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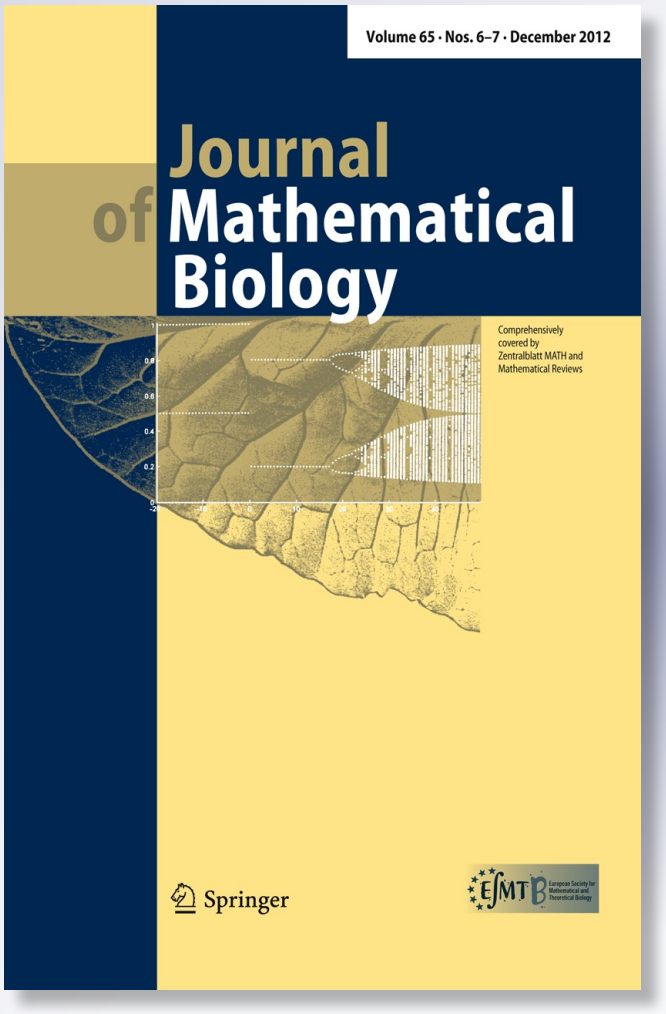
Journal of Mathematical Biology

ISSN 0303-6812
Volume 65
Combined 6-7

J. Math. Biol. (2012) 65:1387-1410
DOI 10.1007/s00285-011-0500-y



Springer

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DOI 10.1007/s00285-011-0500-y

Mathematical Biology

Threshold dynamics of an infective disease model with a fixed latent period and non-local infections

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Received: 23 February 2011 / Revised: 27 November 2011 / Published online: 15 December 2011

Abstract In this paper, we derive and analyze an infectious disease model containing a fixed latency and non-local infection caused by the mobility of the latent individuals in a continuous bounded domain. The model is given by a spatially non-local reaction–diffusion system carrying a discrete delay associated with the zero-flux condition on the boundary. By applying some existing abstract results in dynamical systems theory, we prove the existence of a global attractor for the model system. By appealing to the theory of monotone dynamical systems and uniform persistence, we show that the model has the global threshold dynamics which can be described either by the principal eigenvalue of a linear non-local *scalar* reaction diffusion equation or equivalently by the basic reproduction number $\mathcal{R}_0$ for the model. Such threshold dynamics predicts whether the disease will die out or persist. We identify the next generation operator, the spectral radius of which defines basic reproduction number. When all model parameters are constants, we are able to find explicitly the principal eigenvalue and $\mathcal{R}_0$. In addition to computing the spectral radius of the next generation operator, we also discuss an alternative way to compute $\mathcal{R}_0$.

Keywords Infectious disease model · Reaction–diffusion equation · Non-local infection · Delay · Principal eigenvalue · Basic reproduction number

Mathematics Subject Classification (2000) 35K57 · 37N25 · 92D30

Research partially supported by NNSF of China (No. 10871053), by NSERC and MITACS of Canada, and by a Premier's Research Excellence Award of Ontario.

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1 Introduction and derivation of the model

When studying the transmission of an infectious disease that has a latency, the mobility of the individuals in the latent period will result in non-local infection. This is because an individual infected by the disease in one location can be at any location in the domain when this individual becomes infectious. To address such non-local infections, Li and Zou (2010, 2009a) derived and analyzed some mathematical models for spatially discrete environments (patch models), and Li and Zou (2009b) derived a non-local model for a spatially continuous domain. The model in Li and Zou (2009b) is in the form of reaction diffusion equations in the one dimensional whole space $\mathbb{R}$, and the main concern is the existence of traveling wave fronts accounting for spatial spread of the disease.

In the real world, a domain in which a host population habitats is bounded, and this raises the issue of modeling the dynamics of a disease with latency that is transmitted in a host population living in spatially *bounded* domain. This work intends to address this issue and thus, can be considered as a continuation of the work (Li and Zou 2009b). Because of this, it will be natural and convenient to adopt those notations and concepts used in Li and Zou (2009b), as proceeded below.

Assume that a population lives in an environment that is spatially heterogeneous yet continuous. Let Ω denote this spatial habitat with smooth boundary $\partial\Omega$. Suppose that an infectious disease with a latency τ is brought into this population, resulting in division of the full population into four sub-populations: susceptible, latent, infectious and recovered classes, denoted by $S = S(x,t), L = L(x,t), I = I(x,t), R = R(x,t)$ respectively.

To incorporate the latency into the model properly, we introduce the notion of infection age denoted by the variable a. Let $E(x,t,a)$ be the density (with respect to the infection age a) of infected population at location x and time t with infection age a. A standard argument on structured population and spatial diffusion (see e.g. Metz and Diekmann 1986) leads to

$$\begin{aligned}\frac{\partial E(x,t,a)}{\partial t} + \frac{\partial E(x,t,a)}{\partial a} &= \nabla\cdot[D(x,a)\nabla E(x,t,a)]\\ &-(\sigma(x,a)+\gamma(x,a)+d(x))E(x,t,a),\end{aligned} \tag{1.1}$$

where $a \geq 0$, $\nabla E(x,t,a)$ is the gradient of $E(x,t,a)$ with respect to the spatial variable x and hence $\nabla\cdot[D(x,a)\nabla E(x,t,a)]$ represents the divergence of $D(x,a)\nabla E(x,t,a)$. Here $D(x,a), \sigma(x,a)$ and $\gamma(x,a)$ are the diffusion rate, the disease-induced mortality rate and the recovery rate at location x and age a, respectively; $d(x)$ is the natural death rate which is independent of the infection age. Unlike in Li and Zou (2009b) where all model parameters are assumed to be independent of the spatial variable x, here we allow spatial heterogeneity for the model parameters. In the real world, spatial heterogeneity is ubiquitous due to the variance in environmental conditions such as temperature, humidity and availability of resources etc. Therefore, considering such a more general setting is meaningful and important, although it may bring in new challenges and difficulties when analysing the model as will be seen in later sections.

We consider a closed environment in the sense that the fluxes for each of these four sub-populations are zero. Corresponding to this, we propose the following no flux condition for $E(x,t,a)$ on the boundary:

$$[D(x,a)\nabla E(x,t,a)]\cdot \nu = 0, \quad x\in\partial\Omega,\ t>0, \tag{1.2}$$

where ν is the outward normal to $\partial\Omega$.

By the meaning of τ and the density, it is easy to see that

$$L(x,t)=\int_0^\tau E(x,t,a)da, \quad I(x,t)=\int_\tau^\infty E(x,t,a)da. \tag{1.3}$$

To make the model mathematically tractable yet without losing the main features, we make the following assumptions on those rate functions:

$$D(x,a)=\begin{cases} D_L(x), & \text{for } x\in\Omega,\ a\in[0,\tau],\\ D_I(x), & \text{for } x\in\Omega,\ a\in[\tau,\infty),\end{cases} \tag{1.4}$$

$$\sigma(x,a)=\begin{cases} \sigma_L(x), & \text{for } x\in\Omega,\ a\in[0,\tau],\\ \sigma_I(x), & \text{for } x\in\Omega,\ a\in[\tau,\infty),\end{cases} \tag{1.5}$$

$$\gamma(x,a)=\begin{cases} \gamma_L(x), & \text{for } x\in\Omega,\ a\in[0,\tau],\\ \gamma_I(x), & \text{for } x\in\Omega,\ a\in[\tau,\infty).\end{cases} \tag{1.6}$$

Differentiating (1.3) with respect to t and making use of (1.1), we obtain

$$\begin{aligned}\frac{\partial I(x,t)}{\partial t} &= \nabla\cdot[D_I(x)\nabla I(x,t)]-(\sigma_I(x)+\gamma_I(x)+d(x))I(x,t)\\ &\quad +E(x,t,\tau)-E(x,t,\infty).\end{aligned} \tag{1.7}$$

$$\begin{aligned}\frac{\partial L(x,t)}{\partial t} &= \nabla\cdot[D_L(x)\nabla L(x,t)]-(\sigma_L(x)+\gamma_L(x)+d(x))L(x,t)\\ &\quad -E(x,t,\tau)+E(x,t,0).\end{aligned} \tag{1.8}$$

For biological reasons, we assume that $E(x,t,\infty)=0$, which can be implied by (1.1) and the assumption $d(x)>0$, $x\in\Omega$. Also, note that the new infections are due to the contacts of infectious and susceptible individuals. Thus, adopting mass action infection mechanism leads to the following condition:

$$E(x,t,0)=r(x)I(x,t)S(x,t). \tag{1.9}$$

We further assume that in the absence of disease, the population would settle in a steady state. Since our emphasis it not on demography, we will use the following simplest demographic equation for a population $N(x,t)$ that supports such a dynamics of global convergence to an equilibrium:

$$\frac{\partial N(x,t)}{\partial t}=\mu(x)+\nabla\cdot[D_N(x)\nabla N(x,t)]-d(x)N(x,t), \tag{1.10}$$

where $\mu(x)$ is the recruiting rate, $D_N(x)$ is the diffusion rate and $d(x)$ is the natural death rate.

With all these assumptions, the disease dynamics can be described by the following system of differential equations:

$$\begin{cases} \frac{\partial S(x,t)}{\partial t} = \nabla \cdot [D_S(x)\nabla S(x,t)] + \mu(x) - d(x)S(x,t) - r(x)I(x,t)S(x,t), \\ \frac{\partial L(x,t)}{\partial t} = \nabla \cdot [D_L(x)\nabla L(x,t)] - [\sigma_L(x) + \gamma_L(x) + d(x)]L(x,t) \\ \qquad\quad + r(x)I(x,t)S(x,t) - E(x,t,\tau), \ \ x \in \Omega, \ \ t \geq 0. \\ \frac{\partial I(x,t)}{\partial t} = \nabla \cdot [D_I(x)\nabla I(x,t)] - [(\sigma_I(x) + \gamma_I(x) + d(x))I(x,t)] + E(x,t,\tau), \\ \frac{\partial R(x,t)}{\partial t} = \nabla \cdot [D_R(x)\nabla R(x,t)] + \int_0^\tau \gamma(x,a)E(x,t,a)da \\ \qquad\quad + \gamma_I(x)I(x,t) - d(x)R(x,t). \end{cases} \tag{1.11}$$

In this paper, we always assume that all space dependent parameters in (1.11) are continuous and strictly positive.

In the following, we shall determine $E(x,t,\tau)$ by the method of characteristics. For any $\xi \geq 0$, consider solutions of (1.1) along the characteristic line $t = a + \xi$ by letting $v(x,\xi,a) = E(x,a+\xi,a)$. Then, for $a \in [0,\tau]$, we have

$$\begin{cases} \frac{\partial v(x,\xi,a)}{\partial a} = \left[\frac{\partial E(x,t,a)}{\partial t} + \frac{\partial E(x,t,a)}{\partial a}\right]_{t=a+\xi} \\ \quad = \nabla \cdot [D(x,a)\nabla E(x,a+\xi,a)] - (\sigma(x,a) + \gamma(x,a) + d(x))E(x,a+\xi,a) \\ \quad = \nabla \cdot [D_L(x)\nabla v(x,\xi,a)] - (\sigma_L(x) + \gamma_L(x) + d(x))v(x,\xi,a), \\ v(x,\xi,0) = r(x)I(x,\xi)S(x,\xi). \end{cases} \tag{1.12}$$

Regarding ξ as a parameter and solving the above equation, we obtain

$$v(x,\xi,a) = \int_\Omega \Gamma(x,y,a)[r(y)I(y,\xi)S(y,\xi)]dy,$$

where Γ is the Green function of the operator $\nabla \cdot [D_L(\cdot)\nabla] - \beta_L(\cdot)$ associated with the zero flux boundary condition (see e.g. Friedman 1964) and $\beta_L(\cdot) = \sigma_L(\cdot) + \gamma_L(\cdot) + d(\cdot)$. Evaluating the above at $a = \tau$ (hence $\xi = t - \tau$) gives

$$E(x,t,\tau) = v(x,t-\tau,\tau) = \int_\Omega \Gamma(x,y,\tau)r(y)S(y,t-\tau)I(y,t-\tau)dy. \tag{1.13}$$

Plugging (1.13) into the second and third equations of (1.11) respectively, and dropping the $L(x,t)$ and $R(x,t)$ equations from (1.11) (as they are decoupled from the $S(x,t)$ and $I(x,t)$ equations) results in the following system containing $S(x,t)$ and $I(x,t)$ only:

$$\begin{cases} \frac{\partial S(x,t)}{\partial t} = \nabla \cdot [D_S(x)\nabla S(x,t)] + \mu(x) - d(x)S(x,t) - r(x)I(x,t)S(x,t), \\ \frac{\partial I(x,t)}{\partial t} = \nabla \cdot [D_I(x)\nabla I(x,t)] - \beta_I(x)I(x,t) \\ \qquad + \int_\Omega \Gamma(x,y,\tau)r(y)S(y,t-\tau)I(y,t-\tau)dy, \ x \in \Omega, \ t > 0, \end{cases} \tag{1.14}$$

where $\beta_I(x) = \sigma_I(x) + \gamma_I(x) + d(x)$. Corresponding to (1.2), there are the following boundary conditions:

$$[D_S(x)\nabla S(x,t)] \cdot \nu = 0 = [D_I(x)\nabla I(x,t)] \cdot \nu. \tag{1.15}$$

For convenience, we let $(u_1, u_2) = (S, I)$, $(D_1(\cdot), D_2(\cdot)) = (D_S(\cdot), D_I(\cdot))$ and $\beta = \beta_I$. Then, (1.14) and(1.15) lead to the following model system with time delay and spatially non-locality

$$\begin{cases} \frac{\partial u_1(x,t)}{\partial t} = \nabla \cdot [D_1(x)\nabla u_1(x,t)] + \mu(x) - d(x)u_1(x,t) - r(x)u_1(x,t)u_2(x,t), \\ \frac{\partial u_2(x,t)}{\partial t} = \nabla \cdot [D_2(x)\nabla u_2(x,t)] - \beta(x)u_2(x,t) \\ \qquad + \int_\Omega \Gamma(x,y,\tau)r(y)u_1(y,t-\tau)u_2(y,t-\tau)dy, \ x \in \Omega, \ t > 0, \\ [D_i(x)\nabla u_i(x,t)] \cdot \nu = 0, \quad i = 1,2, \quad x \in \partial\Omega, \ t > 0. \end{cases} \tag{1.16}$$

In Sect. 2, we explore the dynamics of (1.16). As is well known, for a disease model, the basic reproduction number is a biologically and clinically important quantity. For an ODE model, identification of this number done by calculating the spectral radius of the next generation matrix. For a PDE model, finding this number becomes a challenging job, and we will investigate this topic in Sect. 3. In Sect. 4, we illustrate our general results obtained in Sects. 2 and 3 to a special case when all model parameters are positive constants. In Sect. 5, in addition to a summary and some remarks, we also provide an alternative approach, which is motivated by the recent work (Krkosek and Lewis 2010), to calculate the basic reproduction number.

2 Threshold dynamics of (1.16)

Let $\mathbb{X} := C(\bar{\Omega}, \mathbb{R}^2)$ be the Banach space with the supremum norm $\|\cdot\|_{\mathbb{X}}$. Let $\tau \geq 0$ and $C_\tau := C([-\tau, 0], \mathbb{X})$ with the norm $\|\phi\| := \max_{\theta\in[-\tau,0]} \|\phi(\theta)\|_{\mathbb{X}}, \forall\ \phi \in C_\tau$. Define $\mathbb{X}^+ := C(\bar{\Omega}, \mathbb{R}^2_+)$ and $C^+_\tau := C([-\tau, 0], \mathbb{X}^+)$, then $(\mathbb{X}, \mathbb{X}^+)$ and (C_τ, C^+_τ) are strongly ordered spaces (for detailed definition, see, e.g., Smith 1995). For $\sigma > 0$ and a given function $u(t) : [-\tau, \sigma) \to \mathbb{X}$, we define $u_t \in C_\tau$ by

$$u_t(\theta) = u(t+\theta), \quad \forall\ \theta \in [-\tau, 0].$$

Suppose that $T_1(t),\ T_2(t) : C(\bar{\Omega}, \mathbb{R}) \to C(\bar{\Omega}, \mathbb{R})$ are the C_0 semigroups associated with $\nabla \cdot [D_1(\cdot)\nabla] - d(\cdot)$ and $\nabla \cdot [D_2(\cdot)\nabla] - \beta(\cdot)$ subject to the zero flux boundary condition, respectively. It then follows that for any $\varphi \in C(\bar{\Omega}, \mathbb{R})$, $t \geq 0$,

$$(T_i(t)\varphi)(x) = \int_\Omega \Gamma_i(x, y, t)\varphi(y)dy, \quad i = 1, 2, \tag{2.1}$$

where Γ_1 and Γ_2 are the Green functions associated with $\nabla \cdot [D_1(\cdot)\nabla] - d(\cdot)$ and $\nabla \cdot [D_2(\cdot)\nabla] - \beta(\cdot)$ respectively, subject to the zero flux boundary condition. From Smith (1995, Section 7.1 and Corollary 7.2.3), it follows that $T_i(t) : C(\bar{\Omega}, \mathbb{R}) \to C(\bar{\Omega}, \mathbb{R})$ is compact and strongly positive, $\forall\, t > 0$ and $i = 1, 2$. Furthermore, $T(t) := (T_1(t), T_2(t)) : \mathbb{X} \to \mathbb{X}$, $t \geq 0$, is a C_0 semigroup (see, e.g., Pazy 1983).

Let $A_i : \mathfrak{D}(A_i) \to C(\bar{\Omega}, \mathbb{R})$ be the generator of $T_i(t)$, $i = 1, 2$. Then $T(t) : \mathbb{X} \to \mathbb{X}$ is a C_0 semigroup generated by the operator $A := (A_1, A_2)$ defined on $\mathfrak{D}(A) := \mathfrak{D}(A_1) \times \mathfrak{D}(A_2)$. Define $F = (F_1, F_2) : C_\tau^+ \to \mathbb{X}$ by

$$\begin{aligned} F_1(\phi)(x) &= \mu(x) - r(x)\phi_1(x, 0)\phi_2(x, 0), \\ F_2(\phi)(x) &= \int_\Omega \Gamma(x, y, \tau)r(y)\phi_1(y, -\tau)\phi_2(y, -\tau)dy, \end{aligned}$$

for $x \in \bar{\Omega}$ and $\phi = (\phi_1, \phi_2) \in C_\tau^+$. Then (1.16) can be rewritten as the following abstract differential equation

$$\begin{cases} \frac{du}{dt} = Au + F(u_t), & t > 0, \\ u_0 = \phi \in C_\tau^+, \end{cases} \tag{2.2}$$

or it can be rewritten as the following integral equation

$$u(t) = T(t)\phi + \int_0^t T(t - s)F(u_s)ds, \tag{2.3}$$

where $u := (u_1, u_2)$.

Lemma 2.1 *For every initial value function $\phi \in C_\tau^+$, system (1.16) has a unique mild solution $u(\cdot, t, \phi)$ on its maximal interval of existence $[0, \tilde{t}_\phi)$ with $u_0 = \phi$, where $\tilde{t}_\phi \leq \infty$. Furthermore, $u(\cdot, t, \phi) \in C_\tau^+, \forall\, t \in [0, \tilde{t}_\phi)$ and $u(x, t, \phi)$ is a classical solution of (1.16), $\forall\, t > \tau$.*

Proof It is obvious that $F(\phi)$ is locally Lipschitz. By Martin and Smith (1990, Corollary 4) or Smith (1995, Theorem 7.3.1), it suffices to show that

$$\lim_{h \to 0^+} dist(\phi(0) + hF(\phi), C_\tau^+) = 0, \quad \forall\, \phi \in C_\tau^+. \tag{2.4}$$

For any $\phi \in C_\tau^+$ and $h \geq 0$, we have

$$\begin{aligned} \phi(x, 0) + hF(\phi)(x) &= \begin{pmatrix} \phi_1(x, 0) + h[\mu(x) - r(x)\phi_1(x, 0)\phi_2(x, 0)] \\ \phi_2(x, 0) + h[\int_\Omega \Gamma(x, y, \tau)r(y)\phi_1(y, -\tau)\phi_2(y, -\tau)dy] \end{pmatrix} \\ &\geq \begin{pmatrix} \phi_1(x, 0)[1 - h\bar{r}\phi_2(x, 0)] \\ \phi_2(x, 0) \end{pmatrix}, \quad \text{for } x \in \Omega, \end{aligned}$$

where $\bar{r} := \max_{x\in\bar{\Omega}} r(x)$. The above inequalities imply that $\phi + hF(\phi) \in C_\tau^+$ when h is sufficiently small, confirming (2.4). □

To proceed further, we need some information on the following scalar reaction–diffusion equation:

$$\begin{cases} \frac{\partial w}{\partial t} = \nabla \cdot [D(x)\nabla w] + g(x) - d(x)w, & x \in \Omega,\ t > 0, \\ (D(x)\nabla w)\cdot \nu = 0, & x \in \partial\Omega,\ t > 0, \end{cases} \tag{2.5}$$

where $D(x)$, $d(x)$ and $g(x)$ are continuous and positive functions on $\bar{\Omega}$.

Lemma 2.2 *System (2.5) admits a unique positive steady state $w^*(x)$ which is globally attractive in $C(\bar{\Omega}, \mathbb{R}_+)$. Furthermore, if $D(x) \equiv D$, $d(x) \equiv d$ and $g(x) \equiv g$, $\forall\, x \in \Omega$, then $w^*(x) \equiv g/d$, $\forall\, x \in \Omega$.*

The proof is similar to that of Lou and Zhao (2011, Lemma 1) with some minor modifications, and is given in the appendix for readers' convenience.

We are now in the position to address the well-posedness of system (1.16) in the sense of the following theorem.

Theorem 2.1 *For every initial function $\phi \in C_\tau^+$, system (1.16) has a unique solution $u(\cdot, t, \phi)$ on $[0, \infty)$ with $u_0 = \phi$. Moreover, the semiflow $\Phi(t) = u_t(\cdot) : C_\tau^+ \to C_\tau^+$ generated by (1.16), i.e.,*

$$(\Phi(t)\phi)(x, \theta) = u(x, t + \theta, \phi), \quad \forall\, x \in \bar{\Omega},\ t \geq 0,\ \theta \in [-\tau, 0].$$

has a global compact attractor in C_τ^+, $\forall\, t \geq 0$.

Proof It is easy to see that the u_1 equation of (1.16) is dominated by the following scalar equation:

$$\begin{cases} \frac{\partial w(x,t)}{\partial t} = \nabla \cdot [D_1(x)\nabla w(x, t)] + \mu(x) - d(x)w(x, t), & x \in \Omega,\ t > 0, \\ (D_1(x)\nabla w(x, t))\cdot \nu = 0, & x \in \partial\Omega,\ t > 0. \end{cases} \tag{2.6}$$

From Lemma 2.2, it follows that the system (2.6) admits a unique positive steady state $u_1^*(x)$ which is globally asymptotically stable in $C(\bar{\Omega}, \mathbb{R})$. This implies that there is $B_1 > 0$ such that for any $\phi \in C_\tau^+$, there exists a $t_1 = t_1(\phi) > 0$ with $u_1(\cdot, t, \phi) \leq B_1$ for all $t \geq t_1$.

Making use of the boundedness of $u_1(x, t)$ and the property of $\Gamma(\cdot, \cdot, \cdot)$ in the u_2 equation in (1.16), we know that

$$\frac{\partial u_2(x, t)}{\partial t} \leq \nabla \cdot [D_2(x)\nabla u_2(x, t)] - \beta(x)u_2(x, t) + c\bar{u}_2(t - \tau), \tag{2.7}$$

for some constant $c > 0$, where $\bar{u}_2(t) = \int_\Omega u_2(x,t)dx$. We now show that $\bar{u}_2(t)$ is bounded. To this end, we integrate the first equation in (1.16) to obtain

$$\begin{aligned}\frac{d\bar{u}_1(t)}{dt} &= \int_\Omega \mu(x)dx - \int_\Omega d(x)u_1(x,t)dx - \int_\Omega r(x)u_1(x,t)u_2(x,t)dx,\\ &\le \mu_0 - d_0\bar{u}_1(t) - \int_\Omega r(x)u_1(x,t)u_2(x,t)dx, \quad t \ge 0,\end{aligned}$$

where $\bar{u}_1(t) = \int_\Omega u_1(x,t)dx$, $\mu_0 = \int_\Omega \mu(x)dx$ and $d_0 = \min_{x\in\Omega} d(x)$. Thus,

$$\int_\Omega r(x)u_1(x,t)u_2(x,t)dx \le \mu_0 - d_0\bar{u}_1(t) - \frac{d\bar{u}_1(t)}{dt}, \quad t > 0. \tag{2.8}$$

Similarly, integrating the second equation of (1.16) with respect to $x \in \Omega$ and making use of (2.8), we obtain

$$\frac{d\bar{u}_2(t)}{dt} \le -\beta_0\bar{u}_2(t) - k_1\bar{u}_1(t-\tau) - k_2\frac{d\bar{u}_1(t-\tau)}{dt} + k_3, \quad \forall\, t \ge \tau, \tag{2.9}$$

where $\beta_0 = \min_{x\in\bar{\Omega}} \beta(x)$ and k_1, k_2 and k_3 are some positive numbers independent of ϕ. We can choose $k_1 \le \beta_0 k_2$ in (2.9), so that

$$\begin{aligned}\frac{d}{dt}[\bar{u}_2(t) + k_2\bar{u}_1(t-\tau)] &\le -\beta_0\bar{u}_2(t) - k_1\bar{u}_1(t-\tau) + k_3\\ &\le -\frac{k_1}{k_2}\bar{u}_2(t) - k_1\bar{u}_1(t-\tau) + k_3\\ &= -\frac{k_1}{k_2}[\bar{u}_2(t) + k_2\bar{u}_1(t-\tau)] + k_3, \quad \forall\, t \ge \tau.\end{aligned} \tag{2.10}$$

This implies that there are a positive constant k_4 depending on ϕ and a positive constant independent of ϕ, such that

$$\bar{u}_2(t) \le \bar{u}_2(t) + k_2\bar{u}_1(t-\tau) \le k_4(\phi)e^{-(k_1/k_2)t} + k_5, \quad \forall\, t \ge \tau, \tag{2.11}$$

confirming the boundedness of $\bar{u}_2(t)$. Now combining this with (2.7) and by comparison theorem for delayed parabolic equation (see, e.g., Smith 1995; Wu 1996), we conclude that there exists a positive number B_2, independent of ϕ, and $t_2 = t_2(\phi) > t_1(\phi) + \tau$ such that $u_2(\cdot, t, \phi) \le B_2$, $\forall\, t \ge t_2$. Therefore, the existence of the solution $u(\cdot, t, \phi)$ claimed in Lemma 2.1 is indeed global (i.e., $\tilde{t}_\infty = \infty$), and the solution semiflow $\Phi(t) : C_\tau^+ \to C_\tau^+$ is point dissipative. Moreover, $\Phi(t) : C_\tau^+ \to C_\tau^+$ is compact for each $t > \tau$ by Wu (1996, Theorem 2.1.8). By Hale (1988, Theorem 3.4.8), $\Phi(t) : C_\tau^+ \to C_\tau^+$, $t \ge 0$ has a global compact attractor. □

The following results will play an important role in establishing the persistence of (1.16).

Lemma 2.3 *Suppose $u(x,t,\phi)$ is the solution of system (1.16) with $u_0(\phi)=\phi\in C_\tau^+$.*

(i) *If there exists some $t_0\geq 0$ such that $u_2(\cdot,t_0,\phi)\not\equiv 0$, then*

$$u_2(\cdot,t,\phi)>0,\quad \forall\, t>t_0;$$

(ii) *For any $\phi\in C_\tau^+$, we always have $u_1(\cdot,t,\phi)>0,\ \forall\, t>0$ and*

$$\liminf_{t\to\infty} u_1(x,t,\phi)\geq Q\quad \textit{uniformly for } x\in\bar{\Omega},$$

where Q is a positive constant.

Proof By Lemma 2.1 and the second equation of (1.16), it follows that

$$\begin{cases}\nabla\cdot[D_2(x)\nabla u_2(x,t)]-\frac{\partial u_2(x,t)}{\partial t}-\beta(x)u_2(x,t)\leq 0, & x\in\Omega,\ t>0,\\ [D_2(x)\nabla u_2(x,t)]\cdot\nu=0, & x\in\partial\Omega,\ t>0.\end{cases}\tag{2.12}$$

Thus, Part (i) follows from the strong maximum principle (see, e.g., Protter and Weinberger 1984, p. 172, Theorem 4) and the Hopf boundary lemma (see, e.g., Protter and Weinberger 1984, p. 170, Theorem 3), with the initial time at $t=t_0$ instead of $t=0$.

By the proof of Theorem 2.1, there is a $B_2>0$, such that $u_2(x,t)\leq B_2$ for $t>t_2=t_2(\phi)$. From the first equation of (1.16), it follows that

$$\begin{cases}\frac{\partial u_1(x,t)}{\partial t}\geq\nabla\cdot[D_1(x)\nabla u_1(x,t)]+\mu(x)-[d(x)+B_2r(x)]u_1(x,t), & x\in\Omega,\ t\geq t_2,\\ [D_1(x)\nabla u_1(x,t)]\cdot\nu=0, & x\in\partial\Omega,\ t\geq t_2.\end{cases}\tag{2.13}$$

From Lemma 2.2, the following system

$$\begin{cases}\frac{\partial v(x,t)}{\partial t}=\nabla\cdot[D_1(x)\nabla v(x,t)]+\mu(x)-[d(x)+B_2r(x)]v(x,t), & x\in\Omega,\ t\geq t_2,\\ [D_1(x)\nabla v(x,t)]\cdot\nu=0, & x\in\partial\Omega,\ t\geq t_2,\end{cases}$$

admits a unique positive steady state $v^*(x)$ which is globally asymptotically stable in $C(\bar{\Omega},\mathbb{R})$. By the standard parabolic comparison theorem, it follows that $\liminf_{t\to\infty}u_1(\cdot,t,\phi)\geq v^*(\cdot)$. Thus, part (ii) is proved. □

In order to find the disease free steady state, we set $u_2=0$ in (1.16), leading to (2.6) for the density of susceptible host population. From Lemma 2.2, it follows that the system (2.6) admits a unique positive steady state $u_1^*(x)$ which is globally asymptotically stable in $C(\bar{\Omega},\mathbb{R})$. Linearizing system (1.16) at the disease-free equilibrium $(u_1^*,0)$, we get the following system for the infectious component u_2:

$$\begin{cases}\frac{\partial u_2(x,t)}{\partial t}=\nabla\cdot[D_2(x)\nabla u_2(x,t)]-\beta(x)u_2(x,t)\\ \qquad\qquad +\int_\Omega\Gamma(x,y,\tau)r(y)u_1^*(y)u_2(y,t-\tau)dy, & x\in\Omega,\ t>0,\\ [D_2(x)\nabla u_2(x,t)]\cdot\nu=0, & x\in\partial\Omega,\ t>0.\end{cases}\tag{2.14}$$

We first consider the following generalized version of system (2.14):

$$\begin{cases} \frac{\partial u_2(x,t)}{\partial t} = \nabla \cdot [D_2(x)\nabla u_2(x,t)] - \beta(x)u_2(x,t) \\ \qquad\qquad + \int_\Omega \Gamma(x,y,\tau)r(y)m(y)u_2(y,t-\tau)dy, \quad x \in \Omega,\ t > 0, \\ [D_2(x)\nabla u_2(x,t)] \cdot \nu = 0, \quad x \in \partial\Omega,\ t > 0, \end{cases} \tag{2.15}$$

where $m(x) > 0,\ \forall\, x \in \bar{\Omega}$. Substituting $u_2(x,t) = e^{\lambda t}\psi(x)$ into (2.15) results in the following eigenvalue problem:

$$\begin{cases} \lambda\psi(x) = \nabla \cdot [D_2(x)\nabla\psi(x)] - \beta(x)\psi(x) \\ \qquad\qquad + e^{-\lambda\tau}\int_\Omega \Gamma(x,y,\tau)r(y)m(y)\psi(y)dy, \quad x \in \Omega,\ t > 0, \\ [D_2(x)\nabla\psi(x)] \cdot \nu = 0, \quad x \in \partial\Omega,\ t > 0. \end{cases} \tag{2.16}$$

Note that (2.16) is nonlinear in terms of λ, and the nonlinearity is caused by the presence of the delay τ in $u_2(y,t-\tau)$ in (2.15). Noticing that the linear delay equation (2.15) is monotone, the general results on monotone delay equations suggest that the delay τ play no role in determining the stability of the trivial solution of (2.15). This motivates us to consider the following associated linear nonlocal system resulting from dropping τ in (2.15):

$$\begin{cases} \frac{\partial u_2(x,t)}{\partial t} = \nabla \cdot [D_2(x)\nabla u_2(x,t)] - \beta(x)u_2(x,t) \\ \qquad\qquad + \int_\Omega \Gamma(x,y,\tau)r(y)m(y)u_2(y,t)dy, \quad x \in \Omega,\ t > 0, \\ [D_2(x)\nabla u_2(x,t)] \cdot \nu = 0, \quad x \in \partial\Omega,\ t > 0. \end{cases} \tag{2.17}$$

Substituting $u_2(x,t) = e^{\lambda t}\psi(x)$ into (2.22) leads to the following eigenvalue problem:

$$\begin{cases} \lambda\psi(x) = \nabla \cdot [D_2(x)\nabla\psi(x)] - \beta(x)\psi(x) \\ \qquad\qquad + \int_\Omega \Gamma(x,y,\tau)r(y)m(y)\psi(y)dy, \quad x \in \Omega,\ t > 0, \\ [D_2(x)\nabla\psi(x)] \cdot \nu = 0, \quad x \in \partial\Omega,\ t > 0. \end{cases} \tag{2.18}$$

The following lemma gives some useful information eigenvalue problems (2.16) and (2.18),

Lemma 2.4 *Let $m(x) > 0,\ \forall\, x \in \bar{\Omega}$ be given. Then,*

(i) *the eigenvalue problem (2.16) has a principal eigenvalue $\bar{\lambda}(m)$, corresponding to which, there is a unique strongly positive eigenfunction;*
(ii) *the eigenvalue problem (2.18) has a principal eigenvalue $\lambda(m)$, corresponding to which, there is a unique strongly positive eigenfunction;*
(iii) *$\lambda(m)$ and $\bar{\lambda}(m)$ have the same sign.*

The proof is similar to that of Thieme and Zhao (2001, Theorem 2.2) with some minor modifications corresponding to the more general second order partial differential operator in (2.15)–(2.18). For readers' convenience, we give the proof in the appendix.

The following theorem indicates that $\bar{\lambda}(u_1^*)$ is a threshold index for disease extinction/persistence.

Theorem 2.2 *Let $u_1^*(x)$ be the positive steady state of (2.6). Then, the following statements hold.*

(i) *If $\bar{\lambda}(u_1^*) < 0$, then the disease free equilibrium $(u_1^*(x), 0)$ is globally attractive in C_τ^+;*

(ii) *If $\bar{\lambda}(u_1^*) > 0$, then $(u_1^*(x), 0)$ is unstable; moreover, system (1.16) admits at least one endemic steady state $\hat{u}(x)$ and there exists an $\eta > 0$ such that for any $\phi \in C_\tau^+$ with $\phi_2(\cdot, 0) \not\equiv 0$, we have*

$$\liminf_{t\to\infty} u_i(x,t) \geq \eta, \quad \forall\, i = 1, 2,$$

uniformly for all $x \in \bar{\Omega}$.

Proof We first assume that $\bar{\lambda}(u_1^*) < 0$. Since $\bar{\lambda}(m)$ is continuous in m, it follows from $\bar{\lambda}(u_1^*) < 0$ that there is a $\rho_0 > 0$ such that $\bar{\lambda}(u_1^* + \rho_0) < 0$. From the first equation of (1.16), it follows that

$$\begin{cases} \frac{\partial u_1(x,t)}{\partial t} \leq \nabla \cdot [D_1(x)\nabla u_1(x,t)] + \mu(x) - d(x)u_1(x,t), & x \in \Omega,\ t > 0, \\ [D_1(x)\nabla u_1(x,t)] \cdot \nu = 0, & x \in \partial\Omega,\ t > 0. \end{cases} \tag{2.19}$$

From Lemma 2.2, (2.19) and the comparison principle, it follows that there is a $t_0 = t_0(\phi)$ such that

$$u_1(x,t,\phi) \leq u_1^*(x) + \rho_0, \quad \forall\, t \geq t_0,\ x \in \bar{\Omega}.$$

Thus,

$$\begin{cases} \frac{\partial u_2(x,t)}{\partial t} \leq \nabla \cdot [D_2(x)\nabla u_2(x,t)] - \beta(x)u_2(x,t) \\ \qquad\qquad + \int_\Omega \Gamma(x,y,\tau) r(y)(u_1^*(y) + \rho_0)u_2(y, t-\tau)dy, \quad x \in \Omega,\ t \geq t_0, \\ [D_2(x)\nabla u_2(x,t)] \cdot \nu = 0, \quad x \in \partial\Omega,\ t \geq t_0. \end{cases} \tag{2.20}$$

Let $\hat{\psi}$ be the strongly positive eigenfunction $\hat{\psi}$ corresponding to $\bar{\lambda}(u_1^* + \rho_0)$ for (2.16) with $m(x) = u_1^*(x) + \rho_0$ (see Lemma 2.4). Then, for any given $\phi \in C_\tau^+$, there exists some $\alpha > 0$ such that $u_2(x,t,\phi) \leq \alpha e^{\bar{\lambda}(u_1^*+\rho_0)t}\hat{\psi}(x)$, $\forall\, x \in \bar{\Omega},\ t \in [t_0 - \tau, t_0]$. Note that the following linear system

$$\begin{cases} \frac{\partial \bar{u}_2(x,t)}{\partial t} = \nabla \cdot [D_2(x)\nabla \bar{u}_2(x,t)] - \beta(x)\bar{u}_2(x,t) \\ \qquad\qquad + \int_\Omega \Gamma(x,y,\tau) r(y)(u_1^*(y) + \rho_0)\bar{u}_2(y, t-\tau)dy, \quad x \in \Omega,\ t \geq t_0, \\ [D_2(x)\nabla \bar{u}_2(x,t)] \cdot \nu = 0, \quad x \in \partial\Omega,\ t \geq t_0, \end{cases} \tag{2.21}$$

admits a solution $\alpha e^{\bar{\lambda}(u_1^*+\rho_0)t}\hat{\psi}(x),\ \forall\, t \geq t_0$. The comparison principle implies

$$u_2(x,t,\phi) \leq \alpha e^{\bar{\lambda}(u_1^*+\rho_0)t}\hat{\psi}(x), \quad \forall\, t \geq t_0,$$

leading to $\lim_{t\to\infty} u_2(x,t,\phi) = 0$ uniformly for $x \in \bar{\Omega}$. Then, the equation for u_1 in (1.16) is asymptotic to the reaction–diffusion equation (2.6). Hence, by the theory for asymptotically autonomous semiflows (see, e.g., Thieme 1992, Corollary 4.3), we conclude that $\lim_{t\to\infty} u_1(x,t,\phi) = u_1^*(x)$ uniformly for $x \in \bar{\Omega}$, completing the proof of Part (i).

We now consider the case $\bar{\lambda}(u_1^*) > 0$. Let

$$\mathbb{W}_0 = \{\phi \in C_\tau^+ : \phi_2(\cdot,0) \not\equiv 0\},$$

and

$$\partial\mathbb{W}_0 = C_\tau^+ \backslash \mathbb{W}_0 = \{\phi \in C_\tau^+ : \phi_2(\cdot,0) \equiv 0\}.$$

By Lemma 2.3, it follows that for any $\phi \in \mathbb{W}_0$, we have $u_2(x,t,\phi) > 0,\ \forall\, x \in \bar{\Omega},\ t > 0$. In other words, $\Phi(t)\mathbb{W}_0 \subseteq \mathbb{W}_0,\ \forall\, t \geq 0$. Let

$$M_\partial := \{\phi \in \partial\mathbb{W}_0 : \Phi(t)\phi \in \partial\mathbb{W}_0, \forall\, t \geq 0\},$$

and $\omega(\phi)$ be the omega limit set of the orbit $O^+(\phi) := \{\Phi(t)\phi : t \geq 0\}$.

Claim 1 $\omega(\psi) = \{(u_1^*, 0)\},\ \forall\, \psi \in M_\partial$.
Since $\psi \in M_\partial$, we have $\Phi(t)\psi \in M_\partial,\ \forall\, t \geq 0$. Thus $u_2(\cdot,t,\psi) \equiv 0,\ \forall\, t \geq 0$. Then u_1 satisfies the reaction–diffusion equation (2.6) and hence we get $\lim_{t\to\infty} u_1(x,t,\phi) = u_1^*(x)$ uniformly for $x \in \bar{\Omega}$. Hence, $\omega(\psi) = \{(u_1^*, 0)\},\ \forall\, \psi \in M_\partial$.

Since $\bar{\lambda}(u_1^*) > 0$, by the continuity of $\bar{\lambda}(m)$ on m, there exists a sufficiently small positive number ζ_0 such that $\bar{\lambda}(u_1^* - \zeta_0) > 0$.

Claim 2 *$(u_1^*, 0)$ is a uniform weak repeller for $\mathbb{W}_0$ in the sense that*

$$\limsup_{t\to\infty} \|\Phi(t)\phi - (u_1^*, 0)\| \geq \zeta_0,\ \forall\, \phi \in \mathbb{W}_0.$$

Suppose, by contradiction, there exists $\phi_0 \in \mathbb{W}_0$ such that

$$\limsup_{t\to\infty} \|\Phi(t)\phi_0 - (u_1^*, 0)\| < \zeta_0.$$

Then, there exists $t_1 > 0$ such that $u_1(x,t,\phi_0) > u_1^*(x) - \zeta_0, \forall\, t \geq t_1, x \in \bar{\Omega}$. Thus $u_2(x,t,\phi_0)$ satisfies

$$\begin{cases} \frac{\partial u_2(x,t)}{\partial t} \geq \nabla\cdot[D_2(x)\nabla u_2(x,t)] - \beta(x)u_2(x,t) \\ \qquad\qquad + \int_\Omega \Gamma(x,y,\tau)r(y)(u_1^*(y) - \zeta_0)u_2(y,t-\tau)dy, \quad x \in \Omega,\ t \geq t_1, \\ [D_2(x)\nabla u_2(x,t)]\cdot\nu = 0, \quad x \in \partial\Omega,\ t \geq t_1. \end{cases} \tag{2.22}$$

By Lemma 2.4, we can let $\tilde{\psi}$ be the strongly positive eigenfunction corresponding to $\bar{\lambda}(u_1^* - \zeta_0)$. Since $u_2(x,t,\phi_0) > 0,\ \forall\, x \in \bar{\Omega},\ t > 0$, there exists $\epsilon_0 > 0$ such that $u_2(x,t,\phi_0) \geq \epsilon_0 e^{\bar{\lambda}(u_1^* - \zeta_0)t}\tilde{\psi},\ \forall\, x \in \bar{\Omega},\ t \in [t_1 - \tau, t_1]$. Note that $\epsilon_0 e^{\bar{\lambda}(u_1^* - \zeta_0)t}\tilde{\psi}$ is a solution of the following linear system:

$$\begin{cases} \frac{\partial v(x,t)}{\partial t} = \nabla \cdot [D_2(x)\nabla v(x,t)] - \beta(x)v(x,t) \\ \qquad\qquad + \int_\Omega \Gamma(x,y,\tau)r(y)(u_1^*(y) - \zeta_0)v(y,t-\tau)dy, \quad x \in \Omega,\ t \geq t_1, \\ [D_2(x)\nabla v(x,t)] \cdot \nu = 0, \quad x \in \partial\Omega,\ t \geq t_1. \end{cases} \tag{2.23}$$

The comparison principle implies that

$$u_2(x,t,\phi_0) \geq \epsilon_0 e^{\bar{\lambda}(u_1^* - \zeta_0)t}\tilde{\psi}, \quad \forall\, t > t_1,\ x \in \bar{\Omega}.$$

Since $\bar{\lambda}(u_1^* - \zeta_0) > 0$, it follows that $u_2(x,t,\phi_0)$ is unbounded, contradicting Theorem 2.2. This contradiction proves Claim 2.

Define a continuous function $p : C_\tau^+ \to [0,\infty)$ by

$$p(\phi) := \min_{x \in \bar{\Omega}} \phi_2(x,0), \quad \forall \phi \in C_\tau^+.$$

By Lemma 2.3, it follows that $p^{-1}(0,\infty) \subseteq \mathbb{W}_0$ and p has the property that if $p(\phi) > 0$ or $\phi \in \mathbb{W}_0$ with $p(\phi) = 0$, then $p(\Phi(t)\phi) > 0,\ \forall\, t > 0$. That is, p is a generalized distance function for the semiflow $\Phi(t) : C_\tau^+ \to C_\tau^+$ (see, e.g., Smith and Zhao 2001). From the above claims, it follows that any forward orbit of $\Phi(t)$ in M_∂ converges to $(u_1^*, 0)$ which is isolated in C_τ^+ and $W^S(u_1^*, 0) \cap \mathbb{W}_0 = \emptyset$, where $W^S(u_1^*, 0)$ is the stable set of $(u_1^*, 0)$ (see Smith and Zhao 2001). It is obvious that there is no cycle in M_∂ from $(u_1^*, 0)$ to $(u_1^*, 0)$. By Smith and Zhao (2001, Theorem 3), it follows that there exists an $\tilde{\eta} > 0$ such that

$$\min_{\psi \in \omega(\phi)} p(\psi) > \tilde{\eta}, \quad \forall \phi \in \mathbb{W}_0.$$

Hence, $\liminf_{t\to\infty} u_2(\cdot,t,\phi) \geq \tilde{\eta},\ \forall\, \phi \in \mathbb{W}_0$. From Lemma 2.3, there exists an $0 < \eta \leq \tilde{\eta}$ such that

$$\liminf_{t\to\infty} u_i(\cdot,t,\phi) \geq \eta, \quad \forall \phi \in \mathbb{W}_0,\ i = 1,2.$$

Hence, the uniform persistence stated in the conclusion (ii) are valid. By Magal and Zhao (2005, Theorem 3.7 and Remark 3.10), it follows that $\Phi(t) : \mathbb{W}_0 \to \mathbb{W}_0$ has a global attractor A_0. It then follows from Magal and Zhao (2005, Theorem 4.7) that $\Phi(t)$ has an equilibrium $\tilde{u}(\cdot) \in \mathbb{W}_0$ which, by Lemma 2.3, is a positive steady state of (1.16). The proof is completed. □

Corollary 2.1 *Let $u_1^*(x)$ be the positive steady state of (2.6). Then, the following statements hold.*

(i) *If* $\lambda(u_1^*) < 0$, *then the disease free equilibrium* $(u_1^*(x), 0)$ *is globally attractive in* C_τ^+;

(ii) *If* $\lambda(u_1^*) > 0$, *then* $(u_1^*(x), 0)$ *is unstable; moreover, system* (1.16) *admits at least one endemic steady state* $\hat{u}(x)$ *and there exists an* $\eta > 0$ *such that for any* $\phi \in C_\tau^+$ *with* $\phi_2(\cdot, 0) \not\equiv 0$, *we have*

$$\liminf_{t\to\infty} u_i(x,t) \geq \eta, \quad \forall\, i = 1, 2,$$

uniformly for all $x \in \bar{\Omega}$.

3 The basic reproduction number of the model

In Sect. 2, we have seen that the principal eigenvalue $\bar{\lambda}(u_1^*)$ is a threshold index for the model in determining whether the disease will die out or remain endemic. In epidemiology, there is another index, called basic reproduction number and usually denoted by $\mathcal{R}_0$, which also plays a similar threshold role: if $\mathcal{R}_0 < 1$ then the disease will die out; if $\mathcal{R}_0 < 1$ then the disease will remain endemic. Considering the fact the basic reproduction number is more commonly used in the communities of health sciences and clinics, identifying the right formula of $\mathcal{R}_0$ for an epidemic model is necessary and important. For models described by ordinary differential equations (finite dimensions), van den Driessche and Watmough (2002) provides a standard procedure for defining and computing $\mathcal{R}_0$ by using the next generation matrix. For infinite dimensional models, including the models with age and spatial structures, the works Diekmann et al. (1990) and Thieme (2009) contribute fundamental and useful recipe for defining $\mathcal{R}_0$, which is based on the idea of next generation operator. In this section, we mainly follow notions and procedure (Thieme 2009) to identify the basic reproduction number $\mathcal{R}_0$ for (1.16).

Assume that host population is near the disease free equilibrium $(u_1^*, 0)$ with $u_2(\theta) = 0$ for $\theta \in [-\tau, 0)$, before some infectious individuals with a spatial distribution $\varphi(x)$ are brought into the population at $t = 0$ (i.e., $u_2(x, 0) = \varphi(x)$). Let V be the positive linear operator on $C(\bar{\Omega}, \mathbb{R})$ defined by

$$V(\varphi)(x) := \int_\Omega \Gamma(x, y, \tau) r(y) u_1^*(y) \varphi(y) dy, \quad \forall\, \varphi \in C(\bar{\Omega}, \mathbb{R}),\ x \in \bar{\Omega}. \tag{3.1}$$

Note that if $\varphi(x)$ is a spatial distribution of infectious individuals, then $r(y)u_1^*(y)\varphi(y)$ accounts for new *infections* at location y. By the meaning of the Green function $\Gamma(x, y, \tau)$, $V(\varphi)(x)$ sums up the new infections in the whole domain Ω that can survive the latent period, and thus, gives the distributions of new *infectious* individuals caused by the distribution $\varphi(x)$. Combining this explanation and the meaning of the semigroup $T_2(t)$ with (2.14), we know that there will be no new infectious individuals available at any time $t \in [0, \tau)$; and for $t \geq \tau$, the new infectious individuals become available with the spatial distribution given by $V((T_2(t-\tau)\phi)(x)$. Thus, the distribution of the total new infectious individuals caused by the initial distribution $\varphi(x)$ of infectious individuals is

$$\int_{\tau}^{\infty} V((T_2(t-\tau)\phi)(x)dt = \int_{0}^{\infty} V((T_2(t)\phi)(x)dt. \tag{3.2}$$

Therefore, (3.2) defines the next generation operator:

$$\mathbf{L}(\varphi) := \int_{0}^{\infty} V(T_2(t)\varphi)dt = V(\int_{0}^{\infty} T_2(t)\varphi\, dt). \tag{3.3}$$

By Diekmann et al. (1990) and Thieme (2009), the spectral radius of $\mathbf{L}$ is the basic reproduction number for the model (1.16), that is,

$$\mathcal{R}_0 := r(\mathbf{L}). \tag{3.4}$$

Then, we have the following result. By the general results in Thieme (2009) and the same arguments as in Wang and Zhao (2011, Lemma 2.2), we have the following result.

Lemma 3.1 *$\mathcal{R}_0 - 1$ and $\lambda(u_1^*)$ have the same sign.*

The proof by a general results in Thieme (2009) and a similar argument to that in the proof of Wang and Zhao (2011, Lemma 2.2), and is given in the appendix for readers' convenience.

By Lemma 3.1 and Corollary 2.1, we have the following natural results which are expected in light of the biological meaning of the basic reproduction number $\mathcal{R}_0$.

Theorem 3.1 *Suppose $u(x,t,\phi)$ is the solution of system (1.16) with $u_0 = \phi \in C_\tau^+$. Then the following statements hold.*

(i) *If $\mathcal{R}_0 < 1$, then the disease free equilibrium $(u_1^*(x), 0)$ is globally attractive in C_τ^+;*

(ii) *If $\mathcal{R}_0 > 1$, then system (1.16) admits at least one positive steady state $\hat{u}(x)$ and there exists an $\eta > 0$ such that for any $\phi \in C_\tau^+$ with $\phi_2(\cdot, 0) \not\equiv 0$, we have*

$$\liminf_{t\to\infty} u_i(x,t) \geq \eta, \ \forall\, i = 1, 2,$$

uniformly for all $x \in \bar{\Omega}$.

4 A special case

In general, calculating the principal eigenvalue $\lambda(u_1^*)$ and spectral radius of the operator $\mathbf{L}$ is very difficult and challenging, if not impossible. Below, we shall discuss a special case where all the coefficients in (1.16) are independent of the spatial variable x, that is,

$$\begin{aligned} &D_1(x) \equiv D_1,\ D_2(x) \equiv D_2,\ D_L(x) \equiv D_L,\\ &\mu(x) \equiv \mu,\ d(x) \equiv d,\ r(x) \equiv r,\ \beta(x) \equiv \beta,\ \beta_L(x) \equiv \beta_L, \end{aligned} \quad \text{for } x \in \Omega. \tag{4.1}$$

By Lemma 2.2, it follows that

$$u_1^*(x) \equiv \frac{\mu}{d}, \quad \forall\, x \in \Omega.$$

Then $\lambda(u_1^*) \equiv \lambda(\frac{\mu}{d})$ satisfies

$$\begin{cases} \lambda\psi(x) = D_2\Delta\psi(x) - \beta\psi(x) + \frac{r\mu}{d}\int_\Omega \Gamma(x, y, \tau)\psi(y)dy, & x \in \Omega,\ t > 0, \\ \frac{\partial\psi(x)}{\partial\nu} = 0, & x \in \partial\Omega,\ t > 0. \end{cases} \tag{4.2}$$

Substituting $\psi(x) \equiv 1 > 0$ into (4.2) and using the equality: $\int_\Omega \Gamma(x, y, \tau)dy = e^{-\beta_L\tau}$, one obtains the principal eigenvalue of (4.1)

$$\lambda\left(\frac{\mu}{d}\right) = -\beta + \frac{r\mu}{d}e^{-\beta_L\tau}, \tag{4.3}$$

corresponding to which there is the unique (up to constant multiple) positive eigenfunction $\psi(x) \equiv 1 > 0$. Thus, in such a special case, the threshold dynamics can be described *explicitly* in terms of the model parameters as below.

Corollary 4.1 *Assume* (4.1) *for the model system* (1.16). *Then,*

(i) *if* $\beta d > r\mu e^{-\beta_L\tau}$, *then the disease free equilibrium* $(\frac{\mu}{d}, 0)$ *is globally attractive for the model system* (1.16);
(ii) *if* $\beta d < r\mu e^{-\beta_L\tau}$, *the system* (1.16) *is uniformly persistent (hence the disease remains endemic), admits at least one positive steady state (endemic steady state).*

Next, we identify the basic reproduction number $\mathcal{R}_0$ by computing the spectral radius of $\mathbf{L}$ under (4.1). To avoid the main feature being hidden by the complexity in the Green functions caused by higher dimension, we only consider one dimensional spatial space. Without loss of generality, we take $\Omega = (0, \pi)$. In such a special case, $T_2(t)\varphi$ represents the solution of the following system:

$$\begin{cases} \frac{\partial u}{\partial t} = D_2\Delta u(x, t) - \beta u(x, t), & x \in (0, \pi),\ t > 0, \\ \frac{\partial u(0,t)}{\partial x} = \frac{\partial u(\pi,t)}{\partial x} = 0, & t > 0, \\ u(x, 0) = \varphi(x), & x \in (0, \pi). \end{cases}$$

Thus $T_2(t)\varphi$ may be given by

$$(T_2(t)\varphi)(x) = e^{-\beta t}\int_0^\pi K_2(x, z, t)\varphi(z)dz,$$

where

$$K_2(x, y, t) = \frac{1}{\pi} + \frac{2}{\pi}\sum_{n=1}^{\infty} e^{-n^2D_2t}\cos nx\cos ny \tag{4.4}$$

is the Green function associated with $D_2\Delta$ subject to the homogeneous Neumann boundary condition.

Also in this case, (3.1) becomes

$$V(\varphi)(x) := e^{-\beta_L \tau}\frac{r\mu}{d}\int_0^\pi K(x, y, \tau)\varphi(y)dy, \quad \forall\, \varphi \in C([0,\pi],\mathbb{R}),\ x \in [0,\pi],$$

where

$$K(x, y, t) = \frac{1}{\pi} + \frac{2}{\pi}\sum_{n=1}^{\infty} e^{-n^2 D_L t}\cos nx \cos ny \tag{4.5}$$

is the Green functions associated with $D_L\Delta$ subject to the Neumann boundary conditions. Thus

$$V(T_2(t)\varphi)(x) := e^{-\beta_L \tau}\frac{r\mu}{d}\int_0^\pi K(x, y, \tau)(T_2(t)\varphi)(y)dy, \quad \forall\, \varphi \in C([0,\pi],\mathbb{R}),\ x \in [0,\pi].$$

Hence,

$$V(T_2(t)\varphi)(x) := e^{-\beta t}e^{-\beta_L \tau}\frac{r\mu}{d}\int_0^\pi\int_0^\pi K(x, y, \tau)K_2(y, z, t)\varphi(z)dydz.$$

By (3.3), it follows that

$$\mathbf{L}(\varphi)(x) = e^{-\beta_L \tau}\frac{r\mu}{d}\int_0^\infty e^{-\beta t}\int_0^\pi\int_0^\pi K(x, y, \tau)K_2(y, z, t)\varphi(z)dydzdt. \tag{4.6}$$

Substituting (4.4) and (4.5) into (4.6) and after doing some routine computations, we obtain

$$\mathbf{L}(\varphi))(x) = e^{-\beta_L \tau}\frac{r\mu}{d}\left[\frac{1}{\pi\beta}\int_0^\pi \varphi(z)dz + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{e^{-n^2 D_2 \tau}}{\beta + n^2 D_L}\cos nx\int_0^\pi \cos nz\varphi(z)dz\right].$$

For any $k \in N$, iterating this operator k times results in

$$\mathbf{L}^k(\varphi)(x) = \left[e^{-\beta_L \tau}\frac{r\mu}{\beta d}\right]^k\left[\frac{1}{\pi}\int_0^\pi \varphi(z)dz\right] + \mathbf{H}_k(x),$$

where $\mathbf{H}_k(x)$ satisfies $\int_0^\pi \mathbf{H}_k(x)dx = 0,\ \forall\, k \in N$. Hence,

$$\|\mathbf{L}^k\| = \sup \frac{\int_0^\pi \mathbf{L}^k(\varphi)(x)dx}{\int_0^\pi \varphi(x)dx} = \left[e^{-\beta_L \tau}\frac{r\mu}{\beta d}\right]^k.$$

Note that $\mathbf{L}$ is a bounded operator and it is well-known that

$$r(\mathbf{L}) = \lim_{k\to\infty} \|\mathbf{L}^k\|^{1/k} = \frac{r\mu}{\beta d}e^{-\beta_L \tau}.$$

This together with (3.4) leads to

$$\mathcal{R}_0 = \frac{r\mu}{\beta d}e^{-\beta_L \tau}. \tag{4.7}$$

Obviously, the two formulas (4.3) and (4.7) agree with Theorem 3.1 and Corollary 4.1.

From (4.3) and Theorem 3.1, or (4.7) and Corollary 4.1, the threshold dynamics of the model under (4.1) can be explicitly determined by the model parameters and is independent of all constant diffusion rates. This is reasonable since in an homogeneous environment with homogeneous Neumann boundary conditions, solutions generically tend to be homogenized.

5 Discussion

We have derived a model, given by (1.16), to describe the dynamics of a disease that has a latent period τ and spreads in a host population that habitats in a continuous bounded domain Ω. We have shown that the model has a threshold dynamics which can be expressed either in terms of the principal eigenvalue $\lambda(u_1^*)$ of the linear scalar equation (2.22) (Corollary 2.1), or in terms of the basic reproduction number $\mathcal{R}_0$ (Corollary 4.1) which is identified as the spectral radius of the next generation operator $\mathbf{L}$ given by (3.3).

For the special case when all model parameters are positive constants, we are able to compute $\lambda(u_1^*)$ and $\mathbf{L}$ analytically, as is done in Sect. 3. However, in general situations, computations of $\lambda(u_1^*)$ and $\mathcal{R}_0$ are mathematically difficult and challenging, if not impossible. In such a case, as in Wang and Zhao (2011), numeric computations becomes a natural alternative. Here we will not explore numeric computations in this paper. Instead, we shall investigate the possibility of calculating $\mathcal{R}_0$ by using the idea in Krkosek and Lewis (2010), that is, combining the biological meanings of $\mathcal{R}_0$ and $\mathbf{L}$. To this end, we consider a case which is a slightly more general than (4.1) by assuming the following:

$$\begin{aligned} &D_1(x) \equiv D_1,\ D_2(x) \equiv D_2,\ D_L(x) \equiv D_L,\\ &\mu(x) \equiv \mu,\ d(x) \equiv d,\ \beta(x) \equiv \beta,\ \beta_L(x) \equiv \beta_L, \end{aligned} \qquad \text{for } x \in \Omega, \tag{5.1}$$

but allowing $r(x)$ to be space dependent. Then, similar to (4.5), we can obtain the following formula for the next generation operator

$$\mathbf{L}(\varphi)(x) = e^{-\beta_L \tau} \frac{\mu}{d} \int_0^\infty e^{-\beta t} \int_\Omega \int_\Omega K(x, y, \tau) K_2(y, z, t) r(y) \varphi(z) dy dz dt, \tag{5.2}$$

where $K(x, y, t)$ and $K_2(x, y, t)$ are the Green functions associated to the Laplacian operators $D_L \Delta$ and $D_2 \Delta$, respectively, with the homogeneous Neumann boundary condition.

Now, assume that *a single* infectious individual is brought into the the domain, with the probability of landing at the location x being $\varphi(x)$. Then $\varphi(x)$ is a non-negative function defined on Ω satisfying $\int_\Omega \varphi(x)\, dx = 1$. By the meaning of $\mathbf{L}$, the total number of infectious individuals caused by this single infectious individual with the probability distribution $\varphi(x)$ for his/her initial location is then given by

$$\begin{aligned} &\int_\Omega (\mathbf{L}\varphi)(x)\, dx \\ &\quad = e^{-\beta_L \tau} \frac{\mu}{d} \int_0^\infty e^{-\beta t} \int_\Omega \int_\Omega \int_\Omega K(x, y, \tau) K_2(y, z, t) r(y) \varphi(z)\, dy\, dz\, dt\, dx. \end{aligned} \tag{5.3}$$

Noting that $\int_\Omega K(x, y, \tau) dx = 1$ for all $y \in \Omega$ and $\tau \geq 0$, we can further compute the above as

$$\begin{aligned} \int_\Omega (\mathbf{L}\varphi)(x)\, dx &= e^{-\beta_L \tau} \frac{\mu}{d} \int_0^\infty e^{-\beta t} \int_\Omega \int_\Omega K_2(y, z, t) r(y) \varphi(z)\, dy\, dz\, dt \\ &= e^{-\beta_L \tau} \frac{\mu}{d} \int_0^\infty e^{-\beta t} \int_\Omega g(z, t) \varphi(z)\, dz\, dt, \end{aligned} \tag{5.4}$$

where

$$g(z, t) = \int_\Omega K_2(y, z, t) r(y)\, dy. \tag{5.5}$$

Note that the total number give by (5.4) depends on the initial distribution $\varphi(x)$. Taking the worst scenario, the maximal number that a single infectious individual can cause, which should be nothing but the basic production number of the model by its biological definition, is given by

$$\mathcal{R}_0 = \sup_{\varphi\geq 0,\int_\Omega \phi(x)dx=1} \int_\Omega (\mathbf{L}\varphi)(x)\,dx = e^{-\beta_L\tau}\frac{\mu}{d}\int_0^\infty e^{-\beta t} \sup_{\varphi\geq 0,\int_\Omega \phi(x)dx=1} \int_\Omega g(z,t)\varphi(z)\,dz\,dt$$

$$= e^{-\beta_L\tau}\frac{\mu}{d}\int_0^\infty e^{-\beta t} \max_{z\in\bar{\Omega},t\geq 0} g(z,t)\,dt. \tag{5.6}$$

Now, $g(x,t)$ can be considered as the solution to the following initial-boundary value problem:

$$\begin{cases} \frac{\partial u}{\partial t} = D_2\Delta u(x,t) - \beta u(x,t), & x\in\Omega,\ t>0,\\ \frac{\partial u(x,t)}{\partial \nu}\Big|_{\partial\Omega} = 0, & t>0,\\ u(x,0) = r(x), & x\in\Omega. \end{cases}$$

By the maximal principle for this problem, one concludes that

$$\max_{x\in\bar{\Omega},t\geq 0} g(z,t) = \max_{x\in\bar{\Omega}} r(x) := \hat{r}$$

and hence,

$$\mathcal{R}_0 = \sup_{\varphi\geq 0,\int_\Omega \varphi(x)dx=1} \int_\Omega (\mathbf{L}\varphi)(x)\,dx = e^{-\beta_L\tau}\frac{\mu\hat{r}}{d}\int_0^\infty e^{-\beta t}\,dt = \frac{\mu\hat{r}}{\beta d}e^{-\beta_L\tau}. \tag{5.7}$$

It is obvious that the formula (4.6) is a direct result of (5.7) when $r(x)$ is also constant.

We point out that the results obtained under (4.1) or (5.1) show that the magnitudes of the positive constant diffusion rates have no impact on the threshold dynamics of the disease. This is in contrast to the results in Li and Zou (2010) where a similar model but on the one dimensional whole spatial space $\mathbb{R}$ was considered and traveling wave fronts were investigated; and it was seen in Li and Zou (2010) that the constant diffusion rates can affect the speed of traveling wave front and the spatial spread speed of the disease. However, a reader should not be misled by this conclusion for these two very special cases. Indeed, as can be seen from the definition of $\lambda(u_1^*)$ and $\mathcal{R}_0$, they both depend on the diffusion coefficients in general case, although such dependence is not easy to analyze. For a similar model, Wang and Zhao (2011) numerically explored the impact of diffusion rates on the basic reproduction number $\mathcal{R}_0$ when the spatially periodic transmission functions are adopted, and the results show that $\mathcal{R}_0$ is a deceasing function of the diffusion rate for susceptible host population. Rigorous study of the impact of the space dependent parameters on $\lambda(u_1^*)$ and $\mathcal{R}_0$ seems to be meaningful and worthwhile project and we will have to leave it as a future work.

Finally, when $\mathcal{R}_0 > 1$ or equivalently $\lambda(u_1^*) > 0$, the model allows an endemic steady state $\hat{u}(x) = (\hat{u}_1(x), \hat{u}_2(x))$. The stability of $\hat{u}(x)$ is an important but very difficult problem. When the demographic function in (1.11) is replace by the logistic function with constant coefficients: $S(x,t)[c_0 - c_1 S(x,t)]$, the mass action infection

incidence function is replaced by $rSI/(b_0 + b_2 I)$ with $b_0 > 0$ and $b_2 > 0$ and all other model parameters are also constants, the stability of $\hat{u}(x)$ (constant in this case) can be obtained by employing Theorem 4.1-(i) in Thieme and Zhao (2001) under some conditions on the constant parameters. But these conditions will not hold when $b_2 = 0$ and thus Theorem 4.1-(i) does not apply to our model (1.16). Whether or not the method used in proving Theorem 4.1-(i) can be extended to (1.16) with location dependent parameters remains an open problem worthy of exploring.

We conclude the paper by remarking that infectivity of a disease usually depends on the infection age. Since our focus here is on spatial issues, we choose not to include this factor in this work.

Acknowledgments The first author is grateful to the University of Western Ontario for the support from its Visiting University Scholar Program, and he would like to thank the staff in the Department of Applied Mathematics at UWO for their friendly and helpful assistance during his visit. The authors would like to thank the two anonymous referees and the handling editor for their helpful comments which have led to an improvement of the presentation of the paper.

6 Appendix

In this Appendix we give the detailed proofs of Lemmas 2.2, 2.4, and 3.1.

Proof of Lemma 2.2 By standard theory of parabolic equations (see, e.g., Pao 1992; Pazy 1983), we know that for any $\phi \in C(\bar{\Omega}, \mathbb{R}_+)$, the linear equation (2.5) has a unique solution $w(x, t, \phi)$ defined for $t \in [0, \infty)$ with $w(\cdot, 0, \phi) = \phi(\cdot)$, implying that (2.5) generate a semiflow (solution semiflow), denoting it by $\Pi(t)$, i.e., $(\Pi(t)\phi)(\cdot) = w(\cdot, t, \phi)$. Let $\bar{g} = \max\{g(x) : x \in \bar{\Omega}\}$, $\underline{g} = \min\{g(x) : x \in \bar{\Omega}\}$, $\bar{d} = \max\{d(x) : x \in \bar{\Omega}\}$, $\underline{d} = \min\{d(x) : x \in \bar{\Omega}\}$. It follows from the standard comparison theorem and maximal principle (see, e.g., Pao 1992; Protter and Weinberger 1984) that for any $\phi \in C(\bar{\Omega}, \mathbb{R}_+)$ with $\phi \not\equiv 0$, its omega limit set $\omega(\phi)$ satisfies $\omega(\phi) \subset \{\psi : \underline{g}/\bar{d} \le \psi \le \bar{g}/\underline{d}\} =: M_0$. This implies that $\Pi(t)$ has a global compact attractor M contained in M_0. From Hirsch (1984, Theorems 3.2) (also see Martin and Smith 1991, Theorem 4.1), it follows that M contains a steady state of (2.5), which is of course positive. Assume that there are two steady states, $w_1(x)$ and $w_2(x)$ for (2.5). Then $u(x) = w_2(x) - w_1(x)$ satisfies $\nabla \cdot [D(x)\nabla u] - d(x)u = 0$ for $x \in \Omega$ and $(D(x)\nabla u) \cdot \nu = 0$, $x \in \partial\Omega$. The strong maximal principle (see, e.g.,Protter and Weinberger 1984) would lead to $u(x) = 0$ for $x \in \Omega$. Thus, (2.5) has exactly one steady state, denoting it by $w^*(x)$. By Hirsch (1984, Theorems 3.3) (also see Martin and Smith 1991, Theorem 4.1), $w^*(x)$ attracts every solution $w(x, t, \phi)$ with $\phi \in C(\bar{\Omega}, \mathbb{R}_+)$ and $\phi \not\equiv 0$, completing the proof.

Proof of Lemma 2.4 Let $\mathbb{Y} = C(\bar{\Omega}, \mathbb{R})$, $\mathbb{Y}^+ = C(\bar{\Omega}, \mathbb{R}_+)$, $\mathbb{E} = C([-\tau, 0], \mathbb{Y})$, $\mathbb{E}^+ = C([-\tau, 0], \mathbb{Y}^+)$ and $\mathbb{B} = A_2 : \mathfrak{D}(A_2) \to \mathbb{Y}$ be the generator of $\mathbb{T}(t) := T_2(t)$, where T_2 is defined in (2.1). Clearly, $\mathbb{Y}$ is a Banach lattice. Define $\mathbb{L} : \mathbb{E} \to \mathbb{Y}$ by

$$\mathbb{L}\phi(x) = \int_\Omega \Gamma(x, y, \tau) r(y) m(y) \phi(y, -\tau) dy, \quad x \in \Omega, \ \phi \in \mathbb{E}.$$

It is easy to see that $\mathbb{L}$ is positive, that is, $\mathbb{L}(\mathbb{E}^+) \subset \mathbb{Y}^+$. For each $\lambda \in \mathbb{R}$, we define $\mathbb{L}_\lambda : \mathbb{Y} \to \mathbb{Y}$ by

$$\mathbb{L}_\lambda(\varphi) = \mathbb{L}(e^{\lambda \cdot}\varphi), \quad \varphi \in \mathbb{Y},$$

where $e^{\lambda \cdot}\varphi \in \mathbb{E}$ is defined by

$$(e^{\lambda \cdot}\varphi)(\theta, x) = e^{\lambda\theta}\varphi(x), \quad \theta \in [-\tau, 0],\ x \in \bar{\Omega},\ \varphi \in \mathbb{Y}.$$

Then the system (2.15) is equivalent to

$$\begin{cases} \frac{dv(t)}{dt} = \mathbb{B}v(t) + \mathbb{L}v_t, \quad t > 0, \\ v_0 = \phi \in \mathbb{E}, \end{cases} \tag{6.1}$$

and the system (2.17) is equivalent to

$$\begin{cases} \frac{dv(t)}{dt} = \mathbb{B}v(t) + \mathbb{L}_0 v(t) = (\mathbb{B} + \mathbb{L}_0)v(t), \quad t > 0, \\ v(0) \in \mathbb{Y}, \end{cases} \tag{6.2}$$

Let $U(t) : \mathbb{E} \to \mathbb{E},\ t \geq 0$ be the solution semiflow associated with the abstract delay equation (6.1) and let $A_U : \mathfrak{D}(A_U) \to \mathbb{E}$ be its generator (see, e.g., Wu 1996). Then $U(t) : \mathbb{E} \to \mathbb{E}$ is positive (see, e.g., Kerscher and Nagel 1984, Section 4). Let $\sigma(A_U)$ be the spectral set and $s(A_U) = \{Re\lambda : \lambda \in \sigma(A_U)\}$ (i.e., the spectral bound of A_U). By Kerscher and Nagel (1984, Section 4), we indeed have $s(A_U) \in \sigma(A_U)$.

We now prove that $s(A_U)$ is a *point* spectral value of A_U and $s(A_U)$ has a strongly positive eigenvector $\psi \in int(\mathbb{E}^+)$. To this end, we first show that the operators $U(t)$ are eventually strongly positive. For any $\phi = \phi(\theta, x) \in \mathbb{E}^+$ with $\phi \not\equiv 0$, let $v(x, t) = v(x, t, \phi),\ x \in \Omega,\ t \geq 0$ be the solution of (6.1) (i.e. (2.15)), that is, $U(t)\phi = v(\cdot, t, \phi)$.

Claim $v(x, t) > 0,\ \forall x \in \bar{\Omega},\ t > \tau$.
To prove this claim, we only need consider two cases: Case(I). $\phi(0, \cdot) \not\equiv 0$. In this case, the strong maximum principle (see, e.g., Protter and Weinberger 1984, p. 172, Theorem 4) and the Hopf boundary lemma (see, e.g., Protter and Weinberger 1984, p. 170, Theorem 3) imply that $v(x, t) > 0,\ \forall x \in \bar{\Omega},\ t > 0$. Case(II). There is a $\theta_0 \in (0, \tau)$ such that $\phi(-\theta_0, \cdot) \not\equiv 0$. In this case, we first show that $v(\cdot, \tau - \theta_0) \not\equiv 0$, by contradiction. If $v(\cdot, \tau - \theta_0) \equiv 0$, it follows from (6.1) (or (2.15)) that

$$\begin{aligned} \frac{\partial v(x, \tau - \theta_0)}{\partial t} &= \mathbb{B}v(x, \tau - \theta_0) + \int_\Omega \Gamma(x, y, \tau) r(y) m(y) v(y, -\theta_0) dy \\ &= 0 + T_2(\tau)(r(\cdot)m(\cdot)v(\cdot, -\theta_0))(x) > 0, \quad x \in \Omega, \end{aligned}$$

where T_2 is defined in (2.1) and we have used the strong positivity of $T_2(t) > 0$ for $t > 0$. On the other hand, since $v(\cdot, t) \geq 0,\ \forall\ t \geq 0$ and $v(\cdot, \tau - \theta_0) \equiv 0$, it follows that $\frac{\partial v(x, \tau - \theta_0)}{\partial t} \leq 0$, which is a contradiction. Thus, $v(\cdot, \tau - \theta_0) \not\equiv 0$.

It then follows from the strong maximum principle and the Hopf boundary lemma that $v(x,t) > 0,\ \forall x \in \bar{\Omega},\ t > \tau - \theta_0$. Combining Case (I) and Case (II), we have proved that $v(x,t) > 0,\ \forall x \in \bar{\Omega}$ and $t > \tau$, and hence, $v(x,t+\theta) > 0,\ \forall x \in \bar{\Omega},\ t > 2\tau$. This implies that $U(t) : \mathbb{E} \to \mathbb{E}$ is strongly positive for each $t > 2\tau$.

Moreover, $U(t) : \mathbb{E} \to \mathbb{E}$ is compact for each $t > \tau$. Thus, for any $t > 2\tau$, by Krein–Rutman theorem (see, e.g., Smith 1995, Theorem 2.4.1), the spectral radius $r(U(t)) := \sup\{|\ \lambda\ |: \lambda \in \sigma(U(t))\}$ is a positive eigenvalue of $U(t)$ and its corresponding eigenvector is strongly positive. By the point spectral mapping theorem (Pazy 1983, Theorem 2.2.4), there is a point spectral value $\bar{\lambda}$ of A_U such that $r(U(t)) = e^{\bar{\lambda}t}$. Clearly, $\bar{\lambda}$ is also real and $\bar{\lambda} \le s(A_U)$. Moreover, by the fact that $s(A_U) \in \sigma(A_U)$ and the spectral mapping theorem (Pazy 1983, Theorem 2.2.3), it follows that $e^{ts(A_U)} \in \sigma(U(t))$. Thus, $e^{ts(A_U)} \le r(U(t)) = e^{\bar{\lambda}t}$. This implies that $s(A_U) \le \bar{\lambda}$. Therefore, $\bar{\lambda} = s(A_U)$ (also denoting it by $\bar{\lambda}(m)$ to indicate that it depends on $m(x)$), is a point spectral value (or eigenvalue) of A_U. Let $\psi \in \mathbb{E}$ be a nonzero eigenvector of A_U associated with $s(A_U)$. Then $U(t)\psi = e^{ts(A_U)}\psi = r(U(t))\psi$ and hence, again by the Krein–Rutman theorem, it follows that $\psi \in int(\mathbb{E}^+)$. Therefore $s(A_U)$ is the principal eigenvalue of A_U corresponding to which, there is an eigenvector $\psi \in int(\mathbb{E}^+)$, proving (i).

The proof of (ii) is included in the proof of (i), as a special case indeed, confirming the existence of the principal eigenvelue $\lambda = \lambda(m)$. Finally, by Kerscher and Nagel (1984, Section 4) again, $s(A_U)$ has the same sign as $s(\mathbb{B} + \mathbb{L}_0) = \lambda(m)$ does, that is $\bar{\lambda}(m)$ and $\lambda(m)$ have the same sign, completing the proof. □

Proof of Lemma 3.1 Recall that $\mathbb{B} = A_2 : \mathfrak{D}(A_2) \to \mathbb{Y}$ is the generator of $\mathbb{T}(t) := T_2(t)$, where T_2 is defined in (2.1). Since $\mathbb{T}(t)$ is a positive semigroup in the sense that $\mathbb{T}(t)\mathbb{Y}^+ \subset \mathbb{Y}^+,\ \forall\, t \ge 0$, Thieme 2009, Theorem 3.12 implies that $\mathbb{B}$ is resolvent positive, and

$$(\lambda I - \mathbb{B})^{-1}\varphi = \int_0^\infty e^{\lambda t}\mathbb{T}(t)\varphi, \quad \forall\, \lambda > s(\mathbb{B}),\ \varphi \in \mathbb{Y}. \tag{6.3}$$

By (2.1) and the fact that $\mathbb{T}(t) := T_2(t)$, it is easy to see that there exists some $\epsilon_0 > 0$ such that

$$\lim_{t\to\infty} e^{\epsilon_0 t}\mathbb{T}(t)\varphi = 0, \quad \forall\, \varphi \in \mathfrak{D}(\mathbb{B}).$$

By Thieme (2009, Theorem 3.13) (see also Engel and Nagel 2000, section V.1), it follows that $s(\mathbb{B}) < 0$.

Letting $\lambda = 0$ in (6.3), we obtain $-\mathbb{B}^{-1}\varphi = \int_0^\infty \mathbb{T}(t)\varphi,\ \forall\, \varphi \in \mathbb{Y}$. Thus, we have $\mathbf{L} = -\mathbb{C}\mathbb{B}^{-1}$, where $\mathbf{L}$ and $\mathbb{C} := V$ are defined in (3.3) and (3.1), respectively. From the linear system (2.22) with $m(\cdot) = u_1^*(\cdot)$, we further see that the linear operator $\mathbb{A} := \mathbb{B} + \mathbb{C}$ generates a positive C_0-semigroup, and hence $\mathbb{A}$ is also resolvent positive. Clearly, $\lambda(u_1^*) = s(\mathbb{A})$. It then follows from Thieme (2009, Theorem 3.5) that $s(\mathbb{A})$ has the same sign as $r(-\mathbb{C}\mathbb{B}^{-1}) - 1 = r(\mathbf{L}) - 1 = \mathcal{R}_0 - 1$.

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