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New generic quasi-convergence principles with applications ☆

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ABSTRACT

In this paper, essentially strongly order-preserving and conditionally set-condensing semiflows are considered. Obtained is a new type of generic quasi-convergence principles implying the existence of an open and dense set of stable quasi-convergent points when the state space is order bounded. The generic quasi-convergence principles are then applied to essentially cooperative and irreducible systems in the forms of ordinary differential equations and delay differential equations, giving some results of theoretical and practical significance.

1. Introduction

In recent years there have been many works dealing with generic properties of strongly monotone dynamical systems (see [4–8,10–12,15,17–19,21,24–26]). As is known, two key ingredients in the proof of these principal results for monotone dynamical systems are *monotonicity and compactness*. Consequently, one can improve these results by weakening the monotonicity or/and compactness assumptions. Recently, inspired by the work of Smith and Thieme [19], Yi and Huang [24] derived some results on generic quasi-convergence by replacing the classic strongly order-preserving condition with the *essentially strongly order-preserving* condition and applied the results to a quasimonotone system of delay differential equations. An advantage of this new order-preserving property is that it does not require delicate choice of the state space and the technical *ignition* assumption required in the classical work.

On the compactness aspect, Hirsch and Smith [7,8] introduced the assumption that the limit sets have infima and suprema in the state space and proved that the generic quasiconvergence principle holds provided that the strong compactness assumption is replaced by this assumption. As pointed out by Hirsch and Smith in [7,8], under the standard ordering, this assumption is automatically satisfied in "nice" subsets (every compact subset of such a set has an infimum and supremum in this set) of the Euclidean space or the space of continuous functions on a compact set. Such nice subsets include the whole phase space and standard order cone. Therefore, the results of [7,8] can be *conveniently* applied to systems of delay differential equations defined in the *whole phase space* or in its *nice* subsets, provided the standard ordering is adopted.

However, due to their practical backgrounds, some differential equations may not be defined in the whole space, neither are they defined in "nice" subsets of the space. Even if the equations are defined in the whole space or a "nice" subset of the space, in many cases a *nonstandard ordering* needs to be adopted in order for the solution semi-flow to be monotone

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(e.g. [8,9,13,14,22]). In such cases, a compact subset may not have an infimum and/or supremum in the space where the equations are defined, and hence the replacing assumption for compactness in Hirsch and Smith [7,8] does not automatically hold anymore.

To demonstrate aforementioned possibilities, we consider following system of delay differential equations:

$$x'(t) = f(x_t), \tag{1.1}$$

where $f : \Omega \to R^3$ is a continuous map, where $r > 0$, Ω is the closure of an open subset of $C = C([-r, 0], R^3)$, and C is the Banach space equipped with the usual supremum norm and the usual pointwise standard function ordering. Assume also that for each $\varphi \in \Omega$, (1.1) has a unique solution on R^1_+ satisfying $x_0 = \varphi$. Let

$$\Omega = \{\varphi \in C:\ \varphi : [-r, 0] \to \{(x, y, z) \in R^3:\ y \geqslant |x|,\ z \geqslant 0\}\}$$

and choose $\zeta : [-r, 0] \mapsto (0, 2, 0)$ and $\eta : [-r, 0] \mapsto (-4, 5, 0)$. Now if Ω_1 is a compact subset in Ω such that $\zeta, \eta \in \Omega_1$, then it is clear that Ω_1 cannot attain its infimum in Ω.

Consider (1.1) again with $\Omega = C([-r, 0], R^3)$ (the whole space), but with ordering induced by the cone $C_+ = C([-r, 0], \{(x, y, z) \in R^3:\ z \geqslant \sqrt{x^2 + y^2}\})$. Let $\zeta : [-r, 0] \mapsto (1, 0, 0)$, $\eta : [-r, 0] \mapsto (2, 0, 0)$, $\xi : [-r, 0] \mapsto (1.5, 0, 0.5)$ and $\varrho : [-r, 0] \mapsto (1, 1, \sqrt{2})$. Then $\Omega_1 = \{\zeta, \eta\}$ is a compact subset of Ω. A simple computation shows that $\xi, \varrho \geqslant \Omega_1$. We now claim that ξ is the minimal element of $\{\psi \in \Omega:\ \psi \geqslant \Omega_1\}$. Otherwise, there exists $\varphi \in \Omega$ such that $\varphi \geqslant \Omega_1$ and $\xi > \varphi$, and hence $\xi(r_1) > \varphi(r_1)$ for some $r_1 \in [-r, 0]$. Let $\varphi(r_1) = (a, b, c) \in R^3$. Then by the choice of φ and r_1, we have $\varphi(r_1) \geqslant \zeta(r_1)$, $\varphi(r_1) \geqslant \eta(r_1)$ and $\xi(r_1) > \varphi(r_1)$, that is, $c \geqslant \sqrt{(a-1)^2 + b^2}$, $c \geqslant \sqrt{(a-2)^2 + b^2}$, $0.5 - c \geqslant \sqrt{(a-1.5)^2 + b^2}$ and $c \neq 0.5$. Thus $c \geqslant |a-1|$, $c \geqslant |a-2| \geqslant 1 - |a-1|$ and $0.5 - c > 0$, a contradiction. Since $\varrho - \xi \notin C_+$, Ω_1 cannot have its supremum in Ω.

From the above, one can see that seeking more convenient conditions for the compactness part of the generic quasi-convergence principle still remains an interesting and meaningful problem. Notice that compactness is a concept independent of the ordering of the state space, while the concepts of supremum and infimum are related to the ordering. This motivates us to look for alternative conditions along directions other than that in Hirsch and Smith [7,8].

Recall that when considering a delay differential system of the form (1.1), the following assumptions are typically required (before Hirsch and Smith [7,8], see [17–20,23,24]) to guarantee strong compactness of the solution semi-flow for the generic quasi-convergence principle:

(T1) f maps bounded subsets of Ω to bounded subsets of R^3, and for each $\varphi \in \Omega$, $x_t(\varphi)$ is a bounded solution defined for $t \geqslant 0$.
(T2) For each compact subset A of Ω, there exist $T = T_A > 0$ and a closed, bounded subset $B = B_A$ of Ω such that $x_t(A) \subset B$ for all $t \geqslant T$.

In this paper, we shall show that (T1) *alone* is enough to serve the compactness purpose for the generic quasi-convergence property for (1.1). To this end, in Section 2, we first state some preliminary results; then, we establish a new type of generic quasi-convergence principles for essentially strongly order-preserving and conditionally set-condensing semi-flows. In Section 3, we apply our generic quasi-convergence principles to cooperative and irreducible ordinary differential equations and essentially cooperative and irreducible delay differential equations of the form (1.1). It is amazing and a little bit surprising that the observation that (T2) can be dropped in this context has been overlooked for such a long time.

2. Main results

Assume that X is an ordered complete metric space with a metric d and a closed partial order relation $\leqslant$. For any $x, y \in X$, we write $x < y$ if $x \leqslant y$ and $x \neq y$. Given two subsets A and B of X, we write $A \leqslant B$ ($A < B$, resp.) if $x \leqslant y$ ($x < y$) holds for all $x \in A$ and $y \in B$. For subsets A and B with $A \leqslant B$, denote $[A, B] = \{x \in X:\ A \leqslant x \leqslant B\}$.

We call X an ordered bounded space provided $[a, b]$ is bounded for any $a, b \in X$ with $a \leqslant b$. Clearly, $[A, B]$ is bounded for any subsets A and $B \in X$ such that $A \leqslant B$ provided X is an ordered bounded space.

The metric space X is normally ordered if there exists a constant $k > 0$ such that $d(x, y) \leqslant kd(u, v)$ whenever $u, v \in X$ and $x, y \in [u, v]$.

A semiflow on X is a continuous map $\Phi : X \times R^1_+ \to X$ with $\Phi_t(x) \triangleq \Phi(x, t)$ satisfying

(i) $\Phi_0(x) = x$ for all $x \in X$;
(ii) $\Phi_t(\Phi_s(x)) = \Phi_{t+s}(x)$ for all $x \in X$ and $t, s \in R^1_+$.

For $x \in X$, $O(x) = \{\Phi_t(x):\ t \geqslant 0\}$ is the orbit of x, and $\omega(x) = \bigcap_{t \geqslant 0} \overline{O(\Phi_t(x))}$ is the omega limit set of x. As is well known, if $\overline{O(x)}$ is compact, then $\omega(x)$ is nonempty, compact, connected, and invariant.

Let $\mathbb{E} = \{x \in X:\ \Phi_t(x) = x \text{ for all } t \geqslant 0\}$ be the set of equilibria of Φ. The set of quasi-convergent points is denoted by $\mathbb{Q} = \{x \in X:\ \omega(x) \subset \mathbb{E}\}$ and the set of convergent points by $\mathbb{C} = \{x \in X:\ \omega(x) \text{ is a singleton set}\}$.

Let Φ be a semiflow on X. We say Φ is monotone if $\Phi_t(x) \leqslant \Phi_t(y)$ whenever $x \leqslant y$ and $t \geqslant 0$. For a given monotone semiflow Φ and fixed $t_0 \geqslant 0$, denote $x \prec_{(\Phi,t_0)} y$ if $x \leqslant y$ and $\Phi_{t_0}(x) < \Phi_{t_0}(y)$. We shall write $\prec$ for $\prec_{(\Phi,t_0)}$ when no confusion results. Note that the relations "$\prec_{(\Phi,t_0)}$" is just "$<$" when $t_0 = 0$. A monotone semiflow Φ is said to be *essentially* strongly order-preserving if for any $x, y \in X$ with $x \prec y$, there exist $T_0 \geqslant 0$ and open sets U and V such that $x \in U$, $y \in V$ and $\Phi_{T_0}(U) \leqslant \Phi_{T_0}(V)$. As pointed out in [24], when $t_0 = 0$, an essentially strongly order-preserving semiflow becomes a strongly order-preserving semiflow in the sense of Smith and Thieme [19].

For a given measure β of noncompactness, a continuous map $H : X \to X$ is said to be conditionally set-condensing on X if $\beta(TB) < \beta(B)$ for each bounded subset B in X for which TB is bounded in X and $\beta(B) > 0$.

For convenience of statements, we introduce the following assumptions:

(A1) Φ is an essentially strongly order-preserving semiflow on X.
(A2) There exists $t_1 > 0$ such that Φ_{t_1} is a conditionally set-condensing map on X.
(A3) For each $x \in X$, $O(x)$ is a bounded subset of X.
(A4) X is an ordered bounded space.
(A5) X is a normally ordered space.

Note that the assumptions (A2) and (A3) imply that every orbit has a compact closure in X, due to the definition of the conditionally set-condensing map.

If the metric space X is normally ordered, then there exists a constant $k > 0$ such that $d(x, y) \leqslant kd(u, v)$ whenever $u, v \in X$ and $x, y \in [u, v]$. So, for any order interval $[a, b] \subseteq X$ and $x \in [a, b]$, we have $d(x, a) \leqslant kd(a, b)$ and $[a, b]$ is bounded. Hence, the assumption (A5) implies (A4).

For $x \in X$, we say that x can be essentially approximated from below (resp. above) if there exists a sequence $\{x_n\} \in X$ such that $x_n \prec x_{n+1} \prec x$ (resp. $x \prec x_{n+1} \prec x_n$) and $x_n \to x$ as $n \to \infty$. In this case we say sequence $\{x_n\}$ essentially approximates x from below or above. The sequence $\{x_n\}$ is omega compact if $\bigcup_{n\geqslant 1} \omega(x_n)$ has compact closure contained in X.

For $x \in X$, we say that x is a stable point if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $d(\Phi_t(x), \Phi_t(y)) < \varepsilon$ for $t \geqslant 0$ whenever $y \in X$ and $d(x, y) < \delta$; we say that a stable point x is an asymptotically stable point if there is a neighborhood V of x with the property that for every $\varepsilon > 0$ there exists $T_\varepsilon > 0$ such that $d(\Phi_t(x), \Phi_t(y)) < \varepsilon$ if $t \geqslant T_\varepsilon$ and $y \in V$.

Define

$$X_- = \{x \in X\colon\ x \text{ can be essentially approximated from below by a sequence of } X\},$$

$$X_+ = \{x \in X\colon\ x \text{ can be essentially approximated from above by a sequence of } X\},$$

$$X_-^\omega = \{x \in X\colon\ x \text{ can be essentially approximated from below by an omega compact sequence of } X\},$$

$$X_+^\omega = \{x \in X\colon\ x \text{ can be essentially approximated from above by an omega compact sequence of } X\},$$

$$\mathbb{S} = \{x \in X\colon\ x \text{ is stable point}\},$$

$$\mathbb{A} = \{x \in X\colon\ x \text{ is asymptotically stable point}\}.$$

Obviously, $X_+^\omega \subseteq X_+$, $X_-^\omega \subseteq X_-$ and $\mathbb{A} \subseteq \mathbb{S}$.

We need the following proposition which has been proved in Yi and Huang [24].

Proposition 2.1 *(Limit set dichotomy). Assume that the assumption* (A1) *holds and every orbit has compact closure in X. Let $x, y \in X$ satisfy $x \prec y$. Then, one of the following holds*:

(i) $\omega(x) < \omega(y)$;
(ii) $\omega(x) = \omega(y) \subset \mathbb{E}$.

Proposition 2.2. *Assume that* (A1)–(A4) *hold. Then $X_+ \subseteq X_+^\omega$ and $X_- \subseteq X_-^\omega$, and hence $X_+ = X_+^\omega$ and $X_- = X_-^\omega$.*

Proof. We may assume without loss of generality that $x \in X_-$ and thus there exists a sequence $\{x_n\}$ in X which essentially approximates x from below. By Proposition 2.1, we then have $\omega(x_1) = \omega(x)$ or $\omega(x_1) < \omega(x)$. If the former holds, then by Proposition 2.1 $\omega(x_n) = \omega(x)$ for all positive integers n and hence the sequence $\{x_n\}$ is omega compact. If the latter holds, then $[\omega(x_1), \omega(x)]$ is bounded since X is ordered bounded space. Applying Proposition 2.1, we can deduce $\bigcup_{n\geqslant 2} \omega(x_n) \subseteq [\omega(x_1), \omega(x)]$, and hence $\bigcup_{n\geqslant 1} \omega(x_n)$ is bounded. By the invariance of the omega set, $\bigcup_{n\geqslant 1} \omega(x_n)$ is invariant. In view of the definition of conditionally set-condensing, $\bigcup_{n\geqslant 1} \omega(x_n)$ is pre-compact and thus $\overline{\bigcup_{n\geqslant 1} \omega(x_n)}$ is compact, that is, the sequence $\{x_n\}$ is also omega compact in this case. Consequently, $x \in X_-^\omega$. This completes the proof. □

Now we are in the position to state and prove the following new generic quasi-convergence principles.

Theorem 2.1. *Let* (A1)–(A4) *hold. Suppose Y is an open and dense subset of X. If $Y \subseteq X_- \cup X_+$, then $Y \subseteq \operatorname{Int}\mathbb{Q} \cup \overline{\operatorname{Int}\mathbb{C}}$, and hence $X = \overline{Y} = \overline{\operatorname{Int}\mathbb{Q}}$.*

Proof. By Proposition 2.2, $Y \subseteq X_-^\omega \cup X_+^\omega$. Suppose that $x_0 \in Y \setminus \mathrm{Int}\,\mathbb{Q}$. Then there exists $x_n \in Y \setminus \mathbb{Q}$ such that $x_n \to x_0$ as $n \to \infty$. Without loss of generality, we may assume that $x_n \in X_-^\omega$ holds for all $n \geqslant 1$. Since $x_n \notin \mathbb{Q}$, it follows from Propositions 4.1 and 4.3 in [24] that $x_n \in \overline{\mathrm{Int}\,\mathbb{C}}$ for all $n \geqslant 1$. Therefore, $x_0 \in \overline{\mathrm{Int}\,\mathbb{C}}$. This completes the proof. □

Theorem 2.2. *Let* (A1)–(A3) *and* (A5) *hold. If there exists an open and dense subset* Y *of* X *with* $Y \subseteq X_- \cap X_+$, *then* $\overline{\mathbb{A} \cup \mathrm{Int}(\mathbb{S} \cap \mathbb{C})} = X$, *and hence* $X = \overline{\mathrm{Int}\,\mathbb{S}}$.

Proof. By Proposition 2.2, we obtain $Y \subseteq X_-^\omega \cap X_+^\omega$. For the sake of contradiction, assume $\overline{\mathbb{A} \cup \mathrm{Int}(\mathbb{S} \cap \mathbb{C})} \neq X$. Then there exists an open subset U of X such that $U \cap \mathbb{A} = \emptyset$ and $U \cap \mathrm{Int}(\mathbb{S} \cap \mathbb{C}) = \emptyset$. Without loss of generality we may assume $U \in Y$ since Y is an open, dense subset of X. By $Y \subseteq X_-^\omega \cap X_+^\omega$, $U \cap \mathbb{A} = \emptyset$ and $U \subseteq Y$, it follows from Propositions 4.1 and 4.4 in [24] that $U \subset \mathbb{S} \cap \mathbb{C}$. Therefore, $U \cap \mathrm{Int}(\mathbb{S} \cap \mathbb{C}) \neq \emptyset$, a contradiction. This completes the proof. □

3. Applications

In this section, we first introduce some concepts and notations which will be used in the rest of this paper. Let $K \subset R^n$ be a cone with nonempty interior and denote by $\mathrm{Int}\,K$ the interior of K in R^n, and by ∂K the boundary of K in R^n. In what follows, K^* will be used to denote the dual cone of K, i.e., $K^* = \{\lambda \in R^n\colon\ \lambda(x) \geqslant 0 \text{ for all } x \in K\}$. Let n be a positive integer, r be a positive real number and $C = C([-r,0], R^n)$ denote the Banach space of all continuous mappings $\varphi : [-r,0] \to R^n$. Also let $C_+ = C([-r,0], K)$ denote all continuous mappings $\varphi : [-r,0] \to K$.

For $x, y \in R^n$, we denote

(i) $x \leqslant_K y$ iff $y - x \in K$;
(ii) $x <_K y$ iff $x \leqslant_K y$ and $x \neq y$; and
(iii) $x \ll_K y$ iff $y - x \in \mathrm{Int}\,K$.

For $\varphi, \psi \in C$, we denote

(i) $\varphi \leqslant_K \psi$ iff $\psi - \varphi \in C_+$;
(ii) $\varphi <_K \psi$ iff $\varphi \leqslant_K \psi$ and $\varphi \neq \psi$; and
(iii) $\varphi \ll_K \psi$ iff $\psi - \varphi \in \mathrm{Int}(C_+)$.

In the following discussion, we shall use $\leqslant$ (resp. $<$, $\ll$) to replace $\leqslant_K$ (resp. $<_K$, $\ll_K$) when no confusion results.

We need the following elementary results from [1,2,8,9,22].

Lemma 3.1. *Let* $x \in K$. *Then* $x \in \mathrm{Int}\,K$ *if and only if* $\lambda(x) > 0$ *for all* $\lambda \in K^* \setminus \{0\}$.

Lemma 3.2. *Let* $x \in K \setminus \{0\}$. *Then there exists* $\lambda \in K^*$ *such that* $\lambda(x) > 0$.

Lemma 3.3. *The ordered Banach space* R^n (*with cone* K) *and* C (*with cone* C_+) *are ordered bounded and normally ordered space. Hence, so is every subspace of* R^n *or* C.

The conclusions in the following lemma are obvious.

Lemma 3.4. *Let* $X \subseteq R^n$ (*or* C) *be an open subset. Then for any* $x \in X$ *there exists a sequence* $\{x_n\}$ *in* X *such that* $x_n \ll x_{n+1}$ *and* $x_n \to x$ *as* $n \to \infty$. *Similarly, for any* $x \in X$ *there exists a sequence* $\{x_n\}$ *in* X *such that* $x_n \gg x_{n+1}$ *and* $x_n \to x$ *as* $n \to \infty$.

Remark 3.1. In the following discussion, we shall find that for any open subset X of R^n (or C), every point of X can be essentially approximated from below and above since "$\gg$" (resp. "$\ll$") implies "$\succ$" (resp. "$\prec$").

3.1. Ordinary differential equations

Consider the autonomous systems of ordinary differential equations of the form

$$x'(t) = f\bigl(x(t)\bigr), \tag{3.1}$$

where $f : U \to R^n$ is continuous and $U \subset R^n$ with the ordering in R^n generated by the cone K. We assume that for every $x_0 \in U$, (3.1) has a unique solution $x(t)$ for $t \in R_+^1$ with $x(0) = x_0$. Write $\phi_t(x_0)(\phi(t, x_0))$ for the solution of the initial value problem (3.1) with $\phi(0, x_0) = x_0$. In this subsection, we assume that U is the closure of an open subset D of R^n, and for $x, y \in U$ with $x < y$ there exist sequences $x_n, y_n \in D$ satisfying $x_n < y_n$, $x_n \to x$ and $y_n \to y$ as $n \to \infty$.

We need the following cooperative and irreducible assumptions.

(M) For every $x, y \in U$ and $\lambda \in K^*\backslash\{0\}$, $x \leqslant y$ and $\lambda(x) = \lambda(y)$ imply $\lambda(f(x)) \leqslant \lambda(f(y))$.
(I) If $x, y \in U$ with $x \leqslant y$ and $y - x \in \partial K \setminus \{0\}$, then there exists $\lambda \in K^*$ such that $\lambda(y - x) = 0$ and $\lambda(f(y) - f(x)) > 0$.

Definition 3.1. We say that f is cooperative and/or irreducible in U, if f satisfies the assumption (M) and/or (I).

Theorem 3.1. *Let f satisfy the assumption* (M). *Then ϕ is a monotone semiflow. If, in addition, the assumption* (I) *also holds, then ϕ is a strongly monotone semiflow.*

Proof. The first assertion follows from Proposition 1.5 in [22] and the continuity of ϕ. □

We next prove the second assertion. Suppose that $x, y \in U$ with $x > y$ and set $I_t = \{\lambda \in K^*\colon\ \lambda(\phi_t(x) - \phi_t(y)) > 0\}$ for $t \geqslant 0$. Then by Lemma 3.2, we know that I_t is not empty for every $t \geqslant 0$. We will show that $I_t = K^*\backslash\{0\}$ for all $t \geqslant 0$. Otherwise, there exists $\delta > 0$ such that $K^*\backslash I_\delta \neq \{0\}$, and thus by the assumption (I), $\lambda(f(\phi(\delta, x)) - f(\phi(\delta, y))) > 0$ for some $\lambda \in K^*\backslash I_\delta$. It follows from (3.1) that

$$\lambda\big(\phi'(\delta, x) - \phi'(\delta, y)\big) = \lambda\big(f\big(\phi(\delta, x)\big) - f\big(\phi(\delta, y)\big)\big) > 0.$$

Hence, by $\lambda(\phi(\delta, x) - \phi(\delta, y)) = 0$, there exists sufficiently small $\varepsilon > 0$ such that

$$\lambda\big(\phi(\delta - \varepsilon, x) - \phi(\delta - \varepsilon, y)\big) < 0,$$

from which one can conclude that $\phi(\delta - \varepsilon, x) - \phi(\delta - \varepsilon, y) \notin K$, a contradiction to the first assertion. Thus, the second assertion follows from Lemma 3.1.

Remark 3.2. If $f : U \to R^n$ is *continuously differentiable*, Theorem 3.1 has already been obtained in [8,9].

In order to apply Theorem 2.2 to (3.1), we also need the following assumption which corresponds to (T1) for the case of ordinary differential equations.

(TO1) The positive semiorbit of every solution of (3.1) is bounded.

The main result of this subsection is the following.

Theorem 3.2. *Assume that f satisfies the assumptions* (M), (I) *and* (TO1). *Then the set of stable quasiconvergent points for* (3.1) *contains a subset which is open and dense in U.*

Proof. Let $\phi : R^1_+ \times U \to U$ be the solution semiflow for (3.1). It follows from Theorem 3.1 that Φ is a strongly order-preserving semiflow in U. By the assumption (TO1), Φ is compact. It follows from Lemma 3.4 and Remark 3.1 that $D \subseteq U_- \cap U_+$, and thus Theorem 3.2 follows from Theorem 2.2. This completes the proof. □

Remark 3.3. According to Proposition 2.2 and Theorem 3.1, we know that the assumption (TO1) actually implies the strong compactness for the solution semiflow generated by system (3.1) provided the assumptions (M) and (I) hold. Hence, the results of Theorem 3.14 [8] may hold when we drop the condition "$U = AC \cup BC$" in Theorem 3.14 [8].

3.2. Delay differential equations

In this subsection, for a given $r > 0$ and cone K in R^n, denote $C = C([-r, 0], R^n)$ and consider the ordering in C induced by the cone $C([-r, 0], K)$.

Consider the following autonomous systems of delay differential equations

$$x'(t) = F(x_t), \tag{3.2}$$

where $F : \Omega \to R^n$ is continuous and $\Omega \subset C$ is the closure of an open subset $\Pi \subseteq C$. We assume that a solution of (3.2) with $x_0 = \varphi$, for $\varphi \in \Omega$, exists for R^1_+ and is unique. We write $x_t(\varphi)(x(t, \varphi))$ for the solution of the initial value problem (3.2) with $x_0 = \varphi$. In the following, we shall assume that for $\varphi, \psi \in \Omega$ with $\varphi < \psi$ there exist sequences $\varphi_n, \psi_n \in \Pi$ satisfying $\varphi_n \ll \psi_n$, $\varphi_n \to \varphi$ and $\psi_n \to \psi$ as $n \to \infty$.

We begin by introducing the following monotonicity condition.

(MD) For every $\varphi, \psi \in \Omega$ and $\lambda \in K^* \setminus \{0\}$, $\varphi \leqslant \psi$ and $\lambda(\varphi(0)) = \lambda(\psi(0))$ imply $\lambda(F(\varphi)) \leqslant \lambda(F(\psi))$.

The following comparison principle for (3.2) follows from the continuity of $x_t(\varphi)$ and Theorem 4.1 in [8].

Theorem 3.3. *Assume that F satisfies assumption* (MD)*. Then for any $\varphi, \psi \in \Omega$ with $\varphi \leqslant \psi$, we have $x_t(\varphi) \leqslant x_t(\psi)$ for all $t \in R^1_+$.*

Remark 3.4. Assumption (MD) reduces to the assumption (H) in [16] when $K = R^n_+$.

We will introduce an assumption ensuring that whenever $\lambda(x(t_1, \psi) - x(t_1, \varphi)) > 0$ for some $t_1 \in [0, \infty)$ and some $\lambda \in K^*$, it follows that $\lambda(x(t, \psi) - x(t, \varphi)) > 0$ for all $t \in [t_1, \infty)$, that is,

(PD) For $\lambda \in K^*$, there exists a continuous mapping $\alpha_\lambda : \Omega \times \Omega \to R^1$ such that $\lambda(F(\psi) - F(\varphi)) \geqslant \alpha_\lambda(\varphi, \psi)\lambda(\psi(0) - \varphi(0))$.

Lemma 3.5. *Let* (PD) *hold. Then, for any $\varphi, \psi \in \Omega$ and $\lambda \in K^*$ with $\psi \geqslant \varphi$ and $\lambda(\psi(0) - \varphi(0)) > 0$, we have $\lambda(x(t, \psi) - x(t, \varphi)) > 0$ for all $t \in R^1_+$.*

Proof. Clearly, assumption (PD) implies assumption (MD). Then, by Theorem 3.3, we obtain $x_t(\psi) \geqslant x_t(\varphi)$ for all $t \in R^1_+$. It follows from (3.2) and assumption (PD) that

$$\lambda\big(x'(t, \psi) - x'(t, \varphi)\big) = \lambda\big(F\big(x_t(\psi)\big) - F\big(x_t(\varphi)\big)\big) \geqslant \alpha\big(x_t(\varphi), x_t(\psi)\big)\lambda\big(x(t, \varphi) - x(t, \psi)\big).$$

Therefore,

$$\lambda\big(x(t, \psi) - x(t, \varphi)\big) \geqslant \lambda\big(\psi(0) - \varphi(0)\big)e^{\int_0^t \alpha(x_s(\varphi), x_s(\psi))\,ds} > 0$$

for all $t \in R^1_+$. This completes the proof. □

We introduce the following irreducibility assumption.

(ID) Suppose that $\varphi, \psi \in \Omega$ with $\varphi \leqslant \psi$. Denote $D \equiv \{\lambda \in K^*\colon\ \lambda(\psi(\theta) - \varphi(\theta)) > 0,\ \theta \in [-r, 0]\}$. If $D \neq \emptyset$ and $D \neq K^*\backslash\{0\}$, then there exists $\lambda \in K^*\backslash D$ such that either $\lambda(\psi(0) - \varphi(0)) > 0$ or $\lambda(F(\psi) - F(\varphi)) > 0$.

Theorem 3.4. *Assume that the assumptions* (PD) *and* (ID) *hold. Then for any $\psi, \varphi \in \Omega$ with $\psi > \varphi$, either $x(t, \psi) = x(t, \phi)$ for all $t \in R^1_+$ or $x(t, \psi) \gg x(t, \varphi)$ for all $t \in [(n+1)r, \infty)$. Hence, $x_t(\cdot)$ is an essentially strongly order-preserving semiflow with $t_0 = (n+2)r$.*

Proof. By Theorem 3.3, we then have $x(t, \psi) \geqslant x(t, \varphi)$ for all $t \in [-r, \infty)$ and thus either $x(t, \psi) = x(t, \varphi)$ for all $t \in [0, r]$ or $x(s_0, \psi) > x(s_0, \varphi)$ for some $s_0 \in [0, r]$ holds. If the former holds, then $x(t, \psi) = x(t, \psi)$ for all $t \in [0, \infty)$ and the conclusion follows.

We next assume the latter case. Let

$$M_t = \big\{\lambda \in K^*\colon\ \lambda\big(x(t, \psi) - x(t, \varphi)\big) > 0\big\},$$

for all $t \in [0, \infty)$. By Lemma 3.5, we have $M_t \subseteq M_s$, $0 \leqslant t \leqslant s$. Again by the choice of s_0, $M_{s_0} \neq \emptyset$ and hence $M_t \neq \emptyset$ for all $t \in [s_0, \infty)$. We claim that if $t_1 \geqslant 0$ and $M_{t_1} \neq K^* \setminus \{0\}$, then $M_{t_1} \subsetneq M_{t_1+r}$. If this is not true, then $M_{t_1+r+\theta} = M_{t_1}$ for all $\theta \in [-r, 0]$ and thus by (ID) there exists $\lambda \in K^*\backslash M_{t_1+r}$ such that

$$\lambda\big(x(t_1 + r, \psi) - x(t_1 + r, \varphi)\big) = 0$$

and

$$\lambda\big(F\big(x_{t_1+r}(\psi)\big)\big) > \lambda\big(F\big(x_{t_1+r}(\varphi)\big)\big).$$

From (3.2) and the continuity of F, it follows that there exists $\varepsilon > 0$ such that

$$\lambda\big(x(t_1 + r - \varepsilon, \psi) - x(t_1 + r - \varepsilon, \varphi)\big) < 0.$$

This yields a contradiction and thus, our claim follows.

Let $V_k = M_{kr}$ for all nonnegative integer k. We shall show $V_{n+1} = K^*\backslash\{0\}$. Otherwise,

$$\emptyset \neq V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_{n+1} \neq K^*\backslash\{0\}.$$

Let $V^0_k = (K^*\backslash\{0\})\backslash V_k$, where $0 \leqslant j \leqslant n$. Then

$$\emptyset \neq V^0_{n+1} \subsetneq V^0_n \subsetneq \cdots \subsetneq V^0_2 \subsetneq V^0_1.$$

Choose $\lambda_1 \in V^0_{n+1}$, $\lambda_2 \in V^0_n\backslash V^0_{n+1}$, $\ldots$, $\lambda_n \in V^0_2\backslash V^0_3$, $\lambda_{n+1} \in V^0_1\backslash V^0_2$. Clearly, $\{\lambda_i\}_{i=1}^{n+1}$ is linearly dependent, that is, there exist $c_1, c_2, \ldots, c_{n+1}$ not all zero such that $\sum_{i=1}^{n+1} c_i\lambda_i = 0$. Let $i_0 = \sup\{i\colon\ c_i \neq 0\}$. Then

$$\lambda_{i_0} = \frac{c_{i_0-1}}{-c_{i_0}}\lambda_{i_0-1} + \frac{c_{i_0-2}}{-c_{i_0}}\lambda_{i_0-2} + \cdots + \frac{c_2}{-c_{i_0}}\lambda_2 + \frac{c_1}{-c_{i_0}}\lambda_1.$$

From $\lambda_{i_0-1}, \lambda_{i_0-2}, \ldots, \lambda_1 \in V^0_{n+2-i_0}$ and the definition of $V^0_{n+2-i_0}$, it follows that $\lambda_{i_0} \in V^0_{n+2-i_0}$, a contradiction to the choice of λ_{i_0}. Therefore, the theorem follows from Lemma 3.1. This completes the proof. □

Remark 3.5. If $P = R^n_+$, then Theorem 3.4 reduces to [24, Theorem 2.2].

Remark 3.6. Note that Hirsch and Smith in [8] propose a stronger assumption than (ID) and introduce some sufficient conditions to guarantee the property of strongly order-preserving of (3.2).

We next apply Theorem 2.2 to (3.2). To this end, we also need the following assumption corresponding to (T1) for the case of n-dimensional delay differential equations (3.2).

(TD1) F maps bounded subsets of Ω to bounded subsets of R^n, and the positive semiorbit of every solution of (3.2) is bounded.

The main result of this subsection is the following.

Theorem 3.5. *Assume that F satisfies the assumptions* (PD), (ID) *and* (TD1)*. Then Ω contains an open and dense subset of stable quasi-convergent points.*

Proof. Define $\Phi : R^1_+ \times \Omega \to \Omega$ by $\Phi(t,\varphi) = x_t(\varphi)$. It follows from Theorem 3.4 that Φ is an essentially strongly monotone semiflow in Ω with $t_0 = (n+2)|r|$. By the assumption (TD1), Theorem 3.6.1 in Hale [3] implies that Φ_{r+1} is a conditionally completely continuous map and hence Φ_{r+1} is a conditionally set-condensing map. Note that Lemma 3.4 and Remark 3.1 imply $\Pi \subseteq \Omega_- \cap \Omega_+$, and thus Theorem 3.5 follows from Theorem 2.2. This completes the proof. □

Remark 3.7. Theorem 3.5 does not require that F satisfies the technical *ignition* assumption required in the classical work (see [16,17]).

Remark 3.8. By applying Proposition 2.2 to (3.2), we know that the assumption (TD1) alone can actually assure the strong compactness for the solution semiflow generated by system (3.2). It is amazing and a little bit surprising that the fact that the condition corresponding to (T2) can be dropped in this context has been overlooked for such a long time.

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