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Co-Existence of Chaos and Stable Periodic Orbits in a Simple Discrete Neural Network

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Summary. We show that a simple discrete network of two identical neurons can demonstrate chaotic behavior near the origin. This is complementary to the results in Wu and Zhang (*Disc. Contin. Dynam. Syst. Series B*, **4** (2004), 853–865), where it was shown that the same system can have a large capacity of stable periodic orbits in a region away from the origin.

Key words. neural network, discrete, delay, chaos, capacity, associate memory, periodic solutions

1. Introduction

In designing a neural network for content-addressable memory of spatial patterns, large capacity is desirable. In view of dynamical systems, it requires that the model system for the network have large number of stable (retrievable) periodic solutions. Recently, Wu and Zhang [21] considered the following simple discrete network of two identical neurons with excitatory interactions:

$$\begin{cases} x(n) = \beta x(n-1) + \alpha f(y(n-k)), \\ y(n) = \beta y(n-1) + \alpha f(x(n-k)), \end{cases} \tag{1.1}$$

where $n \in N$, $\alpha > 0$, $\beta \in (0, 1)$, and $k \geq 1$ is a fixed integer, and $f\colon \mathbb{R} \to \mathbb{R}$ satisfies the following conditions:

(H_1) There are some constants $\varepsilon > 0$ and $R > r > 0$ such that

$$\begin{cases} |f(x) - 1| \leq \varepsilon, & \text{if } x \in (r, R], \\ |f(x) + 1| \leq \varepsilon, & \text{if } x \in [-R, -r). \end{cases} \tag{1.2}$$

(H_2) There is a constant $L > 0$ such that

$$|f(x) - f(y)| \leq L|x - y|, \qquad \forall x, y \in [-R, -r) \quad \text{or} \quad x, y \in (r, R]. \tag{1.3}$$

Note that (H2) is the standard Lipschitz condition and (H1) allows a wide range of activation functions, including the frequently used McCulloch-Pitts step function and sigmoid functions with large gain $f'(0)$. Such a system describes the dynamics of a network of two identical neurons, updated discretely, where the information processing of a neuron involves the internal decay and feedback from the other neuron with delay. For this simple network, it was shown in [21] that under certain technical conditions, for every positive integer p with $p|2k$, system (1.1) has $N(p)/p$ distinct asymptotically stable p-periodic solutions. Here the integer $N(p)$ is related to the well-known Möbius inversion formula as below. Let $p = \prod_{i=1}^{l} p_i^{m_i}$, where p_i, $i = 1, 2, \ldots, l$, are primes. For every subset I of the set $\{1, \ldots, l\}$, let $p_I = \prod_{i \in I} p_i$. Then

$$N(p) = \sum_{I \subset \{1, \ldots, l\}} (-1)^{|I|} 2^{p/p_I}. \tag{1.4}$$

Similar results have also been obtained for networks with more than two distinct neurons (more choices for the parameters), but with some special connection structure. For example, [22] deals with a ring structure and [20] considers a block structure. These results reveal that networks updated discretely can allow amazingly large capacity and therefore suggest the adoption of discrete networks, as far as memory for spatial patterns is concerned.

We notice that all periodic solutions obtained in [21] are located in the region $\Omega = \{(x, y) \in R^2\colon\ a \leq |x| \leq b, a \leq |y| \leq b\}$ for some positive constants $a < b$ depending on r, R, α, and β. Thus, one naturally wonders how system (1.1) would behave outside the region Ω. Noting that it is well known that difference equations generally have bigger chances to support chaos than differential equations, it is reasonable for one to expect chaotic behavior of (1.1) outside Ω. In this paper, we will investigate the possibility of chaos for system (1.1) near the origin. We point out that the obtained results in [21] for multiplicity and stability of periodic solutions of (1.1) is independent of the behavior of f on the interval $[-r, r]$, allowing much flexibility for f near the origin. Taking advantage of this flexibility, we will show that indeed there are many periodic orbits (but usually unstable) and even chaotic vibration near the origin for (1.1) under some assumptions on f.

We would like to mention that chaotic neural networks have potential applications to various practical problems, for example, to the problem of combinational optimization (see,. e.g., [8], [9], [10], [13], [14], [17], [19]) and to the problem of dynamical associative memory (see, e.g., [1], [2]). Taking the traveling salesman problem (TSP) as an example, the earlier application of neural networks to TSP by Hopfield and Tank [12] suffered from the existence of numerous local minima, and the probability of convergence to the global optimization is very low. However, it has been shown that the global searching ability of a neural network can be greatly enhanced by incorporating the chaotic simulated annealing in a discrete neural network, in [1] (see [11] for a review on this topic).

The rest of the paper is organized as follows. In Section 2, we will consider a simple case of (1.1): $k = 1$, meaning that there is no time delay in the system. In this case, system (1.1) is two-dimensional; however, there is an invariant subset for (1.1) on which the system (1.1) is equivalent to a one-dimensional dynamical system. By appealing to some existing results on chaos of one-dimensional dynamical systems, we will show that even in this simplest case, the system still has chaos if the nonlinearity f satisfies some conditions. In Section 3, we consider the general $k \geq 2$, for which (1.1) is $2k$ dimensional in essence. We first show that, on the one hand, the dynamics of (1.1) on a domain *outside* the ball $B(0, r)$ is topologically semiconjugate to a simple dynamical system $(\sum, \pi)$, which will be specified later. As is known, topological semiconjugacy is an important notion in dynamical systems. On the other hand, we will identify a symmetric k-dimensional invariant subset for (1.1) in $\mathbb{R}^{2k}$, and will employ the modified Morotto Theorem to prove that system (1.1) may have infinitely many (unstable) periodic orbits and chaos on this subset near the origin under some conditions on f near the origin. We point out that the reason we separate the case $k = 1$ from the cases $k \geq 2$ is that the existing results on chaos for one-dimensional maps only require C^0 property, while in the higher dimensional cases, smoothness is required, as will be seen in Section 3.

2. The Case $k = 1$

In this section, we consider the system (1.1) with $k = 1$. In this case, the system can be written as

$$\begin{cases} x(n) = \beta x(n-1) + \alpha f(y(n-1)), \\ y(n) = \beta y(n-1) + \alpha f(x(n-1)), \end{cases} \tag{2.1}$$

which is a dynamical system on $\mathbb{R}^2$. Here we are concerned only with the dynamics on the region $D = [-r, r] \times [-r, r] \subset \mathbb{R}^2$.

It is easy to see that the diagonal line $\mathcal{L}_1 \triangleq \{(x, x); x \in \mathbb{R}\}$ is an invariant set for system (2.1). On $\mathcal{L}_1$, the dynamics of (2.1) is the same as that of the following one-dimensional dynamical system:

$$x(n) = \beta x(n-1) + \alpha f(x(n-1)) \triangleq g(x(n-1)). \tag{2.2}$$

Obviously, if (2.2) is chaotic, so is (2.1); if (2.2) has a periodic orbit $\{p_i\}_1^l$ with period l, then $\{(p_i, p_i)\}_1^l$ is a periodic orbit of (2.1) with the same period l.

To study the discrete dynamical system (2.2), we first review some results about one-dimensional dynamical systems. For details, see, for example, [4]–[6].

Let I denote a closed interval on the real line and $C^0(I, I)$ the set of all continuous maps from I into itself. Let $g \in C^0(I, I)$. For any positive integer n, we define g^n inductively by $g^0 = I_d$, the identity map of I, and $g^n = g \circ g^{n-1}$.

Let p be a fixed point of g. The *unstable manifold* $W^u(p, g)$ is defined by

$$W^u(p, g) = \cap_{\varepsilon>0} \cup_{n=0}^{\infty} g^n((p - \varepsilon,\ p + \varepsilon)).$$

That is, $x \in W^u(p, g)$ if for every neighborhood V of p, $x \in g^n(V)$ for some positive integer n.

A point x is said to be a *homoclinic point* of g if there exists a periodic point p of period n with $x \neq p$ and $x \in W^u(p, g^n)$ such that $g^{nm}(x) = p$ for some positive integer m.

Definition 2.1. Let $g \in C^0(I, I)$. g is said to be chaotic in Li-Yorke's sense if there exists an uncountable set $S \subset I$ with

$$\limsup_{n\to\infty} |g^n(x) - g^n(y)| > 0, \qquad \forall x, y \in S, \quad x \neq y,$$

and

$$\liminf_{n\to\infty} |g^n(x) - g^n(y)| = 0, \qquad \forall x, y \in S.$$

Note that this definition is just one version of the original definition of chaos by Li and Yorke [16] for scalar maps. The following lemma gives some equivalent conditions guaranteeing chaos in Li-Yorke's sense.

Lemma 2.1 ([5]). *Let $g \in C^0(I, I)$. Then the following statements are equivalent:*

(i) *g has a periodic point whose period is not a power of 2;*
(ii) *g has a homoclinic point;*
(iii) *g has positive topological entropy (see [3] for a definition of topological entropy).*

Moreover, each of the above conditions implies that g is chaotic in Li-Yorke's sense.

From Lemma 2.1, we know that if one of the conditions (i)–(iii) holds, then g has very complicated dynamical behavior. However, in practice, for a given $g \in C^0(I, I)$, it is generally very difficult, if not impossible, to verify these conditions. Based on our experience, we feel that (ii) is relatively more convenient. Thus, in the following theorem, we will establish a chaos result for equation (2.2) by verifying condition (ii) for the nonlinear map g associated with (2.2).

Theorem 2.1. *Assume that f is continuous and satisfies the following conditions: There exist $0 < r_1 < r_2 < r$, where r is given by (H_1), such that*

$$f(0) = f(r_1) = 0, \tag{2.3}$$

$$f(x) > 0, \quad 0 < x < r_1, \tag{2.4}$$

$$f(x) < 0, \quad r_1 < x < r_2. \tag{2.5}$$

Then there exists a constant $\alpha_0 > 0$ such that for $\alpha > \alpha_0$, system (2.1) is chaotic in Li-Yorke's sense on $\{[-r, r] \times [-r, r]\} \cap \mathcal{L}_1$.

Proof. Let $g_\alpha(x) \triangleq \beta x + \alpha f(x)$. By (2.3) and (2.5), one knows that $g_\alpha(0) = 0$ and there exists $\alpha_1 > 0$ such that for any $\alpha > \alpha_1$, the equation $g_\alpha(x) = 0$ has a solution $r_\alpha \in (r_1, r_2)$ and $r_\alpha \to r_1$ as $\alpha \to \infty$. Moreover, $g_\alpha(x) > 0$, $0 < x < r_\alpha$.

On the other hand, since

$$\lim_{\alpha\to+\infty}\max_{0\le x\le r_1} g_\alpha(x)=\infty,$$

there exists α_2 such that for $\alpha > \alpha_2$,

$$\max_{0\le x\le r_1} g_\alpha(x)\ge r_2. \tag{2.6}$$

Let $\alpha_0=\max\{\alpha_1,\alpha_2\}$. For any $\alpha>\alpha_0$, let $x_\alpha\in[0,r_1]$ be the maximum point of g_α on $[0,r_1]$, and let p_α be the largest fixed point on $[0,x_\alpha]$ of g_α. Then, in terms of [5], we have

$$W^u(p_\alpha,g_\alpha)\supset[0,r_2].$$

Since $g_\alpha(r_\alpha)=0$, $g_\alpha(x_\alpha)\ge r_2$ by (2.6) and $x_\alpha>p_\alpha$, we have

$$g_\alpha([x_\alpha,r_\alpha])\supset[0,r_2].$$

Thus there exists $z_\alpha\in(x_\alpha,r_\alpha]\subset[0,r_2]\subset W^u(p_\alpha,g_\alpha)$ such that $g_\alpha(z_\alpha)=p_\alpha$. Therefore z_α is a homoclinic point of g_α. This completes the proof by Lemma 2.1(ii). $\square$

Remark 2.1. In fact, we can give an estimation of α_0 for a given signal transmission function f. For example, if $f(0)=0$, $\beta x_0+f(x_0)=0$ for some $0\neq x_0\in(0,r)$ and the maximum of f on $[0,x_0]$ is larger than or equal to x_0, then $g(x)=\beta x+\alpha f(x)$ has a homoclinic point, and thus the corresponding system (2.1) is chaotic in the Li-Yorke sense for any $\alpha>1$.

Remark 2.2. Under the assumption of Theorem 2.1, from Lemma 2.1, one knows that system (2.1) has a periodic point whose period is not a power of 2. Let g have a periodic orbit with period $m_0\cdot 2^{l_0}$ for some odd integer $m_0>1$ and positive integer l_0. Then g indeed has infinitely many periodic orbits, whose periods can be any positive integer that is arranged behind the integer $m_0\cdot 2^{l_0}$ according to Sarkovskii's ordering. More precisely, if we let $PP(g)$ denote the set of periods of all periodic points of g, then from the well-known Sarkovskii Theorem, we have

$$PP(g)\supset\{p\in N^+\mid p=m2^l,\quad \text{or}\quad p=2^{l'},\ m\ge1,\ l'\ge0,\ l\ge l_0\}.$$

Thus, system (2.1) is chaotic and has infinitely many periodic orbits on $[-r,r]\times[-r,r]$.

3. General $k\ge 2$

In this section, we consider general $k\ge2$, for which system (1.1) is equivalent to a $2k$-dimensional dynamical system as is shown below. Letting $w_j(n)=x(n+j-k-1)$ and $w_{k+j}(n)=y(n+j-k-1)$ for $j=1,\dots,k$, we then rewrite (1.1) as the following discrete dynamical system on $\mathbb{R}^{2k}$:

$$w(n+1)=F(w(n)), \tag{3.1}$$

where $F\colon \mathbb{R}^{2k} \to \mathbb{R}^{2k}$ is given by

$$F\begin{pmatrix} w_1 \\ \vdots \\ w_{k-1} \\ w_k \\ w_{k+1} \\ \vdots \\ w_{2k-1} \\ w_{2k} \end{pmatrix} = \begin{pmatrix} w_2 \\ \vdots \\ w_k \\ \beta w_k + \alpha f(w_{k+1}) \\ w_{k+2} \\ \vdots \\ w_{2k} \\ \beta w_{2k} + \alpha f(w_1) \end{pmatrix}. \tag{3.2}$$

We shall denote by $w(n, w^0)$ the solution of (3.1) with initial condition $w(0) = w^0 \in \mathbb{R}^{2k}$, where N denotes the set of all nonnegative integer numbers.

For $w = (w_1, \dots, w_{2k})^T \in \mathbb{R}^{2k}$, its norm is defined by

$$\|w\| = \max_{1\le j\le 2k}\{|w_j|\}.$$

It is easy to see that F is continuous under this norm. Thus we can take (3.1) as a discrete dynamical system $(\mathbb{R}^{2k}, F)$ with finite dimension $2k$.

Let $S = \{-1, 1\}$ be endowed with the discrete topology. Its $2k$ times product,

$$\sum = S^{2k} = S \times S \times \cdots \times S, \tag{3.3}$$

is a compact metric space. One of its equivalent distances is

$$\rho(\sigma^1, \sigma^2) = \max_{1\le j\le 2k}\{|\sigma_j^1 - \sigma_j^2|\}, \quad \sigma^i = (\sigma_1^i, \dots, \sigma_{2k}^i) \in \sum, \quad i = 1, 2. \tag{3.4}$$

Define the cycle shift π on $\sum$ by

$$\pi(\sigma_1, \dots, \sigma_{2k}) = (\sigma_2, \dots, \sigma_{2k}, \sigma_1), \tag{3.5}$$

for $(\sigma_1, \dots, \sigma_{2k}) \in \sum$. Thus, we obtain another compact topological dynamical system $(\sum, \pi)$. The dynamics of the system $(\sum, \pi)$ is very simple, e.g., all points in $\sum$ are periodic points of π.

We are now in the position to prove that the system F restricted to some invariant set in $\mathbb{R}^{2k}$ is topologically semiconjugate to $(\sum, \pi)$. Recall that the sign function is defined by

$$\operatorname{sgn}(x) = \begin{cases} 1, & x \ge 0 \\ -1, & x < 0 \end{cases} \quad \text{for} \quad x \in \mathbb{R}.$$

Let $\beta \in (0, \frac{1}{2})$ and define

$$a^* = \alpha(1-\varepsilon) - \beta b^*, \qquad b^* = \frac{\alpha}{1-\beta}(1+\varepsilon), \tag{3.6}$$

where $\varepsilon \in (0, \varepsilon_\beta)$ with $\varepsilon_\beta = 1 - 2\beta$. If $r < a^*$, $R > b^*$, let $r_* = \min\{a^* - r,\ R - b^*\}$, and

$$a_c = a^* - c, \qquad b_c = b^* + c, \tag{3.7}$$

for a given $c \in [0, r_*)$. For $\sigma \in \sum$, define

$$\Omega(\sigma, c) = \{w \in \mathbb{R}^{2k} \mid |w_j| \in [a_c, b_c],\ \operatorname{sign}\{w_j\} = \sigma_j, \quad j = 1, \ldots, 2k\}, \tag{3.8}$$

$$\Omega(c) = \bigcup_{\sigma \in \sum} \Omega(\sigma, c). \tag{3.9}$$

Lemma 3.1. *Assume that (H_1) holds with $\beta \in (0, \frac{1}{2})$ and $r < a^*$, $R > b^*$, where a^* and b^* are defined by (3.6). Then, for any $c \in [0, r_*)$, the compact set $\Omega(c)$ given by (3.9) is an invariant set of F. That is, $F(\Omega(c)) \subset \Omega(c)$.*

The proof is included in the proof of Theorem 2.2 in [21].

Theorem 3.1. *Under the assumptions of Lemma 3.1, for any $c \in [0, r_*)$, the subsystem $(\Omega(c), F)$ is topologically semiconjugate to the system $(\sum, \pi)$. That is, there exists a continuous and onto mapping h from $\Omega(c)$ to $\sum$ such that*

$$h \circ F = \pi \circ h. \tag{3.10}$$

Proof. For a fixed $c \in [0, r_*)$ and $w = (w_1, w_2, \ldots, w_{2k}) \in \Omega(c)$, define

$$h(w) = (\operatorname{sign}(w_1), \operatorname{sign}(w_2), \ldots, \operatorname{sign}(w_{2k})). \tag{3.11}$$

It is routine to check that h is a continuous and onto (but not necessarily 1-1) mapping from $\Omega(c)$ to $\sum$, since the set $\Omega(c)$ is away from any axes. By the definitions of F, π and F and the set $\Omega(c)$ (see also [21]), we then obtain

$$h \circ F(w) = \pi \circ h(w), \qquad \forall w \in \Omega(c),$$

completing the proof. □

We now explore the dynamics of (3.1) on the set $D = \{w \in \mathbb{R}^{2k} \mid \|w\| \leq r\}$. It is easy to see that the set

$$\mathcal{L}_k = \{w = (w_1, \ldots, w_{2k})^T \in \mathbb{R}^{2k} \mid w_j = w_{k+j},\ j = 1, \ldots, k\}$$

is invariant under F. On this set, the dynamics of F is determined by that of the k-dimensional dynamical system $(\mathbb{R}^k, G_\alpha)$, where

$$G_\alpha \begin{pmatrix} w_1 \\ \vdots \\ w_{k-1} \\ w_k \end{pmatrix} = \begin{pmatrix} w_2 \\ \vdots \\ w_k \\ \beta w_k + \alpha f(w_1) \end{pmatrix}. \tag{3.12}$$

To investigate the system $(\mathbb{R}^k, G_\alpha)$, we first introduce the concepts of expanding fixed point and snap-back repeller for a map as below.

Definition 3.1. Let F be a continuous mapping from $\mathbb{R}^N$ into itself, and $B(z_0, r)$ be the open ball of radius r centered at z_0 with respect to a given norm $|\cdot|$ in $\mathbb{R}^N$.

(i) Let F be differentiable in $B(z_0, r)$. The point z_0 is an *expanding fixed point* of F in $B(z_0, r)$ with respect to the norm $|\cdot|$ if $F(z_0) = z_0$ and there exists a constant $s > 1$ such that for any $x, y \in B(z_0, r)$,

$$|F(x) - F(y)| \geq s|x - y|. \tag{3.13}$$

(ii) Assume that z_0 is an expanding fixed point of F in $B(z_0, r)$ for some $r > 0$. Then z_0 is said to be a *snap-back repeller* of F if there exists a point $x^0 \in B(z_0, r)$ with $x^0 \neq z_0$, such that $F^m(x^0) = z_0$ and the determinant $|DF^m(x^0)| \neq 0$ for some positive integer m.

The following theorem shows that the notion of snap-back repeller indeed characterizes chaotic behavior for multidimensional discrete dynamical systems.

Lemma 3.2. *If F possesses a snap-back repeller, then the system $(\mathbb{R}^N, F)$ is chaotic in the sense of Li-Yorke. That is,*

(i) *there is a positive integer K such that for each integer $p \geq K$, F has a periodic point of period p;*

(ii) *there is a "scrambled set" of F, i.e., an uncountable set S containing no periodic points of F, such that*

(ii$_1$) $F(S) \subset S$,

(ii$_2$) *for every $x_S, \ y_S \in S$ with $x_S \neq y_S$,*

$$\limsup_{k\to\infty} |F^k(x_S) - F^k(y_S)| > 0,$$

(ii$_3$) *for every $x_S \in S$ and any periodic point y_{per} of F,*

$$\limsup_{k\to\infty} |F^k(x_S) - F^k(y_{\text{per}})| > 0;$$

(iii) *there is an uncountable subset S_0 of S such that for every $x_{S_0}, \ y_{S_0} \in S_0$,*

$$\liminf_{k\to\infty} |F^k(x_{S_0}) - F^k(y_{S_0})| = 0.$$

Remark 3.1. We point out that Definition 3.1(i) is a modification of the original definition of the expanding fixed point in Marotto [18], where the fixed point z_0 of F was said to be expanding if every eigenvalue λ of $DF(z_0)$ satisfies $|\lambda| > 1$. Lemma 3.2 was initially proved in Marotto [18] in his sense of an expanding fixed point, where the author took it for granted that his "expanding property" of a fixed point z_0 would imply (3.13) under the same norm. However, this is not true, as was pointed out and demonstrated by examples in [7] and [15]. The work [7] shows that redefining the expanding fixed point as in our Definition 3.1(i) is a remedy, the cost is that verifying (3.13) is usually much harder than verifying that every eigenvalue λ of $DF(z_0)$ satisfies $|\lambda| > 1$. Fortunately, for system (3.1), we are able to verify (3.13) by a limiting and continuity argument (see Lemma 3.4).

The following assumption on the nonlinear activation function f near the origin will be needed in our main results.

(H_3) $f\colon \mathbb{R} \to \mathbb{R}$ is continuous differentiable, and there exists $\bar{x} \in (-r, r)$ with $\bar{x} \neq 0$ such that

$$f(0) = f(\bar{x}) = 0, \qquad f'(0) \neq 0, \qquad \text{and} \qquad f'(\bar{x}) \neq 0, \tag{3.14}$$

where $r > 0$ is given by (1.2).

We are now in the position to state and prove our main result.

Theorem 3.2. *Assume (H_3) holds. Then there exists a positive constant α_0 such that for any $\alpha > \alpha_0$, the system (3.12) is chaotic in the sense of Li-Yorke (and thus, so is (3.1)). In particular, in this case, (3.1) has infinitely many periodic orbits with different periods near the origin.*

We divide the proof of this theorem into the following lemmas under the same assumption.

Lemma 3.3. *There exists r_1 such that* $\det G'_\alpha(w) \neq 0$ *for any $\alpha > 0$ and $w \in B(0, r_1) \cap B(\bar{w}, r_1)$, where $G'(w)$ is the Jacobian matrix of G_α at w, $B(w, a)$ denotes the ball in $\mathbb{R}^k$ with the center at w and radius a and $\bar{w} = (\bar{x}, 0, \ldots, 0)^T \in \mathbb{R}^k$.*

Proof. A direct calculation shows that for $w = (w_1, \ldots, w_k)^T \in \mathbb{R}^k$,

$$\det G'_\alpha(w) = (-1)^{k+1} \alpha f'(w_1). \tag{3.15}$$

The conclusion follows from the assumption (H_3) and (3.15). □

Lemma 3.4. *There exist $s > 1$, $\alpha_1 > 0$, $r_2 > 0$, and a norm in $\mathbb{R}^k$ such that*

$$|G_\alpha(w) - G_\alpha(u)| \geq s|w - u|, \tag{3.16}$$

for any $\alpha > \alpha_1$ and any $w, u \in B(0, r_2)$.

Proof. The Jacobian matrix of G_α at 0 is

$$G'_\alpha(0) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdots & \cdot\cdot \\ 0 & 0 & 0 & \cdots & 1 \\ \alpha f'(0) & 0 & 0 & \cdots & \beta \end{pmatrix}.$$

Its eigenvalues are determined by

$$\lambda^k - \beta \lambda^{k-1} = \alpha f'(0). \tag{3.17}$$

Let $\alpha_1 = \frac{1+\beta}{|f'(0)|}$. From (3.17), one easily knows that for $\alpha > \alpha_1$, all the eigenvalues of $G'_\alpha(0)$ are strictly larger than 1 in norm. By Chen et al. [7], there exists a norm $|\cdot|$ in $\mathbb{R}^k$ such that

$$|G'_\alpha(0)w| \geq s|w|, \qquad \forall w \in \mathbb{R}^k,$$

for some $s = 1+\delta > 1$. By continuity of $f'(\cdot)$, there exists $r_2 > 0$ such that for $\alpha > \alpha_1$,

$$\|G'_\alpha(w) - G'_\alpha(0)\|_2 < \frac{\delta}{2}, \qquad \forall w \in B(0, r_2).$$

Therefore

$$\begin{aligned} |G_\alpha(w) - G_\alpha(u)| \; &= \; \left| \int_0^1 G'_\alpha(u + t(w-u))(w-u)dt \right| \\ &\geq \; |G'_\alpha(0)(w-u)| - \frac{\delta}{2}|w-u| \\ &\geq \; \left(1 + \frac{\delta}{2}\right)|w-u|, \qquad \forall w,\ u \in B(0, r_2), \end{aligned}$$

completing the proof. □

Lemma 3.5. *For a sufficiently small neighborhood $O(0)$ of 0 and any bounded interval I of $\mathbb{R}$, there exists $\alpha_3 = \alpha_3(O(0), I)$ such that the equation $\alpha f(x) = y$ has at least one solution $x \in O(0)$ for any $\alpha > \alpha_3$ and any $y \in I$.*

Proof. Since $f(0) = 0$ and $f'(0) \neq 0$, for any neighborhood $O(0)$ of the origin, there exist two neighborhoods U and V of 0 such that $U, V \subset O(0)$ and $f\colon U \to V$ is a homeomorphism. For any bounded interval I, there exists $\mu_0 > 0$ such that

$$\mu_0 I \triangleq \{\mu_0 y \mid y \in I\} \subset V.$$

Take $\alpha_3 = \mu_0$. Then

$$\frac{1}{\alpha} I \subset V,$$

for any $\alpha > \alpha_3$. Thus, for any $y \in I$ and $\alpha > \alpha_3$, the equation $f(x) = \frac{1}{\alpha}y$ has as a solution $x \in U \subset O(0)$. □

Lemma 3.6. *Let $k \geq 2$. For any sufficiently small neighborhood W of the origin in $\mathbb{R}^k$, there exists a constant α_4 such that for any $\alpha > \alpha_4$ there is a $0 \neq w_\alpha \in W$ with*

$$G_\alpha^{k+1}(w_\alpha) = 0. \tag{3.18}$$

Proof. Let W be any small neighborhood of the origin in $\mathbb{R}^k$. There are small intervals U of 0 such that $U \times U \times \cdots \times U \subset W$. For $k = 2$, by Lemma 3.5, there exists α'_4 such that for any $\alpha > \alpha'_4$, there are $x_1, x_2 \in U$ with

$$\begin{aligned} \alpha f(x_2) \; &= \; -\beta \bar{x}, \\ \alpha f(x_1) \; &= \; \bar{x} - \beta x_2, \end{aligned}$$

where $\bar{x}$ is such that $f(\bar{x}) = 0$. Let $w_\alpha = (x_1, x_2)^T$. Then $w_\alpha \neq 0$, $w_\alpha \in W$, and

$$G_\alpha^3(w) = 0.$$

For $k > 2$, again by Lemma 3.5, there exists $\alpha_4'' > 0$ such that for any $\alpha > \alpha_4''$, there are $x_1, x_2 \in U$ with

$$\alpha f(x_1) = \bar{x}, \qquad \alpha f(x_2) = -\beta\bar{x}.$$

Take $w_\alpha = (x_1, x_2, 0, \ldots, 0)^T$. Then $0 \neq w_\alpha \in W$ and

$$G_\alpha^{k+1}(w_\alpha) = 0.$$

Letting $\alpha_4 = \max\{\alpha_4',\ \alpha_4''\}$, we obtain the result. □

We are now ready for the following:

Proof of Theorem 3.2. Let W be a neighborhood of 0 in $\mathbb{R}^k$, small enough such that $W \in B(0, r_2)$, which is a ball centered at 0 with radius r_2 with respect to the norm $|\cdot|$ given in Lemma 3.4. Let $\alpha_0 = \max\{\alpha_2, \alpha_4\}$. By the Chain Rule, (3.15) and the fact that $f'(x) \neq 0$ at $x = 0, x_1, x_2$, and $\bar{x}$, one easily sees that $|DG_\alpha^{k+1}(w_\alpha)| \neq 0$, where $\alpha > \alpha_0$ and w_α is as in Lemma 3.6. This, together with Lemmas 3.4 and 3.6, shows that G_α has a snap-back repeller for any $\alpha > \alpha_0$, completing the proof of the theorem by Lemma 3.2.

Remark 3.2. In the proof of Theorem 3.2, we have shown only the existence of chaos in the k-dimensional invariant subspace $\mathcal{L}_k$ near the origin by showing that the k-dimensional dynamical system G_α possesses a snap-back repeller, which then implies the occurrence of chaos in the sense of Li-Yorke by Lemma 3.2 in this subspace. However, we do not have to confine ourselves to $\mathcal{L}_k$, and the reason we choose $\mathcal{L}_k$ is simply that we want to avoid the mathematical ideas from being hidden behind the complexity caused by the high dimension. Indeed, calculation shows that the Jacobian matrix of F at $w = (w_1, \ldots, w_k, w_{k+1}, \ldots, w_{2k}) \in \mathbb{R}^{2k}$ is

$$F'(w) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & \beta & \alpha f'(w_{k+1}) & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & 1 \\ \alpha f'(w_1) & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & \beta \end{pmatrix}. \tag{3.19}$$

From (3.19), corresponding to (3.15) for G_α, we can easily compute for F

$$\det F'(w) = (-1)^{2k+1}\alpha^2 f'(w_1) f'(w_{k+1}). \tag{3.20}$$

Thus, Lemma 3.3 is also valid for F if we choose $\bar{w} = (\bar{x}, 0, \ldots, 0, \bar{x}, 0, \ldots, 0) \in \mathbb{R}^{2k}$. Also from (3.19), we can obtain the characteristic equation of $F'(0)$ as

$$(\lambda^k - \beta\lambda^{k-1})^2 = (\alpha f'(0))^2, \tag{3.21}$$

Table 1. $n(k)$ for some k.

k	2	3	4	5	10	15	20
$p{:}p \mid k$	1, 2	1, 3	1, 2, 4	1, 5	1, 2, 5, 10	1, 3, 5, 15	1, 2, 4, 5, 10, 20
$n(k)$	3	4	6	11	108	2192	52488

from which we can similarly establish the expanding property for F near the origin, as is done for G_α in Lemma 3.4. As for Lemma 3.6, if we replace $w_\alpha = (x_1, x_2, 0, \dots, 0) \in \mathbb{R}^k$ in the proof by $w_\alpha = (x_1, x_2, 0, \dots, 0, x_1, x_2, 0, \dots, 0) \in \mathbb{R}^{2k}$, the proof can easily be carried over for F as well. Now, since Lemma 3.5 is only for the scalar function f, we can actually prove that F also has a snap-back repeller for $\alpha > \alpha_0$, implying the existence of chaos in $\mathbb{R}^{2k}$ near the origin.

4. Conclusion

Recent work ([21], [22], and [20]) has shown that discrete neural networks with delay can admit a large capacity of periodic solutions. To gain some sense of such large capacity, we provide the following table, showing how the total number $n(k)$ of stable periodic orbits of (1.1), located in the region $\Omega(c)$, grows as the delay increases (using the formula (1.4)), where

$$n(k) = \sum_{p|k} \frac{N(p)}{p}.$$

This suggests that as far as associative memory for periodic patterns is concerned, delayed discrete networks may be superior to continuous networks, and thus reveal great potential for delayed discrete networks. How to implement such associate memory networks remains a technical and engineering problem. On the other hand, the dynamics of the network outside the region where the $n(k)$ stable periodic orbits are identified is another interesting, practical, and challenging problem. In this paper, we have seen that even the very simple discrete network (1.1) of two identical neurons can demonstrate very complicated (chaotic) behavior near the origin. This is complementary to the results in [21] and [20], [22], and gives some further valuable information for designing networks. The dynamics of system (1.1) in the exterior region of $\Omega(c)$ is also worth investigating. Extending the work to discrete neural networks with more parameters for training is a meaningful job. Also worth exploring are the applications of such neural networks with both chaos and large-number stable periodic solutions co-existing.

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