



# Effect of impulsive controls in a model system for age-structured population over a patchy environment

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**Abstract** In this paper, a very general model of impulsive delay differential equations in  $n$ -patches is rigorously derived to describe the impulsive control of population of a single species over  $n$ -patches. The model allows an age structure consisting of immatures and matures, and also considers mobility and culling of both matures and immatures. Conditions are obtained for extinction and persistence of the model system under three special scenarios: (1) without impulsive control; (2) with impulsive culling of the immatures only; and (3) with impulsive culling of the matures only, respectively. In the case of persistence, the persistence level is also estimated for the systems in the case of identical  $n$  patches, by relating the issue to the dynamics of multi-dimensional maps. Two illustrative examples and their numerical simulations are given to show the effectiveness of the results. Based on the theoretical results, some strategies of impulsive culling are provided to eradicate the population of a pest species.

**Keywords** Delay differential equation · Impulsive culling · Age structure · Patch · Extinction · Persistence

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## 1 Introduction

In recent years, the study of population dynamics for species with age structure and spatial structure has attracted increasing attention in the communities of mathematical biology and applied dynamical systems, see, e.g., [Smith and Thieme \(1991\)](#), [So et al. \(2001\)](#), [Li and Zou \(2010\)](#), [Weng et al. \(2010\)](#) and [Xu \(2005\)](#). In the real world, many species have two major life stages: immature and mature stages. Taking a single species as an example, the consideration of maturation time naturally brings a delay into the model, leading to a delay differential equation (DDE); further incorporation of spatial mobility may naturally modify the delayed term to one with both the temporal delay and spatial non-locality. In their earlier work, [Smith and Thieme \(1991\)](#) formulated a system for the population of a single species living in  $n$ -patches with the maturation age and established the generic convergence of the system by the theory of monotone dynamical systems. By similar derivations, [So et al. \(2001\)](#) and [Weng et al. \(2010\)](#) presented two models for a single age-structured species living in *two and three patches, respectively*, and investigated the stability and the Hopf bifurcation of the systems. [Xu \(2005\)](#) further investigated the uniform persistence, global stability of equilibria and the Hopf bifurcation for a structured population in *two identical patches*.

On the other hand, some real world problems involve abrupt changes of populations by human's intervening at some time moments. For instance, in pest control, in order to eradicate a pest species, spraying adulticides or larvicides may be exercised at certain times of the year that coincide with critical stages in the matured pest's development or in the larval's growth. Another good example is harvesting of fish or other animal species in animal farms, in which a population is typically harvested only at some discrete times. Population dynamics with such abrupt removals or culls are described by systems of impulsive differential equations (IDEs). For some fundamental results on IDEs, a reader is referred to [Lakshmikantham et al. \(1989\)](#), [Gopalsamy and Zhang \(1989\)](#), [Bainov and Stamova \(1997\)](#), [Stamova \(2009\)](#), [Liu and Ballinger \(2002\)](#), [Nieto \(2002\)](#), [Yan \(2009\)](#), [Yang and Xu \(2006\)](#) and the references therein.

When applying culling to an age structured population, there can be culling of immatures and culling of matures. [Simons and Gourley \(2006\)](#) proposed a stage-structured population model for species whose *adult members* are subject to impulsive culls, and obtained some extinction criteria for their delayed impulsive systems. Later, [Terry \(2010a, b\)](#) studied a *two patch* model with an impulsive *adult culling* regime to control the pest with different birth functions in one or two patches. [Gourley et al. \(2007\)](#) derived the impulsive control strategy involving culling of both the immature and the mature of the host population, and explored the possibility to eradicate vector-borne diseases via age-structured culling of host. However, to the authors' best knowledge, no existing work has discussed the impulsive control for *age-structured* populations in *multi-patchy environment*. Moreover, in some situations, the goal of population control is not complete eradication of a species, but is, instead, a sustainable harvesting (e.g., in fishery management). In other words, persistence of the age-structured population in a patchy environment should also be of interest and significance when applying impulsive culling. In addition, for resource management purpose, it should also be important to predict or estimate the population in the long time in every patch,

when the population is persistent. Mathematically, this corresponds to the eventual lower and upper bounds of solutions to the model systems, and it turns out to be very challenging since there are few, if not no at all, effective methods in the literature.

Motivated by the above discussions, we rigorously derive a model for the population of a single age-structured species living in  $n$ -patches, incorporated with impulsive culling control of the population. The model is very general in the sense that it allows the control strategy to be implemented in some chosen patches or all patches, and it allows culling of both immatures and matures. For three special cases, we offer detailed analysis which leads to some more explicit conditions for extinction and persistence: (1) non-impulsive control; (2) impulsive culling of the immatures; and (3) impulsive culling of the matures. In these three cases, the model system reduces to an *autonomous DDE* [for (1)], a *non-autonomous DDE* [for (2)] and a *true impulsive DDE* [for (3)] respectively.

The rest of the paper is organized as below. We derive the general model in Sect. 2 and give some preliminaries in Sect. 3. In Sect. 4, we establish the threshold dynamics on the persistence and extinction for the model. The criteria are helpful in determining whether or not the culling is needed. In Sect. 5, we discuss extinction when the impulsive culling is implemented to immature only; and in the case of periodic culling, we obtain conditions for persistence of the population. Moreover, by some results on the dynamics of multi-dimension maps, we give an estimate of persistence level for the model system in the case of identical patches and with or without culling of the immatures. In Sect. 6, we derive some results on extinction and instability in the case of culling of the mature only, and show that the extinction (eradication) can be achieved by choosing some patches to cull. The main ideas used to deal with extinction and persistence in these sections are closely related to the theories of persistence, monotone dynamical systems and map dynamics, e.g. in Thieme (1993), Smith (1995), Zhao (2003), Zhao (2017), Yi and Zou (2010), etc. In Sect. 7, we give two illustrative examples, and present two practical schedules for eradicating a pest species by culling immatures or immatures respectively. To make the reading smoother, we leave some proofs to the two appendices.

## 2 Model description

Consider the population of a single species with age-structure that lives in  $n$  patches. Let  $u_i(t, a)$  be the population density at time  $t$  with age  $a$ ,  $d_i(a) > 0$  denote the natural death rate at age  $a$ ,  $p_{ij}(a) \geq 0$  be to the migration rate of the individuals at age  $a$  from patch  $j$  to patch  $i$  for  $i = 1, 2, \dots, n$ . In the absence of culling, by the standard von-Foester equation, we have

$$\begin{aligned} \frac{\partial u_i(t, a)}{\partial t} + \frac{\partial u_i(t, a)}{\partial a} &= -d_i(a)u_i(t, a) + \sum_{j \neq i} p_{ji}(a)u_j(t, a) \\ &\quad - \sum_{j \neq i} p_{ij}(a)u_i(t, a), t, a \geq 0. \end{aligned} \quad (2.1)$$

Now, assume that a culling strategy is implemented to control the population of the species by removing some individuals at some pre-scheduled times (see, e.g. Simons

and Gourley 2006). The culling strength is typically assumed to be proportional to the population of the present time, meaning that

$$\Delta u_i(t_k, a) := u_i(t_k^+, a) - u_i(t_k^-, a) = -c_{ik}(a)u_i(t_k^-, a), \quad a \geq 0, k = 1, 2, \dots, \tag{2.2}$$

with  $c_{ik}(a)$  being the culling rate at the  $k$ th culling time for the population of age  $a$  in patch  $i$ . Thus, the population dynamics under culling can be governed by a system obtained by incorporating the above culling terms into (2.1), leading to the following impulsive system

$$\begin{aligned} \frac{\partial u_i(t, a)}{\partial t} + \frac{\partial u_i(t, a)}{\partial a} = & -d_i(a)u_i(t, a) + \sum_{j \neq i} p_{ji}(a)u_j(t, a) - \sum_{j \neq i} p_{ij}(a)u_i(t, a) \\ & - \sum_{0 < t_k \leq t} c_{ik}(a)u_i(t, a)\delta(t - t_k), \quad t, a \geq 0, \quad i = 1, 2, \dots, n. \end{aligned} \tag{2.3}$$

where  $\delta(t)$  is the Dirac delta function. For convenience, we assume  $u_i(t_k, \cdot) = u_i(t_k^+, \cdot)$  throughout this paper.

Next, let  $r > 0$  be the age at which an individual becomes a producing adult. For simplicity, we follow (So et al. 2001) to assume that for  $i = 1, \dots, n, k = 1, 2, \dots,$

$$\begin{aligned} d_i(a) &= \begin{cases} d(a), & 0 \leq a \leq r, \\ d_i, & a > r, \end{cases} & p_{ij}(a) &= \begin{cases} p(a), & 0 \leq a \leq r, \\ p_{ij}, & a > r, \end{cases} \\ c_{ik}(a) &= \begin{cases} c_k(a), & 0 \leq a \leq r, \\ C_{ik}, & a > r. \end{cases} \end{aligned}$$

The adult population in the  $i$ th patch is then given by  $x_i(t) = \int_r^\infty u_i(t, a)da$ . In addition to the condition  $u_i(t, \infty) = 0$ , there is also the condition at  $a = 0$ :  $u_i(t, 0) = b_i(x_i(t))$  since the new born individuals are produced by the adults, where  $b_i(\cdot)$  is birth function for the species in patch  $i$ . Integrating (2.3) w.r.t.  $a$  from 0 to  $r$  gives

$$\begin{aligned} \frac{dx_i(t)}{dt} &= \int_r^\infty \left[ -\frac{\partial u_i(t, a)}{\partial a} - d_i(a)u_i(t, a) + \sum_{j \neq i} p_{ji}(a)u_j(t, a) \right. \\ &\quad \left. - \sum_{j \neq i} p_{ij}(a)u_i(t, a) - \sum_{0 < t_k \leq t} c_{ik}(a)u_i(t, a)\delta(t - t_k) \right] da \\ &= u_i(t, r) - d_i x_i(t) + \sum_{j \neq i} p_{ji}x_j(t) - \sum_{j \neq i} p_{ij}x_i(t) - \sum_{0 < t_k \leq t} C_{ik}x_i(t)\delta(t - t_k). \end{aligned}$$

For any given  $s \geq -r$ , setting  $U_i^s(t) = u_i(t, t-s), s \leq t \leq s+r$  and  $a = t-s \in [0, r]$ , we have

$$\begin{aligned} \frac{dU_i^s(t)}{dt} &= -d(t-s)U_i^s(t) + \sum_{j \neq i} p(t-s)U_j^s(t) - \sum_{j \neq i} p(t-s)U_i^s(t) \\ &\quad - \sum_{0 < t_k \leq t} c_k(t-s)U_i^s(t)\delta(t-t_k). \end{aligned}$$

Denoting  $U^s(t) = U_1^s(t) + \dots + U_n^s(t)$ , we get

$$\frac{dU^s(t)}{dt} = -d(t-s)U^s(t) - \sum_{0 < t_k \leq t} c_k(t-s)U^s(t)\delta(t-t_k). \tag{2.4}$$

Solving the above linear ODE and noting that  $U_j^s(s) = u_j(s, 0) = b_j(x_j(s))$ , we obtain

$$\begin{aligned} U^s(t) &= U^s(s) \exp \left\{ - \int_0^{t-s} d(a)da \right\} \prod_{s < t_k \leq t} [1 - c_k(t_k - s)] \\ &= \exp \left( - \int_0^{t-s} d(a)da \right) \prod_{s < t_k \leq t} [1 - c_k(t_k - s)] \sum_{j=1}^n b_j(x_j(s)). \end{aligned}$$

Setting  $\hat{d}(t-s) = d(t-s) + np(t-s)$ ,  $U_i^s(t)$  then satisfies

$$\frac{dU_i^s(t)}{dt} = -\hat{d}(t-s)U_i^s(t) + p(t-s)U^s(t) - \sum_{0 < t_k \leq t} c_k(t-s)U_i^s(t)\delta(t-t_k). \tag{2.5}$$

Applying the impulsive version of the constant-variation-formula (see, e.g. [Lakshmikantham et al. 1989](#)) to the linear inhomogeneous system (2.5), we then obtain

$$\begin{aligned} U_i^s(t) &= U_i^s(s) \exp \left( - \int_s^t \hat{d}(t-\xi)d\xi \right) \prod_{s < t_k \leq t} [1 - c_k(t_k - s)] \\ &\quad + \int_s^t \exp \left( - \int_\xi^t \hat{d}(\theta-s)d\theta \right) p(\xi-s)U^s(\xi) \prod_{s < t_k \leq \xi} [1 - c_k(t_k - s)]d\xi \\ &= \exp \left( - \int_0^{t-s} \hat{d}(a)da \right) \prod_{s < t_k \leq t} [1 - c_k(t_k - s)] b_i(x_i(s)) \\ &\quad + \int_s^t \exp \left( - \int_{\xi-s}^{t-s} \hat{d}(a)da \right) p(\xi-s) \exp \left( - \int_0^{\xi-s} d(a)da \right) \\ &\quad \times \prod_{\xi < t_k \leq t} [1 - c_k(t_k - s)] \prod_{s < t_k \leq \xi} [1 - c_k(t_k - s)]d\xi \sum_{j=1}^n b_j(x_j(s)) \\ &= \exp \left( - \int_0^{t-s} \hat{d}(a)da \right) \prod_{s < t_k \leq t} [1 - c_k(t_k - s)] b_i(x_i(s)) \end{aligned}$$

$$\begin{aligned}
 & + \exp\left(-\int_0^{t-s} d(a)da\right) \int_0^{t-s} \exp\left(-\int_\zeta^{t-s} np(a)da\right) p(\zeta)d\zeta \\
 & \times \prod_{s < t_k \leq t} [1 - c_k(t_k - s)] \sum_{j=1}^n b_j(x_j(s)).
 \end{aligned}$$

Evaluating at  $s = t - r$  yields

$$\begin{aligned}
 u_i(t, r) = U_i^{t-r}(t) & = \exp\left(-\int_0^r \hat{d}(a)da\right) \prod_{t-r < t_k \leq t} [1 - c_k(r - (t - t_k))] b_i(x_i(t - r)) \\
 & + \exp\left(-\int_0^r d(a)da\right) \left(\int_0^r \exp\left(-\int_\zeta^r np(a)da\right) p(\zeta)d\zeta\right) \\
 & \times \prod_{t-r < t_k \leq t} [1 - c_k(r - (t - t_k))] \sum_{j=1}^n b_j(x_j(t - r)).
 \end{aligned}$$

Note that

$$\begin{aligned}
 \int_0^r \exp\left(-\int_\zeta^r np(a)da\right) p(\zeta)d\zeta & = \frac{1}{n} \left[ \exp\left\{-\int_\zeta^r np(a)da\right\} \right]_{\zeta=0}^{\zeta=r} \\
 & = \frac{1}{n} \left[ 1 - \exp\left(-\int_0^r np(a)da\right) \right].
 \end{aligned}$$

Thus,

$$\begin{aligned}
 u_i(t, r) & = \frac{1}{n} \exp\left(-\int_0^r d(a)da\right) \prod_{t-r < t_k \leq t} [1 - c_k(r - (t - t_k))] \\
 & \times \left[ 1 + (n - 1) \exp\left\{-\int_0^r np(a)da\right\} \right] b_i(x_i(t - r)) \\
 & + \frac{1}{n} \exp\left(-\int_0^r d(a)da\right) \prod_{t-r < t_k \leq t} [1 - c_k(r - (t - t_k))] \\
 & \times \left[ 1 - \exp\left\{-\int_0^r np(a)da\right\} \right] \sum_{j \neq i}^n b_j(x_j(t - r)).
 \end{aligned}$$

Therefore, we obtain the impulsive delay model

$$\begin{aligned}
 \frac{dx_i(t)}{dt} & = -d_i x_i(t) + \sum_{j \neq i} p_{ji} x_j(t) - \sum_{j \neq i} p_{ij} x_i(t) \\
 & \quad + \frac{1}{n} \alpha \gamma(t) [1 + (n - 1) \beta] b_i(x_i(t - r))
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{1}{n} \alpha \gamma(t) [1 - \beta] \sum_{j \neq i}^n b_j(x_j(t - r)) \\
 & - \sum_{0 < t_k \leq t} C_{ik} x_i(t) \delta(t - t_k),
 \end{aligned} \tag{2.6}$$

where

$$\begin{aligned}
 \alpha & = \exp\left(-\int_0^r d(a) da\right), \quad \beta = \exp\left(-\int_0^r np(a) da\right), \\
 \gamma(t) & = \prod_{t-r < t_k \leq t} [1 - c_k(r - (t - t_k))].
 \end{aligned} \tag{2.7}$$

The right hand side of (2.6) can be biologically explained as below. The first line on the right side accounts for instantaneous death rate and migration in and out rates for matured individuals in patch  $i$ , and the third line explains the total loss rate of matured individuals in patch  $i$  up to time  $t$  due to culling of the matures. As for the second line on the right side of (2.6), it is the rate at which patch  $i$  gains matured individuals. Rewriting it as

$$\beta \gamma(t) \alpha b_i(x_i(t - r)) + \sum_{j=1}^n \frac{1}{n} (1 - \beta) \gamma(t) \alpha b_j(x_j(t - r)),$$

we find that it consists of two parts: the first part is nothing but the individuals born  $r$  time units ago in patch  $i$  who have survived immature period (including natural death and culling) and have remained in patch  $i$  (with probability  $\beta$ ); and the second part adds up all individuals born  $r$  time units ago in all patches survived immature period who have once left the birth patch (with probability  $1 - \beta$ ), but find themselves in patch  $i$  (with probability  $1/n$ ) when becoming mature (i.e., at age  $a = r$ ).

Equation (2.6) can be rewritten as the matrix form

$$\begin{cases} \frac{d}{dt} x(t) = -(D + L)x(t) + \alpha \gamma(t) S b(x(t - r)), & t \neq t_k, \\ \Delta x(t_k^+) = x(t_k^+) - x(t_k^-) = -C_k x(t_k^-), & k \in N, \end{cases} \tag{2.8}$$

where  $x(t) = (x_1(t), x_2(t), \dots, x_n(t))^T$ ,  $b(x(t)) = (b_1(x_1(t)), b_2(x_2(t)), \dots, b_n(x_n(t)))^T$ ,  $L = P_0 - P$ ,

$C_k = \text{diag}\{C_{1k}, C_{2k}, \dots, C_{nk}\}$ ,  $D = \text{diag}\{d_1, d_2, \dots, d_n\}$ ,

$P_0 = \text{diag}\{\sum_{j \neq 1}^n p_{1j}, \sum_{j \neq 2}^n p_{2j}, \dots, \sum_{j \neq n}^n p_{nj}\}$ , and

$$P = \begin{bmatrix} 0 & p_{21} & \cdots & p_{n1} \\ p_{12} & 0 & \cdots & p_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ p_{1n} & p_{2n} & \cdots & 0 \end{bmatrix},$$

$$S = \frac{1}{n} \begin{bmatrix} 1 + (n - 1)\beta & 1 - \beta & \cdots & 1 - \beta \\ 1 - \beta & 1 + (n - 1)\beta & \cdots & 1 - \beta \\ \vdots & \vdots & \ddots & \vdots \\ 1 - \beta & 1 - \beta & \cdots & 1 + (n - 1)\beta \end{bmatrix}.$$

Throughout this paper, we will assume that the functions  $b_i(\cdot)$ ,  $i = 1, 2, \dots, n$ , satisfy those common properties for a birth function (see e.g., [Rost and Wu 2007](#); [Liz and Rost 2010](#)):

- (H1)**  $b_i(0) = 0$  and  $b_i(s) > 0$  for  $s > 0$ ; there is a unique  $\eta_i > 0$  such that  $b'_i(s) > 0$  if  $0 \leq s < \eta_i$ ,  $b'_i(\eta_i) = 0$ ,  $b'_i(s) < 0$  if  $s > \eta_i$ ,  $b''_i(s) < 0$  if  $0 \leq s \leq \eta_i$ ; and  $\lim_{s \rightarrow \infty} b_i(s) = 0$ ;

In the rest of this paper,  $z = (z_1, z_2, \dots, z_n)^T \in \mathbb{R}_+^n$ , we will use  $B'(z)$  to denote the diagonal matrix  $\text{diag}\{b'_1(z_1), b'_2(z_2), \dots, b'_n(z_n)\}$  and use  $b'(z)$  to denote the vector  $(b'_1(z), b'_2(z), \dots, b'_n(z))^T$ .

There are some familiar examples of birth functions satisfying (H1), among which are the birth functions used in the Nicholson's blowflies model ([Gurney et al. 1980](#)) and the Mackey–Glass model ([Mackey and Glass 1977](#)). For the culling terms, based on the biological meanings, we assume the following

- (H2)** the culling rates satisfy  $c_k(\cdot)$ ,  $C_{ik} \in [0, 1)$ ,  $c_k$  is continuous function defined on  $[0, r]$ , and culling moments  $\{t_k, k \in \mathbb{N}\}$  satisfy  $0 < t_1 < \dots < t_k < t_{k+1} < \dots$ ,  $\lim_{k \rightarrow \infty} t_k = \infty$ .

*Remark 2.1* The parameters in (2.7) have their biological meanings.  $d_i$ ,  $p_{ij}$  and  $C_{ik}$  are the natural death rate, migration rate and culling rate of the mature in patch  $i$ , respectively.  $\alpha$  is the probability that an immature individual can survive natural death,  $\beta$  is the probability that an individual immature is in the patch of birth when becoming mature, while  $\gamma(t)$  is probability of an immature individual surviving the culling during the time  $[t - r, t]$ . Clearly,  $\alpha, \beta, \gamma(\cdot) \in (0, 1]$ .

*Remark 2.2* In the absence of culling, that is,  $c_k(\cdot) = 0$  and  $C_k \equiv 0$ , the model (2.8) reduces to a system of continuous *autonomous* delay differential equations which has been explored by some researchers, see e.g., [Ivanov and Sharkovsky \(1992\)](#), [Rost and Wu \(2007\)](#), [Liz and Rost \(2010\)](#), [Xu \(2005\)](#), and [Smith and Thieme \(1991\)](#). When  $C_k \equiv 0$ , (2.8) becomes a system of *non-autonomous* delay differential equations with *piecewise continuous parameter*  $\gamma(t)$ . When  $c_k(\cdot) = 0$ , (2.8) remains an impulsive system of delay differential equations, and for the special cases of  $n = 1, 2$ , the extinction problem has been studied in [Simons and Gourley \(2006\)](#) and [Terry \(2010a, b\)](#) respectively.

### 3 Preliminaries

In this section, we first introduce some definitions and notations that will be used in the sequel. Then, we present the preliminary results.

As usual,  $\mathbb{R}^n$  is the  $n$ -dimensional Euclidian space and  $\mathbb{R}^{m \times n}$  denotes the set of all  $m \times n$  real matrices and  $\mathbb{N}$  consists of all natural numbers. Let  $\mathbb{R}_+ = [0, \infty)$  and  $\mathbb{R}_+^n = \mathbb{R}_+ \times \dots \times \mathbb{R}_+$ . Also denote by  $E$  the  $n \times n$  identity matrix, and let  $e_n = (1, \dots, 1)^T \in \mathbb{R}^n$ .

Let  $C := C[-r, 0, \mathbb{R}^n]$  be the Banach space of all continuous functions on  $[-r, 0]$  equipped with the sup norm given by  $\|\phi\| = \max_{1 \leq i \leq n} \sup_{-r \leq s \leq 0} |\phi_i(s)|$ . The space



$C$  contains the cone  $C_+ := \{\phi \in C : \phi_i(s) \geq 0, s \in [-r, 0], i = 1, 2, \dots, n\}$  and its interior  $C_+^\circ := \{\phi \in C_+ : \phi_i(s) > 0, s \in [-r, 0], i = 1, 2, \dots, n\}$ .

For the purpose of dealing with impulses at the  $t_k, k = 1, \dots$ , we need the following set:

$$PC[I, R^n] \triangleq \left\{ \psi : I \rightarrow R^n : \begin{array}{l} \psi(t) \text{ is right continuous at all } t \in I; \psi(t^-) \text{ exists for all} \\ t \in (t_0, \infty) \text{ with } \psi(t^-) = \psi(t) \text{ for all but points } t_k \in (t_0, \infty) \end{array} \right\}$$

where  $I \subset R$  is an interval. Especially when  $I = [-r, 0]$ , we denote  $PC := PC([-r, 0], R^n)$  with the same norm as in  $C$ .

For  $R^n$ , we use the natural entry-wise partial ordering: for  $x = (x_1, \dots, x_n)^T, y = (y_1, \dots, y_n)^T \in R^n, x \leq y$  means  $x_i \leq y_i$  for  $i = 1, \dots, n; x < y$  means  $x_i < y_i$  for all  $i = 1, \dots, n$ . The partial ordering in  $R^{m \times n}$  is defined similarly.

For any  $z \in R^n$ , we use  $z_*$  to denote the constant function on  $[-\tau, 0]$  taking value  $z$ . For  $z, Z \in R^n$  with  $z \leq Z$ , define the order interval  $[z, Z] = \{x \in R^n : z \leq x \leq Z\}$  and  $[z, Z]_* := \{\phi \in C : \phi(s) \in [z, Z], s \in [-r, 0]\}$ . In  $PC$ , order interval is defined in a similar way.

A matrix  $A = (a_{ij})_{n \times n}$  is said to be (1) a non-negative matrix if  $A \geq 0$ ; (2) a quasi-positive matrix if  $A \neq 0$  and  $a_{ij} \geq 0, i \neq j$ ; (3) a row (column) stochastic matrix if it is non-negative and its row (column) sum is 1; (4) a Laplacian matrix if it is quasi-positive and its row (column) sum is 0; (5) a nonsingular  $M$ -matrix if  $-A$  is quasi-positive and all the leading principle minors of  $A$  are positive.

As is customary, we use  $\rho(\cdot)$  to denote the spectral radius of a matrix (or one of a linear operator if no confusion). The following lemma can be found in [Berman and Plemmons \(1979\)](#).

**Lemma 3.1** *Let  $V$  be a nonsingular  $M$ -matrix and  $W$  be a non-negative matrix. Then  $H = W - V$  is a non-singular  $M$ -matrix if and only if one of the following conditions holds*

- (1)  $\rho(V^{-1}W) = \rho(WV^{-1}) < 1$ ;
- (2)  $H^{-1}$  exists and  $H^{-1} \geq 0$ ;
- (3) there exists a positive vector  $\xi \in R^n$  such that  $H\xi > 0$  (or  $\xi^T H > 0$ ).

Especially,  $H$  is a nonsingular  $M$ -matrix if it is row or column strictly dominant diagonal, that is,  $He_n > 0$  or  $H^T e_n > 0$ .

Next, we introduce some notations related to multi-dimensional maps. For an interval  $I = [a, b]$  in  $R$ , the hyper-square generated by  $I$  is denoted by  $\mathbb{I} = [a, b]^n = [a, b] \times \dots \times [a, b]$ . For  $F(\cdot) = (f_1(\cdot), f_2(\cdot), \dots, f_n(\cdot))^T : \mathbb{I} \rightarrow R^n$ , denote

$$F(\mathbb{I}) = \bigcup_{x \in \mathbb{I}} F(x), \quad f_i(\mathbb{I}) = \bigcup_{x \in \mathbb{I}} f_i(x), \quad i = 1, 2, \dots, n,$$

$$F[\mathbb{I}] = [\bar{a}, \bar{b}]^n, \quad \text{where } \bar{a} = \min_{1 \leq i \leq n} \inf\{f_i(\mathbb{I})\}, \quad \bar{b} = \max_{1 \leq i \leq n} \sup\{f_i(\mathbb{I})\},$$

$$\begin{aligned}
 F^j(\mathbb{I}) &= \overbrace{F(F(\cdots F(\mathbb{I})\cdots))}^j, & F^j\langle\mathbb{I}\rangle &= \overbrace{F\langle F(\cdots F(\mathbb{I})\cdots)\rangle}^j, \\
 \mathbb{I}_{(F)} &= \bigcap_{j=1}^{\infty} \overline{F^j(\mathbb{I})}, & \mathbb{I}_{\langle F \rangle} &= \bigcap_{j=1}^{\infty} \overline{F^j\langle\mathbb{I}\rangle}.
 \end{aligned}$$

$\mathbb{I}_{(F)}$  is called the  $\omega$ -limit set of the multi-dimensional map  $F$  on  $\mathbb{I}$ . Clearly,  $F(\mathbb{I}) \subset F\langle\mathbb{I}\rangle$ ,  $\mathbb{I}_{(F)} \subset \mathbb{I}_{\langle F \rangle}$ . In fact,  $F\langle\mathbb{I}\rangle$  can be viewed as the “minimal” hyper-square that includes  $F(\mathbb{I})$ , and  $\mathbb{I}_{(F)}$  is an estimate of  $\mathbb{I}_{\langle F \rangle}$  for  $F$  on  $\mathbb{I}$ .

Consider the following system of delay differential equations

$$\begin{aligned}
 \frac{d}{dt}u(t) &= -Au(t) + \tilde{A}F(u(t-r)), \quad t \geq 0, \\
 u(s) &= \phi(s), \quad s \in [-r, 0], \quad \phi \in C_+,
 \end{aligned} \tag{3.1}$$

where  $F(\cdot) = (f_1(\cdot), f_2(\cdot), \dots, f_n(\cdot))^T : R^n \rightarrow R^n$  is continuous,  $A$  is a non-singular  $M$  matrix,  $\tilde{A}$  is non-negative and  $A^{-1}\tilde{A}$  is a row stochastic matrix. We assume that (3.1) has a unique solution which exists for all  $t \geq 0$  and is denoted by  $u(t, \phi)$  in  $R^n$  and  $u_t(\phi)$  in  $C$ .

Following the main idea developed in Yi and Zou (2010) for the case of one-dimensional map (see also Ivanov and Sharkovsky 1992; Rost and Wu 2007; Liz and Rost 2010), we can obtain the following lemmas on dynamics of multi-dimensional map  $F$  on a hyper-square versus dynamics of (3.1). The proofs are given in “Appendix A”.

**Lemma 3.2** *Let  $\mathbb{I} = [a, b]^n \subset R^n$  be invariant for  $F$  (i.e.,  $F(\mathbb{I}) \subset \mathbb{I}$ ), and  $u(t, \phi)$  be the solution of (3.1) with the initial function  $\phi \in C$ . If  $\phi \in C([-r, 0], \mathbb{I})$ , then  $u_t(\phi) \in C([-r, 0], \mathbb{I})$ ,  $t \geq 0$ .*

**Lemma 3.3** *Let  $\mathbb{I} = [a, b]^n \subset R^n$  and  $u(t, \phi)$  be the solution of (3.1) with the initial function  $\phi \in C$ . If  $\lim_{t \rightarrow \infty} \text{dist}(u(t, \phi), \mathbb{I}) = 0$ , then  $\lim_{t \rightarrow \infty} \text{dist}(u(t, \phi), \mathbb{I}_{(F)}) = 0$ , where the distance  $\text{dist}(u, \mathbb{I}) = \inf_{y \in \mathbb{I}} \{\|u - y\|\}$  for  $u \in R^n$ .*

*Remark 3.1* When  $A$  is a diagonal matrix with positive diagonal elements and  $\tilde{A} = A$  in (3.1), one may estimate  $\mathbb{I}_{(F)}$  by some hyper-rectangles by a similar proof. The above results show that the (eventual) boundedness of systems can be estimated by dynamics of multi-dimensional map  $F$ .

For any given  $\phi \in PC$ , a function  $x(t) \in PC[[-r, +\infty), R^n]$  is called a solution of (2.8) through  $(0, \phi)$ , if  $x(t)$  satisfies the initial condition  $x(s) = \phi(s)$ ,  $s \in [-r, 0]$ , and satisfies Eq. (2.8) for  $t \geq 0$ . For the fundamental theory on the existence and uniqueness of solutions to an initial value problem of impulsive type, a reader is referred to Lakshmikantham et al. (1989), Stamova (2009) and the reference therein. As usual, if (2.8) has a unique solution, we shall also use  $x(t, \phi)$  or  $x_t(\phi)$  or simply  $x(t)$  to denote the solution, if no confusion occurs. An element  $z_*$  in  $C$  is called an equilibrium of (2.8), if  $x(t) = z$  is a solution of (2.8).

Considering the biological background, we first address the well-posedness of (2.8).

**Theorem 3.1** *Let (H1)–(H2) hold. If the initial conditions satisfy*

$$x(t) = \phi(t) \geq 0, \quad t \in [-r, 0], \quad \phi \in PC, \tag{3.2}$$

*then (2.8) has a unique solution which exists globally and is non-negative and (eventually uniformly) bounded.*

*Proof* By (H1)–(H2) and the fundamental theory in Lakshmikantham et al. (1989), Stamova (2009), we know that (2.8) has a unique solution  $x(t, \phi)$  existing in a maximum interval  $J = [0, t_{\max})$ . Note that  $D + L$  is quasi-positive, and hence  $e^{-(D+L)(t-s)}$  is a non-negative for  $t \geq s$ . For any  $t \in [0, t_1) \cap J$ , we have

$$x(t) = e^{-(D+L)t} \phi(0) + \int_0^t e^{-(D+L)(t-s)} \alpha \gamma(s) S b(x(s-r)) ds. \tag{3.3}$$

Combining this with the nonnegative properties of  $\alpha, S, \gamma(\cdot), b(\cdot)$  and  $\phi(\cdot)$ , we then conclude that  $x(t, \phi) \geq 0$  for  $t \in [0, t_1) \cap J$ . When  $t_{\max} > t_1, x(t_1^+) = (E - C_1)x(t_1^-) \geq 0$  from (H2). Similarly, when  $t \in [t_1, t_2) \cap J$ ,

$$x(t) = e^{-(D+L)(t-t_1)} x(t_1^+) + \int_{t_1}^t e^{-(D+L)(t-s)} \alpha \gamma(s) S b(x(s-\tau)) ds \geq 0.$$

By an induction, we have  $x(t, \phi) \geq 0$  for any  $t \in J$ . Set  $\tilde{x}(t) = x_1(t) + x_2(t) + \dots + x_n(t), \hat{d} = \max_{1 \leq i \leq n} \{d_i\}, \hat{b} = \max_{1 \leq i \leq n} \{b_i(\eta_i)\}$ . Note that  $\alpha, \gamma(\cdot) \in (0, 1], L$  is a Laplacian matrix and  $S$  is an stochastic matrix. Thus, for  $t \in J$  we have

$$\frac{d\tilde{x}(t)}{dt} \leq -\hat{d}\tilde{x}(t) + \hat{b}, \quad t \neq t_k, \quad \tilde{x}(t_k^+) \leq \tilde{x}(t_k^-).$$

By the impulsive-type comparison theorem with  $V(x(t)) = \tilde{x}(t)$  in Stamova (2009, Theorem 1.23),  $\tilde{x}(t)$  is (eventually uniformly) bounded. This implies the nonnegative solution  $x(t, \phi)$  is (eventually uniformly) bounded, and therefore,  $J = [0, \infty)$  (see, e.g., Stamova 2009; Liu and Ballinger 2002). The proof is completed.  $\square$

*Remark 3.2* From the proof, we can see that if we further assume  $\phi_i(0) > 0, i = 1, 2, \dots, n$ , then the solution actually remains strictly positive:  $x(t) > 0, t \geq 0$ . Obviously, the non-negativity preserving property of (2.8) also holds when the initial functions is continuous:  $\phi \in C_+ \subset PC$ . For convenience of discussion, we will assume the initial function  $\phi$  is continuous unless otherwise specified.

### 4 Extinction and persistence when there is no culling

When determining whether or not an impulsive culling should be implemented, it is necessary to investigate the dynamics of system (2.6) in the absence of culling, i.e.,

when  $c_k(\cdot) = C_{ik} = 0, k \in N$ . In this case, (2.8) reduces to the following system of autonomous delay differential equations

$$\dot{x}(t) = -(D + L)x(t) + \alpha S b(x(t - r)). \tag{4.1}$$

The following proposition identifies an invariant and attracting interval for (4.1).

**Proposition 4.1** *For any  $z = (z_1, \dots, z_n)^T \geq \alpha S b(\eta) = \alpha S (b_1(\eta_1), \dots, b_n(\eta_n))^T$ , the set  $[0, (D + L)^{-1}z]_*$  is a positively invariant set and also globally attracting set for (4.1).*

*Proof* Let  $x(t) = x(t, \phi)$  be a solution of (4.1) with  $x(s) = \phi(s), s \in [-r, 0], \phi \in C_+$ . From the unimodal property of  $b$ , we have  $\dot{x}(t) \leq -(D + L)x(t) + z$ . Consider its comparison system

$$\dot{y}(t) = -(D + L)y(t) + z, \quad y(0) = \left( \sup_{s \in [-r, 0]} \phi_1(s), \dots, \sup_{s \in [-r, 0]} \phi_n(s) \right)^T.$$

which has the solution  $y(t) = e^{-(D+L)t}[y(0) - (D + L)^{-1}z] + (D + L)^{-1}z$ . Since  $L$  is a Laplacian matrix, all the eigenvalues of  $D + L$  are positive and  $(D + L)^{-1} \geq 0$ . Then

$$\limsup_{t \rightarrow \infty} x(t) \leq \lim_{t \rightarrow \infty} y(t) = (D + L)^{-1}z.$$

Moreover, if  $\phi(s) \in [0, (D + L)^{-1}z]$  for  $s \in [-r, 0]$ , we have  $x(t) \leq y(t) = (D + L)^{-1}z$ . The above implies the conclusion, completing the proof.  $\square$

It turns out that system (4.1) has the threshold dynamics with respect to species extinction and persistence, as is shown in the next theorem.

**Theorem 4.1** *Assume that  $L$  is irreducible. Let  $\mathcal{R} = \rho(\alpha(D + L)^{-1}S B'(0))$ .*

- (i) *If  $\mathcal{R} < 1$ , then the equilibrium  $0_*$  of (4.1) is globally asymptotically stable in  $C_+$ .*
- (ii) *If  $\mathcal{R} > 1$ , the trivial equilibrium  $0_*$  of (4.1) is unstable. Moreover, system (4.1) has at least one positive equilibrium and is uniformly persistent in  $C_+^\circ$ , that is, there is a positive number  $\delta > 0$  such that for any  $\phi \in C_+^\circ$*

$$\liminf_{t \rightarrow \infty} x_i(t, \phi) \geq \delta, \quad i = 1, 2, \dots, n. \tag{4.2}$$

The above result is a special case of Theorem 5.1 in Sect. 5 and we omit the proof here.

*Remark 4.1* From Theorem 4.1,  $\mathcal{R}$  plays a threshold role in determining whether the population goes to extinction or remains persistent under the assumption that the dispersion matrix  $L$  is irreducible. In fact,  $\mathcal{R}$  is nothing but the so-called basic reproduction number for the species governed by (4.1) [see, e.g., Van den Driessche and Watmough (2002) for a discussion on this topic].

We point out that the above theorem does not cover the critical value  $\mathcal{R} = 1$ ; moreover, it gives no information about the persistence level (i.e., the lower eventual bound  $x(t)$ ) or even an estimate for  $\delta$  in (4.2), although an estimate of eventual upper bound is obtained in Proposition 4.1, which helps to obtain the attracting region for solutions in the interior  $C_+^o$ .

To gain some more insights, we consider a special case of (4.1): the case with identical patches, i.e., when

$$d_1 = d_2 = \dots = d_n, \quad L^T = L, \quad b_1(s) = b_2(s) = \dots = b_n(s). \tag{4.3}$$

In such a case, (4.1) further reduces to

$$\dot{x}(t) = -(D + L)x(t) + DF(x(t - r)), \quad t \geq 0, \tag{4.4}$$

where  $D = d_1E$  and  $F(u) = d_1^{-1}\alpha Sb(u)$ . Note that  $(D + L)$  is a non-singular  $M$ -matrix since it is column strictly dominant diagonal. Moreover,  $(D + L)e_n = d_1e_n = De_n$ , thus  $(D + L)^{-1}De_n = e_n$  implying that  $(D + L)^{-1}D$  is indeed a stochastic matrix. In order to apply the results in Lemmas 3.2 and 3.3 to (4.4), we need to identify appropriate hyper-square(s) for this multi-dimensional map  $F$ . Note that  $F(u) = (f_1(u), f_2(u), \dots, f_n(u))^T$ , where  $u = (u_1, u_2, \dots, u_n)^T$  and  $f_i(u) = d_1^{-1}\alpha \sum_{j=1}^n S_{ij}b_1(u_j) = \sum_{j=1}^n S_{ij}g(u_j)$ . This suggest that we focus on the corresponding one-dimension map  $g$  [due to (4.3)]:

$$g(s) = d_1^{-1}\alpha b_1(s), \quad s \geq 0. \tag{4.5}$$

Let  $I = [a, b] \subset R_+$  and  $\mathbb{I} = [a, b]^n \subset R_+^n$ . From the continuity of  $g$ , we assume  $g$  attains the minimum and the maximum at  $m$  and  $M$  on  $[a, b]$ , respectively. Noting  $S = (S_{ij})$  is the stochastic matrix, then for all  $i$

$$\min_{u \in \mathbb{I}} f_i(u) = \sum_{j=1}^n S_{ij}g(m) = g(m), \quad \max_{u \in \mathbb{I}} f_i(u) = \sum_{j=1}^n S_{ij}g(M) = g(M).$$

From the continuity of  $f_i$  on  $R_+^n$ ,  $f_i(\mathbb{I}) = g([a, b])$ ,  $i = 1, 2, \dots, n$ , and so

$$F(\mathbb{I}) = \overbrace{g([a, b]) \times g([a, b]) \cdots \times g([a, b])}^n,$$

$$F^j(\mathbb{I}) = \overbrace{g^j([a, b]) \times g^j([a, b]) \cdots \times g^j([a, b])}^n.$$

Therefore,

$$\mathbb{I}_{(F)} = \bigcap_{j=1}^{\infty} g^j([a, b]) \times \bigcap_{j=1}^{\infty} g^j([a, b]) \cdots \times \bigcap_{j=1}^{\infty} g^j([a, b])$$

$$= I_{(g)} \times I_{(g)} \cdots I_{(g)} = [I_{(g)}]^n. \tag{4.6}$$

According to [Rost and Wu \(2007\)](#), [Liz and Rost \(2010\)](#), and [Ivanov and Sharkovsky \(1992\)](#), we have the following proposition for map  $g$  on interval  $I$  since  $g$  actually satisfies the same unimodal condition (H1) as  $b_1$  does.

**Proposition 4.2** *Let  $g$  be defined in (4.5) and  $\hat{\mathcal{R}} := g'(0) = d_1^{-1}\alpha b_1'(0)$ ,  $a_0 = g(g(\eta_1))$ ,  $b_0 = g(\eta_1)$ .*

- (i) *If  $\hat{\mathcal{R}} \leq 1$ , then  $I_{(g)} = \bigcap_{j=1}^{\infty} g^j(I) = \{0\}$  for any closed interval  $I = [0, b] \subset R_+$  with  $g(I) \subset I$ .*
- (ii) *If  $\hat{\mathcal{R}} > 1$  and  $w_0 \leq \eta_1$ , where  $w_0$  is the unique positive fixed point of  $g$ , then  $I_{(g)} = \{w_0\}$  for any closed interval  $I = [a, b] \subset R_+ \setminus \{0\}$  with  $w_0 \in [a, b]$  and  $g(I) \subset I$ , provided that  $g(a) \leq g(b)$ .*
- (iii) *If  $\hat{\mathcal{R}} > 1$  and  $w_0 > \eta_1$ , then  $[a_0, b_0]$  is invariant for  $g$  and  $I_{(g)} \subset [a_0, b_0]$  for any closed interval  $I = [a, b] \subset R_+ \setminus \{0\}$  containing  $[a_0, b_0]$ .*

Combining Lemmas 3.2, 3.3, Proposition 4.2 and (4.6), we have

**Theorem 4.2** *Assume that  $L$  is irreducible. Let  $\hat{\mathcal{R}}$ ,  $a_0, b_0, w_0, g$  be defined in Proposition 4.2.*

- (i) *If  $\hat{\mathcal{R}} \leq 1$ , then the equilibrium  $0_*$  of (4.4) is globally asymptotically stable in  $C_+$ ;*
- (ii) *If  $\hat{\mathcal{R}} > 1$ , then the equilibrium  $0_*$  of (4.4) is unstable. Moreover,*
  - (ii-1) *when  $w_0 \leq \eta_1$ , then the equilibrium  $W_* = (w_0, w_0, \dots, w_0)_*^T$  of (4.4) is globally asymptotically stable in  $C_+^o$ ;*
  - (ii-2) *when  $w_0 > \eta_1$ , then  $[a_0, b_0]_*^n$  is an invariant and attracting set for system (4.4) in  $C_+^o$ .*

*Proof* Firstly by (4.3) and Proposition 4.1, we observe that for given  $\phi \in C_+$ , we have

$$\limsup_{t \rightarrow \infty} x(t, \phi) \leq (D + L)^{-1}\alpha Sb(\eta) = d_1^{-1}\alpha b_1(\eta_1)e_n = b_0e_n, \tag{4.7}$$

since  $Se_n = e_n$  and  $(D + L)^{-1}e_n = d_1^{-1}e_n$ .

Case (i):  $\hat{\mathcal{R}} \leq 1$ . Let  $I = [0, b_0]$  and  $\mathbb{I} = [0, b_0]^n$ . By (4.7),  $\phi \in C_+$ ,  $\lim_{t \rightarrow \infty} \text{dist}(x(t, \phi), \mathbb{I}) = 0$ . It follows from Lemma 3.3 and (4.6) that

$$\lim_{t \rightarrow \infty} \text{dist}(x(t, \phi), \mathbb{I}_{(F)}) = \text{dist}(x(t, \phi), [I_{(g)}]^n) = 0.$$

From Proposition 4.2-(i),  $[I_{(g)}]^n = (0, 0, \dots, 0)^T$ , and thus,  $0_*$  is globally attractive in  $C_+$ . In addition, for any  $\epsilon > 0$ ,  $[0, \epsilon]$  is an invariant set of  $g$ , and hence  $[0, \epsilon]^n$  is invariant for  $F$  since

$$F([0, \epsilon]^n) \subset F([0, \epsilon]^n) = [g([0, \epsilon])]^n \subset [0, \epsilon]^n.$$

Then, by Lemma 3.2,  $x_t(\phi) \in [0, \epsilon]_*^n$  for  $t \geq 0$  if  $\phi \in [0, \epsilon]_*^n$ , which implies  $0_*$  is stable. Therefore,  $0_*$  is globally asymptotically stable.

Case (ii):  $\hat{\mathcal{R}} > 1$ . Since  $\hat{\mathcal{R}}$  is a positive eigenvalue of  $(D + L)^{-1}\alpha SB'(0)$  with the eigenvector  $e_n$ ,  $\mathcal{R} \geq \hat{\mathcal{R}} > 1$ . By Theorem 4.1-(ii), the system (4.1) is uniform persistent and (4.2) holds with some  $\delta > 0$ . Let  $I = [\delta, b_0] \subset R_+ \setminus \{0\}$  and  $\mathbb{I} = [\delta, b_0]^n$ . Then,  $\lim_{t \rightarrow \infty} \text{dist}(x(t, \phi), \mathbb{I}) = 0$  for  $\phi \in C_+^\circ$ . We have the following sub-cases.

Case (ii-1):  $w_0 \leq \eta_1$ . Clearly,  $W_*$  is an equilibrium of (4.4) and  $w_0 \in [\delta, b_0]$ . From Proposition 4.2-(ii),  $[I_{(g)}]^n = W = (w_0, w_0, \dots, w_0)^T$ . It follows from Lemma 3.3 that

$$\lim_{t \rightarrow \infty} \text{dist}(x(t, \phi), \mathbb{I}_{(F)}) = \text{dist}(x(t, \phi), W) = 0, \quad \phi \in C_+^\circ.$$

So, the equilibrium  $W_*$  is globally attractive in  $C_+^\circ$ . Next we prove that  $W_*$  is stable. When  $w_0 < \eta_1$ ,  $[w_0 - \epsilon, w_0 + \epsilon]^n$  is an invariant set of  $F$  for sufficiently small  $\epsilon > 0$ , according to Proposition 4.2-(ii) and (4.6). This implies  $W_*$  is stable by Lemma 3.2. When  $w_0 = \eta_1$ , for any given  $\epsilon \in (0, w_0)$ , one can find  $\delta_1, \delta_2$  (by (H1)) satisfying

$$\begin{aligned} 0 < w_0 - \epsilon \leq \delta_1 < w_0 < \delta_2 \leq w_0 + \epsilon, \quad g(\delta_1) = g(\delta_2) \\ \geq \max\{g(w_0 - \epsilon), g(w_0 + \epsilon)\}. \end{aligned}$$

Take  $\delta_0 := \min\{w_0 - \delta_1, \delta_2 - w_0\} > 0$ . Noting that  $[\delta_1, \delta_2]$  is an invariant set of  $g$  according to Proposition 4.2-(ii),  $[\delta_1, \delta_2]^n$  is an invariant set of  $F$ . For any  $\phi \in [w_0 - \delta_0, w_0 + \delta_0]_*^n$ , we have  $\phi \in [\delta_1, \delta_2]_*^n$ . It follows from Lemma 3.2 that  $x_t(\phi) \in [\delta_1, \delta_2]_*^n \subset [w_0 - \epsilon, w_0 + \epsilon]_*^n, t \geq 0$ , implying that  $W_*$  is stable.

Case (ii)-2:  $w_0 > \eta_1$ . By Proposition 4.2-(iii),  $[I_{(g)}]^n \subset [a_0, b_0]^n$ . It follows from Lemma 3.3 that

$$\lim_{t \rightarrow \infty} \text{dist}(x(t, \phi), \mathbb{I}_{(F)}) = \text{dist}(x(t, \phi), [a_0, b_0]^n) = 0, \quad \phi \in C_+^\circ,$$

which implies that  $[a_0, b_0]_*^n$  is a globally attracting set in  $C_+^\circ$ . Since  $[a_0, b_0]$  is invariant for  $g$ ,

$$F([a_0, b_0]^n) \subset F([a_0, b_0]^n) = [g([a_0, b_0])]^n \subset [a_0, b_0]^n,$$

It follows from Lemma 3.2 that  $x_t(\phi) \subset [a_0, b_0]_*^n, t \geq 0$  whenever  $\phi \in [a_0, b_0]_*^n$ . So,  $[a_0, b_0]_*^n$  is also an invariant set of (4.4). The proof is completed.  $\square$

*Remark 4.2* For the persistent case (ii) in Theorem 4.2, we have actually obtained estimates for the eventual bound of solutions; particularly the lower bound serves as an estimate of the persistence level. The method of finding estimates may also be applied when other types of birth functions are adopted.

### 5 Extinction and persistence when culling immatures only

In this section, we consider the scenario of culling immatures only. This is represented by the conditions  $c_k \in (0, 1)$  but  $C_k = 0, k \in N$ , which reduce (2.8) to

$$\dot{x}(t) = -(D + A)x(t) + \alpha\gamma(t)Sb(x(t - r)). \tag{5.1}$$

This is a *non-autonomous* system of DDEs, as  $\gamma(t)$  given in (2.7) is time dependent. Clearly,  $\gamma \in PC[R_+, (0, 1]]$  and  $\gamma^\infty := \limsup_{t \rightarrow \infty} \gamma(t)$ ,  $\gamma_\infty := \liminf_{t \rightarrow \infty} \gamma(t)$  exist. The following proposition establishes more information about  $r(t)$ , and its proof is given in ‘‘Appendix B’’.

**Proposition 5.1** *The following statement about  $\gamma(t)$  hold.*

- (i) *Assume that there are  $0 < \tau_1 < \tau_2$  and  $0 \leq \sigma_1 < \sigma_2$  such that  $\tau_1 \leq t_k - t_{k-1} \leq \tau_2$  and  $0 \leq \sigma_1 \leq c_k(\cdot) \leq \sigma_2 < 1$ . Then  $(1 - \sigma_2)^{r/\tau_1+1} \leq \gamma_\infty \leq \gamma^\infty \leq (1 - \sigma_1)^{r/\tau_2-1}$ .*
- (ii) *If  $c_k(\cdot) \equiv c(\text{constant})$  and the impulsive moments have equidistance, that is  $t_{k+1} = t_k + T, k \in N$ , then the mean value of  $\gamma(t), t \in [0, T]$*

$$\bar{\gamma} = \begin{cases} 1 - \frac{r}{T}c, & T > r, \\ 1 - c, & T = r, \\ (1 - c)^{\lfloor \frac{r}{T} \rfloor + 1} (\frac{r}{T} - \lfloor \frac{r}{T} \rfloor) + (1 - c)^{\lfloor \frac{r}{T} \rfloor} (1 - \frac{r}{T} + \lfloor \frac{r}{T} \rfloor), & T < r, \end{cases}$$

where, for any real number  $\eta$ ,  $\lfloor \eta \rfloor$  denotes the maximal integer that is no larger than  $\eta$ .

In reality, *periodic culling* is often a common practice. For (5.1), this can be achieved by assuming that

$$t_{k+q} = t_k + T, c_{k+q}(a) = c_k(a), \quad \forall a \in [0, r], \quad \forall k \in N, \tag{5.2}$$

where constants  $T > 0$  and  $q \in N$ . Indeed, condition (5.2) implies that  $\gamma(t)$  is a  $T$ -periodic function. To see this, we assume that there are  $m + 1$  impulsive moments  $t_{s_1}, t_{s_1+1}, \dots, t_{s_1+m}$  in  $(t - r, t]$ , and  $m + 1$  impulsive moments  $t_{s_1+q}, t_{s_1+1+q}, \dots, t_{s_1+m+q}$  in  $(t + T - r, t + T]$ . Then,

$$\begin{aligned} \gamma(t + T) &= \prod_{t+T-r < t_k \leq t+T} [1 - c_k(r - (t + T - t_k))] \\ &= \prod_{j=0}^m [1 - c_{t_{s_1+q+j}}(r - (t + T - t_{s_1+q+j}))] \\ &= \prod_{j=0}^m [1 - c_{t_{s_1+j}}(r - (t + T - t_{s_1+j} - T))] = \gamma(t). \end{aligned}$$

Especially, when  $T = r$  and  $c_k(\cdot) = c$ , we have  $\gamma(t) \equiv (1 - c)^q$ , and (5.1) becomes autonomous.

For (5.1) with (5.2), we shall show that there is a threshold between extinction and persistence. We will achieve this by employing the results on basic reproduction ratio for a class of periodic delay systems recently obtained in Zhao (2017).

Linearizing (5.1) at its trivial equilibrium, we obtain a  $T$ -periodic linear delay system

$$\dot{u}(t) = -(D + L)u(t) + \alpha\gamma(t)SB'(0)u(t - r). \tag{5.3}$$



As in Zhao (2017), we let  $X = C_+$ ,  $X_0 = C_+^o$ ,  $\partial X_0 = X \setminus X_0$ , and  $C_T$  be the ordered Banach space of all continuous and  $T$ -periodic functions from  $R$  to  $R^n$ . Setting

$$F(t)\phi = \alpha\gamma(t)SB'(0)\phi(-r), \phi \in X, \quad V(t) \equiv D + L,$$

we easily see that  $F$  and  $V$  satisfy assumptions (H1) and (H2) given in Zhao (2017) if  $L$  is irreducible. Following the procedure in Zhao (2017), we define a linear operators on  $C_T$  by

$$[\mathcal{L}\phi](t) = \int_0^\infty e^{-(D+L)s} \alpha\gamma(t-s)SB'(0)\phi(t-s-\tau)ds, t \in R, \phi \in C_T. \quad (5.4)$$

Furthermore, we define the Poincaré map  $\mathcal{U} : X \rightarrow X$  associated with (5.3) by  $\mathcal{U}(\phi) = u_T(\phi)$ , where  $u_T(\phi) = u(T+s, \phi)$  and  $u(\cdot, \phi)$  is the unique solution of (5.3) satisfying  $u_0 = \phi \in X$ . Denote the spectral radius of  $\mathcal{U}$  by  $\rho(\mathcal{U})$ . By a similar proof to that for Xu and Zhao (2005, Proposition 2.1), we then have

**Proposition 5.2** *Assume that  $L$  is irreducible and the periodic culling (5.2) is adopted. Then  $\rho(\mathcal{U})$  is a positive eigenvalue of  $\mathcal{U}$ , and there is a  $T$ -periodic positive function  $v(t)$  such that  $v(t)e^{\frac{\ln \rho(\mathcal{U})}{T}t}$  is a solution of (5.3).*

Now, denote the spectral radius of the operator  $\mathcal{L}$  by  $\bar{\mathcal{R}} = \rho(\mathcal{L})$ . Combining with Proposition 5.2 and Theorem 2.1 in Zhao (2017), then we have the threshold dynamics for (5.1) as follows.

**Theorem 5.1** *Assume that  $L$  is irreducible and the periodic culling (5.2) is adopted.*

- (i) *If  $\bar{\mathcal{R}} < 1$ , then the trivial equilibrium  $0_*$  of (5.1) is globally asymptotically stable in  $C_+$ .*
- (ii) *If  $\bar{\mathcal{R}} > 1$ , then the system (5.1) is uniformly persistence in  $C_+^o$  and there exists a positive  $T$ -periodic solution.*

*Proof* In the case where  $\bar{\mathcal{R}} < 1$ , it follows from Theorem 2.1 in Zhao (2017) that  $\rho(\mathcal{U}) < 1$ , where  $\mathcal{U}$  is the Poincaré map associated with (5.3). For any given  $\phi \in X$ , let  $x(t) = x(t, \phi) \geq 0_*$  be the solution of (5.1) with the initial condition  $x(s) = \phi(s)$ ,  $s \in [-r, 0]$ . From (H1),  $b(z) = B'(\xi)z \leq B'(0)z$  for  $z \in R_+^n$  where  $\xi \in (0, z)$ , and hence,

$$\dot{x}(t) \leq -(D + L)x(t) + \gamma(t)\alpha SB'(0)x(t-r). \quad (5.5)$$

According to Proposition 5.2, there is a positive  $T$ -periodic function  $v(t) = (v_1(t), v_2(t), \dots, v_n(t))$  such that  $\bar{u}(t) = e^{-\mu t}v(t)$  is a positive solution of (5.3), where  $\mu = -\frac{\ln(\rho(\mathcal{U}))}{T} > 0$ . Let  $v_{\min} = \min_i \{\min_{s \in [0, T]} v_i(s)\}$ ,  $v_{\max} = \max_i \{\max_{s \in [0, T]} v_i(s)\}$ . Note that (5.3) is cooperative and irreducible, and  $\phi(s) \leq \frac{\|\phi\|}{v_{\min}} \bar{u}(s)$ ,  $s \in [-r, 0]$ . By the comparison theorem, we have  $0 \leq x(t) \leq \frac{\|\phi\|}{v_{\min}} \bar{u}(t) \leq \frac{v_{\max}}{v_{\min}} \|\phi\| e^{-\mu t} e_n$ ,  $t \geq 0$ . This implies that the origin  $x = 0$  is globally asymptotically (exponentially) stable when  $\bar{\mathcal{R}} < 1$ .

In the case where  $\bar{R} > 1$ , it follows from Theorem 2.1 in Zhao (2017) that  $\rho(\mathcal{U}) > 1$ . Let  $x(t, \phi)$  be the unique solution of the  $T$ -periodic system (5.1) with  $\phi \in C_+$ . Then  $x(t, \phi)$  satisfies

$$x(t, \phi) = e^{-(D+L)t} \phi(0) + \int_0^t e^{-(D+L)(t-s)} \alpha \gamma(s) S b(x(s-r)) ds.$$

Let  $\Phi(t)$  be the solution semi-flow on  $X$  generated by (5.1) which is  $T$ -periodic, i.e.,  $\Phi(t)\phi = x_t(\phi)$  for  $\phi \in X$ . The Poincaré map corresponding to  $\Phi(t) : X \rightarrow X$  is then given by  $\mathcal{P}\phi = \Phi(T)\phi$ . Thus,  $\mathcal{P}^m(\phi) = \Phi(mT)$ ,  $m \in \mathbb{N}$ . Note that  $\gamma(t) \in (0, 1]$  is  $T$ -periodic,  $\mathcal{P}$  is eventually uniformly bounded (point dissipative, by Theorem 3.1 with  $C_k = 0$ ). Clearly,  $X_0$  is open in  $X$  and forward invariant under  $\Phi$ , and the boundary  $\partial X_0$  consists of functions  $\phi \in X$  with  $\phi_i(0) = 0$  for at least  $i \in \{1, 2, \dots, n\}$ . Next, we verify the following two claims.

*Claim 1* Let  $M_\partial := \{\phi \in \partial X_0 : \mathcal{P}^j(\phi) \in \partial X_0, \forall j \geq 0\}$ , then  $M_\partial = \{0_*\}$ . It suffices to show that  $\phi \equiv 0_*$  if  $\phi \in M_\partial$ . Assume for the sake of contradiction that  $\phi \in M_\partial$  but  $\phi \not\equiv 0_*$ . Let  $x(t) := x(t, \phi)$  and consider the scalar function  $\tilde{x}(t) := x_1(t) + x_2(t) + \dots + x_n(t)$ . Then,  $\dot{\tilde{x}}(t) \geq -d_M \tilde{x}(t) + f(t), t \geq 0$ , where  $f(t) := \alpha \gamma(t) \sum_{j=1}^n b_j(x_j(t-r))$  and  $d_M := \max_{1 \leq j \leq n} \{d_j\}$ . Noting that  $\gamma(t) > 0, \alpha > 0$  and  $\phi \not\equiv 0$ , we have  $f(s_0) > 0$  for some  $s_0 \in [0, r]$ . Thus,  $\tilde{x}(r) \geq e^{-d_M r} \tilde{x}(0) + \int_0^r e^{-d_M(r-s)} f(s) ds > 0$ . Furthermore,

$$\tilde{x}(t) \geq e^{-d_M(t-r)} \tilde{x}(r) + \int_r^t e^{-d_M(t-s)} f(s) ds > 0, \quad t \geq r. \tag{5.6}$$

However, we can show  $x(mT) = 0$  for all  $m = 1, 2, \dots$ . If not so, there is an integer  $k \in \{1, 2, \dots, n\}$  and  $m_0$  such that  $x_k(m_0T) > 0$  and thus, the set  $S_2 = \{i : x_i(m_0T, \phi) > 0\}$  is non-empty. On the other hand, by  $x(mT) = [\mathcal{P}^m \phi](0)$  and the fact that  $\phi \in M_\partial$ , we know that the set  $S_1 = \{i : x_i(m_0T, \phi) = 0\}$  is also non-empty. Obviously the  $\{1, 2, \dots, n\} = S_1 \cup S_2$ . Now for any  $j \in S_1$ , there is an  $l \in S_2$  such that  $p_{jl} > 0$  since the matrix  $L$  is irreducible. From the  $j$ th equation in (2.6) with  $C_i = 0$  for  $i = 1, \dots, n$ , we have  $\frac{dx_j(t)}{dt} \geq -[d_j + \sum_{i \neq j} p_{ji}]x_j(t) + p_{lj}x_l(t)$ , which leads to  $x_j(t) \geq \int_{m_0T}^t \exp(-[d_j + \sum_{i \neq j} p_{ji}](t-s)) p_{lj}x_l(s) ds$ , for  $t \geq m_0T$ . This implies that there exists an  $\epsilon_1 > 0$  such that  $x_j(t, \phi) > 0$  for any  $j \in S_1$  and  $t \in (m_0T, m_0T + \epsilon_1)$ . Noting that  $x_i(m_0T, \phi) > 0, i \in S_2$ , we can find an  $\epsilon_2 > 0$  such that  $x_i(t, \phi) > 0$  for any  $i \in S_2, t \in (m_0T, m_0T + \epsilon_2)$ . This implies that  $x(t, \phi) > 0$  for  $t \in (m_0T, m_0T + \epsilon)$  where  $\epsilon = \min\{\epsilon_1, \epsilon_2\}$ , a contradict to  $\phi \in M_\partial$ . Thus,  $x(kT) = [\mathcal{P}^j \phi](0) \equiv 0$  for all  $k = 1, 2, \dots$ , which contradicts (5.6). Therefore, we have  $\phi \equiv 0_*$  and Claim 1 holds.

Let  $\eta = (\eta_1, \eta_2, \dots, \eta_n)^T$  and  $\eta_0 := \min_{1 \leq i \leq n} \eta_i$ , and denote by  $\mathcal{U}_\theta$  the Poincaré map of the following linear periodic system

$$\dot{y}(t) = -(D + L)y(t) + \gamma(t)\alpha S B'(\theta e_n)y(t-r) \tag{5.7}$$

which is a perturbation of (5.3). Since  $\rho(\mathcal{U}_\theta) \rightarrow \rho(\mathcal{U}) > 1$  as  $\theta \rightarrow 0$ , we can choose a small enough  $\theta_0 \in (0, \eta_0)$  such that  $\rho(\mathcal{U}_{\theta_0}) > 1$ . Note that  $\lim_{\|\phi\| \rightarrow 0} \|x_t(\phi)\| = 0$

holds uniformly for  $t \in [0, T]$ . Thus, there is  $\delta \in (0, \eta_0)$  such that  $\|x_t(\phi)\| < \theta_0$  for all  $t \in [0, T]$  whenever  $\|\phi\| < \delta$ . Next, we shall show that

*Claim 2* for any given  $\phi \in X_0$ ,

$$\limsup_{n \rightarrow \infty} \|\mathcal{P}^n \phi\| = \limsup_{n \rightarrow \infty} \|\Phi(nT)\phi\| \geq \delta. \tag{5.8}$$

If this is not true, then there are  $\phi \in X_0$  and  $K > 1$  such that  $\|\Phi(nT)\phi\| < \delta$  for all  $n \geq K$ . Now for any  $t \geq KT$ , there are integer  $m \geq K$  and  $\hat{t} \in [0, T]$  such that  $t = mk + \hat{t}$ . Thus,  $\|\Phi(t)\phi\| = \|\Phi(\hat{t})\Phi(mT)\phi\| < \theta_0$  for all  $t \geq KT$ , implying that  $x(t) := x(t, \phi) < \theta_0 e_n, t \geq KT$  and  $x(t - r) = x(t - r, \phi) < \theta_0 e_n, t \geq KT + r$ . By the fact that  $\theta_0 e_n \leq \eta$  and the monotonicity of  $b'_i(\cdot)$  in  $[0, \eta_i], i = 1, 2, \dots, n$ , we then have  $b(x(t - r)) = b'(\xi)x(t - r) \geq B'(\theta_0 e_n)x(t - r), t \geq KT + r$ , where  $\xi \in (0, \theta_0 e_n)$ . Accordingly,

$$\dot{x}(t) \geq -(D + L)x(t) + \gamma(t)\alpha SB'(\theta_0 e_n)x(t - r), \quad t \geq KT + r,$$

which suggests the following cooperative linear system as an comparison system for (5.1)

$$\dot{y}(t) = -(D + L)y(t) + \gamma(t)\alpha SB'(\theta_0 e_n)y(t - r). \tag{5.9}$$

Applying Proposition 5.2 to (5.9) with  $B'(0)$  replaced by  $B'(\theta_0 e_n)$  in (5.3), we know that there is a positive  $T$ -periodic function  $\hat{v}(t)$  such that  $\hat{u}(t) = e^{vt} \hat{v}(t)$  is a positive solution of (5.9), where  $v = \frac{\ln(\rho(\mathcal{L}_{\theta_0}))}{T} > 0$ . For  $\phi \in X_0$ , we can choose a small enough  $\kappa > 0$  such that  $y(t, \phi) \geq \kappa \hat{u}(t), t \in [KT, KT + r]$ . By the comparison theorem, we have  $x(t, \phi) \geq y(t, \phi) \geq \kappa e^{vt} \hat{v}(t), t \geq KT + r$ . This leads to  $x(t) \rightarrow \infty$  as  $t \rightarrow \infty$ , a contradiction to the boundedness of solution  $x(t)$ . Thus, the claim (5.8) holds.

From the above claims, it follows that  $\{0_*\}$  is one (and the only one) isolated invariant set in  $\partial X_0$ , and  $M_\partial := \{\phi \in \partial X_0 : \mathcal{P}^n(\phi) \in \partial X_0\} = \{0_*\}$  and  $W^s(0_*) \cap X_0 = \emptyset$ . By the acyclicity theorem (Zhao 2003, Theorem 1.3.1),  $\mathcal{P}$  is uniformly persistent with respect to  $X_0$ . Moreover, by Zhao (2003, Theorem 3.1.1), the  $T$ -periodic solution semiflow  $\Phi(t)$  is uniformly persistent with respect to  $X_0 = C_+^0$ . Further more, by Theorem 1.3.6 in Zhao (2003), system (5.1) has at least one positive  $T$ -periodic solution in  $X_0 = C_+^0$ , and the proof is completed.  $\square$

*Remark 5.1* Generally, it is very difficult (if not impossible) to explicitly obtain the spectral radius of a linear operator associated with periodic linear delayed system, and even numeric computation is non trivial at all. According to Corollary 2.1 in Zhao (2017) and the properties of the spectral radius, we can, however, establish the following estimation for  $\bar{\mathcal{R}}$

$$\rho(\alpha\gamma_\infty SB'(0)(D + L)^{-1}) \leq \bar{\mathcal{R}} = \rho(\mathcal{L}) \leq \rho(\alpha\gamma^\infty SB'(0)(D + L)^{-1}).$$

Especially, when  $\gamma(\cdot) = 1$  (i.e.,  $c_{ik}(\cdot) = 0$ ), system (5.1) reduces to (4.1). Hence, Theorem 4.1 can be viewed as a special case of Theorem 5.1.

*Remark 5.2* If the periodic culling strategy (5.2) is adopted with  $T = r$ ,  $c_k(\cdot) = c$ , we have  $\gamma(t) \equiv (1 - c)^q$ , where  $q$  is the number of culling moments within a period  $[0, T)$ . In this case, (5.1) also reduces to an autonomous delay system of the form (4.1) with the parameter  $\alpha$  replaced by  $\tilde{\alpha} := (1 - c)^q \alpha$ . Thus, all results in Sect. 4 hold for this special case of (5.1), with  $\tilde{\mathcal{R}}$  replaced by  $\tilde{\mathcal{R}} := d_1^{-1} \tilde{\alpha} b_1'(0) = (1 - c)^q d_1^{-1} \alpha b_1'(0)$ ,  $g(s)$  replaced by  $\tilde{g}(s) := d_1^{-1} \tilde{\alpha} b_1(s) = (1 - c)^q g(s)$ ,  $b_0$  replaced by  $\tilde{b}_0 := \tilde{g}(\eta_1)$  and  $a_0$  replaced by  $\tilde{a}_0 = \tilde{g}(\tilde{b}_0)$ . In such a scenario, we can easily see the impact of the culling strength  $c$  and the culling frequency  $q$  by looking at the effect of the constant multiple  $(1 - c)^q$ . We point out that although both  $g(s)$  and  $\tilde{g}(s)$  attain their maxima at the same  $\eta_1$  and share the inflection point, their respective unique positive fixed points are different, with that of  $\tilde{g}(s)$  smaller than that of  $g(s)$ . Also, for the conclusion in Theorem 4.2 (ii)-2), the attracting invariant interval  $[a_0, b_0]_*^n$  has been pushed toward the left by the factor  $(1 - c)^q$  to  $[\tilde{a}_0, \tilde{b}_0]_*^n$  which is closer to  $0_*$ . In particular, sufficiently large culling strength  $c \in (0, 1)$  and sufficiently large culling frequency  $q$  can always bring  $\tilde{\mathcal{R}} = (1 - c)^q \mathcal{R}$  to a value that is less than 1, and hence, by Theorem 4.2, the species can be eventually eradicated.

The following theorem gives an *explicit sufficient condition* for the extinction of the species represented by (5.1), meaning success of this general culling strategy (culling immatures only).

**Theorem 5.2** *Assume that*

$$\alpha b'_{\max} \limsup_{t \rightarrow \infty} \int_0^t e^{-\hat{d}(t-s)} \gamma(s) ds < 1, \tag{5.10}$$

where  $\hat{d} = \min_{1 \leq i \leq n} d_i$ ,  $b'_{\max} = \max_{1 \leq i \leq n} b'_i(0)$ . Then the equilibrium  $0_*$  of (5.1) is globally attractive in  $C_+$ . Particularly, if the culling is  $T$ -periodic [i.e., (5.2) holds], then the conclusion holds provided that (5.10) is replaced by

$$(1 - e^{-\hat{d}T})^{-1} T \bar{\gamma} \alpha b'_{\max} < 1, \quad \text{where } \bar{\gamma} := \frac{1}{T} \int_0^T \gamma(s) ds. \tag{5.11}$$

*Proof* The supremum limit on the left side of (5.10) is well-defined since

$$\int_0^t e^{-\hat{d}(t-s)} \gamma(s) ds \leq \int_0^t e^{-\hat{d}(t-s)} \leq \frac{1}{\hat{d}}.$$

For any given  $\phi \in C_+$ , let  $\tilde{x}(t) = \sum_{j=1}^n x_j(t, \phi)$ . Since  $b_i(x_i) = b'_i(\xi)x_i \leq b'_{\max} x_i$  where  $\xi \in [0, x_i]$ , we have  $\dot{\tilde{x}}(t) \leq -\hat{d}\tilde{x}(t) + \gamma(t)\alpha b'_{\max} \tilde{x}(t - r)$ . This leads to  $\tilde{x}(t) \leq e^{-\hat{d}t} \tilde{x}(0) + \int_0^t e^{-\hat{d}(t-s)} \gamma(s) \alpha b'_{\max} \tilde{x}(s - r) ds$ . By Theorem 3.1, we can assume  $\tilde{x}(t) \leq \sigma$  for all  $t \geq -r$  and let  $\hat{\sigma} := \limsup_{t \rightarrow \infty} \tilde{x}(t)$ . Thus, for any  $\epsilon > 0$ , there is an large enough  $T_0$  such that  $\tilde{x}(t) \leq \hat{\sigma} + \epsilon$  for all  $t \geq T$ . Accordingly, when  $t > T + r$ ,

$$\begin{aligned} \tilde{x}(t) &\leq e^{-\hat{\Delta}t} \tilde{x}(0) + \int_0^{T_0} e^{-\hat{\Delta}(t-s)} \gamma(s) \alpha b'_{\max} \tilde{x}(s-r) ds \\ &\quad + \alpha b'_{\max} (\hat{\sigma} + \epsilon) \int_{T_0}^t e^{-\hat{\Delta}(t-s)} \gamma(s) ds \\ &\leq e^{-\hat{\Delta}t} \tilde{x}(0) + \alpha b'_{\max} \sigma \int_0^{T_0} e^{-\hat{\Delta}(t-s)} ds + \alpha b'_{\max} (\hat{\sigma} + \epsilon) \int_0^t e^{-\hat{\Delta}(t-s)} \gamma(s) ds \\ &\leq e^{-\hat{\Delta}t} \tilde{x}(0) + \alpha b'_{\max} \sigma \hat{\Delta}^{-1} (e^{\hat{\Delta}T_0} - 1) e^{-\hat{\Delta}t} + \alpha b'_{\max} (\hat{\sigma} + \epsilon) \int_0^t e^{-\hat{\Delta}(t-s)} \gamma(s) ds. \end{aligned}$$

Taking the supremum limit in the above gives  $\hat{\sigma} = \limsup_{t \rightarrow \infty} \tilde{x}(t) \leq \alpha b'_{\max} (\hat{\sigma} + \epsilon) \limsup_{t \rightarrow \infty} \int_0^t e^{-\hat{\Delta}(t-s)} \gamma(s) ds$ . Letting  $\epsilon \rightarrow 0^+$ , we then obtain  $\hat{\sigma} \leq \hat{\sigma} \alpha b'_{\max} \limsup_{t \rightarrow \infty} \int_0^t e^{-\hat{\Delta}(t-s)} \gamma(s) ds$ . By (5.10), we conclude that  $\hat{\sigma} = 0$  and so  $\lim_{t \rightarrow \infty} \tilde{x}(t) = 0$ .

When  $\gamma$  is  $T$ -periodic, then we find  $k \in N$  such that  $(k - 1)T \leq t < kT$  and

$$\int_0^t e^{-\hat{\Delta}(t-s)} \gamma(s) ds \leq \sum_{j=1}^k \int_{(j-1)T}^{jT} e^{-\hat{\Delta}(t-s)} \gamma(s) ds \leq \sum_{j=1}^k e^{-\hat{\Delta}(j-1)T} \int_0^T \gamma(s) ds$$

Here we have used the identity  $\int_0^T \gamma(t-s) ds = \int_0^T \gamma(s) ds$  since  $\gamma(s)$  is  $T$ -periodic. The above inequality together with (5.11) leads to  $\limsup_{t \rightarrow \infty} \int_0^t e^{-\hat{\Delta}(t-s)} \gamma(s) ds \leq (1 - e^{-\hat{\Delta}T})^{-1} T \bar{\gamma} < (\alpha b'_{\max})^{-1}$ , which implies (5.10) holds, and thus, the conclusion for the case of periodic culling holds. The proof is completed.  $\square$

### 6 Extinction when culling matures only

In this section, we discuss the scenario of culling matures only, as opposed to that of culling immatures only in Sect. 5. This means that we will assume in this section that  $c_k(\cdot) = 0$  but  $C_k \neq 0$  for  $k \in N$ , implying  $\gamma(t) \equiv 1$ . In this case, the model (2.8) reduces to the following system of autonomous delay impulsive differential equations

$$\begin{cases} \frac{d}{dt} x(t) = -(D + P_0)x(t) + Px(t) + \alpha Sb(x(t-r)), & t \neq t_k, \\ x(t_k^+) = [E - C_k]x(t_k^-), & k \in N, \end{cases} \tag{6.1}$$

where  $E$  is the  $n \times n$  identity matrix. In the sequel, for a diagonal matrix  $A = \text{diag}\{a_1, a_2, \dots, a_n\}$  with  $a_i > 0, i = 1, 2, \dots, n$ , we use  $\ln A$  to denote  $\text{diag}\{\ln a_1, \ln a_2, \dots, \ln a_n\}$ .

The following result is on the stability/instability of the extinction equilibrium  $0_*$  for (6.1).

**Theorem 6.1** *Assume that  $L$  is irreducible.*

(i) *If there are a positive number  $\tau_1$  and a diagonal matrices  $\Upsilon_1$  with*

$$t_k - t_{k-1} \leq \tau_1 \text{ and } 0 \leq \Upsilon_1 \leq C_k \text{ for all } k \in N. \tag{6.2}$$

such that  $\rho((D + P_0 - \frac{\ln(E-\Upsilon_1)}{\tau_1})(E - \Upsilon_1)]^{-1}(P + \alpha SB'(0))) < 1$ , then the equilibrium  $0_*$  of (6.1) is globally exponentially stable in  $PC_+$ .

(i) If there are a positive number  $\tau_2$  and a diagonal matrices  $\Upsilon_2$  with

$$0 < \tau_2 \leq t_k - t_{k-1} \text{ and } C_k \leq \Upsilon_2 < E \text{ for all } k \in N. \tag{6.3}$$

such that  $\rho((D + P_0 - \frac{\ln(E-\Upsilon_2)}{\tau_2})^{-1}(E - \Upsilon_2)(P + \alpha SB'(0))) > 1$ , then the equilibrium  $0_*$  is unstable.

*Proof* It can be verified that the following variation of parameter formula is valid:

$$x(t) = K(t, 0)x(0) + \int_0^t K(t, s)(Px(s) + \alpha Sb(x(s - r)))ds, \quad t \geq 0, \tag{6.4}$$

where  $K(t, s), t, s \geq 0$  is Cauchy matrix of linear impulsive system

$$\begin{cases} \dot{x}(t) = -(D + P_0)x(t), & t \geq 0, \\ x(t_k^+) = (E - C_k)x(t_k^-), & k \in N. \end{cases} \tag{6.5}$$

According to the representation of Cauchy matrix (see Lakshmikantham et al. 1989, p. 74), we have

$$K(t, s) = e^{-(D+P_0)(t-s)} \prod_{s < t_k \leq t} (E - C_k), \tag{6.6}$$

which, together with (6.2), leads to  $K(t, s) \leq e^{-(D+P_0)(t-s)}(E - \Upsilon_1)^{\frac{t-s}{\tau_1}-1} \leq e^{-(D+P_0-\frac{\ln(E-\Upsilon_1)}{\tau_1})(t-s)}(E - \Upsilon_1)^{-1}, t \geq s \geq 0$ . Letting  $\bar{D} = D + P_0 - \frac{\ln(E-\Upsilon_1)}{\tau_1}$ , we then have for  $t \geq 0$

$$\begin{aligned} x(t) &\leq e^{-\bar{D}t}(E - \Upsilon_1)^{-1}x(0) + \int_0^t e^{-\bar{D}(t-s)}(E - \Upsilon_1)^{-1}(Px(s) \\ &\quad + \alpha SB'(0)x(s - r))ds. \end{aligned} \tag{6.7}$$

By  $\rho([\bar{D}(E - \Upsilon_1)]^{-1}(P + \alpha SB'(0))) < 1$  and Lemma 3.1, we know that  $\bar{D}(E - \Upsilon_1) - (P + \alpha SB'(0))$  is a non-singular matrix and so is  $\bar{D} - (E - \Upsilon_1)^{-1}(P + \alpha SB'(0))$ . Thus, there exists a vector  $Z = (z_1, z_2, \dots, z_n)^T > 0$  satisfying  $[\bar{D} - (E - \Upsilon_1)^{-1}(P + \alpha SB'(0))]Z > 0$ . By continuity, for small  $\lambda > 0$ , we have  $[\bar{D} - (E - \Upsilon_1)^{-1}P - e^{\lambda\tau}(E - \Upsilon_1)^{-1}\alpha SB'(0) - \lambda E]Z > 0$ , meaning that

$$(E - \Upsilon_1)^{-1}[P + e^{\lambda\tau}\alpha SB'(0)]Z < (\bar{D} - \lambda E)Z. \tag{6.8}$$

Set

$$\kappa = \frac{\max_{1 \leq i \leq n} \{(1 - \Upsilon_1^{(i)})^{-1}\} \|\phi\|}{\min_{1 \leq i \leq n} \{z_i\}},$$

where  $\Upsilon_1 = \text{diag}\{\Upsilon_1^{(1)}, \Upsilon_1^{(2)}, \dots, \Upsilon_1^{(n)}\}$ . From  $0 \leq \Upsilon_1 < E$ , it follows that  $(E - \Upsilon_1)x(t) \leq Ex(t) = x(t) = \phi(t)$  for  $t \in [-r, 0]$ . Thus,

$$x(t) \leq (E - \Upsilon_1)^{-1}\phi(t) \leq \kappa Z \leq \kappa Ze^{-\lambda t}, \quad -r \leq t \leq 0. \tag{6.9}$$

In the following, we shall prove that

$$x(t) < \kappa Ze^{-\lambda t}, \quad t \geq 0. \tag{6.10}$$

If this is not true, by the estimate (6.9) and  $x(t) \in PC[[0, \infty), R_+^n]$ , there must be an integer  $l$  and  $t^* > 0$  satisfying

$$x_l(t^*) \geq E_l(\kappa Ze^{-\lambda t^*}), \quad x(t) < \kappa Ze^{-\lambda t}, \quad t < t^*, \tag{6.11}$$

where the  $n$ -dimension row vector  $E_l = \overbrace{(0, \dots, 0, 1, 0, \dots, 0)}$ .

The latter inequality implies

$$x(s - r) \leq \kappa Ze^{-\lambda(s-r)}, \quad 0 \leq s < t^*. \tag{6.12}$$

By (6.7), (6.8), (6.9) and (6.12), we have

$$\begin{aligned} x_l(t^*) &\leq E_l \left( e^{-\bar{D}t^*} \kappa Z + \int_0^{t^*} e^{-\bar{D}(t^*-s)} (E - \Upsilon_1)^{-1} [PkZe^{-\lambda s} + e^{\lambda\tau} \alpha SB'(0) \kappa Ze^{-\lambda(s-r)}] ds \right) \\ &= E_l \left( e^{-\bar{D}t^*} \kappa Z + \int_0^{t^*} \kappa e^{-\bar{D}t^*} e^{(\bar{D}-\lambda I)s} (E - \Upsilon_1)^{-1} [P + \alpha SB'(0)e^{\lambda\tau}] Z ds \right) \\ &< E_l \left( e^{-\bar{D}t^*} \kappa Z + \int_0^{t^*} \kappa e^{-\bar{D}t^*} e^{(\bar{D}-\lambda E)s} (\bar{D} - \lambda E) Z ds \right) \\ &= E_l \left( e^{-\bar{D}t^*} \kappa Z + \kappa e^{-\bar{D}t^*} [e^{(\bar{D}-\lambda E)t^*} - E] Z \right) = E_l \left( \kappa e^{-\lambda t^*} Z \right). \end{aligned}$$

This contradicts (6.11), and hence, (6.10) holds.

Let

$$M = \frac{\max_{1 \leq i \leq n} \{(1 - \Upsilon_1^{(i)})^{-1}\}}{\min_{1 \leq i \leq n} \{z_i\}} Z.$$

Then, there holds  $x(t) \leq Me^{-\lambda t} \|\phi\|, t \geq 0$ , proving (i).

For (ii), first we note that the local stability of the zero solution of (6.1) is equivalent to that of its linearized impulsive delayed system:

$$\begin{cases} \dot{x}(t) = -(D + P_0)x(t) + Px(t) + \alpha SB'(0)x(t - r), & t \neq t_k, \\ x(t_k^+) = [E - C_k]x(t_k^-), & k \in N. \end{cases} \tag{6.13}$$

The solution of (6.13) can be expressed as  $x(t) = K(t, 0)x(0) + \int_0^t K(t, s)(Px(s) + \alpha SB'(0)x(s - r))ds, t \geq 0$ , where  $K(t, s)$  is given in (6.6), by which and (6.3), we further obtain

$$K(t, s) \geq e^{-(D+P_0)(t-s)}(E - \Upsilon_2)^{\frac{t-s}{\tau_2}+1} \geq e^{-(D+P_0-\frac{\ln(E-\Upsilon_2)}{\tau_2})(t-s)}(E - \Upsilon_2), \quad t \geq s \geq 0.$$

Thus, for  $t \geq 0$ ,

$$\begin{aligned} x(t) &\geq e^{-(D+P_0-\frac{\ln(E-\Upsilon_2)}{\tau_2})t}(E - \Upsilon_2)x(0) \\ &\quad + \int_0^t e^{-(D+P_0-\frac{\ln(E-\Upsilon_2)}{\tau_2})(t-s)}(E - \Upsilon_2)(Px(s) + \alpha SB'(0)x(s-r))ds. \end{aligned} \tag{6.14}$$

Observe that the right-hand side of (6.14) is the solution of

$$\dot{y}(t) = -\left(D + P_0 - \frac{\ln(E - \Upsilon_2)}{\tau_2}\right)y(t) + (E - \Upsilon_2)(Py(t) + \alpha SB'(0)y(t-r)), \tag{6.15}$$

with initial condition  $y(s) = x(s)$ ,  $s \in [-r, 0]$ . Noticing that (6.15) is a cooperative linear DDE system, and hence, the stability/instability of the trivial equilibrium for (6.15) is equivalent to that of the corresponding ODE system (see Smith 1995):

$$\dot{y}(t) = \left[-\left(D + P_0 - \frac{\ln(E - \Upsilon_2)}{\tau_2}\right) + (E - \Upsilon_2)(P + \alpha SB'(0))\right]y(t). \tag{6.16}$$

Now, if  $\rho((D + P_0 - \frac{\ln(E-\Upsilon_2)}{\tau_2})^{-1}(E - \Upsilon_2)(P + \alpha SB'(0))) > 1$ , the trivial equilibrium  $0_*$  is unstable for (6.16), and so is for (6.15). This, together with (6.14) and a comparison argument, implies that  $0_*$  is also unstable for (6.13), and hence for (6.1). The proof is completed.  $\square$

Combining Lemma 3.1 and Theorem 6.1, we immediately obtain the following more explicit sufficient conditions on extinction of population.

**Corollary 6.1** *Assume that impulsive culling satisfies*

$$\begin{aligned} C_{ik} &\geq \delta_i, \delta_i \in [0, 1), \quad t_k - t_{k-1} \leq \tau, \quad i = 1, 2, \dots, n, \quad k \in N, \frac{\ln(1 - \delta_i)}{\tau} \\ &< d_i + \sum_{j \neq i} p_{ji} - \frac{1}{1 - \delta_i} \sum_{j \neq i} p_{ij} \\ &\quad - \frac{\alpha}{n(1 - \delta_i)} \left[ (1 - \beta) \sum_{j \neq i} b'_j(0) + (1 - (n - 1)\beta)b'_i(0) \right]. \end{aligned} \tag{6.17}$$

Then the equilibrium  $0_*$  of (6.1) is globally exponentially stable in  $PC_+$ .

Next, we show that under some mild condition, the impulsive delayed system (6.1) may be reduced to a delayed system without impulses, but with the impulsive effects reflected in some time dependent parameters. To this end, we define



$$\begin{aligned}
 X(t) &= \prod_{0 < t_k \leq t} (E - C_k); \quad \chi(t) = \prod_{0 < t_k \leq t} (1 - \underline{\theta}_k), \\
 \Gamma(t) &= \prod_{0 < t_k \leq t} (1 - \bar{\theta}_k)^{-1} \prod_{0 < t_k \leq t-r} (1 - \underline{\theta}_k),
 \end{aligned} \tag{6.18}$$

where  $\bar{\theta}_k = \max_{1 \leq i \leq n} (C_{ik})$  and  $\underline{\theta}_k = \min_{1 \leq i \leq n} (C_{ik})$ .

**Proposition 6.1** *Assume that  $(D + L)C_k = C_k(D + L), k \in N$ . Then  $x(t)$  is the solution of (6.1) if and only if  $y(t) := X^{-1}(t)x(t)$  is the solution of the following DDE system without impulses:*

$$\dot{y}(t) = -(D + L)y(t) + \alpha X^{-1}(t)Sb(X(t - r)y(t - r)), t \geq 0. \tag{6.19}$$

The proof of this proposition is given in ‘‘Appendix B’’.

Taking advantage of Proposition 6.1, we can derive some further condition that ensures the extinction of the population.

**Theorem 6.2** *Assume that  $(D + L)C_k = C_k(D + L), k \in N$ . Let  $\Gamma(t)$  and  $\chi(t)$  be defined in (6.18), and let  $\hat{d} = \min_{1 \leq i \leq n} d_i, b'_{\max} = \max_{1 \leq i \leq n} b'_i(0)$ . If*

$$-\hat{d}t + \ln \chi(t) + \int_0^t \alpha b'_{\max}(0)\Gamma(s)\chi(s)e^{\hat{d}r} ds \rightarrow -\infty, \text{ as } t \rightarrow \infty, \tag{6.20}$$

then the equilibrium  $0_*$  of (6.1) is globally attractive in  $PC_+$ .

*Proof* Let  $x(t)$  be the solution of (6.1) with the initial  $\phi \in C_+$ . By (6.18),  $y(t) = (X(t))^{-1}x(t)$  satisfies (6.19). Let  $\tilde{x}(t) = x_1(t) + x_2(t) + \dots + x_n(t)$  and  $\tilde{y}(t) := y_1(t) + \dots + y_n(t)$ . Noticing that  $L$  is a Laplacian matrix and  $S$  is a doubly stochastic matrix, we then have the following estimate:

$$\begin{aligned}
 \dot{\tilde{y}}(t) &= e_n^T \dot{y}(t) = -e_n^T (D + L)y(t) + e_n^T \alpha X^{-1}(t)Sb(X(t - r)y(t - r)) \\
 &\leq -\sum_{j=1}^n \hat{d}y_j(t) + e_n^T \alpha X^{-1}(t)SB'(0)X(t - r)y(t - r) \\
 &\leq -\hat{d}\tilde{y}(t) + \Gamma(t)\chi(t)\alpha b'_{\max}(0)\tilde{y}(t - r),
 \end{aligned}$$

which leads to  $\tilde{y}(t) \leq e^{-\hat{d}t}n\|\phi\| + \int_0^t \alpha b'_{\max}(0)\Gamma(s)\chi(s)e^{-\hat{d}(t-s)}\tilde{y}(s - r)ds$ . By the Grownwall inequality with delay, we then obtain  $\tilde{y}(t) \leq n\|\phi\|e^{-\hat{d}t} \exp\{\int_0^t \alpha b'_{\max}(0)\Gamma(s)\chi(s)e^{\hat{d}r} ds\}$ , implying

$$\begin{aligned}
 \tilde{x}(t) &= e_n^T x = e_n^T X(t)y(t) \leq \chi(t)\tilde{y}(t) \\
 &\leq n\|\phi\|\chi(t)e^{-\hat{d}t} \exp \left\{ \int_0^t \alpha b'_{\max}(0)\Gamma(s)\chi(s)e^{\hat{d}r} ds \right\}.
 \end{aligned}$$

Therefore,  $\tilde{x}(t) \rightarrow 0$  as  $t \rightarrow \infty$ , provided that (6.20) holds. The proof is completed.

The following corollary is a direct consequence of Theorem 6.2.

**Corollary 6.2** Assume that  $C_k = \theta_k E$  (scaler matrix) for  $k \in N$  and let  $\hat{d}$  and  $b'_{\max}$  be given in Theorem 6.2. If there are constants  $\theta, \hat{\theta}, \tau, \hat{\tau}$  such that the impulsive cullings satisfy

$$0 \leq \hat{\theta} \leq \theta_k \leq \theta < 1, 0 < \hat{\tau} \leq t_k - t_{k-1} \leq \tau, k \in N, \\ -\hat{d} + \frac{\ln(1-\hat{\theta})}{\tau} + \alpha b'_{\max} e^{\hat{d}\tau} (1 - \hat{\theta})^{-\frac{\tau}{\hat{\tau}}-1} < 0, \tag{6.21}$$

then the equilibrium  $0_*$  of (6.1) is globally exponentially stable in  $PC_+$ .

The above corollary can be obtained if we note that

$$-\hat{d}t + \ln \chi(t) + \int_0^t \alpha b'_{\max}(0) \Gamma(s) e^{\hat{d}s} ds \\ \leq \left[ -\hat{d} + \frac{\ln(1-\hat{\theta})}{\tau} + \alpha b'_{\max} e^{\hat{d}\tau} (1 - \hat{\theta})^{-\frac{\tau}{\hat{\tau}}-1} \right] t - \ln(1 - \hat{\theta}).$$

### 7 Illustration examples and discussion on culling strategies

In this section, we first present two concrete examples and its simulations to illustrate the effectiveness of the results.

*Example 7.1* Consider the adult population in  $n$  identical patches with culling immature

$$\dot{x}_i(t) = -d_1 x_i - \sum_{j=1}^n L_{ij} x_j(t) + \alpha \gamma(t) \sum_{j=1}^n S_{ij} b_1(x_j(t-r)), \tag{7.1}$$

where  $d_1 > 0, r > 0, \alpha \in (0, 1], L = (L_{ij})$  and  $S = (S_{ij})$  are Laplacian matrix and stochastic matrix, respectively, and the birth function  $b_1(s) = \frac{\mu s}{v+s^m}, \mu, v > 0, m > 1$ .

Take the  $T$ -periodic culling with the culling rate  $c_k(\cdot) = c$  and times of culling  $q$  in every periodic  $T = r$ . Then  $\gamma(t) \equiv (1-c)^q$ . Denote  $\gamma := (1-c)^q, \Delta := \frac{\alpha \mu}{d_1 v}, \varrho := \frac{m}{m-1}$ . Since  $b'_1(s) = \frac{\mu v + \mu(1-m)s^m}{(v+s^m)^2}$ , we have the parameters defined in Theorem 5.2 as follows:

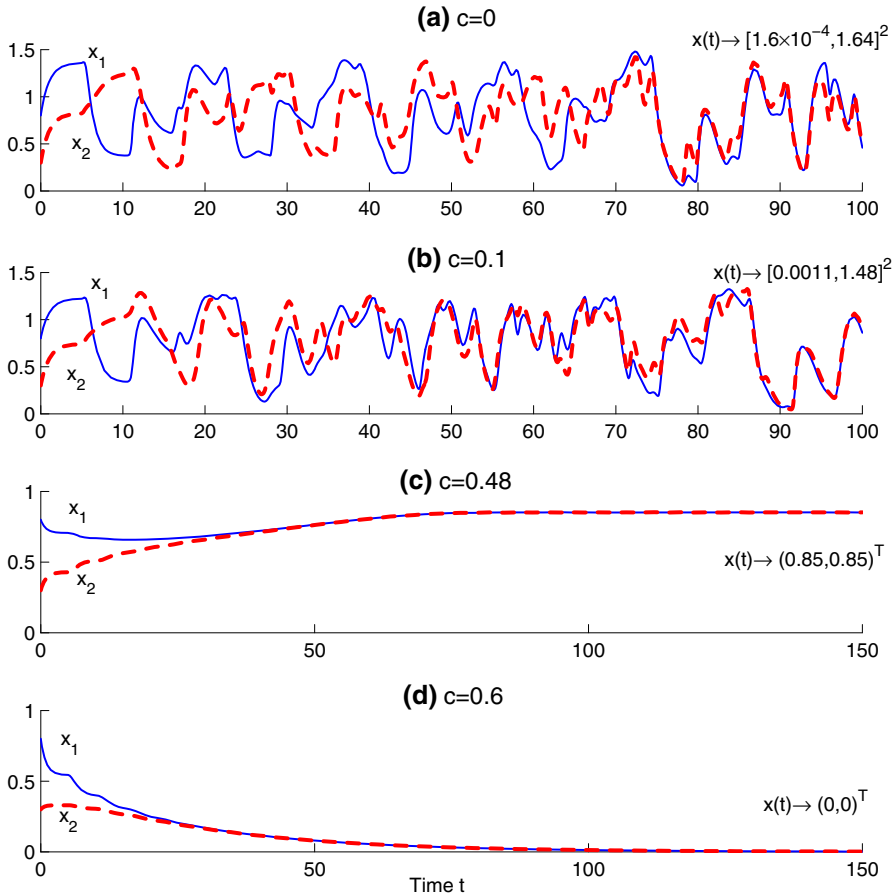
$$\tilde{R} = \gamma \Delta, \quad \eta_1 = \sqrt[m]{\frac{v}{m-1}}, \quad w_0 = \sqrt[m]{(\gamma \Delta - 1)v}, \quad b_0 = \frac{\gamma \Delta \eta_1}{\varrho}, \\ a_0 = \frac{(m-1)(\gamma \Delta)^2 \varrho^{m-1} \eta_1}{(m-1)\varrho^m + (\gamma \Delta)^m},$$

where  $\eta_1$  is the maximum point of  $b_1(s)$ , and  $w_0$  is the unique positive fixed point of  $\tilde{g}(s) := \gamma \alpha d_1^{-1} b_1(s)$  if it exists (i.e.  $\gamma \Delta > 1$ ). Letting  $W = (w_0, w_0, \dots, w_0)^T$  and  $\mathbb{I} = [a_0, b_0]^n$ , the regime of culling the immature and the corresponding dynamical behaviors can be showed in Table 1 by Theorem 5.1 and Remark 5.2.

**Table 1** Dynamical behavior of (7.1) with culling parameters  $\gamma$

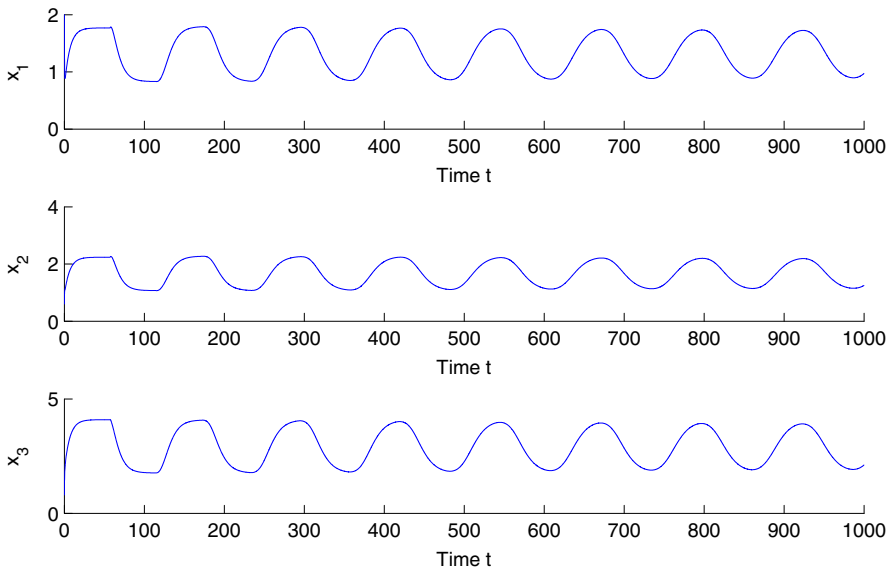
	$x(t) \rightarrow 0$	$x(t) \rightarrow W$	$x(t) \rightarrow \mathbb{I}$
(1) $\Delta \in (0, 1]$	$\gamma \in (0, 1]$	–	–
(2) $\Delta \in (1, \varrho]$	$\gamma \in (0, \frac{1}{\Delta}]$	$\gamma \in (\frac{1}{\Delta}, 1]$	–
(3) $\Delta \in (\varrho, \infty)$	$\gamma \in (0, \frac{1}{\Delta}]$	$\gamma \in (\frac{1}{\Delta}, \frac{\varrho}{\Delta}]$	$\gamma \in (\frac{\varrho}{\Delta}, 1]$

$\gamma = 1$  corresponds to no culling step



**Fig. 1** The time response of state variable  $x(t) = (x_1(t), x_2(t))^T$  of (7.1) with the culling rate: **a**  $c = 0$  (no culling); **b**  $c = 0.1$ ; **c**  $c = 0.48$ ; **d**  $c = 0.6$ ; where  $x(s) = (0.8, 0.3)^T$ ,  $s \in [-5, 0]$ . While **c** and **d** give convergent dynamics, **a** and **b** show oscillations of  $x_1(t)$  and  $x_2(t)$  between  $1.6 \times 10^{-4}$  and  $1.64$

Taking  $n = 2, r = 5, d_1 = 1, \alpha = 0.5, \mu = 4, \nu = 1, m = 20, L_{11} = -L_{12} = -L_{21} = L_{22} = 0.05, S_{11} = S_{22} = 0.8, S_{12} = S_{21} = 0.2$ , we have  $\Delta = 2 > \varrho = 20/19$ . Figure 1a shows the persistence for the system without culling (i.e.  $c = 0$ ). Let times of culling  $q = 1$ , and the culling rate  $c = 0.1, 0.48, 0.6$ , Fig. 1b–d show the corresponding dynamical properties given in Case (3) of Table 1 for the system (7.1) with culling parameters, respectively.



**Fig. 2** The time response of state variable  $x(t) = (x_1(t), x_2(t), x_3(t))^T$  of (7.2) without impulsive control, where  $x(s) = (2, 0.6, 0.8)^T, s \in [-60, 0]$

*Example 7.2* Consider the adult population in non-identical patches

$$\dot{x}_i(t) = -d_i x_i - \sum_{j=1}^n L_{ij} x_j(t) + \alpha \sum_{j=1}^n S_{ij} b_j(x_j(t - r)), \quad (7.2)$$

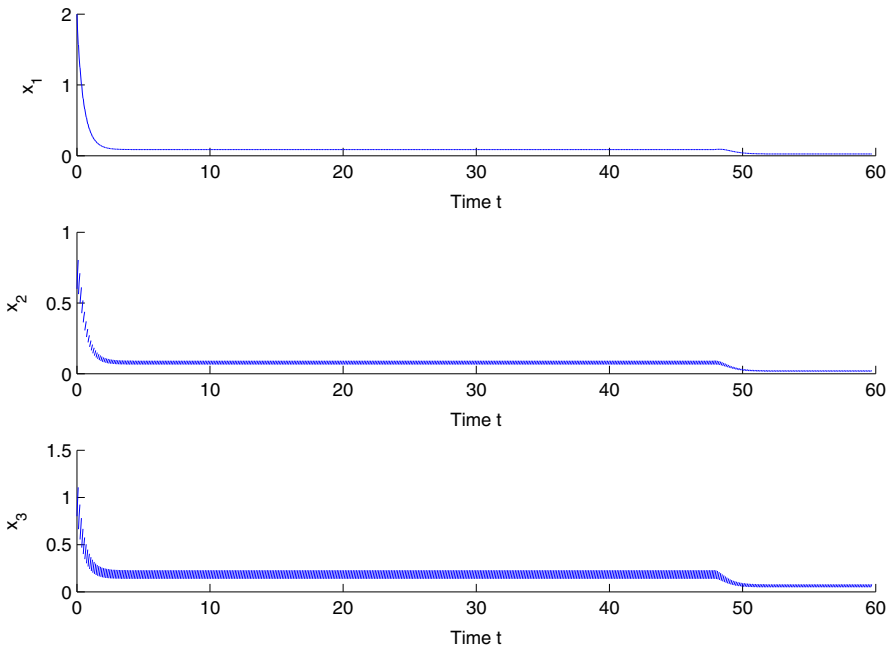
where the birth function  $b_i(s) = \mu_i s e^{-v_i s}$ ,  $\mu_i, v_i > 0$ , and take  $n = 3, \alpha = 0.6, r = 60, d_1 = 0.1, d_2 = 0.2, d_3 = 0.15, \mu_1 = 0.5, \mu_2 = 1, \mu_3 = 4, v_1 = 1, v_2 = 0.8, v_3 = 0.9, L_{11} = 3, L_{12} = -0.15, L_{13} = -0.5, L_{21} = -0.15, L_{22} = 2, L_{23} = -0.5, L_{31} = -0.15, L_{32} = -0.5, L_{33} = 1, S_{ii} = 0.4, S_{ij} = 0.3, i, j = 1, 2, 3, i \neq j$ .

Since  $b'_i(0) = \mu_i$ , we have  $R = \rho(\alpha(D + L)^{-1}B'(0)) = 9.2 > 1$  given in Theorem 4.1. Thus, System (7.2) is persistent, as is confirmed by numeric results in Fig. 2.

Our results can help us design feasible culling strategies, depending on the technologies or means available for culling immatures or matures. For example, assume there is means available for culling immatures. Then, based on the results in Sect. 5 (noting that  $\rho(s\alpha SB'(0)(D + L)^{-1})$  is monotone function of  $s$ ), we may propose the following feasible culling strategy to eradicate the species:

**Strategy 1** (culling immatures for extinction)

- Step 1) Choose a large enough integer  $K$ . compute  $\gamma_j = jh, j = 1, \dots, K - 1$  and  $R_j = \rho(\gamma_j \alpha SB'(0)(D + L)^{-1})$ , where  $h = 1/K$ .
- Step 2) Find the maximum integer  $M := \max\{j : R_j < 1\}$  and  $\gamma_M$ .



**Fig. 3** The time response of state variable  $x(t) = (x_1(t), x_2(t), x_3(t))^T$  of (7.2) with impulsive culling in patch 2 and patch 3,  $C_{2k} = 0.3, C_{3k} = 0.4, t_k - t_{k-1} = 0.12, k \in N$ , where  $x(s) = (2, 0.6, 0.8)^T, s \in [-60, 0]$

Step 3) Let impulsive culling be at  $t_k - t_{k-1} \leq \tau_2$  and choose  $c_k(\cdot) \geq c$ . Determine  $\tau_2$  and  $c$  such that Let impulsive culling be  $t_k = t_{k-1} + T$  and  $c_k(\cdot) = c$ . Determine  $T$  and  $c$  such that  $(1 - c)^{\frac{T}{\tau_2} - 1} \leq \gamma_M$ .

Similarly, if a technology for culling immatures is available, then by the results in Sect. 6, the following culling strategy will be feasible:

**Strategy 2** (culling matures for extinction)

- Step 1) For every  $i$ , computing  $\Delta_i = d_i + \sum_{j \neq i} p_{ji} - \sum_{j \neq i} p_{ij} - \frac{\alpha}{n} [(1 - \beta) \sum_{j \neq i} b'_j(0) + (1 - (n - 1)\beta)b'_i(0)]$ ;
- Step 2) Select the patches  $i \in \Omega := \{i | \Delta_i \leq 0\}$  for culling matures;
- Step 3) Take the impulsive culling rate  $C_{ik} \geq \delta_i, \delta_i \in (0, 1)$  for all  $k \in N, i \in \Omega$ ;
- Step 4) Determinate the impulsive moments  $\{t_k\}$  such that  $t_k - t_{k-1} \leq \tau := \min_{i \in \Omega} \{\bar{\Delta}_i^{-1} \ln(1 - \delta_i)\}$ , where  $\bar{\Delta}_i := d_i + \sum_{j \neq i} p_{ji} - \frac{1}{1 - \delta_i} \sum_{j \neq i} p_{ij} - \frac{\alpha}{n(1 - \delta_i)} [(1 - \beta) \sum_{j \neq i} b'_j(0) + (1 - (n - 1)\beta)b'_i(0)], i \in \Omega$ .

*Remark 7.1* In Step 2), when  $\Omega = \emptyset$ , there is no need to implement culling in any patches. In fact, this means all  $\Delta_i > 0$  for all  $i = 1, 2, \dots, n$ , which implies  $R < 1$  in Theorem 4.1, and so the equilibrium  $0_*$  of (5.1) without impulsive culling is exponentially stable.

*Remark 7.2* In Step 4),  $\tau$  is well-defined since  $\bar{\Delta}_i < 0, i \in \Omega$  from  $\Delta_i \leq 0$  and  $0 < \delta_i < 1$ , which guarantees (6.21) holds for  $i \in \Omega$ . For  $i \notin \Omega$ , (6.21) also holds because  $\Delta_i > 0$  and  $\gamma_i = 0$ , which means that no culling is taken in the patches  $i \notin \Omega$ . According to Corollary 6.1, the extinction of population is achieved by using the above steps.

To demonstrate this, let us implement Strategy 2 to (7.2). We firstly compute  $\Delta_1 = 0.08, \Delta_2 = -0.85, \Delta_3 = -2.08$ . Then, we only choose matures in patch 2 and patch 3 to cull. Taking the culling rates  $C_{2k} \geq 0.3, C_{3k} \geq 0.4$ , then we determine the impulsive moments satisfying  $t_k - t_{k-1} \leq 0.12$ . Figure 3 shows the results of this strategy, confirming the extinction of (7.2) under the given impulsive control.

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## 8 Appendices

### Appendix A

*Proof of Lemma 3.2.* Note that  $e^{-At}$  is non-negative matrix for  $t \geq 0$  since  $A$  is a non-singular  $M$ -matrix. We first show that the conclusion holds for  $t \in [0, r]$ . Noting that  $\phi \in C([-r, 0], \mathbb{I}), F(\mathbb{I}) \subset \mathbb{I}$ , then  $F(\phi(s-r)) \in \mathbb{I}$  for  $s \in [0, r]$ . Combining this with  $A^{-1}\tilde{A}e_n = e_n$  and the fact that  $e^{-As}, s \geq 0$  and  $\tilde{A}$  are non-negative matrices, we obtain

$$\begin{aligned} u(t, \phi) &= e^{-At}\phi(0) + \int_0^t e^{-A(t-s)}\tilde{A}F(\phi(s-r))ds \geq e^{-At}ae_n \\ &\quad + \int_0^t e^{-A(t-s)}ds\tilde{A}ae_n \\ &= e^{-At}ae_n + (E - e^{-At})A^{-1}\tilde{A}ae_n = ae_n, \quad t \in [0, r]. \end{aligned}$$

Similarly, we have  $u(t, \phi) \leq be_n$  for  $t \in [0, r]$ . By mathematical induction, we can deduce that  $u(t) \in \mathbb{I}$  for all  $[(n-1)r, nr], n = 1, 2, \dots$ . The proof is completed.  $\square$

*Proof of Lemma 3.3* It suffices to prove the following claims.

Claim 1. If the solution  $u(t, \phi) \in \mathbb{J}_0$  for all  $t \geq T$ , where some constant  $T \geq 0, \mathbb{J}_0 = [c_0, d_0]^n \subset R^n$ , then  $\lim_{t \rightarrow \infty} dist(u(t, \phi), F(\mathbb{J}_0)) = 0$ .

Assume that  $F(\mathbb{J}_0) = [\bar{c}_0, \bar{d}_0]^n \subset R^n$ . Since  $u(t, \phi) \in C([-r, 0], \mathbb{J}_0)$ , for  $t \geq T \geq 0, F(u(t, \phi)) \subset F(\mathbb{J}_0) \subset [\bar{c}_0, \bar{d}_0]^n$ . Furthermore,

$$\begin{aligned} u(t, \phi) &= e^{-At}\phi(0) + \int_0^t e^{-A(t-s)}\tilde{A}F(u(s-r))ds \geq e^{-At}c_0 + \int_0^t e^{-A(t-s)}ds\tilde{A}\bar{c}_0e_n \\ &= e^{-At}c_0 + (E - e^{-At})A^{-1}\tilde{A}\bar{c}_0e_n = e^{-At}c_0 + (E - e^{-At})\bar{c}_0e_n. \end{aligned}$$

Note  $e^{-At} \rightarrow 0$  as  $t \rightarrow \infty$  since  $A$  is a non-singular  $M$  matrix. Thus  $\liminf_{t \rightarrow \infty} u(t, \phi) \geq \bar{c}_0 e_n$ . Similarly, we have  $\limsup_{t \rightarrow \infty} u(t, \phi) \leq \bar{d}_0 e_n$ . Thus,  $\lim_{t \rightarrow \infty} \text{dist}(u(t, \phi), F\langle \mathbb{J}_0 \rangle) = 0$ .

Claim 2. If  $\lim_{t \rightarrow \infty} \text{dist}(u(t, \phi), \mathbb{J}) = 0$ , where  $\mathbb{J} = [c, d]^n \subset R^n$ , then  $\lim_{t \rightarrow \infty} \text{dist}(u(t, \phi), F\langle \mathbb{J} \rangle) = 0$ .

For any small enough number  $\epsilon$ , we define  $\mathbb{J}_\epsilon = [c + \epsilon, d + \epsilon]^n$ . Since  $\lim_{t \rightarrow \infty} \text{dist}(u(t, \phi), \mathbb{J}) = 0$ , there exists a  $T$  such that  $u(t, \phi) \in \mathbb{J}_\epsilon$  for  $t \geq T$ . Then  $\lim_{t \rightarrow \infty} \text{dist}(u(t, \phi), F\langle \mathbb{J}_\epsilon \rangle) = 0$  from Claim 1. From the continuity of  $F$ , we have  $F\langle \mathbb{J}_\epsilon \rangle \rightarrow F\langle \mathbb{J} \rangle$  as  $\epsilon \rightarrow 0^+$ . Thus  $\lim_{t \rightarrow \infty} \text{dist}(u(t, \phi), F\langle \mathbb{J} \rangle) = 0$ .

From Claim 2,  $\lim_{t \rightarrow \infty} \text{dist}(u(t, \phi), F\langle \mathbb{J} \rangle) = 0$ , and so  $\lim_{t \rightarrow \infty} \text{dist}(u(t, \phi), F^j\langle \mathbb{J} \rangle) = 0$ ,  $j = 1, 2, \dots$ . From the definition of  $\mathbb{I}_{(F)}$ , we obtain that  $\lim_{t \rightarrow \infty} \text{dist}(u(t, \phi), \mathbb{I}_{(F)}) = 0$ . The proof is completed.  $\square$

### Appendix B

*Proof of Proposition 5.1* For  $s < l$ , denote the number of impulsive moments in  $(s, l]$  by  $l(s, t)$ . From  $0 < \tau_1 \leq t_k - t_{k-1} \leq \tau_2$ , we have  $\frac{r}{\tau_2} - 1 \leq l(t - r, t) \leq \frac{r}{\tau_1} + 1$ . By  $0 \leq \sigma_1 \leq c_k(\cdot) \leq \sigma_2 < 1$ , we easily obtain the conclusion (i).

When  $c_k(\cdot) \equiv c$  and  $t_{k+1} = t_k + T, k \in N$ , we have three cases.

Case 1:  $T > r$ . In this case we have  $l(s - r, s) = 1$  if  $s \in [t_k, t_k + r)$  while  $l(s - r, s) = 0$  if  $s \in [t_k + r, t_{k+1})$ . Thus,

$$\gamma(t) = \begin{cases} (1 - c), & t \in [t_k, t_k + r), \\ 1, & t \in [t_k + r, t_{k+1}). \end{cases}$$

Case 2:  $T = r$ . In this case, we have  $l(s - r, s) = 1$  for all  $s$ . Hence,  $\gamma(t) \equiv 1 - c$  and  $\bar{\gamma} = 1 - c$ .

Case 3:  $T < r$ . In this case, when  $t \in [t_k, t_k + r - \lfloor \frac{r}{T} \rfloor T)$ ,  $l(t - r, t) = \lfloor \frac{r}{T} \rfloor + 1$  and  $\gamma(t) = (1 - c)^{\lfloor \frac{r}{T} \rfloor + 1}$ ; when  $t \in [t_k + r - \lfloor \frac{r}{T} \rfloor T, t_{k+1})$ ,  $l(t - r, t) = \lfloor \frac{r}{T} \rfloor$  and  $\gamma(t) = (1 - c)^{\lfloor \frac{r}{T} \rfloor}$ . Therefore,

$$\gamma(t) = \begin{cases} (1 - c)^{\lfloor \frac{r}{T} \rfloor + 1}, & t \in [t_k, t_k + r - \lfloor \frac{r}{T} \rfloor T), \\ (1 - c)^{\lfloor \frac{r}{T} \rfloor}, & t \in [t_k + r - \lfloor \frac{r}{T} \rfloor T, t_{k+1}). \end{cases}$$

From the above cases, we easily obtain the mean as given in the conclusion (ii) of Proposition 5.1. The proof is completed.  $\square$

*Proof of Proposition 6.1.* Since  $D + L$  and  $C_k$  are commutable,  $D + L$  and  $X(t)$  (hence  $X^{-1}(t)$ ) are also commutable. Suppose  $x(t)$  is the solution of (6.1) and let  $y(t) = X^{-1}(t)x(t)$ . Note that  $X(t)$  is piecewise continuous having jumps at  $t_k, k = 1, 2, \dots$ . Thus, for  $t \neq t_k$ ,

$$\begin{aligned} \dot{y}(t) &= (X(t))^{-1} \dot{x}(t) = -(X(t))^{-1} (D + L)x(t) + \alpha (X(t))^{-1} Sb(x(t - r)) \\ &= -(D + L)y(t) + \alpha (X(t))^{-1} Sb(X(t - r)y(t - r)), \end{aligned}$$

and at  $t = t_k$ ,

$$\begin{aligned} y(t_k^+) &= X^{-1}(t_k^+)x(t_k^+) = \prod_{0 < t_j \leq t_k} (E - C_j)^{-1}(E - C_k)x(t_k^-) \\ &= \prod_{0 < t_j < t_k} (E - C_j)^{-1}x(t_k^-) = X^{-1}(t_k^-)x(t_k^+) = y(t_k^-). \end{aligned}$$

Hence,  $y(t) = X^{-1}(t)x(t)$  is continuous at  $t = t_k$  and differentiable in  $(t_{k-1}, t_k)$  and satisfies (6.19). Accordingly,  $y(t) = X^{-1}(t)x(t)$  is the solution of (6.19) provided that  $x(t)$  is the solution of (6.1).

The reversed implication can be similarly proved and thus, the proof is completed.  $\square$

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