), 0)$, the system may have an interior equilibrium $E_2 : (1, y_*)$, where y_* is expressed in terms of parameters a_i s ($i = 1, ..., 6$), under some conditions. They discussed the stability of these equilibria in hyperbolic cases and showed the occurrence of E_2 as E_1 loses its stability. When E_2 is of center type, they discussed the sign of a quantity computed from a formula given in Perko (1996), which is actually equivalent to the focal value of multiplicity 1. They proved the existence of a periodic orbit in the case that E_2 is unstable and explained that the periodic orbit is a result of Hopf bifurcation. They also gave a condition for non-existence of periodic orbits by the Dulac–Bendixson criterion (Perko 1996). Moreover, they observed the existence of two limit cycles *numerically* with the choice of parameters $(r, \delta_1, \delta_2, \delta_3, c, p, q, s) = (0.12, 0.01, 0.05, 0.01, 0.4, 0.5, 0.6, 60)$.

The numerical results in Wang et al. (2016) suggest that the dynamics of (1.1) are far away from complete, and there may be some interesting dynamics un-discovered. In this paper, we revisit system (1.1), hoping to theoretically confirm the extra periodic orbit numerically observed in Wang et al. (2016) and even find more periodic orbits. This will enable us to better understand the structure of global attractors which allows us to identify more possibilities of survival patterns for the predator and prey. To this end, we use a different set of rescaling and change of variables, in terms of a_i s ($i = 1, ..., 6$), to transform (1.1) to a new planar polynomial system (2.2) in Sect. 2, which contains less parameters than (1.3) and hence has certain superiority over (1.3). Corresponding to the boundary equilibria E_0, E_1 and the interior equilibrium E_2 of system (1.3), system (2.2) also has two boundary equilibria $\tilde{E}_0$, $\tilde{E}_1$ and one interior equilibrium $\tilde{E}_2$. In Sect. 2, corresponding to the results on the hyperbolic boundary equilibria E_0 and E_1 for (1.3) in [26, Theorem 4.1], we explore the qualitative properties of the boundary equilibria in the nonhyperbolic case which is complementary to [26, Theorem 4.1]; we also further discuss equilibria at infinity, from which we deduce the existence of a global attractor in the closure of the first quadrant. In Sect. 3, we find small cycles (i.e., limit cycles of small amplitude), which arise via Hopf bifurcations from the equilibrium $\tilde{E}_2$ of center type. After computing focal values, we not only prove that $\tilde{E}_2$ is a *weak focus of multiplicity at most 4* but also give conditions for the occurrence of *exactly one, two, or three small cycles* from Hopf bifurcations, by the stratified resultant elimination. In Sect. 4, by using the Poincaré–Bendixson theorem, we find another periodic orbit which is different from those arising from Hopf bifurcations and is thus referred to as a *large cycle*. Finally in Sect. 5, we briefly summarize the main results on the structure of the global attractor and explain how the structure affects the population dynamics of the involved predator and prey. Our work shows that, in addition to the bistability of node–cycle type (one stable equilibrium coexists with a stable periodic oscillation) found in Wang et al. (2016), this model system can also support the bistability of cycle–cycle type (one stable periodic orbit coexisting with another stable periodic orbit). All these further new results clearly illustrate the complexity of a predator–prey system when a fear effect is incorporated.

2 Equilibria and Global Attractor

In order to avoid the case $cp - \delta_2 q = 0$, in which the rescaling used in (1.2) is not available, we make a different transformation

$$y \mapsto sy, \quad x \mapsto qx, \quad dt \mapsto \frac{\delta_2 dt}{(1+qx)(1+sy)}, \tag{2.1}$$

to transform system (1.1) to another new planar polynomial differential system

$$\begin{cases} \frac{dx}{dt} = x\big[(b_1 - b_2) + (b_1 - b_2 - b_3)x - (b_2 + b_4)y - b_3x^2 \\ \qquad -(b_2 + b_3)xy - b_4y^2 - b_3x^2y\big] \\ \quad =: X(x, y), \\ \frac{dy}{dt} = y(y+1)\big[(b_5 - 1)x - 1\big] \\ \quad =: Y(x, y), \end{cases} \tag{2.2}$$

where $b_1 := r/\delta_2$, $b_2 := \delta_1/\delta_2$, $b_3 := \delta_3/(\delta_2 q)$, $b_4 := p/(s\delta_2)$, and $b_5 := cp/(\delta_2 q)$. The form of (2.2) is of the same degree 4 as (1.3), but it only contains *five* parameters, in contrast with *six* parameters in (1.3), and is thus a bit more convenient to analyze. Moreover, (2.2) is also more convenient than (1.3) when discussing the non-hyperbolicity of the boundary equilibrium, because for (1.3), the expression for the equilibrium E_1 is irrational.

Obviously all b_i's are positive with $b_1 > b_2$ (since $r > \delta_1$). Furthermore, (2.2) is of Gaussian Type, and thus, the component $x(t)$ (resp. $y(t)$) of a solution remains positive provided that $x_0 > 0$ (resp. $y_0 > 0$).

Corresponding to the three equilibria E_0, E_1, and E_2 for system (1.3), system (2.2) also has three possible equilibria $\tilde{E}_0 = (0, 0)$, $\tilde{E}_1 = ((b_1 - b_2)/b_3, 0)$ and $\tilde{E}_2 = (1/(b_5 - 1), \tilde{y}_*)$. Here $\tilde{y}_* := -(\varpi_2 + (\varpi_2^2 - 4\varpi_1\varpi_3)^{1/2})/(2\varpi_1)$ with

$$\varpi_1 := -b_4(b_5 - 1)^2, \quad \varpi_2 := -(b_5 - 1)^2 b_4 - b_5(b_5 - 1)b_2 - b_3b_5,$$
$$\varpi_3 := b_5(-(b_5 - 1)b_1 + (b_5 - 1)b_2 + b_3).$$

Obviously, the two boundary equilibria $\tilde{E}_0$ and $\tilde{E}_1$ always exist (noting that $r > \delta_1$ is pre-assumed) with $\tilde{E}_0$ being a saddle. As for $\tilde{E}_1$, it is a stable node if either [$0 < b_5 \le 1$ and $b_1 > b_2$] or [$b_5 > 1$ and $b_2 < b_1 < \beta_1$], and a saddle if $b_5 > 1$ and $b_1 > \beta_1$. $\tilde{E}_2$ exists if $b_5 > 1$ and $b_1 > \beta_1$, and it is a sink (resp. source) if $b_5 > 1$ and $\beta_1 < b_1 < \beta_2$ (resp. $b_5 > 1$ and $b_1 > \beta_2$), where a sink is a stable node or focus, a source is an unstable node or focus,

$$\beta_1 := b_2 + \frac{b_3}{b_5 - 1} \quad \text{and} \quad \beta_2 := \Big(b_2 + \frac{b_3(b_5+1)}{b_5 - 1}\Big)\Big(1 + \frac{b_3 b_5^2}{b_4(b_5-1)^2}\Big). \tag{2.3}$$

Notice that $\tilde{E}_2$ is of center type (need to identify focus from center) if $b_5 > 1$ and $b_1 = \beta_2$. $\tilde{E}_1$ is degenerate with eigenvalues $-b_3b_5/(b_5-1)^2$ and 0 if $b_5 > 1$ and $b_1 = \beta_1$, where β_1 is given in (2.3). In the following, we give qualitative properties of the boundary equilibrium $\tilde{E}_1$ in degenerate cases or at infinity.

Theorem 2.1 (i) *If $b_5 > 1$ and $b_1 = \beta_1$, equilibrium $\tilde{E}_1$ is a saddle node, which has a parabolic sector in the first quadrant.* (ii) *System* (2.2) *has two equilibria I_x and I_y at infinity in the first quadrant, which locate on the positive half x-axis and the positive half y-axis, respectively, and both are degenerate. Near I_x the system has a unique orbit in each direction* (*including the x-axis and its vertical one*), *which leaves from I_x; near I_y the system has a unique orbit in the direction of the y-axis, which leaves from I_y, and another unique orbit vertical to the y-axis, which approaches I_y.*

Proof We first discuss $\tilde{E}_1$, for which we assumed that $b_5 > 1$ and $b_1 = \beta_1$. With the linear transformation

$$x \mapsto -\tfrac{1}{b_5-1} + x + \left(\tfrac{b_2}{b_3} + \tfrac{b_4(b_5-1)}{b_3b_5} + \tfrac{1}{b_5-1}\right)y, \quad y \mapsto \tfrac{1}{b_3b_5(b_5-1)}y,$$

which translates $\tilde{E}_1$ to the origin and diagonalizes the linear part, we can simplify system (2.2) as

$$\begin{cases} \frac{dx}{dt} = p_{10}x + p_{20}x^2 + p_{11}xy + p_{02}y^2 + O(|(x,y)^3|), \\ \frac{dy}{dt} = \qquad\qquad\qquad\qquad q_{11}xy + q_{02}y^2 + O(|(x,y)^3|), \end{cases} \tag{2.4}$$

where $p_{10} := -b_3b_5/(b_5-1)^2$, $p_{20} := -b_3(b_5+1)/(b_5-1)$, $q_{11} := b_5 - 1$, $q_{02} := -(b_5-1)(b_5(b_5-1)b_2 + (b_5-1)^2b_4 + b_5b_3)$, and p_{11}, p_{02} are given in Appendix. By the Center Manifold Theorem (Guckenheimer and Holmes 1983, Theorem 3.2.1, p.127), system (2.4) has a C^2 center manifold $x = h(y)$ near the origin, which is tangent to the curve $x = 0$ at origin in the (x, y)-space. Clearly, h is of the form $h(y) = \varpi_4 y^2 + o(y^2)$ with indeterminate ϖ_4. By the invariant property, we have the equality $\dot{x} = h_y\dot{y}$. Substituting the equations of (2.4) in the equality and comparing the coefficients of y^2, we obtain ϖ_4, given in Appendix. Thus, restricted to the manifold, system (2.4) becomes the equation

$$\tfrac{dy}{dt} = q_{11}h(y)y + q_{02}y^2 + O(|(h(y), y)^3|) = q_{02}y^2 + O(y^3),$$

which shows that $\tilde{E}_1$ is a saddle node in system (2.4). Since p_{10} and q_{02} are both negative and the stable manifold of $\tilde{E}_1$ lies on the positive half x-axis, $\tilde{E}_1$ has a parabolic sector in the first quadrant.

For possible equilibria at infinity, applying the Poincaré transformation $x = 1/z$, $y = u/z$, we change system (2.2) into the form

$$\begin{cases} \frac{\mathrm{d}u}{\mathrm{d}t_1} = u\{b_3u + b_3z + (b_3+b_2+b_5-1)uz + (b_3+b_2+b_5-b_1 \\ \qquad -1)z^2 + b_4u^2z + (b_4+b_2-1)uz^2 + (b_2-b_1-1)z^3\} \quad =: U(u,z), \\ \frac{\mathrm{d}z}{\mathrm{d}t_1} = z\{b_3u + b_3z + (b_3+b_2)uz + (b_3+b_2-b_1)z^2 \\ \qquad + b_4u^2z + (b_4+b_2)uz^2 + (b_2-b_1)z^3\} \quad =: Z(u,z), \end{cases} \tag{2.5}$$

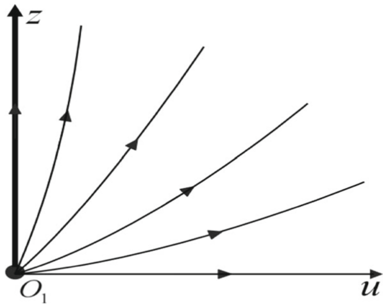


Fig. 1 Dynamics near O_1 of system (2.5)

where $dt_1 = dt/z^3$. As indicated in Zhang et al. (1992, section 5.1, p.325-326) (or Perko 1996, Theorem 2, p.369), equilibria of (2.5) on the u-axis correspond to equilibria of equation (2.2) at infinity in the first quadrant excluding the infinity I_y on the y-axis. Solving equations $U(u, 0) = 0$ and $Z(u, 0) = 0$, we obtain a unique equilibrium $O_1 : (0, 0)$ of system (2.5) on the positive half u-axis, which corresponds to an equilibrium I_x of (2.2) at infinity on the positive half x-axis. Note that O_1 is of fully null degeneracy because the Jacobian matrix $J|_{O_1}$ is zero. Applying the Briot–Bouquet transformation (Briot and Bouquet 1856; Zhang et al. 1992)

$$u = u, \quad z = u\tilde{z} \tag{2.6}$$

to blow up O_1 along the z-axis, we change system (2.5) into the form

$$\begin{cases} \frac{du}{dt_2} = b_3 + b_3\tilde{z} + u\{(b_3 + b_2 + b_5 - 1)\tilde{z} + b_4 u\tilde{z} + (b_3 + b_2 + b_5 - b_1 \\ \qquad - 1)\tilde{z}^2 + (b_4 + b_2 - 1)u\tilde{z}^2 + (b_2 - b_1 - 1)u\tilde{z}^3\} =: \tilde{U}(u, \tilde{z}), \\ \frac{d\tilde{z}}{dt_2} = \tilde{z}^2(1 + \tilde{z})\{(1 - b_5) + u\tilde{z}\} =: \tilde{Z}(u, \tilde{z}), \end{cases} \tag{2.7}$$

where $dt_2 = dt_1/u^2$. Since $\tilde{U}(0, \tilde{z}) = b_3(1 + \tilde{z}) > 0$, we see that (2.7) has no equilibria on the positive half $\tilde{z}$-axis and all orbits on the half-plane $\tilde{z} > 0$ cross the $\tilde{z}$-axis from the left to the right, which by the blowing down (the inverse of (2.6)) gives phase portrait Fig. 1 of (2.5) in the first quadrant except for orbits in the direction of the z-axis because the transformation (2.6) is not invertible for $u = 0$. In order to detect orbits in the direction of the z-axis, we apply another Briot–Bouquet transformation $u = z\hat{u}$, $z = z$ to blow up in the direction of the u-axis, which changes system (2.5) into the form

$$\begin{cases} \frac{d\hat{u}}{dt_3} = \hat{u}(1 + \hat{u})\{(b_5 - 1) - z\} =: \hat{U}(\hat{u}, z), \\ \frac{dz}{dt_3} = b_3 + b_3\hat{u} + z\{(b_3 + b_2 - b_1) + (b_3 + b_2)\hat{u} + (b_2 \\ \qquad - b_1)z + (b_2 + b_4)\hat{u}z + b_4\hat{u}^2 z\} =: \hat{Z}(\hat{u}, z), \end{cases} \tag{2.8}$$

where $dt_3 = dt_1/z^2$. Since $\hat{Z}(0, 0) = b_3 > 0$, we see that system (2.8) does not have an equilibrium at the origin, through which a unique orbit passes from down to up along the z-axis. This, by the corresponding blowing down, implies that there is a unique orbit on the positive half z-axis which leaves O_1 and goes upward (see the bold line in Fig. 1). The above shows that near I_x system (2.2) has a unique orbit in each direction (including the x-axis and its vertical one), which leaves from I_x.

Similar to the Briot–Bouquet transformation (2.6), the Poincaré transformation used before (2.5) also has singularity, i.e., it is not invertible for $x = 0$. We need to discuss orbits on the y-axis near the point at infinity. For this purpose, we apply another Poincaré transformation $x = v/z$, $y = 1/z$ to system (2.2), which leads to the form

$$\begin{cases} \frac{\mathrm{d}v}{\mathrm{d}t_4} = v\big\{-b_4z - b_3v^2 - (b_3+b_2+b_5-1)vz - (b_4+b_2-1)z^2 \\ \qquad\qquad - b_3v^2z + (b_1-b_3-b_2-b_5+1)vz^2 + (b_1-b_2+1)z^3\big\}, \\ \frac{\mathrm{d}z}{\mathrm{d}t_4} = z\big\{(1-b_5)vz + z^2 + (1-b_5)vz^2 + z^3\big\}, \end{cases} \tag{2.9}$$

where $dt_4 = dt/z^3$. We only need to consider the origin $O_2 : (0, 0)$, which actually is an equilibrium of (2.9), i.e., I_y is an equilibrium of (2.9) at infinity correspondingly. This equilibrium is of fully null degeneracy because the Jacobian matrix $J|_{O_2}$ is zero. Applying the transformation $v = v$, $z = \hat{z}v^2$, which blows up O_2 twice along the z-axis, we change system (2.9) into the form

$$\begin{cases} \frac{\mathrm{d}v}{\mathrm{d}t_5} = -b_3v + v\big\{-b_4\hat{z} - (b_3+b_2+b_5-1)v\hat{z} - b_3v^2\hat{z} \\ \qquad\qquad - (b_4+b_2-1)v^2\hat{z}^2 - (b_3-b_1+b_2+b_5 \\ \qquad\qquad - 1)v^3\hat{z}^2 + (b_1-b_2+1)v^4\hat{z}^3\big\} \qquad =: \check{V}(v,\hat{z}), \\ \frac{\mathrm{d}\hat{z}}{\mathrm{d}t_5} = 2b_3\hat{z} + \hat{z}\big\{2b_4\hat{z} + (2b_3+2b_2+b_5-1)v\hat{z} + 2b_3v^2\hat{z} \\ \qquad\qquad + (2b_4+2b_2-1)v^2\hat{z}^2 + (2b_3-2b_1+2b_2 \\ \qquad\qquad + b_5-1)v^3\hat{z}^2 - (2b_1-2b_2+1)v^4\hat{z}^3\big\} \qquad =: \check{Z}(v,\hat{z}), \end{cases} \tag{2.10}$$

where $dt_5 = dt_4/v^2$. Solving $\check{V}(0, \hat{z}) = \check{Z}(0, \hat{z}) = 0$, we see that the origin of system (2.10) is a unique equilibrium on the closure of the positive $\hat{z}$-axis and has eigenvalues $-b_3$ and $2b_3$, implying that it is a saddle. Thus, by blowing down, we obtain phase portrait Fig. 2 of (2.9) in the first quadrant, where it is unknown whether there is an orbit in the direction of the z-axis. For this unknown one, we apply another Briot–Bouquet transformation $v = \breve{v}z$, $z = z$ to blow up in the direction of the v-axis, which changes system (2.9) into the form

$$\begin{cases} \frac{\mathrm{d}\breve{v}}{\mathrm{d}t_6} = -b_4\breve{v} + \breve{v}\big\{-(b_4+b_2)z - (b_3+b_2)\breve{v}z + (b_1-b_2)z^2 \\ \qquad\qquad - b_3\breve{v}^2z + (-b_3+b_1-b_2)\breve{v}z^2 - b_3\breve{v}^2z^2\big\} \quad =: \check{V}(\breve{v},z), \\ \frac{\mathrm{d}z}{\mathrm{d}t_6} = \qquad z\big\{z - (b_5-1)\breve{v}z + z^2 - (b_5-1)\breve{v}z^2\big\} \qquad =: \check{Z}(\breve{v},z), \end{cases} \tag{2.11}$$

where $dt_6 = dt_4/z$. Since $\check{V}(0, 0) = \check{Z}(0, 0) = 0$, we see that the origin of system (2.11) is an equilibrium. One can compute its two eigenvalues $-b_4$ and 0, showing that it has a center manifold $\breve{v} = 0$. The restriction of system (2.11) to the manifold is

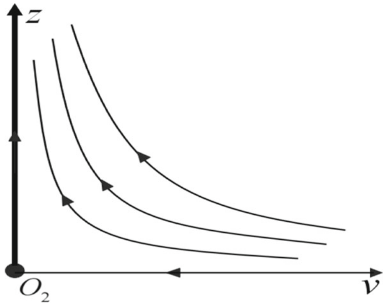


Fig. 2 Dynamics near O_2 of system (2.9)

the equation $dz/dt_6 = z^2 + z^3$, implying that the origin is a saddle node and there is a unique orbit on the positive half z-axis (see the bold line in Fig. 2), but the infinitely many orbits in the direction of z-axis lie in the half-plane $z < 0$ (which cannot be seen in the first quadrant). Thus, near I_y system (2.2) has a unique orbit in the direction of the y-axis, which leaves from I_y, and another unique orbit vertical to the y-axis, which approaches I_y. The proof is completed. □

Next, we apply the results of equilibria obtained in Wang et al. (2016) and Theorem 2.1 to give the existence of global attractors, which is a compact invariant set attracting all bounded subsets in the closure of the first quadrant.

Theorem 2.2 *System* (2.2) *has a global attractor in the closure of the first quadrant.*

Proof Section 4.1 of Wang et al. (2016) shows that system (2.2) has two and three equilibria in the cases **(C1)**: either $0 < b_5 \leq 1$ and $b_1 > b_2$ or $b_5 > 1$ and $b_2 < b_1 \leq \beta_1$, and **(C2)**: $b_5 > 1$ and $b_1 > \beta_1$, respectively. In case **(C1)**, the system has two equilibria $\tilde{E}_0 : (0, 0)$ and $\tilde{E}_1 : ((b_1 - b_2)/b_3, 0)$. Moreover, $\tilde{E}_0$ is a saddle and $\tilde{E}_1$ has a stable parabolic sector in the closure of the first quadrant as indicated in Wang et al. (2016) and Theorem 2.1. From qualitative properties of equilibria at infinity, we similarly see that the union of $\tilde{E}_0$, $\tilde{E}_1$ and the orbit connecting with them is the global attractor.

The most complicated case is **(C2)**, where the system has three equilibria $\tilde{E}_0$, $\tilde{E}_1$ and $\tilde{E}_2 : (1/(b_5 - 1), \tilde{y}_*)$. Moreover, $\tilde{E}_0$ and $\tilde{E}_1$ are both saddles. Note that $\phi^t(P) := \phi(t, P)$, which denotes the solution of system (2.2) initiated from the point $P \in \mathbb{R}^2_+$, where $\mathbb{R}_+$ represents $[0, \infty)$, defines a C^0 semigroup for $t \geq 0$ on the complete metric space $\mathbb{R}^2_+$. In order to find a global attractor, a compact invariant set attracting all bounded subsets of $\mathbb{R}^2_+$, by Corollary 1.1.4 of Cholewa et al. (2000, p.11) or Theorem 9.1 of Hale (2006, p.500) we need to claim that **(K1)** ϕ^t is *asymptotically smooth*, i.e., each nonempty, closed, bounded, positively invariant set in $\mathbb{R}^2_+$ contains a nonempty, compact subset which attracts it, **(K2)** ϕ^t keeps *orbits of bounded sets bounded*, i.e., for any bounded set $\mathcal{B} \subset \mathbb{R}^2_+$, there is a number $t_{\mathcal{B}} \geq 0$, such that $\bigcup_{t \geq t_{\mathcal{B}}} \phi^t(\mathcal{B})$ is bounded in $\mathbb{R}^2_+$, and **(K3)** ϕ^t is *point dissipative*, i.e., there is a bounded set $\mathcal{D} \subset \mathbb{R}^2_+$ such that for any point $P \in \mathbb{R}^2_+$ there is a number $t_P \geq 0$ such that $\phi^t(P) \in \mathcal{D}$ if $t \geq t_P$. Claim **(K1)** is obvious because Hale said "Any ordinary differential equation in $\mathbb{R}^n$ for which the solutions are defined for all $t \geq 0$ defines a dynamical system which is asymptotically smooth" Definition 8.1 of Hale (2006, p.498), which also holds in

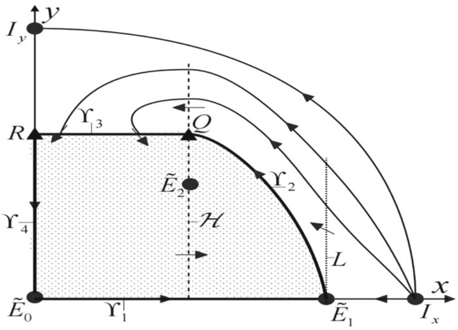


Fig. 3 Construction of $\mathcal{D}$

$\mathbb{R}^2_+$, a closed subset of $\mathbb{R}^2$. **(K2)** is also true; otherwise, there is a point $\tilde{P}$ such that $\bigcup_{t\geq 0}\phi^t(\tilde{P})$ is unbounded in $\mathbb{R}^2_+$, a contradiction to the fact that the positive half x-axis and the positive half y-axis are both invariant and the equilibria at infinity are unstable by Theorem 2.1. Finally, we prove **(K3)**. For this, we denote the unstable manifold of $\tilde{E}_1$ in $\mathbb{R}^2_+$ by $\mathcal{W}^u$ and need to claim that **(W)** $\mathcal{W}^u$ first intersects the horizontal isocline $\mathcal{H}: x = 1/(b_5-1)$ for all $y \geq 0$ at a point, denoted by $Q: (1/(b_5-1), \hat{y})$, and Q does not lie below $\tilde{E}_2$, i.e., $\hat{y} \geq \tilde{y}_*$. We make the auxiliary vertical line $L: x = (b_1-b_2)/b_3$ for all $y > 0$, which connects the point $\tilde{E}_1$ in Fig. 3. On the line L, we can check that all orbits cross from right to left and lower to upper because

$$\frac{dx}{dt}|_{x=(b_1-b_2)/b_3} = -y(b_1-b_2)(b_3b_4y + b_3(b_1+b_4) + b_1(b_1-b_2))/b_3^2 < 0,$$
$$\frac{dy}{dt}|_{x=(b_1-b_2)/b_3} = y(y+1)(b_5-1)(b_1-\beta_1)/b_3 > 0$$

on L as considered in case **(C2)**. Note that no equilibria lie on the right of $\mathcal{H}$ for $y > 0$ and equilibria at infinity are both unstable, which implies that $\mathcal{W}^u$ first intersects $\mathcal{H}$ for all $y \geq 0$ at a point Q. Further, on $\mathcal{H}$ we have $X(1/(b_5-1), y) > 0$ (resp. < 0) if $0 < y < \tilde{y}_*$ (resp. $y > \tilde{y}_*$), implying that each orbit starting from the right of $\mathcal{H}$ for $y > 0$ crosses $\mathcal{H}$ not below $\tilde{E}_2$ to the left. Thus, the intersection Q lies not below $\tilde{E}_2$. The claim **(W)** is proved. Finally, let Υ_1 be the orbit connecting $\tilde{E}_0$ and $\tilde{E}_1$, Υ_2 the orbit connecting $\tilde{E}_1$ and Q, Υ_3 the line segment starting from Q horizontally and intersecting to the y-axis at a point, denoted by R, and Υ_4 the orbit connecting R and $\tilde{E}_0$. Clearly,

$$\Upsilon_1 := \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq \tfrac{b_1-b_2}{b_3}, y = 0\},\ \Upsilon_2 := \{\phi^t(Q) \in \mathbb{R}^2 : -\infty < t \leq 0\},$$
$$\Upsilon_3 := \{(x, y) \in \mathbb{R}^2 : 0 \leq x < \tfrac{1}{b_5-1}, y = \hat{y}\}, \qquad \Upsilon_4 := \{(x, y) \in \mathbb{R}^2 : x = 0, 0 \leq y \leq \hat{y}\},$$

and $\Upsilon := \cup_{i=1}^4 \Upsilon_i$ is a simple closed curve. Note that no orbits cross Υ_i s $(i = 1, 2, 4)$ because of the uniqueness of solutions. On Υ_3, all orbits cross from upper to lower because $\frac{dy}{dt}|_{y=\hat{y}} = \hat{y}(\hat{y}+1)((b_5-1)x-1) < 0$ for any $(x, \hat{y}) \in \Upsilon_3$ since $0 \leq x < 1/(b_5-1)$ and $\hat{y} > 0$ on Υ_3. This shows that the set $\mathcal{D}$ with the boundary Υ is what we need in **(K3)**. □

Remark that a transcritical bifurcation may occur at $\tilde{E}_1$. In fact, introducing a bifurcation parameter $\epsilon := b_1 - \beta_1$, translating $\tilde{E}_1 : ((b_1-b_2)/b_3, 0)$ to the origin, diagonalizing the linear part and suspending system (2.2) with $\dot{\epsilon}=0$, we get

$$\begin{cases} \frac{dx}{dt} = \tilde{p}_{100}x+\tilde{p}_{200}x^2+\tilde{p}_{110}xy+\tilde{p}_{101}x\epsilon+\tilde{p}_{020}y^2+\tilde{p}_{011}y\epsilon+O(|(x,y,\epsilon)^3|), \\ \frac{dy}{dt} = \tilde{q}_{110}xy+\tilde{q}_{020}y^2+\tilde{q}_{011}y\epsilon+O(|(x,y,\epsilon)^3|), \\ \frac{d\epsilon}{dt} = 0, \end{cases} \tag{2.12}$$

where $\tilde{p}_{100}=p_{10}$, $\tilde{p}_{200}=p_{20}$, $\tilde{p}_{101}=-(b_5+1)/(b_5-1)$, $\tilde{q}_{110}=q_{11}$, $\tilde{q}_{020}=q_{02}/(b_5b_3(b_5-1))$, $\tilde{q}_{011}=q_{11}/b_3$, and $\tilde{p}_{110}$, $\tilde{p}_{020}$, $\tilde{p}_{011}$ are given in Appendix. Similar to the proof of Theorem 2.1 on (i), we obtain a C^2 center manifold $x=h(y,\epsilon)=\varpi_5y^2+\varpi_6y\epsilon+o(|(y,\epsilon)^2|)$, where $\varpi_6:=b_3b_5((b_5+1)b_3+(b_5-1)^2)((b_5-1)^2b_4-b_5(b_5+1)b_3)$ and ϖ_5 is given in Appendix. Thus, the restriction of system (2.12) to the manifold is the equation $\frac{dy}{dt}=\check{q}_1(\epsilon)y+\check{q}_2(\epsilon)y^2+O(y^3)$, where

$$\check{q}_1(\epsilon):=\tilde{q}_{011}\epsilon \text{ and } \check{q}_2(\epsilon):=\tilde{q}_{020}+\frac{(b_4+b_5)(b_5-1)^3}{b_3b_5((b_5+1)b_3+(b_5-1)^2)}\epsilon.$$

One can check that $\check{q}_1(0)=0$ and $\check{q}_2(0)=\tilde{q}_{020}<0$. Thus, we use the time-reversing $t\mapsto -t/\check{q}_2(\epsilon)$ to simplify the restricted equation as $\frac{dy}{dt}=\check{q}_1(\epsilon)y-y^2+O(y^3)$, where

$$\check{q}_1(\epsilon):=-\check{q}_1(\epsilon)/\check{q}_2(\epsilon)=-\frac{\tilde{q}_{011}}{\tilde{q}_{020}}\epsilon+O(\epsilon^2).$$

Since $\tilde{q}_{011}/\tilde{q}_{020}<0$, we see that a transcritical bifurcation occurs at $\tilde{E}_1$ by Chow and Hale (1982, p.145). Therefore, if $b_5>1$ and $b_1=\beta_1$, $\tilde{E}_2$ and $\tilde{E}_1$ coincide as a saddle node. When $b_5>1$ and $b_1>\beta_1$, $\tilde{E}_2$ moves to the first quadrant and becomes a stable node. Moreover, $\tilde{E}_1$ becomes a saddle.

3 Small Cycles

As indicated in (4.17) and in (4.33) of Wang et al. (2016), the determinant of the Jacobian matrix at the interior equilibrium E_2 is positive, and its trace is equal to zero as

$$-(a_5+2a_4)y_*+a_2-2a_5=0, \tag{3.1}$$

where y_*, being the ordinate of E_2, is a function of $(a_1,...,a_6)$. In this case, E_2 is of the center type, which has a pair of conjugate pure imaginary eigenvalues. They employed the formula given in Theorem 1 of Perko's book (Perko 1996, p.34) to compute

$$\begin{aligned}\sigma_*:=&-8a_4(a_2-2a_4)^2a_6^2-(a_5+2a_4)(-a_5\\&+6a_4a_5-2a_4+8a_3a_4+4a_4^2)(a_2-2a_4)a_6\\&-a_4(a_5+2a_4)^2(2a_3+a_5)(a_3+a_4+a_5),\end{aligned} \tag{3.2}$$

which is actually equivalent to the focal value of multiplicity 1 and discussed the sign change of the quantity. Use (3.1) to eliminate a_6 in (3.2) and express σ_* as the quadratic function $\sigma_*(a_1) = p_1 a_1^2 + p_2 a_1 + p_3$, where the coefficients p_i $(i = 1, 2, 3)$ are polynomials of $(a_2, ..., a_5)$. Choose values of parameters so as to show that there can be 0, 1 or 2 positive real zeros of the quadratic function, which implies that one limit cycle can arise via the Hopf bifurcation as σ varies to be nonzero. They further found two limit cycles numerically for $(r, \delta_1, \delta_2, \delta_3, c, p, q, s) = (0.12, 0.01, 0.05, 0.01, 0.4, 0.5, 0.6, 60)$. However, for σ_* to be zero, it is not determined yet whether $\tilde{E}_2$ is a weak focus or a center. Moreover, the maximal multiplicity needs to be considered in the case of weak focus.

As known in the first two paragraphs of Sect. 2, the unique interior equilibrium $\tilde{E}_2$ is of center type if

$$(b_1, b_2, b_3, b_4, b_5) \in \Xi_0 := \{(b_1, b_2, b_3, b_4, b_5) \in (0, \infty)^5 : \ b_1 = \beta_2, \ b_5 > 1\}, \tag{3.3}$$

where β_2 is defined in (2.3). In this case, the linearization of system (2.2) at $\tilde{E}_2$ has a pair of conjugate pure imaginary eigenvalues $\pm i\omega$, where

$$\omega := \left\{ \frac{b_3 b_5^2}{(b_5-1)^2}\left(1 + \frac{b_3 b_5^2}{b_4(b_5-1)^2}\right)\left(1 + \frac{b_2 b_5}{b_4(b_5-1)} + \frac{b_3 b_5(2b_5+1)}{b_4(b_5-1)^2}\right) \right\}^{1/2}.$$

In the following, we give the multiplicity of $\tilde{E}_2$ being a weak focus, and conditions for exactly numbers of small cycles bifurcated from $\tilde{E}_2$. For convenience, let ϵ be a sufficiently small perturbation parameter, $B_-(x_0) := (x_0 - \epsilon, x_0]$, $B_+(x_0) := [x_0, x_0 + \epsilon)$, $B_-^o(x_0) := (x_0 - \epsilon, x_0)$, $B_+^o(x_0) := (x_0, x_0 + \epsilon)$, $B^o(x_0) := (x_0 - \epsilon, x_0 + \epsilon)$, and

$$\begin{aligned}
D_1 &:= \{(b_3, b_5) \in \mathbb{R}^2 : 1 < b_5 \le \beta_1^{(33)},\\
&\qquad b_3 = -(2b_5^2 - 6b_5 + 3)(b_5 - 1)^2/(3(2b_5 - 1))\},\\
D_2 &:= \{(b_3, b_5) \in \mathbb{R}^2 : \beta_1^{(33)} < b_5 \le \beta_1^{(34)},\\
&\qquad b_3 = -(2b_5^2 - 6b_5 + 3)(b_5 - 1)^2/(3(2b_5 - 1))\},\\
D_3 &:= \{(b_3, b_5) \in \mathbb{R}^2 : \beta_1^{(34)} < b_5 \le \beta_2^{(36)},\\
&\qquad b_3 = -(2b_5^2 - 6b_5 + 3)(b_5 - 1)^2/(3(2b_5 - 1))\},\\
D_4 &:= \{(b_3, b_5) \in \mathbb{R}^2 : \beta_2^{(36)} < b_5 < \beta_1^{(35)},\\
&\qquad b_3 = -(2b_5^2 - 6b_5 + 3)(b_5 - 1)^2/(3(2b_5 - 1))\},\\
D_5 &:= \{(b_3, b_5) \in \mathbb{R}^2 : \beta_1^{(35)} < b_5 < (3 + \sqrt{3})/2,\\
&\qquad b_3 = -(2b_5^2 - 6b_5 + 3)(b_5 - 1)^2/(3(2b_5 - 1))\},\\
D_6 &:= \{(b_3, b_5) \in \mathbb{R}^2 : b_5 = \beta_1^{(35)}, b_3 = -(2b_5^2 - 6b_5 + 3)(b_5 - 1)^2/(3(2b_5 - 1))\},
\end{aligned}$$

where $\beta_1^{(33)}, \beta_1^{(34)}, \beta_1^{(35)}, \beta_2^{(36)}, \beta_2^{(23)}, \beta_1^{(21)}, \beta_1^{(24)}, \beta_2^{(24)}, \beta_3^{(24)}, \beta_1^{(11)}, \beta_2^{(11)}$, and $\beta_3^{(11)}$, shown in Appendix, are zeros of polynomials.

Theorem 3.1 *For $(b_1, b_2, b_3, b_4, b_5) \in \Xi_0$, the equilibrium $\tilde{E}_2$ of system (2.2) is a weak focus of multiplicity at most 4. Moreover, exactly k small cycles arise from $\tilde{E}_2$ if*

Table 1 Sets $\tilde\Lambda_1^i$, $\tilde\Lambda_2^j$, $\tilde\Lambda_3$ for numbers of small-cycles. Sets $\tilde\Lambda_1^i$, $\tilde\Lambda_2^j$, $\tilde\Lambda_3$ for numbers of small-cycles for $(b_3, b_5) \in D_1$

b_4	b_2	b_1	Number	Label
$b_4 \in (0, \infty)$	$b_2 \in (0, \infty)$	$b_1 \in B_-(\beta_2)$	0	
		$b_1 \in B_+^o(\beta_2)$	1	$\tilde\Lambda_1^{12}$
		$(b_3, b_5) \in D_1$		

Table 2 Sets $\tilde\Lambda_1^i$, $\tilde\Lambda_2^j$, $\tilde\Lambda_3$ for numbers of small-cycles. Sets $\tilde\Lambda_1^i$, $\tilde\Lambda_2^j$, $\tilde\Lambda_3$ for numbers of small-cycles for $(b_3, b_5) \in D_2$

b_4	b_2	b_1	Number	Label
$b_4 \in (0, \beta_2^{(23)})$	$b_2 \in (0, \beta_2^{(11)})$	$b_1 \in B_-^o(\beta_2)$	1	$\tilde\Lambda_1^{15}$
		$b_1 \in B_+(\beta_2)$	0	
	$b_2 \in (\beta_2^{(11)}, \infty)$	$b_1 \in B_-(\beta_2)$	0	
		$b_1 \in B_+^o(\beta_2)$	1	$\tilde\Lambda_1^{16}$
$b_4 \in [\beta_2^{(23)}, \infty)$	$b_2 \in (0, \infty)$	$b_1 \in B_-(\beta_2)$	0	
		$b_1 \in B_+^o(\beta_2)$	1	$\tilde\Lambda_1^{17}$
$b_4 \in (0, \beta_2^{(23)})$	$b_2 \in B_-^o(\beta_2^{(11)})$	$b_1 \in B_-^o(\beta_2)$	2	$\tilde\Lambda_2^{3}$
		$b_1 \in B_+(\beta_2)$	1	$\tilde\Lambda_1^{18}$
	$b_2 \in B_+(\beta_2^{(11)})$	$b_1 \in B_-(\beta_2)$	0	
		$b_1 \in B_+^o(\beta_2)$	1	$\tilde\Lambda_1^{19}$
		$(b_3, b_5) \in D_2$		

$(b_1, b_2, b_3, b_4, b_5) \in \tilde\Lambda_k$, *for any* $k = 1, 2, 3$, *where* $\tilde\Lambda_1 := \cup_{i=1}^{26}\tilde\Lambda_1^i$, $\tilde\Lambda_2 := \cup_{j=1}^{8}\tilde\Lambda_2^j$, *and* $\tilde\Lambda_3$, $\tilde\Lambda_1^i$, $\tilde\Lambda_2^j$ *are shown in Tables 1–5.*

Proof Translating $\tilde E_2$ to the origin and diagonalizing the linear part, we can change system (2.2) into the form

$$\begin{cases} \frac{dx}{dt} = -y + \hat p_{20}x^2 + \hat p_{11}xy + \hat p_{02}y^2 + \hat p_{30}x^3 + \hat p_{21}x^2y + \hat p_{12}xy^2 + \hat p_{31}x^3y, \\ \frac{dy}{dt} = x \qquad + \hat q_{11}xy \qquad + \hat q_{12}xy^2, \end{cases} \tag{3.4}$$

where

$$\hat p_{20} := -\frac{b_3\omega_5}{b_4(b_5-1)^3}, \quad \hat p_{11} := -\frac{\omega(b_5-1)((b_5+1)\omega_4-\omega_5)}{b_5\omega_4}, \quad \hat p_{02} := -\frac{b_3b_5^2\omega_5}{b_4(b_5-1)\omega_4},$$
$$\hat p_{30} := -\frac{b_4\omega^{3/2}(b_5-1)^4}{b_5^2\omega_4}, \quad \hat p_{21} := -\frac{b_3b_5\omega_5(\omega_4-\omega_5+b_3)}{b_4^2(b_5-1)^4}, \quad \hat p_{12} := -\frac{b_3b_5^2\omega\omega_5}{b_4\omega_4},$$
$$\hat p_{31} := -\frac{b_3\omega^{3/2}(b_5-1)^3}{\omega_4}, \quad \hat q_{11} := \frac{b_3b_5^2+\omega_5}{b_4(b_5-1)}, \quad \hat q_{12} := \frac{b_3b_5^2\omega_5}{b_4^2(b_5-1)^2},$$
$$\omega_4 := b_2b_5(b_5-1) + b_3b_5(2b_5+1) + b_4(b_5-1)^2, \quad \omega_5 := b_3b_5^2 + b_4(b_5-1)^2.$$

Using the polar coordinates $(x, y) = (\rho\cos\theta, \rho\sin\theta)$, we can write system (3.4) as the 2π-periodic equation

$$\frac{d\rho}{d\theta} = \frac{G_2(\theta)\rho^2+G_3(\theta)\rho^3+G_4(\theta)\rho^4}{1+H_1(\theta)\rho+H_2(\theta)\rho^2+H_3(\theta)\rho^3} = \sum_{k=2}^{\infty}\rho_k(\theta)\rho^k \tag{3.5}$$

near the origin by the analyticity as shown in Chen et al. (2008), where

$$\begin{aligned}
&G_2(\theta) := (\hat{p}_{02}+\hat{q}_{11})\cos\theta + \hat{p}_{11}\cos^2\theta\sin\theta + (\hat{p}_{20}-\hat{p}_{02}-\hat{q}_{11})\cos^3\theta,\\
&G_3(\theta) := \hat{q}_{12}\cos\theta\sin\theta + \hat{p}_{12}\cos^2\theta + (\hat{p}_{21}-\hat{q}_{12})\cos^3\theta\sin\theta + (\hat{p}_{30}-\hat{p}_{12})\cos^4\theta,\\
&G_4(\theta) := \hat{p}_{31}\cos^4\theta\sin\theta, \quad H_3(\theta) := -\hat{p}_{31}\cos^3\theta\sin^2\theta,\\
&H_1(\theta) := -\hat{p}_{02}\sin\theta - \hat{p}_{11}\cos\theta\sin^2\theta + (\hat{p}_{02}-\hat{p}_{20}+\hat{q}_{11})\cos^2\theta\sin\theta,\\
&H_2(\theta) := -\hat{p}_{12}\cos\theta\sin\theta + (\hat{q}_{12}-\hat{p}_{21})\cos^2\theta\sin^2\theta + (\hat{p}_{12}-\hat{p}_{30})\cos^3\theta\sin\theta,
\end{aligned}$$

$$\rho_k(\theta) = \sum_{i=2}^{4} G_i(\theta)A_{k-i}(\theta), \qquad A_\ell(\theta) = \begin{cases} 0, & \text{if } \ell < 0,\\ 1, & \text{if } \ell = 0,\\ -\sum_{j=1}^{3} H_j(\theta)A_{\ell-j}(\theta), & \text{if } \ell > 0. \end{cases}$$

Let $\rho(\theta, \rho_0)$ be the solution of equation (3.5) associated with $\rho(0, \rho_0) = \rho_0$. Then we can compute the displacement function $d(\rho_0) := \rho(2\pi, \rho_0) - \rho_0 = \sum_{i=1}^{\infty}\mathcal{G}_{2i+1}\rho_0^{2i+1}$, where $\mathcal{G}_{2i+1}$ s are called focal values in Li (2003); Lloyd (1988). The Maple software produces the following expressions

$$\begin{aligned}
\mathcal{G}_3 &:= -\frac{b_3\omega((b_5-1)^2b_4+b_3b_5^2)}{8b_4(b_5-1)^2(b_5(b_5-1)b_2+(b_5-1)^2b_4+b_3b_5(2b_5+1))^2}\cdot\\
&\quad g_3(b_2, b_3, b_4, b_5),\\
\mathcal{G}_5 &:= \frac{b_3\omega((b_5-1)^2b_4+b_3b_5^2)}{768b_4^3(b_5-1)^8(b_5(b_5-1)b_2+(b_5-1)^2b_4+b_3b_5(2b_5+1))^4}\cdot\\
&\quad g_5(b_2, b_3, b_4, b_5),\\
\mathcal{G}_7 &:= -\frac{b_3\omega((b_5-1)^2b_4+b_3b_5^2)}{8847360b_4^5(b_5-1)^{14}(b_5(b_5-1)b_2+(b_5-1)^2b_4+b_3b_5(2b_5+1))^6}\cdot\\
&\quad g_7(b_2, b_3, b_4, b_5),\\
\mathcal{G}_9 &:= -\frac{b_3\omega((b_5-1)^2b_4+b_3b_5^2)}{1070176665600b_4^7(b_5-1)^{20}(b_5(b_5-1)b_2+(b_5-1)^2b_4+b_3b_5(2b_5+1))^8}\cdot\\
&\quad g_9(b_2, b_3, b_4, b_5),
\end{aligned} \tag{3.6}$$

where b_1 is replaced with β_2 in Ξ_0 and

$$\begin{aligned}
&g_3(b_2, b_3, b_4, b_5)\\
&\quad := b_5(2b_5-1)(b_5-1)^2b_2^2 + (b_5-1)\{(4b_5-1)(b_5-1)^2b_4 + b_5((8b_5^2+b_5-2)b_3\\
&\quad - b_5(b_5-1)^2)\}b_2 + \{2(b_5-1)^4b_4^2 + b_3(8b_5^2+3b_5-1)(b_5-1)^2b_4\\
&\quad + b_3b_5((2b_5+1)(4b_5^2+b_5-1)b_3 - b_5(b_5+1)(b_5-1)^2)\}
\end{aligned}$$

but g_5, g_7 and g_9 are much greater polynomials of 1191, 7079, and 23960 terms, respectively. Since the fractions in $\mathcal{G}_i$ s ($i = 3, 5, 7, 9$) are all positive by (3.3), we can use real zeros and signs of g_i s ($i = 3, 5, 7, 9$) to discuss real zeros and signs of $\mathcal{G}_i$ s ($i = 3, 5, 7, 9$).

First, in order to give the multiplicity of the weak focus $\tilde{E}_2$, we claim that

$$V(g_3, g_5, g_7, g_9) \cap \Xi_0 = \emptyset, \tag{3.7}$$

where $V(\phi_1, ..., \phi_m)$ presents the algebraic variety of polynomials $\phi_1, ..., \phi_m$, i.e., the set of common zeros of those polynomials. Taking the order

$$b_2 \prec b_4 \prec b_3 \prec b_5 \tag{3.8}$$

for variables in elimination stratum by stratum, we start from the primary stratum: $\mathcal{G}_0 := \{g_3, g_5, g_7, g_9\}$ and compute

$$\begin{aligned}
r_{12}(b_3, b_4, b_5) &:= \mathrm{res}(g_5, g_3, b_2) = 512 b_3^2 b_5^{13}(2b_5 - 1)(b_5 - 1)^{20} \cdot \\
&\qquad ((b_5 - 1)^2 b_4 + b_3 b_5^2)^5 \tilde{r}_{12}(b_3, b_4, b_5), \\
r_{13}(b_3, b_4, b_5) &:= \mathrm{res}(g_7, g_3, b_2) = 819200 b_3^2 b_5^{18}(2b_5 - 1)(b_5 - 1)^{30} \cdot \\
&\qquad ((b_5 - 1)^2 b_4 + b_3 b_5^2)^7 \tilde{r}_{13}(b_3, b_4, b_5), \\
r_{14}(b_3, b_4, b_5) &:= \mathrm{res}(g_9, g_3, b_2) = 90316800 b_3^2 b_5^{23} \cdot \\
&\qquad (2b_5 - 1)(b_5 - 1)^{40}((b_5 - 1)^2 b_4 + b_3 b_5^2)^9 \tilde{r}_{14}(b_3, b_4, b_5),
\end{aligned}$$

where $\mathrm{res}(\phi_1, \phi_2, x)$ denotes the Sylvester resultant (Gelfand et al. 1994; Mishra 1993) of ϕ_1 and ϕ_2 with respect to the variable x, $\tilde{r}_{12}$, $\tilde{r}_{13}$ and $\tilde{r}_{14}$ are polynomials of degree 13, 38 and 63 having 102, 1260 and 4743 terms, respectively. By Lemma 2 of Chen and Zhang (2009), we get

$$V(\mathcal{G}_0) = V(\mathcal{G}_0, \mathrm{lcff}(g_3, b_2)) \cup V(\tfrac{\mathcal{G}_0, \mathcal{G}_1}{\mathrm{lcff}(g_3, b_2)}), \tag{3.9}$$

where $\mathcal{G}_1 := \{r_{12}, r_{13}, r_{14}\}$ is the first stratum of the primary stratum $\mathcal{G}_0$, the notion $V(\frac{\phi_1, ..., \phi_m}{\psi_1, ..., \psi_n})$ presents $V(\phi_1, ..., \phi_m) \backslash \{\cup_{i=1}^n V(\psi_i)\}$, and $\mathrm{lcff}(\phi, X)$ denotes the leading coefficient of ϕ with respect to the variable X. Then, we compute

$$r_{23}(b_3, b_5) := \mathrm{res}(r_{13}, r_{12}, b_4) \equiv 0, \quad r_{24}(b_3, b_5) := \mathrm{res}(r_{14}, r_{12}, b_4) \equiv 0,$$

because r_{12}, r_{13} and r_{14} have the greatest common factor $512 b_3^2 b_5^{13}(b_5 - 1)^{20}(2b_5 - 1)((b_5 - 1)^2 b_4 + b_3 b_5^2)^5$. Thus, we get the second stratum: $\mathcal{G}_2 := \{r_{23}, r_{24}\} = \{0\}$, which shows that we cannot continue to decompose the variety $V(\mathcal{G}_1)$.

For further decomposition of stratum, we need a lemma for three polynomials $f_1, f_2, f_3 \in \mathbb{K}[x]$ over an algebraic closed field $\mathbb{K}$ with the following irreducible factorization

$$\begin{aligned} f_1(x) &= U_1^{p_{1,1}}(x)\cdots U_r^{p_{1,r}}(x)\tilde{f}_{1,1}^{q_{1,1}}(x)\cdots \tilde{f}_{1,\ell}^{q_{1,\ell}}(x), \\ f_2(x) &= U_1^{p_{2,1}}(x)\cdots U_r^{p_{2,r}}(x)W_1^{h_{1,1}}(x)\cdots W_s^{h_{1,s}}(x)\tilde{f}_{2,1}^{q_{2,1}}(x)\cdots \tilde{f}_{2,m}^{q_{2,m}}(x), \\ f_3(x) &= U_1^{p_{3,1}}(x)\cdots U_r^{p_{3,r}}(x)W_1^{h_{2,1}}(x)\cdots W_s^{h_{2,s}}(x)\tilde{f}_{3,1}^{q_{3,1}}(x)\cdots \tilde{f}_{3,n}^{q_{3,n}}(x), \end{aligned} \tag{3.10}$$

where $x := (x_1, ..., x_\vartheta)$, $U_1, ..., U_r$ are common factors of f_1, f_2 and f_3, $W_1, ..., W_s$ are common factors of f_2 and f_3 but not f_1, $\tilde{f}_{1,1}, ..., \tilde{f}_{1,\ell}$ are factors of f_1 but not f_2 or f_3, $\tilde{f}_{2,1}, ..., \tilde{f}_{2,m}$ are factors of f_2 but not f_1 or f_3, $\tilde{f}_{3,1}, ..., \tilde{f}_{3,n}$ are factors of f_3 but not f_1 or f_2, and $p_{1,1}, ..., p_{1,r}, p_{2,1}, ..., p_{2,r}, p_{3,1}, ..., p_{3,r}, h_{1,1}, ..., h_{1,s}, h_{2,1}, ..., h_{2,s}, q_{1,1}, ..., q_{1,\ell}, q_{2,1}, ..., q_{2,m}, q_{3,1}, ..., q_{3,n}$ are positive integers. For a stratum with more than three polynomials, we select one factor of each polynomial in the stratum, and then combine the selected factors to get a sub-stratum, and further obtain a decomposition of an algebraic variety consisting of the stratum. □

Lemma 3.1 *Let f_1, f_2 and f_3 be polynomials of the form (3.10). Then the stratum $\mathcal{G} := \{f_1, f_2, f_3\}$ contains $r + \ell s + \ell mn$ sub-strata*

$$\begin{aligned} &\mathcal{G}_\kappa := \{U_\kappa\}, && \text{if } \kappa = 1, ..., r, \\ &\mathcal{G}_\kappa := \{\tilde{f}_{1,\alpha}, W_\eta\}, && \text{if } \kappa = r + (\alpha-1)s + \eta, \\ & && \alpha = 1, ..., \ell \text{ and } \eta = 1, ..., s, \\ &\mathcal{G}_\kappa := \{\tilde{f}_{1,\alpha}, \tilde{f}_{2,\beta}, \tilde{f}_{3,\gamma}\}, && \text{if } \kappa = r + \ell s + (\alpha-1)mn + (\beta-1)n + \gamma, \\ & && \alpha = 1, ..., \ell,\ \beta = 1, ..., m \text{ and } \gamma = 1, ..., n, \end{aligned}$$

such that

$$V(\mathcal{G}) = \Big(\bigcup_{\kappa=1}^{r} V(\mathcal{G}_\kappa)\Big) \cup \Big(\bigcup_{\kappa=r+1}^{r+\ell s+\ell mn} V(\mathcal{G}_\kappa, J_\kappa) \cup V\big(\tfrac{\mathcal{G}_\kappa, \mathcal{G}_\kappa^*}{J_\kappa}\big)\Big), \tag{3.11}$$

where J_κ, $\kappa = r+1, ..., r+\ell s+\ell mn$, is the leading coefficient of the first polynomial in the sub-stratum $\mathcal{G}_\kappa$ with respect to X, the main variable in the stratum $\mathcal{G}$, and $\mathcal{G}_\kappa^*$, $\kappa = r+1, ..., r+\ell s+\ell mn$, is the stratum reduced from $\mathcal{G}_\kappa$ by computing resultants, i.e.,

$$\begin{aligned} &\mathcal{G}_\kappa^* := \{\tilde{r}_{\alpha\eta}\}, && \text{if } \kappa = r + (\alpha-1)s + \beta, \\ & && \alpha = 1, ..., \ell \text{ and } \eta = 1, ..., s, \\ &\mathcal{G}_\kappa^* := \{\hat{r}_{\alpha\beta}, \hat{r}_{\alpha\gamma}\}, && \text{if } \kappa = r + \ell s + (\alpha-1)mn + (\beta-1)n + \gamma, \\ & && \alpha = 1, ..., \ell,\ \beta = 1, ..., m \text{ and } \gamma = 1, ..., n \end{aligned}$$

with $\tilde{r}_{\alpha\eta} := \mathrm{res}(W_\eta, \tilde{f}_{1,\alpha}, X)$, $\hat{r}_{\alpha\beta} := \mathrm{res}(\tilde{f}_{2,\beta}, \tilde{f}_{1,\alpha}, X)$ *and* $\hat{r}_{\alpha\gamma} := \mathrm{res}(\tilde{f}_{3,\gamma}, \tilde{f}_{1,\alpha}, X)$.

We leave the proof of Lemma 3.1 after we complete the proof of the theorem. Now, we go back to the proof of Theorem 3.1. Compute

$$\begin{aligned} r_{23}(b_3, b_5) := \mathrm{res}(\tilde{r}_{13}, \tilde{r}_{12}, b_4) = 279936 b_3^7 b_5^{41}(b_5-1)^{74} \cdot \\ \tilde{r}_{23}^{(1)}(b_3, b_5)\tilde{r}_{23}^{(2)}(b_3, b_5)\tilde{r}_{23}^{(3)}(b_3, b_5), \end{aligned}$$

$$r_{24}(b_3, b_5) := \mathrm{res}(\tilde{r}_{14}, \tilde{r}_{12}, b_4) = 2579890176 b_3^{11} b_5^{66} (b_5 - 1)^{120} \cdot \\ \tilde{r}_{23}^{(1)}(b_3, b_5) \tilde{r}_{24}^{(1)}(b_3, b_5) \tilde{r}_{24}^{(2)}(b_3, b_5), \tag{3.12}$$

where $\tilde{r}_{23}^{(1)} := (4b_5 - 1)(2b_5 - 1)b_3 - (b_5 - 1)^3$, and $\tilde{r}_{23}^{(2)}$, $\tilde{r}_{23}^{(3)}$, $\tilde{r}_{24}^{(1)}$ and $\tilde{r}_{24}^{(2)}$ are polynomials of degree 19, 47, 41 and 75 having 137, 594, 534 and 1454 terms, respectively. By Lemma 3.1, the first stratum $\mathcal{G}_1 = \{r_{12}, r_{13}, r_{14}\}$ has the 6 sub-strata:

$$\mathcal{G}_{1,1} := \{b_3\}, \quad \mathcal{G}_{1,2} := \{b_5\}, \quad \mathcal{G}_{1,3} := \{2b_5 - 1\}, \\ \mathcal{G}_{1,4} := \{b_5 - 1\}, \quad \mathcal{G}_{1,5} := \{(b_5 - 1)^2 b_4 + b_3 b_5^2\}, \quad \mathcal{G}_{1,6} := \{\tilde{r}_{12}, \tilde{r}_{13}, \tilde{r}_{14}\},$$

as shown in Fig. 4, which satisfy

$$V(\mathcal{G}_1) = \Big(\bigcup_{\kappa=1}^{5} V(\mathcal{G}_{1,\kappa})\Big) \cup \Big(V(\mathcal{G}_{1,6}, \mathrm{lcff}(\tilde{r}_{12}, b_4)) \cup V\big(\tfrac{\mathcal{G}_{1,6}, \mathcal{G}_2^{(1,6)}}{\mathrm{lcff}(\tilde{r}_{12}, b_4)}\big)\Big),$$

where $\mathcal{G}_2^{(1,6)} := \{r_{23}, r_{24}\}$ is the stratum reduced from the first sub-stratum $\mathcal{G}_{1,6}$ by computing resultant and, for convenience, referred to the second stratum corresponding to $\mathcal{G}_{1,6}$. Noting by (3.3) that $V(\mathcal{G}_{1,\kappa}) \cap \Xi_0 = \emptyset$ for each $\kappa = 1, ..., 5$, we have that

$$V(\mathcal{G}_1) \cap \Xi_0 = \Big(V(\mathcal{G}_{1,6}, \mathrm{lcff}(\tilde{r}_{12}, b_4)) \cup V\big(\tfrac{\mathcal{G}_{1,6}, \mathcal{G}_2^{(1,6)}}{\mathrm{lcff}(\tilde{r}_{12}, b_4)}\big)\Big) \cap \Xi_0. \tag{3.13}$$

Similarly, we compute

$$\begin{aligned} r_{311}(b_5) &:= \mathrm{res}(\tilde{r}_{24}^{(1)}, \tilde{r}_{23}^{(2)}, b_3) \\ &= 854554729679171302784221049153126 4 b_5^{23} (2b_5 - 1)^{16} (b_5 - 1)^{260} \cdot \\ &\quad (16b_5^2 - 9b_5 + 2) \tilde{r}_{311}^{(1)}(b_5) \tilde{r}_{311}^{(2)}(b_5), \\ r_{312}(b_5) &:= \mathrm{res}(\tilde{r}_{24}^{(2)}, \tilde{r}_{23}^{(2)}, b_3) \\ &= -104976 b_5^{18} (2b_5 - 1)^{45} (b_5 - 1)^{390} \tilde{r}_{312}^{(1)}(b_5) \tilde{r}_{312}^{(2)}(b_5), \\ r_{313}(b_5) &:= \mathrm{res}(\tilde{r}_{24}^{(1)}, \tilde{r}_{23}^{(3)}, b_3) \\ &= 2421244566773856 b_5^{29} (2b_5 - 1)^{50} (b_5 - 1)^{492} \tilde{r}_{313}^{(1)}(b_5) \tilde{r}_{313}^{(2)}(b_5), \\ r_{314}(b_5) &:= \mathrm{res}(\tilde{r}_{24}^{(2)}, \tilde{r}_{23}^{(3)}, b_3) \\ &= -2796574286309581231459965835978008491458560 b_5^{45} (5b_5 - 3)^4 \cdot \\ &\quad (2b_5 - 1)^{209} (b_5 - 1)^{744} \tilde{r}_{314}^{(1)}(b_5) \tilde{r}_{314}^{(2)}(b_5) \tilde{r}_{314}^{(3)}(b_5) \tilde{r}_{314}^{(4)}(b_5), \end{aligned} \tag{3.14}$$

where $\tilde{r}_{311}^{(1)}$ is given in Appendix, $\tilde{r}_{311}^{(2)}$, $\tilde{r}_{312}^{(1)}$, $\tilde{r}_{312}^{(2)}$, $\tilde{r}_{313}^{(1)}$, $\tilde{r}_{313}^{(2)}$, $\tilde{r}_{314}^{(2)}$ and $\tilde{r}_{314}^{(3)}$ are polynomials having 146, 276, 93, 120, 358, 141 and 579 terms, respectively, $\tilde{r}_{314}^{(1)} := 10b_5^3 - 16b_5^2 + 4b_5 + 1$ and $\tilde{r}_{314}^{(4)} := 2b_5^2 - 6b_5 + 3$. By Lemma 3.1, the second stratum

$\mathcal{G}_2^{(1,6)} = \{r_{23}, r_{24}\}$ has the 8 sub-strata:

$$\begin{array}{lll} \mathcal{G}_{2,1}^{(1,6)} = \{b_3\}, & \mathcal{G}_{2,2}^{(1,6)} = \{b_5\}, & \mathcal{G}_{2,3}^{(1,6)} = \{b_5 - 1\}, \\ \mathcal{G}_{2,4}^{(1,6)} = \{\tilde{r}_{23}^{(1)}\}, & \mathcal{G}_{2,5}^{(1,6)} = \{\tilde{r}_{23}^{(2)}, \tilde{r}_{24}^{(1)}\}, & \mathcal{G}_{2,6}^{(1,6)} = \{\tilde{r}_{23}^{(2)}, \tilde{r}_{24}^{(2)}\}, \\ \mathcal{G}_{2,7}^{(1,6)} = \{\tilde{r}_{23}^{(3)}, \tilde{r}_{24}^{(1)}\}, & \mathcal{G}_{2,8}^{(1,6)} = \{\tilde{r}_{23}^{(3)}, \tilde{r}_{24}^{(2)}\}, & \end{array}$$

as shown in Fig. 4, which satisfy

$$\begin{aligned} V(\mathcal{G}_2^{(1,6)}) = \Big(\bigcup_{\kappa=1}^{4} V(\mathcal{G}_{2,\kappa}^{(1,6)}) \Big) \cup \Big(\bigcup_{\kappa=5}^{6} V(\mathcal{G}_{2,\kappa}^{(1,6)}, \mathrm{lcff}(\tilde{r}_{23}^{(2)}, b_3)) \cup V\big(\tfrac{\mathcal{G}_{2,\kappa}^{(1,6)}, \mathcal{G}_3^{(2,\kappa)}}{\mathrm{lcff}(\tilde{r}_{23}^{(2)}, b_3)}\big) \Big) \\ \cup \Big(\bigcup_{\kappa=7}^{8} V(\mathcal{G}_{2,\kappa}^{(1,6)}, \mathrm{lcff}(\tilde{r}_{23}^{(3)}, b_3)) \cup V\big(\tfrac{\mathcal{G}_{2,\kappa}^{(1,6)}, \mathcal{G}_3^{(2,\kappa)}}{\mathrm{lcff}(\tilde{r}_{23}^{(3)}, b_3)}\big) \Big), \end{aligned}$$

where $\mathcal{G}_3^{(2,5)} := \{r_{311}\}$, $\mathcal{G}_3^{(2,6)} := \{r_{312}\}$, $\mathcal{G}_3^{(2,7)} := \{r_{313}\}$ and $\mathcal{G}_3^{(2,8)} := \{r_{314}\}$ are reduced from the second sub-strata $\mathcal{G}_{2,5}^{(1,6)}$, $\mathcal{G}_{2,6}^{(1,6)}$, $\mathcal{G}_{2,7}^{(1,6)}$ and $\mathcal{G}_{2,8}^{(1,6)}$ by computing resultant and, for convenience, referred to the third strata corresponding to $\mathcal{G}_{2,5}^{(1,6)}$, $\mathcal{G}_{2,6}^{(1,6)}$, $\mathcal{G}_{2,7}^{(1,6)}$ and $\mathcal{G}_{2,8}^{(1,6)}$, respectively. Noting by (3.3) that $V(\mathcal{G}_{2,\kappa}^{(1,6)}) \cap \Xi_0 = \emptyset$ for each $\kappa = 1, 2, 3$, we have

$$\begin{aligned} V(\mathcal{G}_2^{(1,6)}) \cap \Xi_0 = \Big(V(\mathcal{G}_{2,4}^{(1,6)}) \cup \Big(\bigcup_{\kappa=5}^{6} V(\mathcal{G}_{2,\kappa}^{(1,6)}, \mathrm{lcff}(\tilde{r}_{23}^{(2)}, b_3)) \cup V\big(\tfrac{\mathcal{G}_{2,\kappa}^{(1,6)}, \mathcal{G}_3^{(2,\kappa)}}{\mathrm{lcff}(\tilde{r}_{23}^{(2)}, b_3)}\big) \Big) \\ \cup \Big(\bigcup_{\kappa=7}^{8} V(\mathcal{G}_{2,\kappa}^{(1,6)}, \mathrm{lcff}(\tilde{r}_{23}^{(3)}, b_3)) \cup V\big(\tfrac{\mathcal{G}_{2,\kappa}^{(1,6)}, \mathcal{G}_3^{(2,\kappa)}}{\mathrm{lcff}(\tilde{r}_{23}^{(3)}, b_3)}\big) \Big) \Big) \cap \Xi_0. \end{aligned} \tag{3.15}$$

Similarly, by Lemma 3.1, the third stratum $\mathcal{G}_3^{(2,5)} = \{r_{311}\}$ has the 6 sub-strata:

$$\begin{array}{lll} \mathcal{G}_{3,1}^{(2,5)} := \{b_5\}, & \mathcal{G}_{3,2}^{(2,5)} := \{2b_5 - 1\}, & \mathcal{G}_{3,3}^{(2,5)} := \{b_5 - 1\}, \\ \mathcal{G}_{3,4}^{(2,5)} := \{16b_5^2 - 9b_5 + 2\}, & \mathcal{G}_{3,5}^{(2,5)} := \{\tilde{r}_{311}^{(1)}\}, & \mathcal{G}_{3,6}^{(2,5)} := \{\tilde{r}_{311}^{(2)}\}, \end{array}$$

the third stratum $\mathcal{G}_3^{(2,6)} = \{r_{312}\}$ has the 5 sub-strata:

$$\begin{array}{lll} \mathcal{G}_{3,1}^{(2,6)} := \{b_5\}, & \mathcal{G}_{3,2}^{(2,6)} := \{2b_5 - 1\}, & \mathcal{G}_{3,3}^{(2,6)} := \{b_5 - 1\}, \\ \mathcal{G}_{3,4}^{(2,6)} := \{\tilde{r}_{312}^{(1)}\}, & \mathcal{G}_{3,5}^{(2,6)} := \{\tilde{r}_{312}^{(2)}\}, & \end{array}$$

the third stratum $\mathcal{G}_3^{(2,7)} = \{r_{313}\}$ has the 5 sub-strata:

$$\begin{array}{lll} \mathcal{G}_{3,1}^{(2,7)} := \{b_5\}, & \mathcal{G}_{3,2}^{(2,7)} := \{2b_5 - 1\}, & \mathcal{G}_{3,3}^{(2,7)} := \{b_5 - 1\}, \\ \mathcal{G}_{3,4}^{(2,7)} := \{\tilde{r}_{313}^{(1)}\}, & \mathcal{G}_{3,5}^{(2,7)} := \{\tilde{r}_{313}^{(2)}\}, & \end{array}$$

and the third stratum $\mathcal{G}_3^{(2,8)} = \{r_{314}\}$ has the 8 sub-strata:

$$\begin{array}{llll}\mathcal{G}_{3,1}^{(2,8)} := \{b_5\}, & \mathcal{G}_{3,2}^{(2,8)} := \{5b_5 - 3\}, & \mathcal{G}_{3,3}^{(2,8)} := \{2b_5 - 1\}, & \mathcal{G}_{3,4}^{(2,8)} := \{b_5 - 1\}, \\ \mathcal{G}_{3,5}^{(2,8)} := \{\tilde{r}_{314}^{(1)}\}, & \mathcal{G}_{3,6}^{(2,8)} := \{\tilde{r}_{314}^{(2)}\}, & \mathcal{G}_{3,7}^{(2,8)} := \{\tilde{r}_{314}^{(3)}\}, & \mathcal{G}_{3,8}^{(2,8)} := \{\tilde{r}_{314}^{(4)}\},\end{array}$$

as shown in Fig. 4. Noting by (3.3) that $\mathcal{G}_{3,i}^{(2,\varsigma)} \cap \Xi_0 = \emptyset$ and $\mathcal{G}_{3,j}^{(2,\varrho)} \cap \Xi_0 = \emptyset$ for each $i = 1, 2, 3, 4$, $j = 1, 2, 3$, $\varsigma = 5, 8$ and $\varrho = 6, 7$, we have

$$\begin{aligned}
&V(\mathcal{G}_3^{(2,5)}) \cap \Xi_0 = (\bigcup_{\kappa=5}^{6} V(\mathcal{G}_{3,\kappa}^{(2,5)})) \cap \Xi_0, \quad V(\mathcal{G}_3^{(2,6)}) \cap \Xi_0 = (\bigcup_{\kappa=4}^{5} V(\mathcal{G}_{3,\kappa}^{(2,6)})) \cap \Xi_0, \\
&V(\mathcal{G}_3^{(2,7)}) \cap \Xi_0 = (\bigcup_{\kappa=4}^{5} V(\mathcal{G}_{3,\kappa}^{(2,7)})) \cap \Xi_0, \quad V(\mathcal{G}_3^{(2,8)}) \cap \Xi_0 = (\bigcup_{\kappa=5}^{8} V(\mathcal{G}_{3,\kappa}^{(2,8)})) \cap \Xi_0.
\end{aligned} \tag{3.16}$$

Thus, by (3.9), (3.13), (3.15), and (3.16) we obtain the decomposition

$$V(g_3, g_5, g_7, g_9) \cap \Xi_0 = \left(\bigcup_{i=1}^{17} \mathcal{V}_i\right) \cap \Xi_0, \tag{3.17}$$

where

$$\begin{aligned}
\mathcal{V}_1 &:= V(\mathcal{G}_0, \mathrm{lcff}(g_3, b_2)), \\
\mathcal{V}_2 &:= V\left(\frac{\mathcal{G}_0, \mathcal{G}_{1,6}, \mathrm{lcff}(\tilde{r}_{12}, b_4)}{\mathrm{lcff}(g_3, b_2)}\right), \\
\mathcal{V}_3 &:= V\left(\frac{\mathcal{G}_0, \mathcal{G}_{1,6}, \mathcal{G}_{2,4}^{(1,6)}}{\mathrm{lcff}(g_3, b_2), \mathrm{lcff}(\tilde{r}_{12}, b_4)}\right), \\
\mathcal{V}_4 &:= V\left(\frac{\mathcal{G}_0, \mathcal{G}_{1,6}, \mathcal{G}_{2,5}^{(1,6)}, \mathrm{lcff}(\tilde{r}_{23}^{(2)}, b_3)}{\mathrm{lcff}(g_3, b_2), \mathrm{lcff}(\tilde{r}_{12}, b_4)}\right), \\
\mathcal{V}_5 &:= V\left(\frac{\mathcal{G}_0, \mathcal{G}_{1,6}, \mathcal{G}_{2,6}^{(1,6)}, \mathrm{lcff}(\tilde{r}_{23}^{(2)}, b_3)}{\mathrm{lcff}(g_3, b_2), \mathrm{lcff}(\tilde{r}_{12}, b_4)}\right), \\
\mathcal{V}_6 &:= V\left(\frac{\mathcal{G}_0, \mathcal{G}_{1,6}, \mathcal{G}_{2,7}^{(1,6)}, \mathrm{lcff}(\tilde{r}_{23}^{(3)}, b_3)}{\mathrm{lcff}(g_3, b_2), \mathrm{lcff}(\tilde{r}_{12}, b_4)}\right), \\
\mathcal{V}_7 &:= V\left(\frac{\mathcal{G}_0, \mathcal{G}_{1,6}, \mathcal{G}_{2,8}^{(1,6)}, \mathrm{lcff}(\tilde{r}_{23}^{(3)}, b_3)}{\mathrm{lcff}(g_3, b_2), \mathrm{lcff}(\tilde{r}_{12}, b_4)}\right), \\
\mathcal{V}_8 &:= V\left(\frac{\mathcal{G}_0, \mathcal{G}_{1,6}, \mathcal{G}_{2,5}^{(1,6)}, \mathcal{G}_{3,5}^{(2,5)}}{\mathrm{lcff}(g_3, b_2), \mathrm{lcff}(\tilde{r}_{12}, b_4), \mathrm{lcff}(\tilde{r}_{23}^{(2)}, b_3)}\right), \\
\mathcal{V}_9 &:= V\left(\frac{\mathcal{G}_0, \mathcal{G}_{1,6}, \mathcal{G}_{2,5}^{(1,6)}, \mathcal{G}_{3,6}^{(2,5)}}{\mathrm{lcff}(g_3, b_2), \mathrm{lcff}(\tilde{r}_{12}, b_4), \mathrm{lcff}(\tilde{r}_{23}^{(2)}, b_3)}\right), \\
\mathcal{V}_{10} &:= V\left(\frac{\mathcal{G}_0, \mathcal{G}_{1,6}, \mathcal{G}_{2,6}^{(1,6)}, \mathcal{G}_{3,4}^{(2,6)}}{\mathrm{lcff}(g_3, b_2), \mathrm{lcff}(\tilde{r}_{12}, b_4), \mathrm{lcff}(\tilde{r}_{23}^{(2)}, b_3)}\right),
\end{aligned}$$

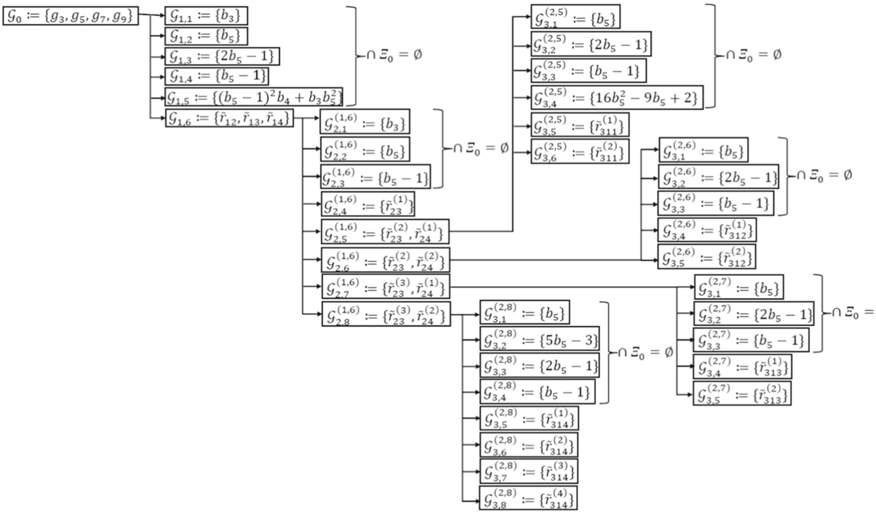


Fig. 4 Decomposition of $\mathcal{G}_0$

$$\mathcal{V}_{11} := V\left(\frac{\mathcal{G}_0, \mathcal{G}_{1,6}, \mathcal{G}_{2,6}^{(1,6)}, \mathcal{G}_{3,5}^{(2,6)}}{\mathrm{lcff}(g_3, b_2), \mathrm{lcff}(\tilde{r}_{12}, b_4), \mathrm{lcff}(\tilde{r}_{23}^{(2)}, b_3)}\right),$$

$$\mathcal{V}_{12} := V\left(\frac{\mathcal{G}_0, \mathcal{G}_{1,6}, \mathcal{G}_{2,7}^{(1,6)}, \mathcal{G}_{3,4}^{(2,7)}}{\mathrm{lcff}(g_3, b_2), \mathrm{lcff}(\tilde{r}_{12}, b_4), \mathrm{lcff}(\tilde{r}_{23}^{(3)}, b_3)}\right),$$

$$\mathcal{V}_{13} := V\left(\frac{\mathcal{G}_0, \mathcal{G}_{1,6}, \mathcal{G}_{2,7}^{(1,6)}, \mathcal{G}_{3,5}^{(2,7)}}{\mathrm{lcff}(g_3, b_2), \mathrm{lcff}(\tilde{r}_{12}, b_4), \mathrm{lcff}(\tilde{r}_{23}^{(3)}, b_3)}\right),$$

$$\mathcal{V}_{14} := V\left(\frac{\mathcal{G}_0, \mathcal{G}_{1,6}, \mathcal{G}_{2,8}^{(1,6)}, \mathcal{G}_{3,5}^{(2,8)}}{\mathrm{lcff}(g_3, b_2), \mathrm{lcff}(\tilde{r}_{12}, b_4), \mathrm{lcff}(\tilde{r}_{23}^{(3)}, b_3)}\right),$$

$$\mathcal{V}_{15} := V\left(\frac{\mathcal{G}_0, \mathcal{G}_{1,6}, \mathcal{G}_{2,8}^{(1,6)}, \mathcal{G}_{3,6}^{(2,8)}}{\mathrm{lcff}(g_3, b_2), \mathrm{lcff}(\tilde{r}_{12}, b_4), \mathrm{lcff}(\tilde{r}_{23}^{(3)}, b_3)}\right),$$

$$\mathcal{V}_{16} := V\left(\frac{\mathcal{G}_0, \mathcal{G}_{1,6}, \mathcal{G}_{2,8}^{(1,6)}, \mathcal{G}_{3,7}^{(2,8)}}{\mathrm{lcff}(g_3, b_2), \mathrm{lcff}(\tilde{r}_{12}, b_4), \mathrm{lcff}(\tilde{r}_{23}^{(3)}, b_3)}\right),$$

$$\mathcal{V}_{17} := V\left(\frac{\mathcal{G}_0, \mathcal{G}_{1,6}, \mathcal{G}_{2,8}^{(1,6)}, \mathcal{G}_{3,8}^{(2,8)}}{\mathrm{lcff}(g_3, b_2), \mathrm{lcff}(\tilde{r}_{12}, b_4), \mathrm{lcff}(\tilde{r}_{23}^{(3)}, b_3)}\right).$$

Next, we claim the following.

Lemma 3.2 ($\mathbf{M}_i$)*:* $\mathcal{V}_i \cap \Xi_0 = \emptyset$*,* $i = 1, ..., 17$.

We will prove this lemma after we complete the proof of the theorem. By Lemma 3.2 and (3.17), we see that claim (3.7) holds, which implies that the equilibrium $\tilde{E}_2$ is a weak focus of multiplicity at most 4 for $(b_1, b_2, b_3, b_4, b_5) \in \Xi_0$.

Further, we give conditions of parameters for exactly numbers of limit cycles. In order to avoid a great deal of complicated symbolic computation, we restrict to the surface

$$\Pi := \{(b_1, b_2, b_3, b_4, b_5) \in \mathbb{R}^5 : b_3 = -\tfrac{(2b_5^2-6b_5+3)(b_5-1)^2}{3(2b_5-1)}\} \tag{3.18}$$

to give sufficient conditions for 1, 2, and 3 limit cycles separately. We will give an explanation for (3.18) after the proof of the theorem. In this case, substituting $b_3 = -(2b_5^2 - 6b_5 + 3)(b_5 - 1)^2/(3(2b_5 - 1))$ in g_3, g_5 and g_7, which are defined just below (3.6), we get

$$\begin{aligned}
g_3 &= (b_5-1)^2\tilde{g}_3(b_2, b_4, b_5)/(9(2b_5-1)^2),\\
g_5 &= -(b_5-1)^{11}\tilde{g}_5(b_2, b_4, b_5)/(6561(2b_5-1)^8),\\
g_7 &= -(b_5-1)^{20}\tilde{g}_5(b_2, b_4, b_5)/(1594323(2b_5-1)^{14}),
\end{aligned}$$

where

$$\begin{aligned}
&\tilde{g}_3(b_2, b_4, b_5)\\
&\quad := 9b_5(2b_5-1)^3 b_2^2 + 3(2b_5-1)(b_5-1)\{3(4b_5-1)(2b_5-1)b_4 - 2b_5\cdot\\
&\qquad\quad (8b_5^4 - 23b_5^3 + 10b_5^2 + 6b_5 - 3)\}b_2\\
&\quad +(b_5-1)^2\{18(2b_5-1)^2 b_4^2 - 3(2b_5-1)(8b_5^2+3b_5-1)(2b_5^2-6b_5+3)b_4\\
&\qquad\quad +b_5(2b_5^2-6b_5+3)(16b_5^5 - 36b_5^4 - 8b_5^3 + 25b_5^2 - 3)\},
\end{aligned}$$

$\tilde{g}_5$ and $\tilde{g}_7$ are polynomials of 34 and 58 degrees having 541 and 2108 terms, respectively. Next, we prove in the following three steps: **(I)** compute $V(\tilde{g}_3) \cap (\Xi_0 \cap \Pi)$, **(II)** compute $V(\tilde{g}_3, \tilde{g}_5) \cap (\Xi_0 \cap \Pi)$, and **(III)** prove $V(\tilde{g}_3, \tilde{g}_5, \tilde{g}_7) \cap (\Xi_0 \cap \Pi) = \emptyset$.

Step (I): Compute $V(\tilde{g}_3) \cap (\Xi_0 \cap \Pi)$.

As indicated in (3.8), we will eliminate the variable in the order $b_2 \prec b_4 \prec b_5$ stratum by stratum because of the removal of b_3 by (3.18). For consistency of notationss, we let φ_{ij} denote the jth occurrence of a polynomial in the $i-1$th stratum of elimination and write the kth occurrence of a real zero of the polynomial φ_{ij} as $\beta_k^{(ij)}$. By (3.3) and (3.18), Ξ_0 requires that $b_3 > 0$ and $b_5 > 1$ and Π requires that $b_3 = -(2b_5^2 - 6b_5 + 3)(b_5 - 1)^2/(3(2b_5 - 1))$. Thus, the intersection of Ξ_0 and Π requires $b_3 = -(2b_5^2 - 6b_5 + 3)(b_5 - 1)^2/(3(2b_5 - 1)) > 0$ and $b_5 > 1$, from which we get $1 < b_5 < (3 + \sqrt{3})/2$. Thus, we need to find real zeros of $\varphi_{11}(\cdot) := \tilde{g}_3(\cdot, b_4, b_5)$ for $b_2 > 0$, $b_4 > 0$ and $1 < b_5 < (3 + \sqrt{3})/2$. In what follows, we use the notation $\Leftrightarrow$ to indicate "if and only if" shortly.

Lemma 3.3 *φ_{11} has two zeros in $(-\infty, \infty)$, denoted by $\beta_1^{(11)} < \beta_2^{(11)}$,*

$$\begin{aligned}
\Leftrightarrow\quad & \textit{either}\ \ 1 < b_5 < (3+\sqrt{3})/2\ \ \textit{and}\ \ b_4 > \beta_2^{(21)}\\
& \textit{or}\ \ \beta_1^{(31)} < b_5 < (3+\sqrt{3})/2\ \ \textit{and}\ \ 0 < b_4 < \beta_1^{(21)},
\end{aligned}$$

and a unique multiple zero in $(-\infty, \infty)$*, denoted by* $\beta_3^{(11)}$*,*

$$\Leftrightarrow \quad \begin{array}{l} \textit{either}\ \ 1 < b_5 < (3+\sqrt{3})/2\ \ \textit{and}\ \ b_4 = \beta_2^{(21)} \\ \textit{or}\ \ \beta_1^{(31)} < b_5 < (3+\sqrt{3})/2\ \ \textit{and}\ \ b_4 = \beta_1^{(21)}; \end{array}$$

otherwise, φ_{11} *has no zeros in* $(-\infty, \infty)$*, where* $\beta_1^{(31)}$*,* $\beta_1^{(21)}$*,* $\beta_2^{(21)}$*,* $\beta_1^{(11)}$*,* $\beta_2^{(11)}$ *and* $\beta_3^{(11)}$ *are given in Appendix. Moreover,*

$$\begin{array}{lll} \beta_1^{(11)} > 0 & \Leftrightarrow & \beta_1^{(34)} < b_5 < (3+\sqrt{3})/2\ \ \textit{and}\ \ \beta_2^{(23)} < b_4 < \beta_1^{(21)}, \\ \beta_2^{(11)} > 0 & \Leftrightarrow & \textit{either}\ \ \beta_1^{(33)} < b_5 \le \beta_1^{(34)}\ \ \textit{and}\ \ 0 < b_4 < \beta_2^{(23)} \\ & & \textit{or}\ \ \beta_1^{(34)} < b_5 < (3+\sqrt{3})/2\ \ \textit{and}\ \ 0 < b_4 < \beta_1^{(21)}, \\ \beta_3^{(11)} > 0 & \Leftrightarrow & \beta_1^{(34)} < b_5 < (3+\sqrt{3})/2\ \ \textit{and}\ \ b_4 = \beta_1^{(21)}, \end{array}$$

where $\beta_1^{(33)}$*,* $\beta_1^{(34)}$ *and* $\beta_2^{(23)}$ *are given in Appendix.*

We will prove this lemma after we complete the proof of the theorem. By Lemma 3.3, we obtain

$$\begin{aligned} & V(\tilde{g}_3) \cap (\Xi_0 \cap \Pi) \\ & = \Big\{(b_1, b_2, b_3, b_4, b_5) \in \mathbb{R}^5 : \beta_1^{(33)} < b_5 \le \beta_1^{(34)},\ 0 < b_4 < \beta_2^{(23)}, \\ & \qquad b_3 = -\frac{(2b_5^2 - 6b_5 + 3)(b_5 - 1)^2}{3(2b_5 - 1)},\ b_2 = \beta_2^{(11)},\ b_1 = \beta_2\Big\} \\ & \cup \Big\{(b_1, b_2, b_3, b_4, b_5) \in \mathbb{R}^5 : \beta_1^{(34)} < b_5 < (3+\sqrt{3})/2,\ 0 < b_4 < \beta_1^{(21)}, \\ & \qquad b_3 = -\frac{(2b_5^2 - 6b_5 + 3)(b_5 - 1)^2}{3(2b_5 - 1)},\ b_2 = \beta_2^{(11)},\ b_1 = \beta_2\Big\} \\ & \cup \Big\{(b_1, b_2, b_3, b_4, b_5) \in \mathbb{R}^5 : \beta_1^{(34)} < b_5 < (3+\sqrt{3})/2,\ \beta_2^{(23)} < b_4 < \beta_1^{(21)}, \\ & \qquad b_3 = -\frac{(2b_5^2 - 6b_5 + 3)(b_5 - 1)^2}{3(2b_5 - 1)},\ b_2 = \beta_1^{(11)},\ b_1 = \beta_2\Big\} \\ & \cup \Big\{(b_1, b_2, b_3, b_4, b_5) \in \mathbb{R}^5 : \beta_1^{(34)} < b_5 < (3+\sqrt{3})/2,\ b_4 = \beta_1^{(21)}, \\ & \qquad b_3 = -\frac{(2b_5^2 - 6b_5 + 3)(b_5 - 1)^2}{3(2b_5 - 1)},\ b_2 = \beta_3^{(11)},\ b_1 = \beta_2\Big\}. \end{aligned} \tag{3.19}$$

The proof of step **(I)** is completed.

Step (II): Compute $V(\tilde{g}_3, \tilde{g}_5) \cap (\Xi_0 \cap \Pi)$.

Similar to the discussion on the decomposition of $\mathcal{G}_0$, we start from the primary stratum $\tilde{\mathcal{G}}_0 := \{\tilde{g}_3, \tilde{g}_5\}$ and compute

$$\begin{aligned}\check{r}_{12}(b_4, b_5) &:= \mathrm{res}(\tilde{g}_5, \tilde{g}_3, b_2)\\ &= 22039921152 b_5^{15}(2b_5^2 - 6b_5 + 3)^2(b_5 - 1)^{10}\\ &\quad (2b_5 - 1)^{18}(3(2b_5 - 1)b_4 + b_5^2(-2b_5^2 + 6b_5 - 3))^5 \check{r}_{12}^{(1)}(b_4, b_5),\end{aligned}$$

where

$$\begin{aligned}\check{r}_{12}^{(1)}(b_4, b_5) := {}& 9(32b_5^3 - 80b_5^2 + 60b_5 - 15)(2b_5 - 1)^2 b_4^2\\ &+6b_5(68b_5^5 - 434b_5^4 + 942b_5^3 - 831b_5^2 + 324b_5 - 45)(2b_5 - 1)^2 b_4\\ &-2b_5^3(2b_5^2 - 6b_5 + 3)(168b_5^6 - 1112b_5^5 + 2738b_5^4 - 3189b_5^3\\ &+1920b_5^2 - 585b_5 + 72).\end{aligned}$$

By Lemma 2 of Chen and Zhang (2009, p. 567), we get

$$V(\tilde{\mathcal{G}}_0) = V(\tilde{\mathcal{G}}_0, \mathrm{lcff}(\tilde{g}_3, b_2)) \cup V\left(\frac{\tilde{\mathcal{G}}_0, \tilde{\mathcal{G}}_1}{\mathrm{lcff}(\tilde{g}_3, b_2)}\right), \tag{3.20}$$

where $\tilde{\mathcal{G}}_1 := \{\check{r}_{12}\}$ is the first stratum of the primary stratum $\tilde{\mathcal{G}}_0$. By Lemma 3.1, the first stratum $\tilde{\mathcal{G}}_1$ has the 6 sub-strata:

$$\begin{aligned}&\tilde{\mathcal{G}}_{1,1} := \{b_5\}, \qquad \tilde{\mathcal{G}}_{1,2} := \{2b_5^2 - 6b_5 + 3\},\\ &\tilde{\mathcal{G}}_{1,3} := \{b_5 - 1\}, \tilde{\mathcal{G}}_{1,4} := \{2b_5 - 1\}, \ \tilde{\mathcal{G}}_{1,5} := \{3(2b_5 - 1)b_4 + b_5^2(-2b_5^2 + 6b_5 - 3)\},\\ &\tilde{\mathcal{G}}_{1,6} := \{\check{r}_{12}^{(1)}\},\end{aligned}$$

which satisfy $V(\tilde{\mathcal{G}}_1) = \left(\bigcup_{\kappa=1}^{6} V(\tilde{\mathcal{G}}_{1,\kappa})\right)$. Noting by (3.3) and (3.18) that $V(\tilde{\mathcal{G}}_{1,\kappa}) \cap (\Xi_0 \cap \Pi) = \emptyset$ for each $\kappa = 1, ..., 5$, we have

$$V(\tilde{\mathcal{G}}_1) \cap (\Xi_0 \cap \Pi) = V(\tilde{\mathcal{G}}_{1,6}) \cap (\Xi_0 \cap \Pi). \tag{3.21}$$

Since $\mathrm{lcff}(\tilde{g}_3, b_2) = 9b_5(2b_5 - 1)^3 > 0$ for $(b_1, b_2, b_3, b_4, b_5) \in \Xi_0$, we see by (3.20) and (3.21) that

$$V(\tilde{\mathcal{G}}_0) \cap (\Xi_0 \cap \Pi) = V(\tilde{\mathcal{G}}_0, \tilde{\mathcal{G}}_{1,6}) \cap (\Xi_0 \cap \Pi). \tag{3.22}$$

Notice that $V(\tilde{\mathcal{G}}_0, \tilde{\mathcal{G}}_{1,6}) \cap (\Xi_0 \cap \Pi) \subset V(\tilde{g}_3) \cap (\Xi_0 \cap \Pi)$. By (3.19), $V(\tilde{g}_3) \cap (\Xi_0 \cap \Pi)$ requires

$$\begin{array}{llll}\text{either} & \beta_1^{(33)} < b_5 < (3 + \sqrt{3})/2 & \text{and} & 0 < b_4 < \beta_2^{(23)}\\ \text{or} & \beta_1^{(34)} < b_5 < (3 + \sqrt{3})/2 & \text{and} & \beta_2^{(23)} \le b_4 \le \beta_1^{(21)}.\end{array} \tag{3.23}$$

Thus, we discuss real zeros of $\varphi_{24}(\cdot) := \check{r}_{12}^{(1)}(\cdot, b_5)$ under the condition (3.23). Similar to Lemma 3.3, we also see that φ_{24} has two zeros in $(-\infty, \infty)$, denoted by $\beta_1^{(24)} <$

$\beta_2^{(24)}$, $\Leftrightarrow$ either $\beta_1^{(33)} < b_5 < \beta_1^{(35)}$ or $\beta_1^{(35)} < b_5 < (3+\sqrt{3})/2$, and that φ_{24} has a unique multiple zero in $(-\infty, \infty)$, denoted by $\beta_3^{(24)}$, $\Leftrightarrow$ $b_5 = \beta_1^{(35)}$, where $\beta_1^{(33)}$, $\beta_1^{(35)}$, $\beta_1^{(24)}$, $\beta_2^{(24)}$ and $\beta_3^{(24)}$ are given in Appendix. Moreover,

$$\begin{array}{lcl} 0 < \beta_1^{(24)} < \beta_2^{(23)} & \Leftrightarrow & \text{either } \beta_2^{(36)} < b_5 < \beta_1^{(35)} \text{ or } \beta_1^{(35)} < b_5 < (3+\sqrt{3})/2, \\ 0 < \beta_2^{(24)} < \beta_2^{(23)} & \Leftrightarrow & \text{either } \beta_1^{(36)} < b_5 < \beta_1^{(35)} \text{ or } \beta_1^{(35)} < b_5 < \beta_1^{(37)}, \\ \beta_2^{(24)} = \beta_2^{(23)} & \Leftrightarrow & b_5 = \beta_1^{(37)}, \\ \beta_2^{(23)} < \beta_2^{(24)} < \beta_1^{(21)} & \Leftrightarrow & \beta_1^{(37)} < b_5 < (3+\sqrt{3})/2, \\ 0 < \beta_3^{(24)} < \beta_2^{(23)} & \Leftrightarrow & b_5 = \beta_1^{(35)}, \end{array}$$

where $\beta_1^{(36)}$, $\beta_2^{(36)}$ and $\beta_1^{(37)}$ are given in Appendix. We can check the common zero of $\tilde{g}_3$ and $\tilde{g}_5$ corresponding to the zero of φ_{24}, i.e.,

$$\begin{array}{ll} \beta_1^{(11)} \text{ is the common zero of } \tilde{g}_3 \text{ and } \tilde{g}_5 \ \Leftrightarrow & \\ & \text{either } \beta_1^{(36)} < b_5 < \beta_1^{(35)} \text{ and } b_4 = \beta_2^{(24)} \\ & \text{or } \quad b_5 = \beta_1^{(35)} \text{ and } b_4 = \beta_3^{(24)} \\ & \text{or } \quad \beta_1^{(35)} < b_5 < (3+\sqrt{3})/2 \text{ and } b_4 = \beta_1^{(24)}, \\ \beta_2^{(11)} \text{ is the common zero of } \tilde{g}_3 \text{ and } \tilde{g}_5 \ \Leftrightarrow & \\ & \text{either } \beta_2^{(36)} < b_5 < \beta_1^{(35)} \text{ and } b_4 = \beta_1^{(24)} \\ & \text{or } \quad b_5 = \beta_1^{(35)} \text{ and } b_4 = \beta_3^{(24)} \\ & \text{or } \quad \beta_1^{(35)} < b_5 < (3+\sqrt{3})/2 \text{ and } b_4 = \beta_2^{(24)}. \end{array}$$

Thus, we obtain

$$\begin{aligned} & V(\tilde{g}_3, \tilde{g}_5) \cap (\Xi_0 \cap \Pi) \\ &\quad = \{(b_1, b_2, b_3, b_4, b_5) \in \mathbb{R}^5 : \beta_2^{(36)} < b_5 < \beta_1^{(35)}, b_4 = \beta_1^{(24)}, \\ &\qquad b_3 = -\frac{(2b_5^2-6b_5+3)(b_5-1)^2}{3(2b_5-1)}, b_2 = \beta_2^{(11)}, \ b_1 = \beta_2\} \\ &\quad \cup \{(b_1, b_2, b_3, b_4, b_5) \in \mathbb{R}^5 : b_5 = \beta_1^{(35)}, \ b_4 = \beta_3^{(24)}, \\ &\qquad b_3 = -\frac{(2b_5^2-6b_5+3)(b_5-1)^2}{3(2b_5-1)}, b_2 = \beta_2^{(11)}, \ b_1 = \beta_2\} \\ &\quad \cup \{(b_1, b_2, b_3, b_4, b_5) \in \mathbb{R}^5 : \beta_1^{(35)} < b_5 < (3+\sqrt{3})/2, \ b_4 = \beta_2^{(24)}, \\ &\qquad b_3 = -\frac{(2b_5^2-6b_5+3)(b_5-1)^2}{3(2b_5-1)}, b_2 = \beta_2^{(11)}, \ b_1 = \beta_2\}. \end{aligned} \tag{3.24}$$

The proof of step **(II)** is completed.

Step (III): Prove $V(\tilde{g}_3, \tilde{g}_5, \tilde{g}_7) \cap (\Xi_0 \cap \Pi) = \emptyset$.

Similar to the discussion on the decomposition of $\mathcal{G}_0$, we compute

$$\begin{aligned} \mathrm{res}(\tilde{g}_7, \tilde{g}_3, b_2) &= 25707364031692800 b_5^{20} \\ &\quad (2b_5^2 - 6b_5 + 3)^2 (b_5 - 1)^{16} (2b_5 - 1)^{26} (3(2b_5 - 1) b_4 \\ &\quad + b_5^2(-2b_5^2 + 6b_5 - 3))^7 \check{r}_{13}^{(1)}(b_4, b_5), \\ \mathrm{res}(\check{r}_{13}^{(1)}, \check{r}_{12}^{(1)}, b_4) &= -3904305912313344 b_5^{24} (2b_5^2 - 6b_5 + 3)^6 \\ &\quad (2b_5 - 1)^{23} \check{r}_{23}^{(1)}(b_5) \check{r}_{23}^{(2)}(b_5) \check{r}_{23}^{(3)}(b_5), \end{aligned}$$

where $\check{r}_{13}^{(1)}$ is a polynomial of degree 38 having 197 terms, $\check{r}_{23}^{(1)}(b_5)$ and $\check{r}_{23}^{(3)}(b_5)$ is given in Appendix, and $\check{r}_{23}^{(2)}(b_5) := 8b_5^3 - 26b_5^2 + 21b_5 - 6$. We get by Lemma 3.1 that

$$V(\tilde{g}_3, \tilde{g}_5, \tilde{g}_7) \cap (\Xi_0 \cap \Pi) = \left(\bigcup_{i=1}^{5} \hat{\mathcal{V}}_i \right) \cap (\Xi_0 \cap \Pi), \tag{3.25}$$

where

$$\hat{\mathcal{V}}_1 := V(\tilde{g}_3, \tilde{g}_5, \tilde{g}_7, \mathrm{lcff}(\tilde{g}_3, b_2)), \quad \hat{\mathcal{V}}_2 := V(\tfrac{\tilde{g}_3, \tilde{g}_5, \tilde{g}_7, \check{r}_{12}^{(1)}, \check{r}_{13}^{(1)}, \mathrm{lcff}(\check{r}_{12}^{(1)}, b_4)}{\mathrm{lcff}(\tilde{g}_3, b_2)}),$$
$$\hat{\mathcal{V}}_3 := V(\tfrac{\tilde{g}_3, \tilde{g}_5, \tilde{g}_7, \check{r}_{12}^{(1)}, \check{r}_{13}^{(1)}, \check{r}_{23}^{(1)}}{\mathrm{lcff}(\tilde{g}_3, b_2), \mathrm{lcff}(\check{r}_{12}^{(1)}, b_4)}), \quad \hat{\mathcal{V}}_4 := V(\tfrac{\tilde{g}_3, \tilde{g}_5, \tilde{g}_7, \check{r}_{12}^{(1)}, \check{r}_{13}^{(1)}, \check{r}_{23}^{(2)}}{\mathrm{lcff}(\tilde{g}_3, b_2), \mathrm{lcff}(\check{r}_{12}^{(1)}, b_4)}),$$
$$\hat{\mathcal{V}}_5 := V(\tfrac{\tilde{g}_3, \tilde{g}_5, \tilde{g}_7, \check{r}_{12}^{(1)}, \check{r}_{13}^{(1)}, \check{r}_{23}^{(3)}}{\mathrm{lcff}(\tilde{g}_3, b_2), \mathrm{lcff}(\check{r}_{12}^{(1)}, b_4)}).$$

Similar to Lemma 3.2, we also have $\hat{\mathcal{V}}_i \cap \tilde{\Xi}_2 = \emptyset$, $i = 1, ..., 5$. By (3.25), $V(\tilde{g}_3, \tilde{g}_5, \tilde{g}_7) \cap \tilde{\Xi}_2 = \emptyset$. This completes the proof of step **(III)**.

At the end of this proof, we discuss the independence of focal values. Restricted to the surface Π as shown in (3.18), we get by (2.2) that

$$X(1/(b_5 - 1), y) = \tilde{X}(y, b_1, b_2, b_4, b_5)/(3(2b_5 - 1)(b_5 - 1)^2),$$

where

$$\begin{aligned}\tilde{X}(y, b_1, b_2, b_4, b_5) &:= -3b_4(b_5 - 1)(2b_5 - 1)y^2 \\ &- (3b_5(2b_5 - 1)b_2 + (b_5 - 1)(3(2b_5 - 1)b_4 - b_5(2b_5^2 - 6b_5 + 3)))y \\ &+ b_5(3(2b_5 - 1)b_1 - (3(2b_5 - 1))b_2 + (b_5 - 1)(2b_5^2 - 6b_5 + 3)),\end{aligned}$$

which implies that the vertical coordinate of $\tilde{E}_2 : (1/(b_5 - 1), \tilde{y}_*)$ satisfies $\tilde{X}(\tilde{y}_*, b_1, b_2, b_4, b_5) = 0$ by (3.3). Also, the trace of the Jacobian matrix at $\tilde{E}_2$ can be computed as

$$\begin{aligned}\mathrm{Tr}(J(\tilde{E}_2)) = &-(3b_4(b_5 - 1)(2b_5 - 1)\tilde{y}_*^2 \\ &+((3(b_5 + 1))(2b_5 - 1)b_2 + (b_5 - 1)(3(2b_5 - 1) \cdot \\ &b_4 - (2b_5 + 1)(2b_5^2 - 6b_5 + 3)))\tilde{y}_* \\ &-(3(b_5 + 1))(2b_5 - 1)b_1 + (3(b_5 + 1))(2b_5 - 1) \cdot \\ &b_2 - (b_5 - 1)(2b_5 + 1)(2b_5^2 - 6b_5 + 3)) \\ &/(3(b_5 - 1)(2b_5 - 1)).\end{aligned}$$

By Kuznetsov (1995, p.67), we need to verify the transversal condition, i.e., a pair of conjugate eigenvalues at $\tilde{E}_2$ crosses the imaginary axis with nonzero speed. Compute

$$\frac{\partial \mathrm{Tr}(J(\tilde{E}_2))}{\partial b_1}\Big|_{b_1=\beta_2}$$

$$= \frac{3(2b_5-1)b_4 - b_5^2(2b_5^2-6b_5+3)}{3b_5(2b_5-1)b_2 + (b_5-1)(3(2b_5-1)b_4 - b_5(2b_5+1)(2b_5^2-6b_5+3))},$$

which is positive by (3.3), implying that there is a perturbation with exactly 1 small cycle near $\tilde{E}_2$ for $(b_1, b_2, b_3, b_4, b_5) \in (\Xi_0 \cap \Pi) \setminus V(\tilde{g}_3)$. By Christopher and Li (2007, p.14), we need to determine the rank of the Jacobian matrix of the vector-valued function constituted by focal values with respect to the perturbation parameters under the condition that the focal values vanish. Compute

$$\begin{aligned}
\frac{\partial \mathcal{G}_3}{\partial b_2} &= -\frac{\sqrt{3}b_5(b_5-1)^{3/2}(-2b_5^2+6b_5-3)^{3/2}((6b_5-3)b_4-b_5^2(2b_5^2-6b_5+3))^{3/2}}{432b_4^2((2b_5-1)(3b_5(2b_5-1)b_2+(b_5-1)(3(2b_5-1)b_4-b_5(2b_5+1)(2b_5^2-6b_5+3))))^{5/2}} \cdot \\
&\quad F_1(b_2, b_4, b_5), \\
\frac{\partial(\mathcal{G}_3, \mathcal{G}_5)}{\partial(b_2, b_4)} &= \frac{b_5^4(b_5-1)^4(2b_5^2-6b_5+3)^4((6b_5-3)b_4-b_5^2(2b_5^2-6b_5+3))^3}{725594112b_4^7(2b_5-1)^{10}(3b_5(2b_5-1)b_2+(b_5-1)(3(2b_5-1)b_4-b_5(2b_5+1)(2b_5^2-6b_5+3)))^6} \cdot \\
&\quad F_2(b_2, b_4, b_5),
\end{aligned} \tag{3.26}$$

where F_1 is given in Appendix and F_2 is a polynomial of degree 41 having 875 terms. Since the fractions in (3.26) are both positive by (3.3) and (3.18), we can use signs of F_1 and F_2 to discuss signs of $\partial\mathcal{G}_3/\partial b_2$ and $\partial(\mathcal{G}_3, \mathcal{G}_5)/\partial(b_2, b_4)$, respectively. Recall that, restricted to the surface $\Xi_0 \cap \Pi$, signs of $\mathcal{G}_i$ s $(i = 3, 5)$ are, respectively, determined by signs of $\tilde{g}_i$ s $(i = 3, 5)$, which are described just below (3.6) and (3.18). Thus, we compute the varieties

$$(\mathbf{U1}) : V(\tfrac{\tilde{g}_3, F_1}{\tilde{g}_5}) \cap (\Xi_0 \cap \Pi) \quad \text{and} \quad (\mathbf{U2}) : V(\tilde{g}_3, \tilde{g}_5, F_2) \cap (\Xi_0 \cap \Pi)$$

separately to discuss signs of two Jacobi determinants in (3.26) under the condition that the focal values vanish. In order to compute $(\mathbf{U1}) : V(\{\tilde{g}_3, F_1\}/\tilde{g}_5) \cap (\Xi_0 \cap \Pi)$, we calculate

$$\begin{aligned}
\mathrm{res}(F_1, \tilde{g}_3, b_2) = -972b_5^4(b_5-1)^4(2b_5-1)^6((6b_5-3)b_4 - b_5^2(2b_5^2-6b_5+3)) \cdot \\
\tilde{R}_{12}^{(1)}(b_4, b_5),
\end{aligned}$$

where $\tilde{R}_{12}^{(1)}(b_4, b_5) := 9(2b_5-1)^2b_4^2 - 12b_5^2(2b_5-1)(13b_5^2-12b_5+3)b_4 + 4b_5^4(25b_5^4 - 96b_5^3 + 114b_5^2 - 54b_5 + 9)$. Similarly, since $\mathrm{lcff}(\tilde{g}_3, b_2) > 0$, by Lemma 2 of Chen and Zhang (2009, p. 567), we get

$$V(\tfrac{\tilde{g}_3, F_1}{\tilde{g}_5}) \cap (\Xi_0 \cap \Pi) = V(\tfrac{\tilde{g}_3, F_1, \tilde{R}_{12}^{(1)}}{\tilde{g}_5}) \cap (\Xi_0 \cap \Pi).$$

Similar to Lemma 3.3, we see that $\varphi_{21}(\cdot) := \tilde{R}_{12}^{(1)}(\cdot, b_5)$ has two real zeros, denoted by $\beta_1^{(21)} < \beta_2^{(21)}$, given in Appendix. By (3.19) and (3.24), we obtain

$$\begin{aligned}
&V(\tfrac{\tilde{g}_3, F_1, \tilde{R}_{12}^{(1)}}{\tilde{g}_5}) \cap (\Xi_0 \cap \Pi) = \{(b_1, b_2, b_3, b_4, b_5) \in \mathbb{R}^5 : \beta_1^{(34)} \\
&< b_5 < (3+\sqrt{3})/2,\ b_4 = \beta_1^{(21)}, \\
&\quad b_3 = -\tfrac{(2b_5^2-6b_5+3)(b_5-1)^2}{3(2b_5-1)},\ b_2 = \beta_3^{(11)},\ b_1 = \beta_2\},
\end{aligned}$$

which implies that there is a perturbation with exactly 2 small cycles near $\tilde{E}_2$ for $(b_1, b_2, b_3, b_4, b_5) \in V(\tilde{g}_3/\{\tilde{g}_5, F_1\}) \cap (\Xi_0 \cap \Pi)$. Moreover, we can check that $\mathcal{G}_5 < 0$ if $(b_1, b_2, b_3, b_4, b_5) \in V(\{\tilde{g}_3, F_1\}/\tilde{g}_5) \cap (\Xi_0 \cap \Pi)$. Also, the discriminant of $\tilde{g}_3$ with respect to the variable b_2 is $9(2b_5-1)^2(b_5-1)^2\varphi_{21}(b_4)$, and that $\mathcal{G}_3 < 0$ if $\beta_1^{(34)} < b_5 < (3+\sqrt{3})/2, b_4 = \beta_1^{(21)}, b_3 = -(2b_5^2-6b_5+3)(b_5-1)^2/(3(2b_5-1))$, $b_2 \neq \beta_3^{(11)}$ and $b_1 = \beta_2$, which implies that there is a perturbation with exactly 1 small cycle near $\tilde{E}_2$ for $(b_1, b_2, b_3, b_4, b_5) \in V(\{\tilde{g}_3, F_1\}/\tilde{g}_5) \cap (\Xi_0 \cap \Pi)$. The discussion on (**U1**) is completed. In order to compute (**U2**) : $V(\tilde{g}_3, \tilde{g}_5, F_2) \cap (\Xi_0 \cap \Pi)$, we calculate

$$\begin{aligned}
\mathrm{res}(F_2, \tilde{g}_3, b_2) &= 10711401679872 b_5^{15}(b_5-1)^{14}(2b_5-1)^{24} \\
&\qquad ((6b_5-3)b_4 - b_5^2(2b_5^2-6b_5+3))^4 \check{R}_{13}^{(1)}(b_4, b_5), \\
\mathrm{res}(\check{R}_{13}^{(1)}, \check{r}_{12}^{(1)}, b_4) &= 2754990144 b_5^{17}(2b_5-1)^{15}(2b_5^2-6b_5+3)^3 \\
&\qquad \hat{R}_{23}^{(1)}(b_5)\hat{R}_{23}^{(2)}(b_5)\hat{R}_{23}^{(3)}(b_5)\hat{R}_{23}^{(4)}(b_5)\hat{R}_{23}^{(5)}(b_5),
\end{aligned}$$

where $\check{R}_{13}^{(1)}$ and $\hat{R}_{23}^{(i)}$ s $(i = 1, ..., 5)$ are given in Appendix. By Lemma 3.1, we similarly get

$$V(\tilde{g}_3, \tilde{g}_5, F_2) \cap (\Xi_0 \cap \Pi) = \left(\bigcup_{i=1}^{7} \tilde{\mathcal{V}}_i\right) \cap (\Xi_0 \cap \Pi), \tag{3.27}$$

where

$$\begin{aligned}
\tilde{\mathcal{V}}_1 &:= V(\tilde{g}_3, \tilde{g}_5, F_2, \mathrm{lcff}(\tilde{g}_3, b_2)), \\
\tilde{\mathcal{V}}_2 &:= V(\frac{\tilde{g}_3, \tilde{g}_5, F_2, \check{r}_{12}^{(1)}, \check{R}_{13}^{(1)}, \mathrm{lcff}(\check{r}_{12}^{(1)}, b_4)}{\mathrm{lcff}(\tilde{g}_3, b_2)}), \\
\tilde{\mathcal{V}}_3 &:= V(\frac{\tilde{g}_3, \tilde{g}_5, F_2, \check{r}_{12}^{(1)}, \check{R}_{13}^{(1)}, \hat{R}_{23}^{(1)}}{\mathrm{lcff}(\tilde{g}_3, b_2), \mathrm{lcff}(\check{r}_{12}^{(1)}, b_4)}), \\
\tilde{\mathcal{V}}_4 &:= V(\frac{\tilde{g}_3, \tilde{g}_5, F_2, \check{r}_{12}^{(1)}, \check{R}_{13}^{(1)}, \hat{R}_{23}^{(2)}}{\mathrm{lcff}(\tilde{g}_3, b_2), \mathrm{lcff}(\check{r}_{12}^{(1)}, b_4)}), \\
\tilde{\mathcal{V}}_5 &:= V(\frac{\tilde{g}_3, \tilde{g}_5, F_2, \check{r}_{12}^{(1)}, \check{R}_{13}^{(1)}, \hat{R}_{23}^{(3)}}{\mathrm{lcff}(\tilde{g}_3, b_2), \mathrm{lcff}(\check{r}_{12}^{(1)}, b_4)}),
\end{aligned}$$

Table 3 Continuation of Table 1—Sets $\tilde{\Lambda}_1^i$, $\tilde{\Lambda}_2^j$, $\tilde{\Lambda}_3$ for numbers of small cycles

b_4	b_2	b_1	Number	Label
$b_4 \in (0, \beta_1^{(21)})$	$b_2 \in B_-^o(\beta_2^{(11)})$	$b_1 \in B_-^o(\beta_2)$	2	$\tilde{\Lambda}_2^2$
		$b_1 \in B_+(\beta_2)$	1	$\tilde{\Lambda}_1^{13}$
	$b_2 \in B_+(\beta_2^{(11)})$	$b_1 \in B_-(\beta_2)$	0	
		$b_1 \in B_+^o(\beta_2)$	1	$\tilde{\Lambda}_1^{14}$
	$(b_3, b_5) \in D_3$			

$$\tilde{\mathcal{V}}_6 := V\left(\frac{\tilde{g}_3, \tilde{g}_5, F_2, \check{r}_{12}^{(1)}, \check{R}_{13}^{(1)}, \hat{R}_{23}^{(4)}}{\mathrm{lcff}(\tilde{g}_3, b_2), \mathrm{lcff}(\check{r}_{12}^{(1)}, b_4)}\right),$$

$$\tilde{\mathcal{V}}_7 := V\left(\frac{\tilde{g}_3, \tilde{g}_5, F_2, \check{r}_{12}^{(1)}, \check{R}_{13}^{(1)}, \hat{R}_{23}^{(5)}}{\mathrm{lcff}(\tilde{g}_3, b_2), \mathrm{lcff}(\check{r}_{12}^{(1)}, b_4)}\right).$$

Similar to Lemma 3.2, we have $\tilde{\mathcal{V}}_i \cap \tilde{\Xi}_2 = \emptyset$, $i = 1, ..., 7$. By (3.27), we get $V(\tilde{g}_3, \tilde{g}_5, F_2) \cap (\Xi_0 \cap \Pi) = \emptyset$, which implies that there is a perturbation with exactly 3 small cycles near $\tilde{E}_2$ for $(b_1, b_2, b_3, b_4, b_5) \in V(\tilde{g}_3, \tilde{g}_5) \cap (\Xi_0 \cap \Pi)$. The discussion on (**U2**) is completed. Finally, checking the signs of focal values, we obtain the numbers of small cycles in Table 1. The proof of this theorem is completed (Table 5). □

Remark 1 Our sufficient condition given in Theorem 3.1 for the occurrence of exactly 1, 2, or 3 small cycles is obtained with the restriction to the surface Π as shown in (3.18). This restriction eliminates b_3, makes the leading coefficient of the cubic $\tilde{r}_{12}$ with respect to b_4 vanish, and simplifies its real zeros to be of a single variable. Without this restriction, it is hopeful to get a weaker condition, but it is hard to check the inequalities which determine whether a real zero, an irrational function in two variables given by the formulae of cubic zeros, exists in the allowed interval, which needs to check the sign of difference between the zero and each endpoint of the interval.

Remark 2 Theorem 3.1 does not give a condition for *4 small cycles* although we proved that the interior equilibrium $\tilde{E}_2$ is a weak focus of multiplicity at most 4 because of the difficulty in determining whether the common zero of two high-degree polynomials with parameters lies in an allowed interval under parameter conditions. Actually, if we want to find 4 small cycles, we need to give conditions for the inequality $V(g_3, g_5, g_7) \cap \Xi_0 \neq \emptyset$. Starting from the primary stratum $\{g_3, g_5, g_7\}$, similar to (3.25), we can find a unique first sub-stratum $\{\tilde{r}_{12}, \tilde{r}_{13}\}$ and three second sub-strata $\{\check{r}_{23}^{(1)}\}$, $\{\check{r}_{23}^{(2)}\}$ and $\{\check{r}_{23}^{(3)}\}$ in Ξ_0, where $\tilde{r}_{12}$ and $\tilde{r}_{13}$ are defined just below (3.8), and $\tilde{r}_{23}^{(1)}$, $\tilde{r}_{23}^{(2)}$ and $\tilde{r}_{23}^{(3)}$ are defined in (3.12). Unlike $\tilde{r}_{23}^{(1)}$, which can be solved with respect to b_3, the polynomials $\tilde{r}_{23}^{(2)}$ and $\tilde{r}_{23}^{(3)}$ are of degrees 8 and 15 with respect to b_3, respectively, without missing terms, zeros of which are functions of b_5 but cannot be expressed in terms of coefficients and rational numbers. This makes difficulties in checking whether the common zeros of two polynomials in the first sub-stratum, which are of 13 and 38

Table 4 Continuation of Table 1—Sets $\tilde{\Lambda}_1^i$, $\tilde{\Lambda}_2^j$, $\tilde{\Lambda}_3$ for numbers of small cycles

b_4	b_2	b_1	Number	Label
$b_4 \in (0, \beta_2^{(23)}]$	$b_2 \in (0, \beta_2^{(11)})$	$b_1 \in B_-^o(\beta_2)$	1	$\tilde{\Lambda}_1^1$
		$b_1 \in B_+(\beta_2)$	0	
	$b_2 \in (\beta_2^{(11)}, \infty)$	$b_1 \in B_-(\beta_2)$	0	
		$b_1 \in B_+^o(\beta_2)$	1	$\tilde{\Lambda}_1^2$
$b_4 \in (\beta_2^{(23)}, \beta_1^{(21)})$	$b_2 \in (0, \beta_1^{(11)})$	$b_1 \in B_-(\beta_2)$	0	
		$b_1 \in B_+^o(\beta_2)$	1	$\tilde{\Lambda}_1^3$
	$b_2 \in (\beta_1^{(11)}, \beta_2^{(11)})$	$b_1 \in B_-^o(\beta_2)$	1	$\tilde{\Lambda}_1^4$
		$b_1 \in B_+(\beta_2)$	0	
	$b_2 \in (\beta_2^{(11)}, \infty)$	$b_1 \in B_-(\beta_2)$	0	
		$b_1 \in B_+^o(\beta_2)$	1	$\tilde{\Lambda}_1^5$
$b_4 = \beta_1^{(21)}$	$b_2 \in (0, \beta_3^{(11)})$	$b_1 \in B_-(\beta_2)$	0	
		$b_1 \in B_+^o(\beta_2)$	1	$\tilde{\Lambda}_1^6$
	$b_2 \in (\beta_3^{(11)}, \infty)$	$b_1 \in B_-(\beta_2)$	0	
		$b_1 \in B_+^o(\beta_2)$	1	$\tilde{\Lambda}_1^7$
$b_4 \in (\beta_1^{(21)}, \infty)$	$b_2 \in (0, \infty)$	$b_1 \in B_-(\beta_2)$	0	
		$b_1 \in B_+^o(\beta_2)$	1	$\tilde{\Lambda}_1^8$
$b_4 \in (\beta_2^{(23)}, \beta_1^{(21)})$	$b_2 \in B_-(\beta_1^{(11)})$	$b_1 \in B_-(\beta_2)$	0	
		$b_1 \in B_+^o(\beta_2)$	1	$\tilde{\Lambda}_1^9$
	$b_2 \in B_+^o(\beta_1^{(11)})$	$b_1 \in B_-^o(\beta_2)$	2	$\tilde{\Lambda}_2^1$
		$b_1 \in B_+(\beta_2)$	1	$\tilde{\Lambda}_1^{10}$
$b_4 = \beta_1^{(21)}$	$b_2 \in B^o(\beta_3^{(11)})$	$b_1 \in B_-(\beta_2)$	0	
		$b_1 \in B_+^o(\beta_2)$	1	$\tilde{\Lambda}_1^{11}$
	$(b_3, b_5) \in D_3 \cup D_4 \cup D_5 \cup D_6$			

degrees with 102 and 1260 terms, respectively, is positive in the case that b_5 is chosen to ensure that $\tilde{r}_{23}^{(2)}$ or $\tilde{r}_{23}^{(3)}$ has a positive zero and b_3 is chosen to be the positive zero exactly.

Finally, we provide the proofs of Lemmas 3.1, 3.2 and 3.3.

Proof of Lemma 3.1 Since $U_1, ..., U_r$ are common factors of polynomials f_1, f_2 and f_3, we have

$$V(\mathcal{G}) = \Big(\bigcup_{\kappa=1}^{r} V(U_\kappa)\Big) \cup V(\tilde{f}_1, \tilde{f}_2, \tilde{f}_3), \tag{3.28}$$

Table 5 Continuation of Table 3—Sets $\tilde{\Lambda}_1^i$, $\tilde{\Lambda}_2^j$, $\tilde{\Lambda}_3$ for numbers of small cycles

b_4	b_2	b_1	Number	Label
$b_4 \in (0, \beta_{\ell-3}^{(24)})$	$b_2 \in B_-(\beta_2^{(11)})$	$b_1 \in B_-^o(\beta_2)$	1	$\tilde{\Lambda}_1^{20}$
		$b_1 \in B_+(\beta_2)$	0	
	$b_2 \in B_+^o(\beta_2^{(11)})$	$b_1 \in B_-(\beta_2)$	1	$\tilde{\Lambda}_1^{21}$
		$b_1 \in B_+^o(\beta_2)$	2	$\tilde{\Lambda}_2^4$
$b_4 \in (\beta_{\ell-3}^{(24)}, \beta_1^{(21)})$	$b_2 \in B_-^o(\beta_2^{(11)})$	$b_1 \in B_-^o(\beta_2)$	2	$\tilde{\Lambda}_2^5$
		$b_1 \in B_+(\beta_2)$	1	$\tilde{\Lambda}_1^{22}$
	$b_2 \in B_+(\beta_2^{(11)})$	$b_1 \in B_-(\beta_2)$	0	
		$b_1 \in B_+^o(\beta_2)$	1	$\tilde{\Lambda}_1^{23}$
$b_4 \in B_-^o(\beta_{\ell-3}^{(24)})$	$b_2 \in B_-(\beta_2^{(11)})$	$b_1 \in B_-^o(\beta_2)$	2	$\tilde{\Lambda}_2^6$
		$b_1 \in B_+(\beta_2)$	1	$\tilde{\Lambda}_1^{24}$
	$b_2 \in B_+^o(\beta_2^{(11)})$	$b_1 \in B_-(\beta_2)$	2	$\tilde{\Lambda}_2^7$
		$b_1 \in B_+^o(\beta_2)$	3	$\tilde{\Lambda}_3$
$b_4 \in B_+(\beta_{\ell-3}^{(24)})$	$b_2 \in B_-^o(\beta_2^{(11)})$	$b_1 \in B_-^o(\beta_2)$	2	$\tilde{\Lambda}_2^8$
		$b_1 \in B_+(\beta_2)$	1	$\tilde{\Lambda}_1^{25}$
	$b_2 \in B_+(\beta_2^{(11)})$	$b_1 \in B_-(\beta_2)$	0	
		$b_1 \in B_+^o(\beta_2)$	1	$\tilde{\Lambda}_1^{26}$
		$(b_3, b_5) \in D_\ell$, $\ell = 4, 5, 6$		

where $\tilde{f}_i := f_i/(U_1^{p_{i,1}} \cdots U_r^{p_{i,r}})$ for each $i = 1, 2, 3$. Since $W_1, \ldots, U_s$ are common factors of polynomials $\tilde{f}_2$ and $\tilde{f}_3$, we have

$$V(\tilde{f}_1, \tilde{f}_2, \tilde{f}_3) = V(\tilde{f}_1) \cap V(\tilde{f}_2, \tilde{f}_3) = V(\tilde{f}_1) \cap \Big((\bigcup_{\eta=1}^{s} V(W_\eta)) \cup V(\hat{f}_2, \hat{f}_3) \Big), \tag{3.29}$$

where $\hat{f}_j := \tilde{f}_j/(W_1^{h_{j-1,1}} \cdots W_s^{h_{j-1,s}})$ for each $j = 2, 3$. Since a polynomial vanishes if and only if one of its factors is zero, we have

$$V(\tilde{f}_1) = \bigcup_{\alpha=1}^{\ell} \tilde{f}_{1,\alpha}, \quad V(\hat{f}_2) = \bigcup_{\beta=1}^{m} \tilde{f}_{2,\beta}, \quad V(\hat{f}_3) = \bigcup_{\beta=1}^{n} \tilde{f}_{3,\gamma}. \tag{3.30}$$

Thus, by (3.29) and (3.30), we can see that

$$V(\tilde{f}_1, \tilde{f}_2, \tilde{f}_3) = \Big(\bigcup_{\eta=1}^{s} \bigcup_{\alpha=1}^{\ell} V(\tilde{f}_{1,\alpha}, W_\eta) \Big) \cup \Big(\bigcup_{\gamma=1}^{n} \bigcup_{\beta=1}^{m} \bigcup_{\alpha=1}^{\ell} V(\tilde{f}_{1,\alpha}, \tilde{f}_{2,\beta}, \tilde{f}_{3,\gamma}) \Big). \tag{3.31}$$

On the other hand, we also see by Theorem 1 of Chen and Zhang (2009) that

$$V(\tilde{f}_1, \tilde{f}_2, \tilde{f}_3) = V(\tilde{f}_1, \tilde{f}_2, \tilde{f}_3, \tilde{J}_1) \cup V(\tfrac{\tilde{f}_1, \tilde{f}_2, \tilde{f}_3, R_{12}, R_{13}}{\tilde{J}_1}), \tag{3.32}$$

where $\tilde{J}_1$ is the leading coefficient of $\tilde{f}_1$ with respect to X and $R_{1j} := \operatorname{res}(\tilde{f}_j, \tilde{f}_1, X)$, $j = 2, 3$. It is obvious that

$$\tilde{J}_1 = \prod_{\alpha=1}^{\ell} \tilde{J}_{1,\alpha}^{q_{1,\alpha}}, \tag{3.33}$$

where $\tilde{J}_{1,\alpha}$ is the leading coefficient of $\tilde{f}_{1,\alpha}$ with respect to X for each $\alpha = 1, ..., \ell$. Note that from Mishra (1993, p. 227), the resultant has three properties (**i**) : $\operatorname{res}(fg, h, x) = \operatorname{res}(f, h, x)\operatorname{res}(g, h, x)$, (**ii**) : $\operatorname{res}(f, g, x) = (-1)^{\deg(f,x)\deg(g,x)}\operatorname{res}(g, f, x)$, (**iii**) : $\operatorname{res}(f^m, g^n, x) = (\operatorname{res}(f, g, x))^{mn}$, where f, g and h are polynomials with respect to x and $\deg(\varphi, x)$ is the degree of the polynomial φ with respect to x. Those properties enable us to see

$$
\begin{aligned}
(\mathbf{I}):\ & \operatorname{res}(\tilde{f}_2, \tilde{f}_1, X) = \textstyle\prod_{\eta=1}^{s} \operatorname{res}(W_\eta^{h_{1,\eta}}, \tilde{f}_1, X) \prod_{\beta=1}^{m} \operatorname{res}(\tilde{f}_{2,\beta}^{q_{2,\beta}}, \tilde{f}_1, X),\\
& \operatorname{res}(\tilde{f}_1, W_\eta^{h_{1,\eta}}, X) = \textstyle\prod_{\alpha=1}^{\ell} \operatorname{res}(\tilde{f}_{1,\alpha}^{q_{1,\alpha}}, W_\eta^{h_{1,\eta}}, X),\ \ \operatorname{res}(\tilde{f}_1, \tilde{f}_{2,\beta}^{q_{2,\beta}}, X)\\
& = \textstyle\prod_{\alpha=1}^{\ell} \operatorname{res}(\tilde{f}_{1,\alpha}^{q_{1,\alpha}}, \tilde{f}_{2,\beta}^{q_{2,\beta}}, X),\\
(\mathbf{II}):\ & \operatorname{res}(W_\eta^{h_{1,\eta}}, \tilde{f}_1, X) = (-1)^{h_{1,\eta}\omega_\eta \varrho_*}\operatorname{res}(\tilde{f}_1, W_\eta^{h_{1,\eta}}, X),\ \ \operatorname{res}(\tilde{f}_{2,\beta}^{q_{2,\beta}}, \tilde{f}_1, X)\\
& = (-1)^{q_{2,\beta}\varsigma_\beta \varrho_*}\operatorname{res}(\tilde{f}_1, \tilde{f}_{2,\beta}^{q_{2,\beta}}, X),\\
& \operatorname{res}(\tilde{f}_{1,\alpha}, W_\eta, X) = (-1)^{\varrho_\alpha \omega_\eta}\operatorname{res}(W_\eta, \tilde{f}_{1,\alpha}, X),\ \ \operatorname{res}(\tilde{f}_{1,\alpha}, \tilde{f}_{2,\beta}, X)\\
& = (-1)^{\varrho_\alpha \varsigma_\beta}\operatorname{res}(\tilde{f}_{2,\beta}, \tilde{f}_{1,\alpha}, X),\\
(\mathbf{III}):\ & \operatorname{res}(\tilde{f}_{1,\alpha}^{q_{1,\alpha}}, W_\eta^{h_{1,\eta}}, X) = (\operatorname{res}(\tilde{f}_{1,\alpha}, W_\eta, X))^{q_{1,\alpha}h_{1,\eta}},\ \ \operatorname{res}(\tilde{f}_{1,\alpha}^{q_{1,\alpha}}, \tilde{f}_{2,\beta}^{q_{2,\beta}}, X)\\
& = (\operatorname{res}(\tilde{f}_{1,\alpha}, \tilde{f}_{2,\beta}, X))^{q_{1,\alpha}q_{2,\beta}},
\end{aligned}
$$

where $\omega_\eta := \deg(W_\eta, X)$, $\varrho_\alpha := \deg(\tilde{f}_{1,\alpha}, X)$, $\varsigma_\beta := \deg(\tilde{f}_{2,\beta}, X)$, and $\varrho_* := \sum_{\alpha=1}^{\ell} q_{1,\alpha}\varrho_\alpha$. Thus,

$$
\begin{aligned}
R_{12} = & \prod_{\eta=1}^{s} (-1)^{h_{1,\eta}\omega_\eta \varrho_*} \prod_{\alpha=1}^{\ell} (-1)^{\varrho_\alpha \omega_\eta q_{1,\alpha} h_{1,\eta}} \tilde{r}_{\alpha\eta}^{q_{1,\alpha}h_{1,\eta}}\\
& \prod_{\beta=1}^{m} (-1)^{q_{2,\beta}\varsigma_\beta \varrho_*} \prod_{\alpha=1}^{\ell} (-1)^{\varrho_\alpha \varsigma_\beta q_{1,\alpha} q_{2,\beta}} \hat{r}_{\alpha\beta}^{q_{1,\alpha}q_{2,\beta}},
\end{aligned}
$$

where $\tilde{r}_{\alpha\eta} := \text{res}(W_\eta, \tilde{f}_{1,\alpha}, X)$ and $\hat{r}_{\alpha\beta} := \text{res}(\tilde{f}_{2,\beta}, \tilde{f}_{1,\alpha}, X)$. Similarly, we see

$$R_{13} = \prod_{\eta=1}^{s} (-1)^{h_{1,\eta}\omega_\eta \varrho_*} \prod_{\alpha=1}^{\ell} (-1)^{\varrho_\alpha \omega_\eta q_{1,\alpha} h_{1,\eta}} \tilde{r}_{\alpha\eta}^{q_{1,\alpha}h_{1,\eta}}$$
$$\prod_{\gamma=1}^{n} (-1)^{q_{3,\gamma}\zeta_\gamma \varrho_*} \prod_{\alpha=1}^{\ell} (-1)^{\varrho_\alpha \zeta_\gamma q_{1,\alpha} q_{3,\gamma}} \hat{r}_{\alpha\gamma}^{q_{1,\alpha}q_{3,\gamma}},$$

where $\hat{r}_{\alpha\gamma} := \text{res}(\tilde{f}_{3,\gamma}, \tilde{f}_{1,\alpha}, X)$ and $\zeta_\gamma := \deg(\tilde{f}_{3,\gamma}, X)$. Thus, from (3.31), (3.32), (3.33) and the expression of R_{12} and R_{13}, we obtain

$$V(\tilde{f}_1, \tilde{f}_2, \tilde{f}_3) = \big(V(\tilde{f}_1, \tilde{f}_2, \tilde{f}_3) \cap V(\tilde{J}_1)\big) \cup \big(V(\tilde{f}_1, \tilde{f}_2, \tilde{f}_3) \cap V(R_{12}, R_{13}) \backslash V(\tilde{J}_1)\big)$$
$$= \big(((\bigcup_{\eta=1}^{s} \bigcup_{\alpha=1}^{\ell} V(\tilde{f}_{1,\alpha}, W_\eta)) \cup (\bigcup_{\gamma=1}^{n} \bigcup_{\beta=1}^{m} \bigcup_{\alpha=1}^{\ell} V(\tilde{f}_{1,\alpha}, \tilde{f}_{2,\beta}, \tilde{f}_{3,\gamma})))$$

$$\cap(\bigcup_{\alpha=1}^{\ell} V(\tilde{J}_{1,\alpha}))\big) \cup \big(((\bigcup_{\eta=1}^{s} \bigcup_{\alpha=1}^{\ell} V(\tilde{f}_{1,\alpha}, W_\eta)) \cup (\bigcup_{\gamma=1}^{n} \bigcup_{\beta=1}^{m} \bigcup_{\alpha=1}^{\ell} V(\tilde{f}_{1,\alpha}, \tilde{f}_{2,\beta}, \tilde{f}_{3,\gamma})))$$
$$\cap((\bigcup_{\eta=1}^{s} \bigcup_{\alpha=1}^{\ell} V(\tilde{r}_{\alpha\eta})) \cup (\bigcup_{\gamma=1}^{n} \bigcup_{\beta=1}^{m} \bigcup_{\alpha=1}^{\ell} V(\hat{r}_{\alpha\beta}, \hat{r}_{\alpha\gamma})))\backslash V(\bigcup_{\alpha=1}^{\ell} V(\tilde{J}_{1,\alpha}))\big)$$
$$= \big(\bigcup_{\eta=1}^{s} \bigcup_{\alpha=1}^{\ell} V(\tilde{f}_{1,\alpha}, W_\eta, \tilde{J}_{1,\alpha}) \cup V(\tfrac{\tilde{f}_{1,\alpha}, W_\eta, \tilde{r}_{\alpha\eta}}{\tilde{J}_{1,\alpha}})\big)$$
$$\cup\big(\bigcup_{\gamma=1}^{n} \bigcup_{\beta=1}^{m} \bigcup_{\alpha=1}^{\ell} V(\tilde{f}_{1,\alpha}, \tilde{f}_{2,\beta}, \tilde{f}_{3,\gamma}, \tilde{J}_{1,\alpha}) \cup (\tfrac{\tilde{f}_{1,\alpha}, \tilde{f}_{2,\beta}, \tilde{f}_{3,\gamma}, \hat{r}_{\alpha\beta}, \hat{r}_{\alpha\gamma}}{\tilde{J}_{1,\alpha}})\big).$$

It follows by (3.28) that

$$V(\mathcal{G}) = \big(\bigcup_{\kappa=1}^{r} V(U_\kappa)\big) \cup \big(\bigcup_{\eta=1}^{s} \bigcup_{\alpha=1}^{\ell} V(\tilde{f}_{1,\alpha}, W_\eta, \tilde{J}_{1,\alpha}) \cup V(\tfrac{\tilde{f}_{1,\alpha}, W_\eta, \tilde{r}_{\alpha\eta}}{\tilde{J}_{1,\alpha}})\big)$$
$$\cup\big(\bigcup_{\gamma=1}^{n} \bigcup_{\beta=1}^{m} \bigcup_{\alpha=1}^{\ell} V(\tilde{f}_{1,\alpha}, \tilde{f}_{2,\beta}, \tilde{f}_{3,\gamma}, \tilde{J}_{1,\alpha}) \cup (\tfrac{\tilde{f}_{1,\alpha}, \tilde{f}_{2,\beta}, \tilde{f}_{3,\gamma}, \hat{r}_{\alpha\beta}, \hat{r}_{\alpha\gamma}}{\tilde{J}_{1,\alpha}})\big),$$

which shows that the conclusion (3.11) holds. This proves the lemma. □

Proof of Lemma 3.2 Claim ($\mathbf{M}_1$) is obvious because (3.3) requires $b_5 \in (1, \infty)$, but the equality $\text{lcff}(g_3, b_2) = b_5(2b_5 - 1)(b_5 - 1)^2 = 0$ gives $b_5 = 0$, $1/2$ or 1, none of which lie in $(1, \infty)$, which shows that $\text{lcff}(g_3, b_2) \neq 0$ for $(b_1, b_2, b_3, b_4, b_5) \in \Xi_0$, i.e., $\mathcal{V}_1 \cap \Xi_0 \subset V(\text{lcff}(g_3, b_2)) \cap \Xi_0 = \emptyset$.

In order to prove claim ($\mathbf{M}_2$), we first try to solve the equation $\text{lcff}(\tilde{r}_{12}, b_4) = (b_5 - 1)^6(3(2b_5 - 1)b_3 + (2b_5^2 - 6b_5 + 3)(b_5 - 1)^2) = 0$, which is equivalent to $\varphi_{31}(b_3) := 3(2b_5 - 1)b_3 + (2b_5^2 - 6b_5 + 3)(b_5 - 1)^2 = 0$ by (3.3). By solving $\varphi_{31} = 0$, we get

$$b_3 = (-2b_5^2 + 6b_5 - 3)(b_5 - 1)^2/(3(2b_5 - 1)) \tag{3.34}$$

because $\mathrm{lcff}(\varphi_{31}, b_3) = 3(2b_5 - 1)$ is positive by (3.3). We restrict $\tilde{r}_{12}$, $\tilde{r}_{13}$ and $\tilde{r}_{14}$ to (3.34) to get

$$\begin{aligned}
\tilde{r}_{12} &= b_5^2(b_5-1)^8\hat{r}_{12}(b_4,b_5)/(27(2b_5-1)^2),\\
\tilde{r}_{13} &= b_5^2(b_5-1)^{24}\hat{r}_{13}(b_4,b_5)/(177147(2b_5-1)^{10}),\\
\tilde{r}_{14} &= b_5^2(b_5-1)^{40}\hat{r}_{14}(b_4,b_5)/(1162261467(2b_5-1)^{17}),
\end{aligned} \tag{3.35}$$

where

$$\begin{aligned}
\hat{r}_{12}(b_4,b_5) := {}& 9(32b_5^3-80b_5^2+60b_5-15)(2b_5-1)^2b_4^2\\
&+6b_5(68b_5^5-434b_5^4+942b_5^3-831b_5^2+324b_5-45)(2b_5-1)^2b_4\\
&-2b_5^3(2b_5^2-6b_5+3)(168b_5^6-1112b_5^5+2738b_5^4-3189b_5^3\\
&+1920b_5^2-585b_5+72),
\end{aligned}$$

and $\hat{r}_{13}$ and $\hat{r}_{14}$ are polynomials of 38 and 64 degrees having 194 and 493 terms, respectively. Similarly, we compute

$$\begin{aligned}
&\mathrm{res}(\hat{r}_{13},\hat{r}_{12},b_4)\\
&\quad= -3904305912313344b_5^{24}(2b_5-1)^{23}\hat{r}_{23}^{(1)}(b_5)\hat{r}_{23}^{(2)}(b_5)\hat{r}_{23}^{(3)}(b_5)\hat{r}_{23}^{(4)}(b_5),\\
&\quad\mathrm{res}(\hat{r}_{14},\hat{r}_{12},b_4)\\
&\quad= -215126486151683819193630 72b_5^{39}(2b_5-1)^{37}\hat{r}_{23}^{(2)}(b_5)\hat{r}_{23}^{(4)}(b_5)\hat{r}_{24}^{(1)}\\
&\quad(b_5)\hat{r}_{24}^{(2)}(b_5),
\end{aligned} \tag{3.36}$$

where $\hat{r}_{23}^{(2)} := 8b_5^3 - 26b_5^2 + 21b_5 - 6$, $\hat{r}_{23}^{(4)} := 2b_5^2 - 6b_5 + 3$, $\hat{r}_{24}^{(1)}$ is a polynomial having 52 terms and $\hat{r}_{23}^{(1)}$, $\hat{r}_{23}^{(3)}$ and $\hat{r}_{24}^{(2)}$ are given in Appendix, and get by Lemma 3.1 that

$$\mathcal{V}_2 \cap \Xi_0 = \left(\bigcup_{i=1}^{7} \mathcal{V}_{2i}\right) \cap \Xi_0, \tag{3.37}$$

where

$$\begin{aligned}
\mathcal{V}_{21} &:= V(g_3, g_5, g_7, g_9, \tilde{r}_{12}, \tilde{r}_{13}, \tilde{r}_{14}, \varphi_{31}, \hat{r}_{12}, \hat{r}_{13}, \hat{r}_{14}, \mathrm{lcff}(\hat{r}_{12}, b_4)),\\
\mathcal{V}_{22} &:= V(\frac{g_3, g_5, g_7, g_9, \tilde{r}_{12}, \tilde{r}_{13}, \tilde{r}_{14}, \varphi_{31}, \hat{r}_{12}, \hat{r}_{13}, \hat{r}_{14}, \hat{r}_{23}^{(2)}}{\mathrm{lcff}(\hat{r}_{12}, b_4)}),\\
\mathcal{V}_{23} &:= V(\frac{g_3, g_5, g_7, g_9, \tilde{r}_{12}, \tilde{r}_{13}, \tilde{r}_{14}, \varphi_{31}, \hat{r}_{12}, \hat{r}_{13}, \hat{r}_{14}, \hat{r}_{23}^{(4)}}{\mathrm{lcff}(\hat{r}_{12}, b_4)}),\\
\mathcal{V}_{24} &:= V(\frac{g_3, g_5, g_7, g_9, \tilde{r}_{12}, \tilde{r}_{13}, \tilde{r}_{14}, \varphi_{31}, \hat{r}_{12}, \hat{r}_{13}, \hat{r}_{14}, \hat{r}_{23}^{(1)}, \hat{r}_{24}^{(1)}}{\mathrm{lcff}(\hat{r}_{12}, b_4)}),\\
\mathcal{V}_{25} &:= V(\frac{g_3, g_5, g_7, g_9, \tilde{r}_{12}, \tilde{r}_{13}, \tilde{r}_{14}, \varphi_{31}, \hat{r}_{12}, \hat{r}_{13}, \hat{r}_{14}, \hat{r}_{23}^{(1)}, \hat{r}_{24}^{(2)}}{\mathrm{lcff}(\hat{r}_{12}, b_4)}),\\
\mathcal{V}_{26} &:= V(\frac{g_3, g_5, g_7, g_9, \tilde{r}_{12}, \tilde{r}_{13}, \tilde{r}_{14}, \varphi_{31}, \hat{r}_{12}, \hat{r}_{13}, \hat{r}_{14}, \hat{r}_{23}^{(3)}, \hat{r}_{24}^{(1)}}{\mathrm{lcff}(\hat{r}_{12}, b_4)}),
\end{aligned}$$

$$\mathcal{V}_{27} := V(\frac{g_3, g_5, g_7, g_9, \tilde{r}_{12}, \tilde{r}_{13}, \tilde{r}_{14}, \varphi_{31}, \hat{r}_{12}, \hat{r}_{13}, \hat{r}_{14}, \hat{r}_{23}^{(3)}, \hat{r}_{24}^{(2)}}{\text{lcff}(\hat{r}_{12}, b_4)}).$$

Next, we prove that

$$(\mathbf{M}_{2i}) : \mathcal{V}_{2i} \cap \Xi_0 = \emptyset, \quad i = 1, ..., 7. \tag{3.38}$$

In order to prove claim $(\mathbf{M}_{21})$, we try to solve the equation $\text{lcff}(\hat{r}_{12}, b_4) = 9(2b_5 - 1)^2(32b_5^3 - 80b_5^2 + 60b_5 - 15) = 0$, which is equivalent to $\varphi_{51}(b_5) := 32b_5^3 - 80b_5^2 + 60b_5 - 15 = 0$ by (3.3). Using the Maple command "realroot(φ_{51}, $1/10^4$)," we find the polynomial φ_{51} has a unique zero in the interval $(1, \infty)$, covered by the isolated interval $[183457/131072, 91729/65536]$. Compute

$$\text{res}(\text{res}(\hat{r}_{13}, \varphi_{51}, b_5), \text{res}(\hat{r}_{12}, \varphi_{51}, b_5), b_4) \approx 2.0820323236234763745 \times 10^{674},$$

which implies that $\mathcal{V}_{21} \cap \Xi_0 \subset V(\hat{r}_{12}, \hat{r}_{13}, \varphi_{51}) \cap \Xi_0 = \emptyset$. Similar to the discussion on claim $(\mathbf{M}_{21})$, we can prove that $\mathcal{V}_{22} \cap \Xi_0 \subset V(\hat{r}_{12}, \hat{r}_{13}, \hat{r}_{23}^{(2)}) \cap \Xi_0 = \emptyset$, implying that claim $(\mathbf{M}_{22})$ holds. In order to prove claim $(\mathbf{M}_{23})$, using the Maple command "realroot($\hat{r}_{23}^{(4)}$, $1/10^4$)," we find the polynomial $\hat{r}_{23}^{(4)}$ has a unique zero in the interval $(1, \infty)$, covered by the isolated interval $[310119/131072, 38765/16384]$. Compute $\text{res}(\varphi_{31}, \hat{r}_{23}^{(4)}, b_5) = 144b_3^2$, which implies that $\mathcal{V}_{23} \cap \Xi_0 \subset V(\varphi_{31}, \hat{r}_{23}^{(4)}) \cap \Xi_0 = \emptyset$ because (3.3) requires $b_3 \in (0, \infty)$. Claim $(\mathbf{M}_{24})$ is obvious because $\text{res}(\hat{r}_{24}^{(1)}, \hat{r}_{23}^{(1)}, b_5) \approx -1.482870925 \times 10^{662}$, implying that $\mathcal{V}_{24} \cap \Xi_0 \subset V(\hat{r}_{24}^{(1)}, \hat{r}_{23}^{(1)}) = \emptyset$. Similar to the discussion on claim $(\mathbf{M}_{24})$, we prove that

$$\mathcal{V}_{25} \cap \Xi_0 \subset V(\hat{r}_{24}^{(2)}, \hat{r}_{23}^{(1)}) = \emptyset, \quad \mathcal{V}_{26} \cap \Xi_0 \subset V(\hat{r}_{24}^{(1)}, \hat{r}_{23}^{(3)}) = \emptyset$$
$$\text{and} \quad \mathcal{V}_{27} \cap \Xi_0 \subset V(\hat{r}_{24}^{(2)}, \hat{r}_{23}^{(3)}) = \emptyset,$$

implying that claims $(\mathbf{M}_{25})$, $(\mathbf{M}_{26})$ and $(\mathbf{M}_{27})$ hold, respectively. Thus, we get by (3.37) and (3.38) that $\mathcal{V}_2 \cap \Xi_0 = \emptyset$, implying that claim $(\mathbf{M}_2)$ holds. Similar to the discussion on claim $(\mathbf{M}_2)$, we can prove that claim $(\mathbf{M}_3)$ holds.

Claim $(\mathbf{M}_4)$ is obvious because (3.3) requires $b_5 \in (1, \infty)$, but the equality $\text{lcff}(\tilde{r}_{23}^{(2)}, b_3) = 9(2b_5 - 1)^2(3200b_5^4 - 3600b_5^3 + 1480b_5^2 - 250b_5 + 13) = 0$ has no real zeros lie in $(1, \infty)$ by using the Maple command "realroot," showing that $\text{lcff}(\tilde{r}_{23}^{(2)}, b_3) \neq 0$ for $(b_1, b_2, b_3, b_4, b_5) \in \Xi_0$, i.e., $\mathcal{V}_4 \cap \Xi_0 \subset V(\text{lcff}(\tilde{r}_{23}^{(2)}, b_3)) \cap \Xi_0 = \emptyset$. Similar to the discussion on claim $(\mathbf{M}_4)$, we prove that

$$\mathcal{V}_5 \cap \Xi_0 \subset V(\text{lcff}(\tilde{r}_{23}^{(2)}, b_3)) \cap \Xi_0 = \emptyset \quad \text{and}$$
$$(\mathcal{V}_6 \cup \mathcal{V}_7) \cap \Xi_0 \subset V(\text{lcff}(\tilde{r}_{23}^{(3)}, b_3)) \cap \Xi_0 = \emptyset,$$

implying that claims $(\mathbf{M}_5)$, $(\mathbf{M}_6)$ and $(\mathbf{M}_7)$ hold, respectively.

In order to prove claim ($\mathbf{M}_8$), using the Maple command "realroot," we find that the polynomial $\tilde{r}_{311}^{(1)}$ has a unique zero in $(1, \infty)$, covered by the isolated interval

$$I_1 := \Big[\frac{3549434735393734987161122358701976160955684419494937}{7482888383134222941202866343507369060638374620037 12}, \frac{7098869470787469974322244717403952321911368838989875}{14965777676626844588240573268701473812127674924007424}\Big].$$

Then, we try to find the common zero of $\tilde{r}_{23}^{(2)}$ and $\tilde{r}_{24}^{(1)}$ corresponding to the zero of $\tilde{r}_{311}^{(1)}$ covered by I_1. Let $f_0(\cdot) := \tilde{r}_{24}^{(1)}(\cdot, b_5)$ and $f_1(\cdot) := \tilde{r}_{23}^{(2)}(\cdot, b_5)$, and compute the residues

$$\begin{aligned}
& f_2(b_3) := \mathrm{rem}(f_0,\ f_1,\ b_3) \\
& = -\frac{12754584(2b_5-1)^{12}(b_5-1)^{18}Q_1(b_3,b_5)}{(\mathrm{lcff}(f_1,b_3))^9}, \\
& f_3(b_3) := \mathrm{rem}(f_1,\ f_2,\ b_3) \\
& = -\frac{(b_5-1)^4(\mathrm{lcff}(f_1,b_3))^9 Q_2(b_3,b_5)}{43046721(2b_5-1)^{13}(\mathrm{lcff}(Q_1,b_3))^2}, \\
& f_4(b_3) := \mathrm{rem}(f_2,\ f_3,\ b_3) \\
& = \frac{24794911296b_5^2(2b_5-1)^{11}(b_5-1)^{22}(\mathrm{lcff}(Q_1,b_3))^2 Q_3(b_3,b_5)}{(\mathrm{lcff}(f_1,b_3))^9(\mathrm{lcff}(Q_2,b_3))^2}, \\
& f_5(b_3) := \mathrm{rem}(f_3,\ f_4,\ b_3) \\
& = \frac{2(b_5-1)^8(\mathrm{lcff}(Q_2,b_3))^2(\mathrm{lcff}(f_1,b_3))^9 Q_4(b_3,b_5)}{43046721(2b_5-1)^{11}(\mathrm{lcff}(Q_1,b_3))^2(\mathrm{lcff}(Q_3,b_3))^2}, \\
& f_6(b_3) := \mathrm{rem}(f_4,\ f_5,\ b_3) \\
& = \frac{297538935552b_5^3(2b_5-1)^{11}(b_5-1)^{26}(\mathrm{lcff}(Q_1,b_3))^2(\mathrm{lcff}(Q_3,b_3))^2 Q_5(b_3,b_5)}{(\mathrm{lcff}(Q_2,b_3))^2(\mathrm{lcff}(Q_4,b_3))^2(\mathrm{lcff}(f_1,b_3))^9}, \\
& f_7(b_3) := \mathrm{rem}(f_5,\ f_6,\ b_3) \\
& = \frac{4b_5(b_5-1)^{12}(\mathrm{lcff}(Q_2,b_3))^2(\mathrm{lcff}(Q_4,b_3))^2(\mathrm{lcff}(f_1,b_3))^9 Q_6(b_3,b_5)}{43046721(2b_5-1)^{11}(\mathrm{lcff}(Q_1,b_3))^2(\mathrm{lcff}(Q_3,b_3))^2(\mathrm{lcff}(Q_5,b_3))^2}, \\
& f_8(b_3) := \mathrm{rem}(f_6,\ f_7,\ b_3) \\
& = -\frac{10711401679872b_5^5(2b_5-1)^{13}(b_5-1)^{32}(\mathrm{lcff}(Q_1,b_3))^2(\mathrm{lcff}(Q_3,b_3))^2(\mathrm{lcff}(Q_5,b_3))^2 Q_7(b_3,b_5)}{(\mathrm{lcff}(Q_2,b_3))^2(\mathrm{lcff}(Q_4,b_3))^2(\mathrm{lcff}(Q_6,b_3))^2(\mathrm{lcff}(f_1,b_3))^9}, \\
& f_9(b_3) := \mathrm{rem}(f_7,\ f_8,\ b_3) \\
& = \frac{4b_5(b_5-1)^{16}(16b_5^2-9b_5+2)(\mathrm{lcff}(Q_2,b_3))^2(\mathrm{lcff}(Q_4,b_3))^2(\mathrm{lcff}(Q_6,b_3))^2(\mathrm{lcff}(f_1,b_3))^9\tilde{r}_{311}^{(1)}(b_5)\tilde{r}_{311}^{(2)}(b_5)}{4782969(2b_5-1)^8(\mathrm{lcff}(Q_1,b_3))^2(\mathrm{lcff}(Q_3,b_3))^2(\mathrm{lcff}(Q_5,b_3))^2(\mathrm{lcff}(Q_7,b_3))^2},
\end{aligned}$$

where Q_1, Q_2, Q_3, Q_4, Q_5, Q_6, Q_7 are polynomials of 81, 111, 132, 149, 163, 174, 177 degrees having 626, 763, 783, 740, 650, 521, 355 terms, respectively. Computing the resultant

$$\begin{aligned}
\mathrm{res}(\mathrm{lcff}(f_1, b_3), \tilde{r}_{311}^{(1)}, b_5) &\approx -2.490625839 \times 10^{169}, \\
\mathrm{res}(\mathrm{lcff}(Q_1, b_3), \tilde{r}_{311}^{(1)}, b_5) &\approx 1.622313374 \times 10^{2182}, \\
\mathrm{res}(\mathrm{lcff}(Q_2, b_3), \tilde{r}_{311}^{(1)}, b_5) &\approx 4.811873821 \times 10^{3216}, \\
\mathrm{res}(\mathrm{lcff}(Q_3, b_3), \tilde{r}_{311}^{(1)}, b_5) &\approx 1.335161275 \times 10^{3924}, \\
\mathrm{res}(\mathrm{lcff}(Q_4, b_3), \tilde{r}_{311}^{(1)}, b_5) &\approx 5.019118277 \times 10^{4538}, \\
\mathrm{res}(\mathrm{lcff}(Q_5, b_3), \tilde{r}_{311}^{(1)}, b_5) &\approx -7.657363922 \times 10^{5047},
\end{aligned}$$

$$\mathrm{res}(\mathrm{lcff}(Q_6, b_3), \tilde{r}_{311}^{(1)}, b_5) \approx 2.182108063 \times 10^{5458},$$
$$\mathrm{res}(\mathrm{lcff}(Q_7, b_3), \tilde{r}_{311}^{(1)}, b_5) \approx 5.576492737 \times 10^{5672},$$

we can check that $\mathrm{lcff}(f_1, b_3) > 0$, $\mathrm{lcff}(f_2, b_3) < 0$, $\mathrm{lcff}(f_3, b_3) < 0$, $\mathrm{lcff}(f_4, b_3) < 0$, $\mathrm{lcff}(f_5, b_3) > 0$, $\mathrm{lcff}(f_6, b_3) > 0$, $\mathrm{lcff}(f_7, b_3) < 0$ and $\mathrm{lcff}(f_8, b_3) > 0$ for $b_5 \in I_1$. Thus, from a pseudo-remainder formula in Knuth (1969); Mishra (1993), we can see that

$$\begin{aligned}
&V(f_0, f_1, \tilde{r}_{311}^{(1)})\\
&= V(f_0, f_1, \tilde{r}_{311}^{(1)}, \mathrm{lcff}(f_1, b_3)) \cup V(\frac{f_1, f_2, \tilde{r}_{311}^{(1)}, \mathrm{lcff}(f_2, b_3)}{\mathrm{lcff}(f_1, b_3)}) \cup V(\frac{f_2, f_3, \tilde{r}_{311}^{(1)}, \mathrm{lcff}(f_3, b_3)}{\mathrm{lcff}(f_1, b_3), \mathrm{lcff}(f_2, b_3)})\\
&\cup V(\frac{f_3, f_4, \tilde{r}_{311}^{(1)}, \mathrm{lcff}(f_4, b_3)}{\mathrm{lcff}(f_1, b_3), \mathrm{lcff}(f_2, b_3), \mathrm{lcff}(f_3, b_3)}) \cup V(\frac{f_4, f_5, \tilde{r}_{311}^{(1)}, \mathrm{lcff}(f_5, b_3)}{\mathrm{lcff}(f_1, b_3), \mathrm{lcff}(f_2, b_3), \mathrm{lcff}(f_3, b_3), \mathrm{lcff}(f_4, b_3)})\\
&\cup V(\frac{f_5, f_6, \tilde{r}_{311}^{(1)}, \mathrm{lcff}(f_6, b_3)}{\mathrm{lcff}(f_1, b_3), \mathrm{lcff}(f_2, b_3), \mathrm{lcff}(f_3, b_3), \mathrm{lcff}(f_4, b_3), \mathrm{lcff}(f_5, b_3)})\\
&\cup V(\frac{f_6, f_7, \tilde{r}_{311}^{(1)}, \mathrm{lcff}(f_7, b_3)}{\mathrm{lcff}(f_1, b_3), \mathrm{lcff}(f_2, b_3), \mathrm{lcff}(f_3, b_3), \mathrm{lcff}(f_4, b_3), \mathrm{lcff}(f_5, b_3), \mathrm{lcff}(f_6, b_3)})\\
&\cup V(\frac{f_7, f_8, \tilde{r}_{311}^{(1)}, \mathrm{lcff}(f_8, b_3)}{\mathrm{lcff}(f_1, b_3), \mathrm{lcff}(f_2, b_3), \mathrm{lcff}(f_3, b_3), \mathrm{lcff}(f_4, b_3), \mathrm{lcff}(f_5, b_3), \mathrm{lcff}(f_6, b_3), \mathrm{lcff}(f_7, b_3)})\\
&\cup V(\frac{f_8, f_9, \tilde{r}_{311}^{(1)}}{\mathrm{lcff}(f_1, b_3), \mathrm{lcff}(f_2, b_3), \mathrm{lcff}(f_3, b_3), \mathrm{lcff}(f_4, b_3), \mathrm{lcff}(f_5, b_3), \mathrm{lcff}(f_6, b_3), \mathrm{lcff}(f_7, b_3), \mathrm{lcff}(f_8, b_3)})\\
&= V(f_8, \tilde{r}_{311}^{(1)}).
\end{aligned}$$

By solving $f_8 = 0$, we get $b_3 = P_1(b_5) := 3(2b_5 - 1)(b_5 - 1)^2 Q_8(b_5)/\mathrm{lcff}(Q_7, b_3)$, where Q_8 is a polynomial having 175 terms. Thus, the interval I_1 determines a unique zero in $V(f_0, f_1)$. In order to check if the zero lies in Ξ_0, we compute the derivative $P_1'(b_5) = 6(b_5 - 1)Q_9(b_5)/(\mathrm{lcff}(Q_7, b_3))^2$, where Q_9 is a polynomial having 352 terms. Computing the resultant $\mathrm{res}(Q_9, \tilde{r}_{311}^{(1)}, b_5) = 2.614190103 \times 10^{11479}$, we can check that $P_1' > 0$ for $b_5 \in I_1$. It follows that P_1 is monotone on I_1. Thus, corresponding to I_1, we have a unique isolated interval $[-230.0212076, -230.0212068]$ for b_3, which does not lie in $(0, \infty)$, implying that $\mathcal{V}_8 \cap \Xi_0 \subset V(\tilde{r}_{23}^{(2)}, \tilde{r}_{24}^{(1)}, \tilde{r}_{311}^{(1)}) \cap \Xi_0 = \emptyset$. Thus, claim ($\mathbf{M}_8$) holds. Similar to the discussion on claim ($\mathbf{M}_8$), we can prove that claims ($\mathbf{M}_i$) s for $i = 9, ..., 17$ hold. This proves the lemma. □

Proof of Lemma 3.3 First, noting $\mathrm{lcff}(\varphi_{11}, b_2) = 9b_5(2b_5 - 1)^3 > 0$, we compute the discriminant $\Delta(\varphi_{11}) = 9(2b_5 - 1)^2(b_5 - 1)^2\varphi_{21}(b_4)$, where

$$\begin{aligned}
\varphi_{21} := {}& 9(2b_5 - 1)^2 b_4^2 - 12b_5^2(2b_5 - 1)(13b_5^2 - 12b_5 + 3)b_4\\
&+ 4b_5^4(25b_5^4 - 96b_5^3 + 114b_5^2 - 54b_5 + 9).
\end{aligned}$$

Since $\Delta(\varphi_{21}) = 2592b_5^5(2b_5 - 1)^5 > 0$, the polynomial φ_{21} has two real zeros $\beta_1^{(21)} < \beta_2^{(21)}$, given in Appendix. Clearly, $\beta_2^{(21)} > 0$ since $26b_5^2 - 24b_5 + 6 > 0$. We can see that

$$\beta_1^{(21)} > 0 \Leftrightarrow 26b_5^2 - 24b_5 + 6 > 6\sqrt{16b_5^4 - 24b_5^3 + 12b_5^2 - 2b_5},$$

which is equivalent to the inequality $\varphi_{31}(b_5) := 100b_5^4 - 384b_5^3 + 456b_5^2 - 216b_5 + 36 > 0$ since $26b_5^2 - 24b_5 + 6 > 0$. Using the Maple command "realroot," we get that φ_{31} has a zero $\beta_1^{(31)}$, given in Appendix. This implies that $\beta_1^{(21)} < 0 \Leftrightarrow 1 < b_5 < \beta_1^{(31)}$, $\beta_1^{(21)} = 0 \Leftrightarrow b_5 = \beta_1^{(31)}$, and $\beta_1^{(21)} > 0 \Leftrightarrow \beta_1^{(31)} < b_5 < (3+\sqrt{3})/2$. Thus, the conclusion on the number of zeros of φ_{11} in $(0, \infty)$ given in Lemma 3.3 is obtained.

Next, by the expression of $\beta_1^{(11)}$, we can see that $\beta_1^{(11)} > 0 \Leftrightarrow \varphi_{22}(b_4) > \varphi_{21}^{1/2}(b_4)$, which is equivalent to the inequalities

$$\varphi_{22}(b_4) > 0 \ \text{ and } \ \varphi_{23}(b_4) > 0, \tag{3.39}$$

where

$$\begin{aligned}
\varphi_{22} &:= -3(4b_5 - 1)(2b_5 - 1)b_4 + 2b_5(8b_5^4 - 23b_5^3 + 10b_5^2 + 6b_5 - 3), \\
\varphi_{23} &:= 18(2b_5 - 1)^2 b_4^2 - 3(2b_5 - 1)(8b_5^2 + 3b_5 - 1)(2b_5^2 - 6b_5 + 3)b_4 \\
&\quad + b_5(2b_5^2 - 6b_5 + 3)(16b_5^5 - 36b_5^4 - 8b_5^3 + 25b_5^2 - 3).
\end{aligned}$$

The linear φ_{22} has a unique zero $\beta_1^{(22)} := 2b_5(8b_5^4 - 23b_5^3 + 10b_5^2 + 6b_5 - 3)/(3(4b_5 - 1)(2b_5 - 1))$ since $\mathrm{lcff}(\varphi_{22}, b_4) = -3(4b_5 - 1)(2b_5 - 1) < 0$. Similar to the discussion on the sign of $\beta_1^{(21)}$, we can see that $\beta_1^{(22)} < 0 \Leftrightarrow 1 < b_5 < \beta_1^{(32)}$, $\beta_1^{(22)} = 0 \Leftrightarrow b_5 = \beta_1^{(32)}$, and $\beta_1^{(22)} > 0 \Leftrightarrow \beta_1^{(32)} < b_5 < (3+\sqrt{3})/2$, where $\beta_1^{(32)}$ is a zero of the polynomial $\varphi_{32}(b_5) := 8b_5^4 - 23b_5^3 + 10b_5^2 + 6b_5 - 3$, in the interval $(1, (3+\sqrt{3})/2)$, covered by $\left[\frac{299614051283}{137438953472}, \frac{74903512821}{34359738368}\right]$. Thus,

$$\varphi_{22} > 0 \ \Leftrightarrow \ \beta_1^{(32)} < b_5 < (3+\sqrt{3})/2 \ \text{ and } \ 0 < b_4 < \beta_1^{(22)}. \tag{3.40}$$

The quadratic φ_{23} has two real zeros $\beta_1^{(23)} < \beta_2^{(23)}$, given in Appendix, since the discriminant $\Delta(\varphi_{23}) = 9(b_5+1)(2b_5-1)^2(-2b_5^2+6b_5-3)(46b_5^3-20b_5^2+3b_5-3) > 0$. Similar to the discussion on the sign of $\beta_1^{(21)}$, we can see that $\beta_1^{(23)} < 0$, and $\beta_2^{(23)} < 0 \Leftrightarrow 1 < b_5 < \beta_1^{(33)}$, $\beta_2^{(23)} = 0 \Leftrightarrow b_5 = \beta_1^{(33)}$, and $\beta_2^{(23)} > 0 \Leftrightarrow \beta_1^{(33)} < b_5 < (3+\sqrt{3})/2$, where $\beta_1^{(33)}$ is a zero of the polynomial $\varphi_{33}(b_5) := 16b_5^5 - 36b_5^4 - 8b_5^3 + 25b_5^2 - 3$, in the interval $(1, (3+\sqrt{3})/2)$, covered by $\left[\frac{296055361513}{137438953472}, \frac{148027680757}{68719476736}\right]$. Thus,

$$\begin{aligned}
\varphi_{23} > 0 \ \Leftrightarrow \ &\text{either } \ 1 < b_5 \le \beta_1^{(33)} \ \text{ and } \ b_4 > 0 \\
&\text{or } \quad \beta_1^{(33)} < b_5 < (3+\sqrt{3})/2 \ \text{ and } \ b_4 > \beta_2^{(23)}.
\end{aligned} \tag{3.41}$$

We also see by a similar discussion on the sign of $\beta_1^{(21)}$ that

$$\begin{array}{ll} \beta_2^{(21)} - \beta_1^{(22)} > 0, & \beta_2^{(21)} - \beta_2^{(23)} > 0, \\ \beta_2^{(23)} - \beta_1^{(22)} > 0 \;\Leftrightarrow\; 1 < b_5 < \beta_1^{(34)}, & \beta_2^{(23)} - \beta_1^{(22)} = 0 \;\Leftrightarrow\; b_5 = \beta_1^{(34)}, \\ \beta_2^{(23)} - \beta_1^{(22)} < 0 \;\Leftrightarrow\; \beta_1^{(34)} < b_5 < (3+\sqrt{3})/2, & \beta_1^{(21)} - \beta_2^{(23)} = 0 \\ \Leftrightarrow\; b_5 = \beta_1^{(34)}, & \\ \beta_1^{(21)} - \beta_2^{(23)} > 0 \;\Leftrightarrow\; \text{either } 1 < b_5 < \beta_1^{(34)} \text{ or } \beta_1^{(34)} < b_5 < (3+\sqrt{3})/2, & \end{array} \tag{3.42}$$

where $\beta_1^{(34)}$ is a zero of the polynomial $\varphi_{34}(b_5) := 92b_5^6 - 224b_5^5 - 16b_5^4 + 162b_5^3 - 57b_5^2 + 18b_5 - 9$, in the interval $(1, (3+\sqrt{3})/2)$, covered by $[\frac{302676473693}{137438953472}, \frac{151338236847}{68719476736}]$. Thus, by (3.39), (3.40), (3.41) and (3.42), the distribution of zeros of φ_{11} given in Lemma 3.3 is obtained. This proves the lemma. □

4 Large Cycles

In contrast with the last section, where we studied limit cycles arising from the focal center in a small neighborhood of the focal center, in this section we find periodic orbits outside the small neighborhood, called large cycles. Ref. Wang et al. (2016) uses the Poincaré–Bendixson theorem (Meiss 2007) to show the existence of a periodic orbit in the case that interior equilibrium E_2 is unstable and explains that the periodic orbit is resulted from a Hopf bifurcation. In this section, we give conditions for the existence of large cycles which do not arise from the interior equilibrium via Hopf bifurcations. We also show that large cycles may exist even if the interior equilibrium is stable and that k ($k = 0, 1, 2$) small cycles can co-exist together with some large cycles. By Theorem 3.1, we see that depending on the signs of focal values, (2.2) can have (i) no small cycles and the interior equilibrium of system (2.2) is unstable if $(b_1, b_2, b_3, b_4, b_5) \in \tilde{\Lambda}_*^0$; (ii) a unique small cycle which is unstable if $(b_1, b_2, b_3, b_4, b_5) \in \tilde{\Lambda}_*^1$; and (iii) two small cycles with the inner one being stable and the outer one being unstable if $(b_1, b_2, b_3, b_4, b_5) \in \tilde{\Lambda}_*^2$. Here

$$\begin{aligned} \tilde{\Lambda}_*^0 := \{&(b_1, b_2, b_3, b_4, b_5) \in \mathbb{R}^5 : (b_3, b_5) \in D_2, b_4 \in (0, \beta_2^{(23)}), b_2 \in (0, \beta_2^{(11)}), b_1 \in B_+(\beta_2)\} \\ &\cup \{(b_1, b_2, b_3, b_4, b_5) \in \mathbb{R}^5 : (b_3, b_5) \in \bigcup_{i=3}^{6} D_i, b_4 \in (0, \beta_2^{(23)}], b_2 \in (0, \beta_2^{(11)}), b_1 \in B_+(\beta_2)\} \\ &\cup \{(b_1, b_2, b_3, b_4, b_5) \in \mathbb{R}^5 : (b_3, b_5) \in \bigcup_{i=3}^{6} D_i, b_4 \in (\beta_2^{(23)}, \beta_1^{(21)}), b_2 \in (\beta_1^{(11)}, \beta_2^{(11)}), b_1 \in B_+(\beta_2)\} \\ &\cup \{(b_1, b_2, b_3, b_4, b_5) \in \mathbb{R}^5 : (b_3, b_5) \in D_4, b_4 \in (0, \beta_1^{(24)}), \end{aligned}$$

$$b_2 \in B_-(\beta_2^{(11)}), b_1 \in B_+(\beta_2)\}$$
$$\cup\{(b_1, b_2, b_3, b_4, b_5) \in \mathbb{R}^5 : (b_3, b_5) \in D_5, b_4 \in (0, \beta_2^{(24)}),$$
$$b_2 \in B_-(\beta_2^{(11)}), b_1 \in B_+(\beta_2)\}$$
$$\cup\{(b_1, b_2, b_3, b_4, b_5) \in \mathbb{R}^5 : (b_3, b_5) \in D_6, b_4 \in (0, \beta_3^{(24)}),$$
$$b_2 \in B_-(\beta_2^{(11)}), b_1 \in B_+(\beta_2)\},$$
$$\tilde{\Lambda}_*^1 := \{(b_1, b_2, b_3, b_4, b_5) \in \mathbb{R}^5 : (b_3, b_5) \in D_2, b_4 \in (0, \beta_2^{(23)}),$$
$$b_2 \in (0, \beta_2^{(11)}), b_1 \in B_-^o(\beta_2)\}$$
$$\cup\{(b_1, b_2, b_3, b_4, b_5) \in \mathbb{R}^5 : (b_3, b_5) \in$$
$$\bigcup_{i=3}^{6} D_i, b_4 \in (0, \beta_2^{(23)}], b_2 \in (0, \beta_2^{(11)}), b_1 \in B_-^o(\beta_2)\}$$
$$\cup\{(b_1, b_2, b_3, b_4, b_5) \in \mathbb{R}^5 : (b_3, b_5) \in$$
$$\bigcup_{i=3}^{6} D_i, b_4 \in (\beta_2^{(23)}, \beta_1^{(21)}), b_2 \in (\beta_1^{(11)}, \beta_2^{(11)}), b_1 \in B_-^o(\beta_2)\}$$
$$\cup\{(b_1, b_2, b_3, b_4, b_5) \in \mathbb{R}^5 : (b_3, b_5) \in D_4, b_4 \in (0, \beta_1^{(24)}),$$
$$b_2 \in B_-(\beta_2^{(11)}), b_1 \in B_-^o(\beta_2)\}$$
$$\cup\{(b_1, b_2, b_3, b_4, b_5) \in \mathbb{R}^5 : (b_3, b_5) \in D_5, b_4 \in (0, \beta_2^{(24)}),$$
$$b_2 \in B_-(\beta_2^{(11)}), b_1 \in B_-^o(\beta_2)\}$$
$$\cup\{(b_1, b_2, b_3, b_4, b_5) \in \mathbb{R}^5 : (b_3, b_5) \in D_6, b_4 \in (0, \beta_3^{(24)}),$$
$$b_2 \in B_-(\beta_2^{(11)}), b_1 \in B_-^o(\beta_2)\}$$
$$\cup\{(b_1, b_2, b_3, b_4, b_5) \in \mathbb{R}^5 : (b_3, b_5) \in D_4, b_4 \in (0, \beta_1^{(24)}),$$
$$b_2 \in B_+^o(\beta_2^{(11)}), b_1 \in B_-(\beta_2)\}$$
$$\cup\{(b_1, b_2, b_3, b_4, b_5) \in \mathbb{R}^5 : (b_3, b_5) \in D_5, b_4 \in (0, \beta_2^{(24)}),$$
$$b_2 \in B_+^o(\beta_2^{(11)}), b_1 \in B_-(\beta_2)\}$$
$$\cup\{(b_1, b_2, b_3, b_4, b_5) \in \mathbb{R}^5 : (b_3, b_5) \in D_6, b_4 \in (0, \beta_3^{(24)}),$$
$$b_2 \in B_+^o(\beta_2^{(11)}), b_1 \in B_-(\beta_2)\},$$
$$\tilde{\Lambda}_*^2 := \{(b_1, b_2, b_3, b_4, b_5) \in \mathbb{R}^5 : (b_3, b_5) \in D_4, b_4 \in (0, \beta_1^{(24)}),$$
$$b_2 \in B_+^o(\beta_2^{(11)}), b_1 \in B_+^o(\beta_2)\}$$
$$\cup\{(b_1, b_2, b_3, b_4, b_5) \in \mathbb{R}^5 : (b_3, b_5) \in D_5, b_4 \in (0, \beta_2^{(24)}),$$
$$b_2 \in B_+^o(\beta_2^{(11)}), b_1 \in B_+^o(\beta_2)\}$$
$$\cup\{(b_1, b_2, b_3, b_4, b_5) \in \mathbb{R}^5 : (b_3, b_5) \in D_6, b_4 \in (0, \beta_3^{(24)}),$$
$$b_2 \in B_+^o(\beta_2^{(11)}), b_1 \in B_+^o(\beta_2)\},$$

β_2 is defined in (2.3), D_1, ..., D_6 are defined just before Theorem 3.1 and $\beta_2^{(23)}$, $\beta_1^{(21)}$, $\beta_1^{(24)}$, $\beta_2^{(24)}$, $\beta_3^{(24)}$, $\beta_1^{(11)}$, $\beta_2^{(11)}$ and $\beta_3^{(11)}$ are given in Appendix.

The following theorem confirms that large cycles are possible.

Theorem 4.1 *For $(b_1, b_2, b_3, b_4, b_5) \in \tilde{\Lambda}_*$, system (2.2) has at least one large cycle, where $\tilde{\Lambda}_* := \tilde{\Lambda}_*^0 \cup \tilde{\Lambda}_*^1 \cup \tilde{\Lambda}_*^2$ and $\tilde{\Lambda}_*^i s$ $(i = 0, 1, 2)$ are defined just before.*

Proof We will use the Poincaré–Bendixson theorem (Hale 1980; Meiss 2007; Zhang et al. 1992) to prove this theorem in three cases: **(L1)** $(b_1, b_2, b_3, b_4, b_5) \in \tilde{\Lambda}_*^0$, **(L2)** $(b_1, b_2, b_3, b_4, b_5) \in \tilde{\Lambda}_*^1$, and **(L3)** $(b_1, b_2, b_3, b_4, b_5) \in \tilde{\Lambda}_*^2$.

In case **(L1)**, from the proof of Theorem 2.2, we see that the flow ϕ^t, defined by the solution $\phi^t(P) := \phi(t, P)$ of system (2.2) initiated from the point $P \in \mathbb{R}_+^2$, is point dissipative. That is, there is a bounded closed set $\mathcal{D} \subset \mathbb{R}_+^2$, as shown in Fig. 3, such that for any point $P \in \mathbb{R}_+^2$, there is a number $t_P \geq 0$, satisfying $\phi^{t_P}(P) \in \mathcal{D}$. This implies that the bounded closed set $\mathcal{D}$ is positively invariant, i.e., each positive semi-orbit, denoted by $\gamma^+ := \{\phi^t(P) : t \geq 0, P \in \mathcal{D}\}$, is always in $\mathcal{D}$. So the positive semi-orbit γ^+ is bounded and in a bounded closed set $\mathcal{D}$. Moreover, by the discussion just before (2.3), we can see that system (2.2) has three equilibria $\tilde{E}_0$, $\tilde{E}_1$ and $\tilde{E}_2$ in this case, which implies that there are a finite number of equilibria in the bounded closed set $\mathcal{D}$. Thus, by the Poincaré–Bendixson theorem of (Hale 1980, Theorem 1.3, p.54), one of the following is satisfied: **(i):** $\omega(\gamma^+)$ is an equilibrium, **(ii):** $\omega(\gamma^+)$ is a periodic orbit, and **(iii):** there are a finite number of equilibria and a set of orbits in $\omega(\gamma^+)$, every one of which tends to the equilibria as $t \to \pm\infty$, where ω denotes the ω-limit set. In this case, we claim that **(i)** does not occur because all three equilibria of system (2.2) are unstable. Also, we claim that **(iii)** does not occur. In fact, from the proof of Theorem 2.2, we see that the stable manifold of the saddle $\tilde{E}_0$ lies on the y-axis and tends to infinity as $t \to -\infty$, and the stable manifold of the saddle $\tilde{E}_1$ lies on the x-axis and tends to $\tilde{E}_0$ or infinity as $t \to -\infty$, which implies that if **(iii)** occurs, an orbit tends to $\tilde{E}_2$ as $t \to \pm\infty$. By the discussion on the beginning of Sect. 2, we can see that the determinant of the Jacobian matrix of system (2.2) at $\tilde{E}_2$ is positive, which implies that $\tilde{E}_2$ does not have hyperbolic sectors. So there are no homoclinic orbits connecting $\tilde{E}_2$. The claim is proved. From the above analysis, the system has at least one periodic orbit in $\mathcal{D}$. By the discussion on the beginning of this proof, no small cycles arise from Hopf bifurcation. This implies that the periodic orbit given by the Poincaré–Bendixson theorem above is not the small cycle.

In case **(L2)**, the equilibrium $\tilde{E}_2$ is stable and there exists a unique small cycle, denoted as $\tilde{\Gamma}$, which is unstable. The cycle $\tilde{\Gamma}$ intersects the horizontal isocline $\mathcal{H} : x = 1/(b_5 - 1)$ transversely at two points. We denote the intersection that lies above $\tilde{E}_2$ as $\hat{Q}_u : (1/(b_5 - 1), \hat{y}_u)$, where $\hat{y}_u > \tilde{y}_*$. Thus, there is a point $\hat{Q} : (1/(b_5 - 1), \hat{y}_u + \delta)$, $\delta > 0$ small enough, such that the positive semi-orbit with $\hat{Q}$ as an initial point intersects $\mathcal{H}$ for $y \geq \tilde{y}_*$ at a point for the first time, denoted by $\hat{Q}_* : (1/(b_5 - 1), \hat{y}_*) = \phi^{\check{t}}(\hat{Q})$, where $\check{t} > 0$ and $\hat{y}_* - \hat{y}_u > \delta$. We construct the curve $\mathcal{L} := \{\phi^t(\hat{Q}) : 0 \leq t \leq \check{t}\} \cup \{(x, y) \in \mathcal{H} : \hat{y}_u + \delta \leq y \leq \hat{y}_*\}$ and further construct an annular region, denoted by $\mathcal{R}$, with $\mathcal{L}$ as the internal boundary and Υ as the external boundary, where Υ is the boundary of $\mathcal{D}$. Clearly, the bounded closed set $\mathcal{R}$ is positively invariant, which implies that each positive semi-orbit, denoted by $\tilde{\gamma}^+ := \{\phi^t(P) : t \geq 0, P \in \mathcal{R}\}$, is bounded.

Moreover, similar to the discussion on case **(L1)**, we can see that system (2.2) has two equilibria $\tilde{E}_0$ and $\tilde{E}_1$ in $\mathcal{R}$ in this case, which implies that there are a finite number of equilibria in the bounded closed set $\mathcal{R}$. Thus, by the Poincaré–Bendixson theorem of (Hale 1980, Theorem 1.3, p.54), the conclusions **(i)**, **(ii)** and **(iii)** described in case **(L1)** still hold for the replacement of γ^+ by $\tilde{\gamma}^+$. By the same reason, we see that **(i)** and **(iii)** do not occur. From the above analysis, the system has at least one periodic orbit in $\mathcal{R}$. In addition, the way we construct the simple closed curve $\mathcal{L}$ determines that there are no limit cycles arising from Hopf bifurcation in the annular region $\mathcal{R}$. This implies that the periodic orbit given by the Poincaré–Bendixson theorem above is not the small cycle.

In case **(L3)**, the equilibrium $\tilde{E}_2$ is unstable and there exist two small-cycles, where the inner one is stable and the outer one (denoted by $\tilde{\Gamma}$) is unstable. The proof for this case is similar to the proof of case **(L2)** and is thus omitted here. The proof of the theorem is completed. □

Theorem 4.1 states that even if the system has two small cycles, it can also have a large cycle. Moreover, if the system has no or one small cycle, it still has a large cycle in some appropriate cases. Next, we present some numerical results to illustrate our analytical results, confirming the existence of a large cycle as well as the possibility of 0, 1 and 2 small cycles. In Figs. 5, 6, 7 and 8, we show the very last part of each trajectory to avoid plotting a huge amount of data. We also use the solid and dotted curves to represent stable and unstable limit cycles, respectively, and use blue color and green color to mark small cycles and large cycles, respectively.

Firstly, we choose $(b_1, b_2, b_3, b_4, b_5) = \left(\frac{709905987}{250000000}, \frac{2869}{5000}, \frac{92621853}{2888000000}, \frac{3}{100}, \frac{461}{200}\right)$ to simulate two small cycles and find that there is one large cycle, which is stable. In Fig. 5, the equilibrium $\tilde{E}_2$ is unstable because $\mathrm{Tr}(J(\tilde{E}_2)) = 7.62306 \times 10^{-12}$. The orbits from the two initial points $(0.766, 3.33)$ and $(0.77, 3.36)$ spiral outward and inward, respectively, as the time $t \to \infty$, implying that there exists a stable limit cycle, generated by the Hopf bifurcation. The orbits from the two initial points $(0.82, 3)$ and $(0.85, 3)$ spiral outward and inward, respectively, as the time $t \to -\infty$, implying that there exists an unstable limit cycle, generated by the Hopf bifurcation. The orbits from the two initial points $(0.965, 3)$ and $(0.975, 3)$ spiral outward and inward, respectively, as the time $t \to \infty$, implying that there is a stable limit cycle in addition to the one arising from the Hopf bifurcation.

Secondly, we choose $(b_1, b_2, b_3, b_4, b_5) = (\frac{74817}{100000}, \frac{87}{250}, \frac{1859}{54000}, \frac{3}{20}, \frac{23}{10})$ to simulate one limit cycle arising from Hopf bifurcations and find that one large cycle exists, which is stable. In Fig. 6, the equilibrium $\tilde{E}_2$ is stable because $\mathrm{Tr}(J(\tilde{E}_2)) = -2.2487578321 \times 10^{-8}$. Similarly, we choose the initial points $(0.791, 0.74)$ and $(0.791, 0.746)$ (resp. $(0.855, 0.856)$ and $(0.863, 0.845)$) and simulate the orbits separately as the time $t \to -\infty$ (resp. $t \to \infty$). This implies that there exists a small cycle (resp. large cycle), which is unstable (resp. stable).

Thirdly, choose $(b_1, b_2, b_3, b_4, b_5) = (\frac{64}{25}, \frac{1}{25}, \frac{2502839}{31500000}, \frac{3}{100}, \frac{109}{50})$ to simulate no limit cycles arising from Hopf bifurcations and find that there is one large cycle, which is stable. In Fig. 7, the equilibrium $\tilde{E}_2$ is unstable because $\mathrm{Tr}(J(\tilde{E}_2)) = 2.8789414 \times 10^{-3}$. Similarly, choose the initial points $(1.2, 15.6)$ and $(1.17, 16)$ and simulate the

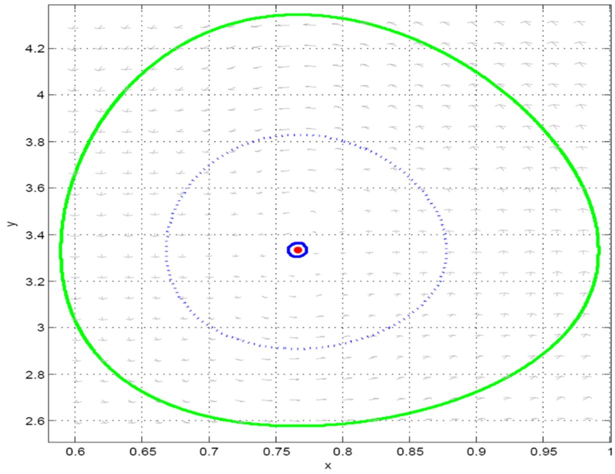


Fig. 5 $\tilde{\Lambda}_*^2$ scenario—two small cycles (blue) and one large cycle (green)

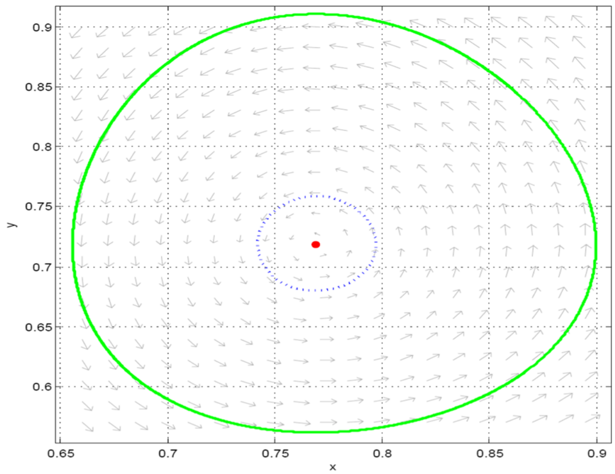


Fig. 6 $\tilde{\Lambda}_*^1$ scenario—one small cycles (blue) and one large cycle (green)

orbits separately as time $t \to \infty$. This implies that there exists a large cycle, which is stable. $\square$

Finally, we simulate 3 small cycles and find that no other periodic orbits exist except the three limit cycles. As a demonstration, we choose $(b_1, b_2, b_3, b_4, b_5) = (\frac{515446001}{125000000}, \frac{1153}{2000}, \frac{3771687633}{112625000000}, \frac{1}{50}, \frac{1151}{500})$. In Fig. 8, the equilibrium $\tilde{E}_2$ is unstable because $\mathrm{Tr}(J(\tilde{E}_2)) = 6.9720317 \times 10^{-11}$. Similarly, choose the initial points $(0.77, 5.25)$, $(0.78, 5.2)$, $(1.1, 6)$ and $(1.1, 6.15)$ (resp. $(0.85, 5.85)$ and $(0.9, 5.5)$) and simulate the orbits separately as the time $t \to \infty$ (resp. $t \to -\infty$). This implies that there exist three small cycles, which are stable, unstable, and stable from inside to outside.

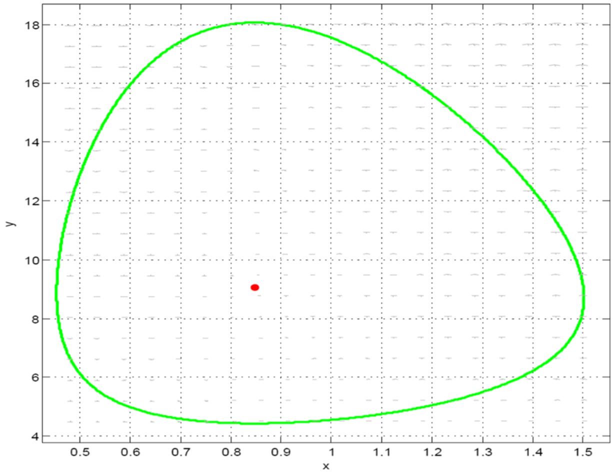


Fig. 7 $\tilde{\Lambda}_*^0$ scenario—no small cycles and one large cycle (green)

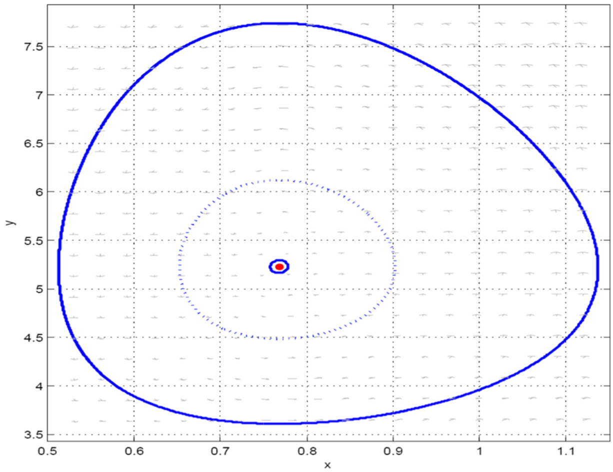
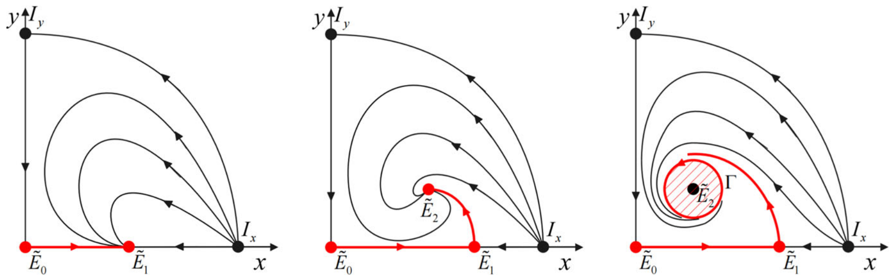


Fig. 8 $\tilde{\Lambda}_3$ scenario—three small cycles (blue) and no large cycles

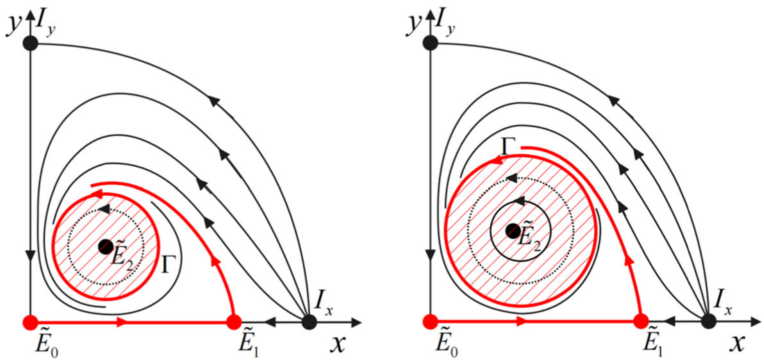
5 Conclusion and Discussion

Unlike an attractor, which is a nonempty invariant set for which there exists an open neighborhood such that all orbits starting from any point in the neighborhood converge to the set, a global attractor is a compact invariant set attracting all bounded subsets in phase space. Therefore, an attractor can be an equilibrium, while a global attractor typically exhibits more complicated internal structure, including equilibria, periodic orbits and homoclinic/heteroclinic orbits. Theorem 2.2 confirms the existence of a global attractor of system (2.2) in the closure of the first quadrant, implying that the populations of both predator and prey will not inflate to infinity. More interesting is the dynamics inside the global attractor, which can have five cases as shown in Fig. 9, and



(a) Case **(S1)**. (b) Case **(S2)**. (c) Case **(S3)**.

Fig. 9 Structure of the global attractor



(a) Case **(S4)**. (b) Case **(S5)**.

Fig. 10 Continuation Fig. 9—structure of the global attractor

in some of these cases, multi-stability within the positive equilibrium $\tilde{E}_2$ and periodic orbits (including small cycles and large cycles) can occur in several different patterns (Fig. 10).

In case **(S1)**, the global attractor is $\tilde{E}_0 \cup \mathcal{W}^u(\tilde{E}_0) \cup \tilde{E}_1$, where $\mathcal{W}^u(\tilde{E}_0)$ is the unstable manifold of $\tilde{E}_0$ in the closure of the first quadrant. In case **(S2)**, the global attractor is $\tilde{E}_0 \cup \mathcal{W}^u(\tilde{E}_0) \cup \tilde{E}_1 \cup \mathcal{W}^u(\tilde{E}_1) \cup \tilde{E}_2$, where $\mathcal{W}^u(\tilde{E}_0)$ is as described above and $\mathcal{W}^u(\tilde{E}_1)$ is the unstable manifold of $\tilde{E}_1$ in the closure of the first quadrant. In cases **(S*i*)** ($i = 3, 4, 5$), the global attractor is $\tilde{E}_0 \cup \mathcal{W}^u(\tilde{E}_0) \cup \tilde{E}_1 \cup \mathcal{W}^u(\tilde{E}_1) \cup \Pi_\Gamma$, where $\mathcal{W}^u(\tilde{E}_i)$, $i = 0, 1$ are as described above, Π_Γ is a bounded closed region with the periodic orbit Γ as the boundary and Γ is the outermost periodic orbit surrounding $\tilde{E}_2$. More specifically, in case **(S3)**, there is the equilibrium $\tilde{E}_2$ in $\text{int}(\Pi_\Gamma)$, in case **(S4)**, there are $\tilde{E}_2$ and one unstable limit cycle in $\text{int}(\Pi_\Gamma)$, and in case **(S5)**, there are $\tilde{E}_2$ and two limit cycles in $\text{int}(\Pi_\Gamma)$, where int represents the interior of the region. Moreover, our global attractor is connected but not locally connected (i.e., a space X is locally connected if for any $x \in X$, and each neighborhood U of x, there is a connected neighborhood V of x which is contained in U, as defined in Armstrong 1983, p.61).

We point out that if we ignore the fear effect (i.e., $s = 0$), Cheng (1981); Hsu et al. (1978) show that system (1.1) has at most one limit cycle, which is stable. This gives a monostable scenario with either the coexistence equilibrium or the positive periodic orbit arising from Hopf bifurcation at the positive equilibrium being *globally asymptotically stable*. If we consider the fear effect, however, Wang et al. (2016) gives a parameter condition for the occurrence of one limit cycle from a Hopf bifurcation while displaying another limit cycle numerically. This indicates that a relatively low level of fear may lead to a situation of bistability in which the positive equilibrium regains its local stability while the limit cycle also remains stable.

Now, our further and more careful analysis of this model incorporated a cost for the prey due to fear effect, shows that the positive equilibrium of (1.1) is indeed a weak focus of *multiplicity of up to 4*, and the case of exactly 3 small cycles leads to another type of bistabiliy: While the positive equilibrium remains unstable, *two stable cycles* may occur which are separated by another cycles between them; see Theorem 3.1. Moreover, Theorem 4.1 also gives certain parameter range for system (1.1) to have a large cycle, showing that even if only 2 small cycles arise from Hopf bifurcations, there can be a large cycle surrounding them which is not a result of Hopf bifurcation. Our theoretical results show that the structure of the global attractor of this model system is rich; there can be a variety of outcomes of the predator–prey interaction when fear effect is considered, as illustrated in Fig. 9. In particular, the demonstrated bistable scenarios indicate that the predator and prey can co-exist in a way that is dependent on their initial populations.

Finally, we would like to discuss a biological implication of our results. There is a well-known paradox in mathematical ecology, called the "Paradox of Enrichment," and it is about the model (1.1) without fear effect with the growth function formulated in the form of logistic growth, that is, the parameters r, δ_1 and δ_3 are absorbed into the form of intrinsic growth rate and carrying capacity. As mentioned in the beginning of this section (citing results from Cheng 1981; Hsu et al. 1978), increasing the carrying capacity of the prey can destabilize the positive equilibrium leading to *a globally stable periodic solution*, and such a periodic solution may stay very close to 0 for the prey population and/or the predator population for a long time (slow dynamics near the boundaries of the first quadrant in the phase plane). This is ecologically not plausible because, by the nature of the predator–prey interaction, increasing the carrying capacity of the prey will benefit the prey species and such a benefit will later be passed to the predator species; yet the slow dynamics near the 0 will put both prey and predator at high risk of extinction by any random negative incidences. This is referred to as the "Paradox of Enrichment." Now, with the fear effect incorporated, we have seen that there are two different types of bistable scenarios for which the solutions with initial populations located in the basin of attraction of the stable positive equilibrium or the stable inner periodic cycle will no longer experience the risky slow dynamics. Thus, for such solutions, the "Paradox of Enrichment" is no longer a paradox.

Appendix: Some Complicated Formulae

In the proof of Theorem 2.1, $p_{11} := (b_5(b_5-1)((b_5+1)b_3+(b_5-1)^2)b_2+(b_5-1)^2((b_5+2)b_3+(b_5-1)^2)b_4+b_5b_3(b_3+(b_5-1)^2))/(b_5-1)$, $p_{02} := -(b_5^2(b_5-1)^4b_2^2+b_5(b_5-1)((b_5-1)^2(b_3+2(b_5-1)^2)b_4-b_5b_3(b_5b_3-2(b_5-1)^2))b_2+(b_5-1)^4(b_3+(b_5-1)^2)b_4^2+b_5b_3(b_5-1)^2(b_3+2(b_5-1)^2)b_4-b_3^2b_5^2(b_5b_3-(b_5-1)^2))/(b_5-1)$ and $\varpi_4 := (1-b_5)(b_5^2(b_5-1)^4b_2^2+b_5(b_5-1)((b_5-1)^2(b_3+2(b_5-1)^2)b_4-b_5b_3(b_5b_3-2(b_5-1)^2))b_2+(b_5-1)^4(b_3+(b_5-1)^2)b_4^2+b_5b_3(b_5-1)^2(b_3+2(b_5-1)^2)b_4-b_3^2b_5^2(b_5b_3-(b_5-1)^2))/(b_3b_5)$.

At the end of Sect. 2, $\tilde{p}_{110} := ((b_5-1)b_5((b_5+1)b_3+(b_5-1)^2)b_2+(b_5-1)^2((b_5+2)b_3+(b_5-1)^2)b_4+b_5b_3(b_3+(b_5-1)^2))/(b_5b_3(b_5-1)^2)$, $\tilde{p}_{020} := (-b_5^2(b_5-1)^4b_2^2+(b_5-1)b_5(-(b_5-1)^2(b_3+2(b_5-1)^2)b_4+b_5b_3(b_5b_3-2(b_5-1)^2))b_2-(b_5-1)^4(b_3+(b_5-1)^2)b_4^2-b_5b_3(b_5-1)^2(b_3+2(b_5-1)^2)b_4+b_5^2b_3^2(b_5b_3-(b_5-1)^2))/(b_5^2b_3^2(b_5-1)3)$, $\tilde{p}_{011} := (b_5(b_5-1)^3b_2+(b_5-1)^2(b_3+(b_5-1)^2)b_4-b_5b_3(b_5b_3-(b_5-1)^2))/(b_5b_3^2(b_5-1)^2)$ and $\varpi_5 := (3(b_5-1)^4b_4^2+b_4b_5(b_5-1)^3(2b_2-b_3+2b_5-2)+b_2b_5^2(b_5-1)((b_5+1)b_3+3(b_5-1)^2)+b_5^2b_3((b_5+2)(b_5+1)b_3+(b_5+4)(b_5-1)^2))(b_5^2b_3(b_5-1)((b_5+1)b_3+(b_5-1)^2))^{-1}$.

Before Theorem 3.1, $\beta_1^{(33)}$, $\beta_1^{(34)}$, $\beta_1^{(35)}$ and $\beta_2^{(36)}$ are zeros of the polynomials $\varphi_{33}(b_5) := 16b_5^5-36b_5^4-8b_5^3+25b_5^2-3$, $\varphi_{34}(b_5) := 92b_5^6-224b_5^5-16b_5^4+162b_5^3-57b_5^2+18b_5-9$, $\varphi_{35}(b_5) := 20b_5^3-68b_5^2+57b_5-15$ and $\varphi_{36}(b_5) := 168b_5^6-1112b_5^5+2738b_5^4-3189b_5^3+1920b_5^2-585b_5+72$ covered by

$$\left[\frac{296055361513}{137438953472}, \frac{148027680757}{68719476736}\right], \quad \left[\frac{302676473693}{137438953472}, \frac{151338236847}{68719476736}\right],$$
$$\left[\frac{316731117455}{137438953472}, \frac{19795694841}{8589934592}\right] \quad \text{and} \quad \left[\frac{157594099533}{68719476736}, \frac{315188199067}{137438953472}\right],$$

respectively, and $\beta_2^{(23)} := (16b_5^4-42b_5^3+4b_5^2+15b_5-3+(-(b_5+1)(2b_5^2-6b_5+3)(46b_5^3-20b_5^2+3b_5-3))^{1/2})/(12(2b_5-1))$, $\beta_1^{(21)} := b_5^2\{26b_5^2-24b_5+6-6(16b_5^4-24b_5^3+12b_5^2-2b_5)^{1/2}\}/(3(2b_5-1))$, $\beta_1^{(24)} := b_5(-136b_5^6+936b_5^5-2318b_5^4+2604b_5^3-1479b_5^2+414b_5-45-(40000b_5^{12}-515200b_5^{11}+2873760b_5^{10}-9142912b_5^9+18444052b_5^8-24884568b_5^7+23115144b_5^6-14958864b_5^5+6713307b_5^4-2038392b_5^3+396036b_5^2-43740b_5+2025)^{1/2})/(3(2b_5-1)(32b_5^3-80b_5^2+60b_5-15))$, $\beta_2^{(24)} := b_5(-136b_5^6+936b_5^5-2318b_5^4+2604b_5^3-1479b_5^2+414b_5-45+(40000b_5^{12}-515200b_5^{11}+2873760b_5^{10}-9142912b_5^9+18444052b_5^8-24884568b_5^7+23115144b_5^6-14958864b_5^5+6713307b_5^4-2038392b_5^3+396036b_5^2-43740b_5+2025)^{1/2})/(3(2b_5-1)(32b_5^3-80b_5^2+60b_5-15))$, $\beta_3^{(24)} := b_5(-136b_5^6+936b_5^5-2318b_5^4+2604b_5^3-1479b_5^2+414b_5-45)/(3(2b_5-1)(32b_5^3-80b_5^2+60b_5-15))$, $\beta_1^{(11)} := (b_5-1)(-3(4b_5-1)(2b_5-1)b_4+2b_5(8b_5^4-23b_5^3+10b_5^2+6b_5-3)-(9(2b_5-1)^2b_4^2-12b_5^2(2b_5-1)(13b_5^2-12b_5+3)b_4+4b_5^4(25b_5^4-96b_5^3+114b_5^2-54b_5+9))^{1/2})/(6b_5(2b_5-1)^2)$, $\beta_2^{(11)} := (b_5-1)(-3(4b_5-1)(2b_5-1)b_4+2b_5(8b_5^4-23b_5^3+10b_5^2+6b_5-3)+(9(2b_5-1)^2b_4^2-12b_5^2(2b_5-1)(13b_5^2-12b_5+$

$3)b_4 + 4b_5^4(25b_5^4 - 96b_5^3 + 114b_5^2 - 54b_5 + 9))^{1/2})/(6b_5(2b_5 - 1)^2)$ and $\beta_3^{(11)} := (b_5-1)(-3(4b_5-1)(2b_5-1)b_4+2b_5(8b_5^4-23b_5^3+10b_5^2+6b_5-3))/(6b_5(2b_5-1)^2)$.

In the proof of Theorem 3.1, $\beta_1^{(31)}, \beta_1^{(36)}, \beta_2^{(36)}$ and $\beta_1^{(37)}$ are zeros of the polynomials $\varphi_{31}(b_5) := 100b_5^4 - 384b_5^3 + 456b_5^2 - 216b_5 + 36$, $\varphi_{36}(b_5)$, $\varphi_{36}(b_5)$ and $\varphi_{37}(b_5) := 1856b_5^{10} - 14816b_5^9 + 44240b_5^8 - 56608b_5^7 + 14036b_5^6 + 40680b_5^5 - 46410b_5^4 + 20016b_5^3 - 2745b_5^2 - 486b_5 + 135$, covered by

$$\left[\frac{147768672059}{68719476736}, \frac{295537344119}{137438953472}\right], \quad \left[\frac{151214956493}{68719476736}, \frac{302429912987}{137438953472}\right],$$
$$\left[\frac{157594099533}{68719476736}, \frac{315188199067}{137438953472}\right]$$
$$\text{and } \left[\frac{323076177543}{137438953472}, \frac{40384522193}{17179869184}\right],$$

respectively, $\beta_2^{(21)} := b_5^2\{26b_5^2 - 24b_5 + 6 + 6(16b_5^4 - 24b_5^3 + 12b_5^2 - 2b_5)^{1/2}\}/(3(2b_5 - 1))$, $\beta_1^{(23)} := (16b_5^4 - 42b_5^3 + 4b_5^2 + 15b_5 - 3 - ((b_5+1)(-2b_5^2 + 6b_5 - 3)(46b_5^3 - 20b_5^2 + 3b_5 - 3))^{1/2})/(12(2b_5 - 1))$, $\hat{R}_{23}^{(1)}(b_5) := 8b_5^3 - 26b_5^2 + 21b_5 - 6$, $\hat{R}_{23}^{(2)}(b_5) := 32b_5^3 - 80b_5^2 + 60b_5 - 15$, $\hat{R}_{23}^{(3)}(b_5) := 168b_5^6 - 1112b_5^5 + 2738b_5^4 - 3189b_5^3 + 1920b_5^2 - 585b_5 + 72$, $\hat{R}_{23}^{(5)}(b_5) := 100b_5^6 - 608b_5^5 + 1324b_5^4 - 1272b_5^3 + 591b_5^2 - 126b_5 + 9$,

$$\begin{aligned}
\check{r}_{23}^{(1)}(b_5) &:= 460800b_5^{16} - 9263360b_5^{15} + 81404288b_5^{14} - 417970176b_5^{13} \\
&\quad + 1410177920b_5^{12} - 3328818080b_5^{11} + 5710818096b_5^{10} \\
&\quad - 7293653136b_5^9 + 7035858828b_5^8 - 5160307968b_5^7 \\
&\quad + 2874517092b_5^6 - 1204489980b_5^5 + 371863629b_5^4 - 81487620b_5^3 \\
&\quad + 11860830b_5^2 - 1008450b_5 + 36450 \equiv \hat{r}_{23}^{(1)}(b_5), \\
F_1(b_2, b_4, b_5) &:= 9b_5^2(2b_5 - 1)^3 b_2^2 + 3b_5(2b_5 - 1)(b_5 - 1) \\
&\quad (3(2b_5 - 1)(4b_5 - 3)b_4 - 2b_5(8b_5^4 - 25b_5^3 + 10b_5^2 + 6b_5 - 3))b_2 \\
&\quad + (b_5 - 1)^2\{18(b_5 - 1)(2b_5 - 1)^2 b_4^2 - 3b_5(2b_5 - 1) \\
&\quad (16b_5^4 - 54b_5^3 + 48b_5^2 + 3b_5 - 9)b_4 + b_5^2(2b_5^2 - 6b_5 \\
&\quad + 3)(16b_5^5 - 44b_5^4 + 12b_5^3 + 13b_5^2 - 3)\}, \\
\tilde{r}_{311}^{(1)}(b_5) &:= 1029220902102630400b_5^{28} \\
&\quad - 18932387359653068800b_5^{27} + 150133535412565606400b_5^{26} \\
&\quad - 687318146007217295360b_5^{25} \\
&\quad + 2068576068140472350720b_5^{24} \\
&\quad - 4443158376724184913920b_5^{23} + 7196576363925899969536b_5^{22} \\
&\quad - 9121489683461357488640b_5^{21} \\
&\quad + 9267554060472935708928b_5^{20} - 7649827649414653180928b_5^{19} \\
&\quad + 5142740566484693410432b_5^{18} \\
&\quad - 2778263794975800256832b_5^{17} + 1149095427108262799776b_5^{16}
\end{aligned}$$

$$
\begin{aligned}
&- 30330598277265680451 2b_5^{15}\\
&- 10397349886853933056b_5^{14} + 6951902885629507138 4b_5^{13}\\
&- 4949309699596729348 4b_5^{12}\\
&+ 2363493983023400130 4b_5^{11} - 881013759124697663 6b_5^{10}\\
&+ 268347107082368601 4b_5^{9} - 67897949283741893 5b_5^{8}\\
&+ 14325275051565509 0b_5^{7} - 2510006424685082 5b_5^{6}\\
&+ 3612691134669890b_5^{5} - 418990199969600b_5^{4} + 37909624759650b_5^{3}\\
&- 2527773384150b_5^{2} + 110982368700b_5 - 2405214000,
\end{aligned}
$$

$$
\begin{aligned}
\check{r}_{23}^{(3)}(b_5) := {}& 13546380119244800b_5^{32}\\
&- 313385065602088960b_5^{31} + 3263312981996863488b_5^{30}\\
&- 19751848519249035264b_5^{29}\\
&+ 72679468424282701824b_5^{28}\\
&- 133523461871425290240b_5^{27} - 149101920498547310592b_5^{26}\\
&+ 1882641718448314736640b_5^{25}\\
&- 6621699129547006033920b_5^{24} + 13498507630764318816256b_5^{23}\\
&- 13482941098231701472256b_5^{22}\\
&- 14206959864832318783488b_5^{21} + 94577448723792819152640b_5^{20}\\
&- 236100973335876209571840b_5^{19}\\
&+ 411229484077109445499968b_5^{18} - 560331770721137677700640b_5^{17}\\
&+ 624012040457000976036960b_5^{16}\\
&- 580309766693303409280872b_5^{15} + 455944991759976539725740b_5^{14}\\
&- 304571657407390634772234b_5^{13}\\
&+ 173472251808073343904432b_5^{12} - 84266698376353021491519b_5^{11}\\
&+ 34844982505233029974422b_5^{10}\\
&- 12216543247894691610648b_5^{9} + 3608974969450679346156b_5^{8}\\
&- 890501080092471181539b_5^{7}\\
&+ 181334416934452161978b_5^{6} - 29975727010020857028b_5^{5}\\
&+ 3930406820462712744b_5^{4}\\
&- 394913924909840880b_5^{3} + 28738434075418656b_5^{2}\\
&- 1360031608961280b_5 + 31754946067200 \equiv \hat{r}_{23}^{(3)}(b_5),
\end{aligned}
$$

$$
\begin{aligned}
\hat{r}_{24}^{(2)}(b_5) := {}& 4527388989849600b_5^{33} - 169013158700646400b_5^{32}\\
&+ 2994766066356387840b_5^{31} - 33606726037831680000b_5^{30}\\
&+ 268784971272203649024b_5^{29}\\
&- 1634299100420448731136b_5^{28} + 7867163007312738791424b_5^{27}\\
&- 30822726164383686107136b_5^{26}\\
&+ 100254986197207612861440b_5^{25} - 274731191154460598347776b_5^{24}\\
&+ 641350279800470279481856b_5^{23}
\end{aligned}
$$

$$
\begin{aligned}
&- 1286253672139043635144704b_5^{22} + 2230312287468288761678208b_5^{21} \\
&- 335940195434675513979168 0b_5^{20} \\
&+ 4410412822398488737801824b_5^{19} - 5058105151614670719632448b_5^{18} \\
&+ 5073758818196560736193360b_5^{17} \\
&- 4453082055862479204251904b_5^{16} + 3417987222327495324285504b_5^{15} \\
&- 2291310711641533906695312b_5^{14} \\
&+ 1338571242090706835590920b_5^{13} - 679291839392409058410756b_5^{12} \\
&+ 298158954238069139646522b_5^{11} \\
&- 112548711365963147598984b_5^{10} + 36267685782348678185583b_5^{9} \\
&- 9881488560145530503934b_5^{8} \\
&+ 2248153774054024161258b_5^{7} - 420123525773087846466b_5^{6} \\
&+ 63071625321665675010b_5^{5} - 7375368278374295220b_5^{4} \\
&+ 642189471423121800b_5^{3} - 38794236987242160b_5^{2} \\
&+ 1433357457285600b_5 - 24083630661600,
\end{aligned}
$$

$$
\begin{aligned}
\check{R}_{13}^{(1)}(b_4, b_5) := {}& 1458(32b_5^3 - 80b_5^2 \\
&+ 60b_5 - 15)(2b_5 - 1)^6 b_4^6 \\
&- 243b_5(8b_5^5 - 1700b_5^4 + 4248b_5^3 - 3729b_5^2 + 1386b_5 \\
&- 180)(2b_5 - 1)^6 b_4^5 + 81b_5^3(32064b_5^8 - 311344b_5^7 \\
&+ 1128064b_5^6 - 2048952b_5^5 + 2133438b_5^4 - 1340694b_5^3 \\
&+ 505071b_5^2 - 105408b_5 + 9396)(2b_5 - 1)^4 b_4^4 \\
&- 54b_5^5(345760b_5^1 0 - 3874112b_5^9 + 18103336b_5^8 \\
&- 46411404b_5^7 + 72707562b_5^6 - 73631646b_5^5 \\
&+ 49420431b_5^4 - 21928833b_5^3 + 6205167b_5^2 - 1017279b_5 \\
&+ 73710)(2b_5 - 1)^3 b_4^3 + 18b_5^7(2b_5^2 - 6b_5 + 3) \\
&(1131776b_5^1 0 - 12161712b_5^9 + 54754912b_5^8 - 135781800b_5^7 \\
&+ 206307750b_5^6 - 202837302b_5^5 + 132158223b_5^4 - 56893860b_5^3 \\
&+ 15610509b_5^2 - 2480868b_5 + 174312)(2b_5 \\
&- 1)^2 b_4^2 - 96b_5^9(2b_5 - 1)(84744b_5^{10} - 894616b_5^9 \\
&+ 3974482b_5^8 - 9769875b_5^7 + 14769558b_5^6 \\
&- 14475357b_5^5 + 9405774b_5^4 - 4036149b_5^3 + 1102680b_5^2 \\
&- 174231b_5 + 12150)(2b_5^2 - 6b_5 + 3)^2 b_4 \\
&+ 256b_5^{11}(168b_5^6 - 1112b_5^5 + 2738b_5^4 - 3189b_5^3 \\
&+ 1920b_5^2 - 585b_5 + 72)(25b_5^4 - 96b_5^3 + 114b_5^2 - 54b_5 \\
&+ 9)(2b_5^2 - 6b_5 + 3)^3,
\end{aligned}
$$

$$
\begin{aligned}
\hat{R}_{23}^{(4)}(b_5) := {}& 23713792000b_5^{24} - 574054969344b_5^{23} \\
&+ 6498506768384b_5^{22} - 45779591628800b_5^{21} + 225333289334784b_5^{20} \\
&- 824940827031040b_5^{19} + 2334874124179072b_5^{18}
\end{aligned}
$$

$$
\begin{aligned}
&- 5242854056898240b_5^{17} + 9509293922106848b_5^{16}\\
&- 14109846206731488b_5^{15} + 17279191287047640b_5^{14}\\
&- 17564967094747932b_5^{13} + 14867581076762958b_5^{12}\\
&- 10485326116792161b_5^{11} + 6150121974821628b_5^{10}\\
&- 2986796712693483b_5^{9} + 1191999779585874b_5^{8}\\
&- 386526987399483b_5^{7} + 100194462899856b_5^{6}\\
&- 20285817268257b_5^{5} + 3102555126996b_5^{4}\\
&- 341160741324b_5^{3} + 24987314808b_5^{2} - 1073904480b_5 + 20995200.
\end{aligned}
$$

Author Contributions W.Z and X.Z. perceived the problem and suggested the ways to tackle the problem; Y.Z did the computations and numerical simulations, and drafted the first version under the supervision of W.Z and X.Z.; W.Z. and X.Z. revised and finalized the manuscript.

Data Availability No datasets were generated or analysed during the current study.

Declarations

Competing interests The authors declare no competing interests.

References

Armstrong, M.A.: Basic Topology. Springer, New York (1983)

Briot, C.A., Bouquet, J.C.: Recherches sur les propriétés des fonctions définies par des équations différentielles. Journal de l'École Polytechnique **21**, 133–197 (1856)

Chen, X., Romanovski, V.G., Zhang, W.: Non-isochronicity of the center at the origin in polynomial Hamiltonian systems with even degree nonlinearities. Nonlinear Anal. Theory Methods Appl. **68**(9), 2769–2778 (2008)

Chen, X., Zhang, W.: Decomposition of algebraic sets and applications to weak centers of cubic systems. J. Comput. Appl. Math. **232**(2), 565–581 (2009)

Cheng, K.: Uniqueness of a limit cycle for predator–prey system. SIAM J. Math. Anal. **12**(4), 541–548 (1981)

Cholewa, J.W., Dlotko, T., Chafee, N.: Global Attractors in Abstract Parabolic Problems. Cambridge University Press, Cambridge (2000)

Chow, S.-N., Hale, J.K.: Methods of Bifurcation Theory. Springer, New York (1982)

Christopher, C., Li, C.: Limit Cycles of Differential Equations. Birkhäuser Verlag, Basel (2007)

Creel, S., Christianson, D.: Relationships between direct predation and risk effects. Trends Ecol. Evol. **23**(4), 194–201 (2008)

Diz-Pita, É., Otero-Espinar, M.V.: Predator–prey models: a review of some recent advances. Mathematics **9**(15), 1783 (2021)

Freedman, H.I.: Deterministic Mathematical Models in Population Ecology. Marcel Dekker, New York (1980)

Gelfand, I.M., Kapranov, M.M., Zelevinsky, A.V.: Discriminants, Resultants, and Multidimensional Determinants. Birkhüser, Boston (1994)

Guckenheimer, J., Holmes, P.: Nonlinear Oscillations. Dynamical Systems and Bifurcations of Vector Fields. Springer, New York (1983)

Hale, J.K.: Ordinary Differential Equations. Robert E. Krieger, New York (1980)

Hale, J.K.: Dissipation and compact attractors. J. Dyn. Differ. Equ. **18**(3), 485–523 (2006)

Hsu, S.-B., Hubbell, S.P., Waltman, P.: Competing predators. SIAM J. Appl. Math. **35**(4), 617–625 (1978)

Iannelli, M., Pugliese, A.: An Introduction to Mathematical Population Dynamics: Along the Trail of Volterra and Lotka. Springer, New York (2014)

Knuth, D.E.: The Art of Computer Programming, Seminumerical Algorithms, vol. 2. Addison-Wesley, London (1969)

Kuznetsov, Y.A.: Elements of Applied Bifurcation Theory, Applied Mathematical Sciences, vol. 112. Springer, New York (1995)

Li, J.: Hilbert's 16th problem and bifurcations of planar polynomial vector fields. Int. J. Bifurc. Chaos **13**, 47–106 (2003)

Lloyd, N.G.: Limit cycles of polynomial systems-some recent developments. In: Bedford, T., Swift, J. (eds.) New Directions in Dynamical Systems, London Mathematical Society Lecture Notes, 127, pp. 192–234. Cambridge Univ. Press, Cambridge (1988)

Meiss, J.D.: Differential Dynamical Systems, Mathematical Modeling and Computation, vol. 14. SIAM, Philadelphia (2007)

Mishra, B.: Algorithmic Algebra. Springer, New York (1993)

Perko, L.: Differential Equations and Dynamical Systems. Springer, New York (1996)

Prestel, A., Delzell, C.N.: Positive Polynomials: From Hilbert's 17th Problem to Real Algebra. Springer, Berlin (2001)

Wang, X., Zanette, L.Y., Zou, X.: Modelling the fear effect in predator–prey interactions. J. Math. Biol. **73**(5), 1179–1204 (2016)

Zanette, L.Y., White, A.F., Allen, M.C., Clinchy, M.: Perceived predation risk reduces the number of offspring songbirds produce per year. Science **334**(6061), 1398–1401 (2011)

Zhang, Z.-F., Ding, T.-R., Huang, W.-Z., Dong, Z.-X.: Qualitative Theory of Differential Equations. Amer. Math. Soc., Providence (1992)