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Traveling wavefronts in diffusive and cooperative Lotka–Volterra system with delays ☆

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Abstract

Existence of traveling wave front solutions is established for diffusive and *cooperative* Lotka–Volterra system with delays. The result is an extension of an existing result for delayed logistic scaler equation to systems, and is somewhat parallel to the existing result for diffusive and *competitive* Lotka–Volterra systems *without delay*. The approach used in this paper is the upper–lower solution technique and the monotone iteration recently developed by Wu and Zou (J. Dynam. Differential Equations 13 (2001) 651–687) for delayed reaction–diffusion systems without the so-called quasimonotonicity. © 2002 Elsevier Science (USA). All rights reserved.

Keywords: Traveling wave front; Non-quasimonotone; Upper solution; Lower solution; Iteration

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1. Introduction

It is well known that the Fisher–KPP equation

$$\frac{\partial u(x,t)}{\partial t} = D\frac{\partial^2 u(x,t)}{\partial x^2} + ru(x,t)\bigl[1 - au(x,t)\bigr] \tag{1.1}$$

allows traveling wave front solutions with speed $c \geqslant 2\sqrt{Dr}$, and $c^* = 2\sqrt{Dr}$ is the minimal wave speed. See Fisher [2], Kolmogorov et al. [5], Fife [1] and Murray [8]. For a proof of this result, one can take advantage of the fact that (1.1) is a scaler equation and thus the corresponding wave equation is a second-order ordinary differential equation for which the phase plane technique can be applied.

Although (1.1) was initially proposed by Fisher [2] to model the advance of a favourable gene in an infinite one-dimensional habitat in which the process of natural selection and random spatial migration were evident, this equation also applies to single species dynamic models with logistic growth and spatial diffusion. In the case there are more than one species, one may use (1.1) as the basic model, but incorporate the interaction between species to obtain the so-called Lotka–Volterra type diffusive model. The interaction can be either competitive or cooperative. Existence of traveling wave fronts of diffusive systems with *competitive* interactions have been considered by Tang and Fife [11] (for two species case) and by van Vuuren [14] (for general n species case), and their main results show that a coupled system of competitive type also admits traveling wave fronts with speed larger than the maximum of all the minimal wave speeds for the uncoupled reaction–diffusion scaler equations. To make this more precise, let us consider the following competitive system:

$$\begin{cases} \frac{\partial u_1(x,t)}{\partial t} = D_1\frac{\partial^2 u_1(x,t)}{\partial x^2} + r_1u_1(x,t)[1 - a_1u_1(x,t) - b_1u_2(x,t)], \\ \frac{\partial u_2(x,t)}{\partial t} = D_2\frac{\partial^2 u_2(x,t)}{\partial x^2} + r_2u_2(x,t)[1 - b_2u_1(x,t) - a_2u_2(x,t)], \end{cases} \tag{1.2}$$

where all parameters are nonnegative constants. By Tang and Fife [11] and van Vuuren [14], (1.2) has traveling wave fronts with speed c if and only if $c \geqslant 2\max\{\sqrt{D_1r_2}, \sqrt{D_2r_2}\}$.

One may naturally ask if a similar conclusion holds for the *cooperative* system

$$\begin{cases} \frac{\partial u_1(x,t)}{\partial t} = D_1\frac{\partial^2 u_1(x,t)}{\partial x^2} + r_1u_1(x,t)[1 - a_1u_1(x,t) + b_1u_2(x,t)], \\ \frac{\partial u_2(x,t)}{\partial t} = D_2\frac{\partial^2 u_2(x,t)}{\partial x^2} + r_2u_2(x,t)[1 + b_2u_1(x,t) - a_2u_2(x,t)]. \end{cases} \tag{1.3}$$

The aim of this paper is to answer this question. Indeed, we will go beyond (1.3) by incorporating time delay into (1.3) and consider the following delayed cooperative and diffusive system:

$$\begin{cases} \frac{\partial u_1(x,t)}{\partial t} = D_1 \frac{\partial^2 u_1(x,t)}{\partial x^2} + r_1 u_1(x,t) \\ \qquad\qquad \times [1 - a_1 u_1(x, t-\tau_1) + b_1 u_2(x, t-\tau_2)], \\ \frac{\partial u_2(x,t)}{\partial t} = D_2 \frac{\partial^2 u_2(x,t)}{\partial x^2} + r_2 u_2(x,t) \\ \qquad\qquad \times [1 + b_2 u_1(x, t-\tau_3) - a_2 u_2(x, t-\tau_4)], \end{cases} \tag{1.4}$$

where $r_i > 0$, $a_i > 0$, $b_i > 0$, $i = 1, 2$, $\tau_j > 0$, $j = 1, 2, 3, 4$. Justifications for incorporating delays into a model can be found, e.g., in Hale and Lunel [4], Kuang [6] and Wu [12].

Note that in their proofs of the main results in Tang and Fife [11] and van Vuuren [14], the competitive property of the systems plays an important role. Thus, their methods cannot be, at least directly if not impossible, applied to the cooperative system (1.3), let alone (1.4) which is infinite dimensional. Thus, we will seek alternative approach to tackle this problem. More precisely, we will use the technique recently developed by Wu and Zou [13] to handle this class of systems.

When $b_1 = b_2 = 0$, (1.4) is decoupled into two delayed logistic scaler equations of the form

$$\frac{\partial u(x,t)}{\partial t} = D \frac{\partial^2 u(x,t)}{\partial x^2} + r u(x,t)\big[1 - a u(x, t-\tau)\big]. \tag{1.5}$$

It is shown in Wu and Zou [13] that for every $c > 2\sqrt{Dr}$ there exists a $\tau_c > 0$ such for $\tau \leqslant \tau_c$ Eq. (1.5) has a traveling wave front with speed c. By using the same technique, Gourley [3] has established a similar result for the diffusive delay equation

$$\frac{\partial u(x,t)}{\partial t} = D \frac{\partial^2 u(x,t)}{\partial x^2} + r u(x,t)\left(\frac{1 - a u(x, t-\tau)}{1 + b u(x, t-\tau)}\right), \tag{1.6}$$

which includes (1.5) as a special case, and which can be used to model the growth of the population of *Daphnia magna* (see Smith [9] and Gourley [3]). We will see in this paper that such a result can be extended to the cooperative system (1.4) by using this technique as well. Note that this technique has also been recently used by So and Zou [10] to the diffusive Nicholson's blowflies equation with delay, and by Wu and Zou [13] and Ma [7] to the Belousov–Zhabotinskii reaction model with delay.

This paper is organized as follows. In Section 2, we will introduce some notations and terminology, and present one of the main theorems from Wu and Zou [13] that will be employed in this paper. Section 3 is devoted to establishing the existence of traveling wave front solution of (1.4).

2. Preliminaries

Consider the following system of reaction–diffusion with time delay:

$$\frac{\partial u(t,x)}{\partial t} = D\frac{\partial^2 u(t,x)}{\partial x^2} + f\big(u_t(x)\big), \tag{2.1}$$

where $t \geqslant 0$, $x \in R$, $u \in R^n$, $D = \mathrm{diag}(D_1, \ldots, D_n)$ with $D_i > 0$, $i = 1, \ldots, n$, $f : C([-\tau, 0]; R^n) \to R^n$ is continuous, and $u_t(x)$ is an element in $C([-\tau, 0]; R^n) \to R^n$ parameterized by $x \in R$ and given by

$$u_t(x)(s) = u(t+s, x), \quad s \in [-\tau, 0],\ t \geqslant 0,\ x \in R.$$

A traveling wave solution of (2.1) is a solution of the form $u(t,x) = \phi(x+ct)$, where $\phi \in C^n(R, R^n)$ and $c > 0$ is a constant corresponding to the wave speed. Substituting $u(t,x) = \phi(x+ct)$ into (2.1) and denoting $x+ct$ still by t, we get

$$D\phi''(t) - c\phi'(t) + f_c(\phi_t) = 0, \quad t \in R, \tag{2.2}$$

where $f_c : X_c = C([-c\tau, 0]; R^n) \to R^n$ is defined by

$$f_c(\psi) = f(\psi^c), \qquad \psi^c(s) = \psi(cs), \quad s \in [-\tau, 0].$$

If for some $c > 0$ (2.2) has a monotone (componentwise) solution defined on R such that

$$\lim_{t\to-\infty} \phi(t) = u_-, \qquad \lim_{t\to\infty} \phi(t) = u_+ \tag{2.3}$$

exist, then $u(t,x) = \phi(x+ct)$ is called a wave front of (2.1) with speed c. Without loss of generality, we assume $u_- = 0$, $u_+ = K$, and seek for traveling wave front solutions connecting these two equilibria. In order to tackle the existence of traveling wave solution, we assume that the following *relaxed* quasimonotonicity condition:

(QM*) There exists a matrix $\beta = \mathrm{diag}(\beta_1, \ldots, \beta_2)$ with $\beta_i \geqslant 0$, $i = 1, \ldots, n$, such that

$$f_c(\phi) - f_c(\psi) + \beta\big[\phi(0) - \psi(0)\big] \geqslant 0$$

for $\phi, \psi \in X_c$ with (i) $0 \leqslant \psi(s) \leqslant \phi(s) \leqslant K$ for $s \in [-c\tau, 0]$, (ii) $e^{\beta s} \times [\phi(s) - \psi(s)]$ nondecreasing in $s \in [-c\tau, 0]$.

Here and in the sequel, an inequality in R^n always corresponds to the standard componentwise partial ordering in R^n. We look for wave front solutions of (2.1) in the following profile set:

$$\Gamma^* = \left\{ \phi \in C(R, R^2);\ \begin{array}{c} \text{(i) } \phi \text{ is nondeceasing in } R, \\ \text{(ii) } \lim_{t\to-\infty}\phi(t) = 0,\ \lim_{t\to\infty}\phi(t) = K, \\ \text{(iii) } e^{\beta t}[\phi(t+s) - \phi(t)] \text{ is nondeceasing in} \\ t \in R \text{ for every } s > 0. \end{array} \right\}.$$

Next we define upper and lower solutions for (2.1):

Definition 2.1. A continuous function $\bar{\phi}: R \to R^2$ is called an upper solution of (2.2) if $\bar{\phi}'$ and $\bar{\phi}''$ exist almost everywhere in R and they are essentially bounded on R, and if the following inequality holds:

$$D\bar{\phi}''(t) - c\bar{\phi}'(t) + f_c(\bar{\phi}_t) \leqslant 0, \quad \text{a.e. in } R. \tag{2.4}$$

A lower solution of (2.2) is defined in a similar way by reversing the inequality in (2.4).

Now we are in the position to state Theorem 4.5* in Wu and Zou [13].

Theorem 2.1. *Assume that* (QM*) *holds and that* $f(0) = f(K) = 0$ *with* $0 < K$. *Suppose that* (2.1) *has an upper solution* $\bar{\phi}(t)$ *in* Γ^* *and a lower solution* $\underline{\phi}$ (*which is not necessarily in* Γ^*) *satisfying*

(H1) $0 \leqslant \underline{\phi}(t) \leqslant \bar{\phi}(t) \leqslant K$, $t \in R$;
(H2) $\underline{\phi}(t) \not\equiv 0$ *in* R, *and there is on other equilibrium of* (2.1) *in* $[\delta, K]$, *where* $\delta = (\delta_1, \ldots, \delta_n)^T$ *with* $\delta_i = \sup_{t\in R} \underline{\phi_i}(t)$, $i = 1, \ldots, n$;
(H3) $e^{\beta t}[\bar{\phi}(t) - \underline{\phi}(t)]$ *is nondecreasing in* R.

Then, (2.2)–(2.3) *with* $c > 1 - \min\{\beta_i D_i;\ i = 1, \ldots, n\}$ *have a solution in* Γ^*. *That is,* (2.1) *has a traveling wave front with speed* $c > 1 - \min\{\beta_i D_i;\ i = 1, \ldots, n\}$.

Note that, if (QM*) is satisfied, we can always choose $\beta_i > 0$ sufficiently large such $c > 1 - \min\{\beta_i D_i; i = 1, \ldots, n\}$. So, in the remainder of this paper, we will ignore this condition assuming (QM*).

3. Existence of traveling wave front solution

Assume that $a_1a_2 - b_2b_1 > 0$. Then (1.4) has four equilibrium points: $(0, 0)$, $(1/a_1, 0)$, $(0, 1/a_2)$ and (k_1, k_2), where

$$k_1 = \frac{b_1 + a_2}{a_1a_2 - b_2b_1}, \qquad k_2 = \frac{a_1 + b_2}{a_1a_2 - b_2b_1}.$$

Substituting $u(x, t) = \phi_1(s)$, $v(x, t) = \phi_2(s)$, $s = x + ct$ into (1.4), and denoting the moving variable s still by t, the responding wave equation becomes

$$\begin{cases} D_1\phi_1''(t) - c\phi_1'(t) + r_1\phi_1(t)[1 - a_1\phi_1(t - c\tau_1) + b_1\phi_2(t - c\tau_2)] = 0, \\ D_2\phi_2''(t) - c\phi_2'(t) + r_2\phi_2(t)[1 + b_2\phi_1(t - c\tau_3) - a_2\phi_2(t - c\tau_4)] = 0. \end{cases} \tag{3.1}$$

We will tackle the existence of solutions of (3.1) with the asymptotic boundary condition

$$\begin{cases} \lim_{t\to-\infty}\phi_1(t)=0, & \lim_{t\to\infty}\phi_1(t)=k_1,\\ \lim_{t\to-\infty}\phi_2(t)=0, & \lim_{t\to\infty}\phi_1(t)=k_2, \end{cases} \tag{3.2}$$

which corresponding traveling wave fronts of (1.4) connecting $(0,0)$ and (k_1,k_2). Comparing (3.2) with (2.2), we know $f_c(\phi)=(f_{c1}(\phi), f_{c2}(\phi))^T$ is defined by

$$\begin{aligned} f_{c1}(\phi)&=r_1\phi_1(0)\big[1-a_1\phi_1(-c\tau_1)+b_1\phi_2(-c\tau_2)\big],\\ f_{c2}(\phi)&=r_2\phi_2(0)\big[1+a_2\phi_1(-c\tau_3)-b_2\phi_2(-c\tau_4)\big]. \end{aligned}$$

We first show that $f_c(\phi)$ satisfies (QM*).

Lemma 3.1. *For any $c>0$, if τ_1 and τ_4 are sufficiently small, then $f_c(\phi)$ satisfies* (QM*).

Proof. Let $\tau=\max\{\tau_1,\tau_2,\tau_3,\tau_4\}$. For any $\phi=(\phi_1,\phi_2)$, $\psi=(\psi_1,\psi_2)\in X_\tau=C([-c\tau,0];R^2)$ with (i) $0\leqslant\psi(s)\leqslant\phi(s)\leqslant K$ for $s\in[-c\tau,0]$, (ii) $e^{\beta s}[\phi(s)-\psi(s)]$ nondecreasing in $s\in[-c\tau,0]$, $i=1,2$, we have

$$\begin{aligned} &f_{c1}(\phi)-f_{c1}(\psi)\\ &\quad=r_1\phi_1(0)\big[1-a_1\phi_1(-c\tau_1)+b_1\phi_2(-c\tau_2)\big]\\ &\qquad-r_1\psi_1(0)\big[1-a_1\psi_1(-c\tau_1)+b_1\psi_2(-c\tau_2)\big]\\ &\quad=r_1\big[\phi_1(0)-\psi_1(0)\big]-a_1r_1\big[\phi_1(0)\phi_1(-c\tau_1)-\psi_1(0)\psi_1(-c\tau_1)\big]\\ &\qquad+r_1b_1\big[\phi_1(0)\phi_2(-c\tau_2)-\psi_1(0)\psi_2(-c\tau_2)\big]\\ &\quad=r_1\big[\phi_1(0)-\psi_1(0)\big]-a_1r_1\big[\phi_1(0)\phi_1(-c\tau_1)-\phi_1(0)\psi_1(-c\tau_1)\\ &\qquad\qquad+\phi_1(0)\psi_1(-c\tau_1)-\psi_1(0)\psi_1(-c\tau_1)\big]\\ &\qquad+r_1b_1\big[\phi_1(0)\phi_2(-c\tau_2)-\psi_1(0)\phi_2(-c\tau_2)+\psi_1(0)\phi_2(-c\tau_2)\\ &\qquad\qquad-\psi_1(0)\psi_2(-c\tau_2)\big]\\ &\quad=\big(r_1-a_1r_1\psi_1(-c\tau_1)+r_1b_1\phi_2(-c\tau_2)\big)\big[\phi_1(0)-\psi_1(0)\big]\\ &\qquad-a_1r_1\phi_1(0)\big[\phi_1(-c\tau_1)-\psi_1(-c\tau_1)\big]\\ &\qquad+r_1b_1\psi_1(0)\big[\phi_2(-c\tau_2)-\psi_2(-c\tau_2)\big]\\ &\quad\geqslant r_1\big[1+b_1\phi_2(-c\tau_2)-a_1\psi_1(0)e^{\beta_1c\tau_1}-a_1\psi_1(-c\tau_1)\big]\big[\phi_1(0)-\psi_1(0)\big]\\ &\quad\geqslant r_1\big[1-a_1k_1e^{\beta_1c\tau_1}-a_1k_1\big]\big[\phi_1(0)-\psi_1(0)\big] \end{aligned}$$

and hence

$$\begin{aligned} &f_{c1}(\phi)-f_{c1}(\psi)+\beta_1\big[\phi_1(0)-\psi_1(0)\big]\\ &\quad\geqslant\big[\beta_1+r_1-r_1a_1k_1-r_1a_1k_1e^{\beta_1c\tau_1}\big]\big[\phi_1(0)-\psi_1(0)\big]. \end{aligned}$$

Similarly, we have

$$\begin{aligned} & f_{c2}(\phi) - f_{c2}(\psi) + \beta_2\big[\phi_2(0) - \psi_2(0)\big] \\ & \quad \geqslant \big[\beta_2 + r_2 - r_2a_2k_2 - r_2a_2k_2e^{\beta_2 c\tau_4}\big]\big[\phi_2(0) - \psi_2(0)\big]. \end{aligned}$$

Therefore, if we choose

$$\begin{cases} \beta_1 > 2r_1a_1k_1 - r_1, \\ \beta_2 > 2r_2a_2k_2 - r_2, \end{cases} \tag{3.3}$$

then by continuity we know that, for τ_1 and τ_4 sufficiently small,

$$\begin{cases} \beta_1 + r_1 - r_1a_1k_1 - r_1a_1k_1e^{\beta_1 c\tau_1} \geqslant 0, \\ \beta_2 + r_2 - r_2a_2k_2 - r_2a_2k_2e^{\beta_2 c\tau_4} \geqslant 0. \end{cases} \tag{3.4}$$

Hence, $f_c(\phi) = (f_{c1}(\phi), f_{c2}(\phi))$ satisfies (QM*). This completes the proof. □

For $c > \max\{2\sqrt{a_1r_1D_1k_1}, 2\sqrt{a_2r_2D_2k_2}\}$, we can introduce the following real positive numbers:

$$\lambda_1 = \frac{c + \sqrt{c^2 - 4D_1r_1a_1k_1}}{2D_1}, \qquad \lambda_2 = \frac{c + \sqrt{c^2 - 4D_2r_2a_2k_2}}{2D_2},$$

$$\lambda_3 = \frac{c + \sqrt{c^2 - 4D_1r_1(1 - a_1k_1)}}{2D_1}, \qquad \lambda_4 = \frac{c + \sqrt{c^2 - 4D_2r_2(1 - a_2k_2)}}{2D_2}.$$

Fix $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ such that

$$\begin{aligned} & \varepsilon_1 < \max\left\{\frac{1}{a_1}, \frac{(\lambda_1 + \beta_1)k_1}{2\beta_1}, \frac{k_1\beta_1}{\beta_1 + \lambda_3}\right\}, \\ & \varepsilon_2 < \max\left\{\frac{1}{a_2}, \frac{(\lambda_2 + \beta_2)k_2}{2\beta_2}, \frac{k_2\beta_2}{\beta_2 + \lambda_4}\right\}. \end{aligned} \tag{3.5}$$

Then, we can choose $\alpha_1 > 0$ and $\alpha_2 > 0$ sufficiently small such that

$$\varepsilon_1 < \frac{k_1}{1 + \alpha_1}, \qquad \varepsilon_2 < \frac{k_2}{1 + \alpha_2}, \tag{3.6}$$

$$\varepsilon_1 < \frac{k_1(\lambda_1 + \beta_1)}{(2 + \alpha_1)\beta_1}, \qquad \varepsilon_2 < \frac{k_2(\lambda_2 + \beta_2)}{(2 + \alpha_2)\beta_2}, \tag{3.7}$$

$$\varepsilon_1 < \frac{k_1\beta_1}{(1 + 2\alpha_1)\beta_1 + \lambda_3}, \qquad \varepsilon_2 < \frac{k_2\beta_2}{(1 + 2\alpha_2)\beta_2 + \lambda_4}, \tag{3.8}$$

$$\varepsilon_1 < \frac{k_1(\beta_1 + \lambda_1)}{\alpha_1(\beta_1 + \lambda_3)}, \qquad \varepsilon_2 < \frac{k_2(\beta_2 + \lambda_2)}{\alpha_2(\beta_2 + \lambda_4)}. \tag{3.9}$$

Define $\Phi(t) = (\phi_1(t), \phi_2(t))$, $\Psi(t) = (\psi_1(t), \psi_2(t))$ by

$$\phi_1(t) = \frac{k_1}{1 + \alpha_1 e^{-\lambda_1 t}}, \quad t \in R, \qquad \phi_2(t) = \frac{k_2}{1 + \alpha_2 e^{-\lambda_2 t}}, \quad t \in R,$$

and

$$\psi_1(t)=\begin{cases}\varepsilon_1 e^{\lambda_3 t}, & t\leqslant 0,\\ \varepsilon_1, & t>0,\end{cases}\qquad \psi_2(t)=\begin{cases}\varepsilon_2 e^{\lambda_4 t}, & t\leqslant 0,\\ \varepsilon_2, & t>0.\end{cases}$$

Since $a_1k_1>1$ and $a_2k_2>1$, we have $0<\varepsilon_1<1/a_1<k_1$, $0<\varepsilon_2<1/a_2<k_2$, and $\lambda_3>\lambda_1$, $\lambda_4>\lambda_2$. We claim that $\phi_1(t)\geqslant\psi_1(t)$ and $\phi_2(t)\geqslant\psi_2(t)$. In fact, if $t>0$, then $\phi_1(t)\geqslant k_1/(1+\alpha_1)>\varepsilon_1=\psi_1(t)$ (by (3.6)). If $t\leqslant 0$, then

$$\begin{aligned}\phi_1(t)-\psi_1(t)&=\frac{k_1}{1+\alpha_1e^{-\lambda_1 t}}-\varepsilon_1e^{\lambda_3 t}=\frac{k_1-\varepsilon_1e^{\lambda_3 t}-\alpha_1\varepsilon_1e^{(\lambda_3-\lambda_1)t}}{1+\alpha_1e^{-\lambda_1 t}}\\ &\geqslant\frac{k_1-\varepsilon_1(1+\alpha_1)}{1+\alpha_1e^{-\lambda_1 t}}>0.\end{aligned}$$

Similarly, we can prove $\phi_2(t)\geqslant\psi_2(t)$ for $t\in R$. Moreover, $\phi_1(t)$ and $\phi_2(t)$ are nondecreasing in $t\in R$ since

$$\phi_1'(t)=\frac{k_1\alpha_1\lambda_1e^{-\lambda_1 t}}{[1+\alpha_1e^{-\lambda_1 t}]^2}>0,\qquad \phi_2'(t)=\frac{k_2\alpha_2\lambda_2e^{-\lambda_2 t}}{[1+\alpha_2e^{-\lambda_2 t}]^2}>0.$$

Lemma 3.2. *If τ_1 and τ_4 are sufficiently small, then $\Phi(t)=(\phi_1(t),\phi_2(t))$ is a upper solution of* (3.1).

Proof. Since $0\leqslant\phi_1(t)\leqslant k_1$ and $0\leqslant\phi_2(t)\leqslant k_2$, then we have

$$\begin{aligned}&D_1\phi_1''(t)-c\phi_1'(t)+r_1\phi_1(t)\big[1-a_1\phi_1(t-c\tau_1)+b_1\phi_2(t-c\tau_2)\big]\\ &\quad\leqslant D_1\phi_1''(t)-c\phi_1'(t)+r_1\phi_1(t)\big[1-a_1\phi_1(t-c\tau_1)+b_1k_2\big]\\ &\quad=D_1\phi_1''(t)-c\phi_1'(t)+a_1r_1\phi_1(t)\big[k_1-\phi_1(t-c\tau_1)\big].\end{aligned}$$

By some straightforward calculations and using $D_1\lambda_1^2=c\lambda_1-r_1a_1k_1$, we get

$$\begin{aligned}&D_1\phi_1''(t)-c\phi_1'(t)+a_1r_1\phi_1(t)\big[k_1-\phi_1(t-c\tau_1)\big]\\ &\quad=\frac{k_1\alpha_1e^{-\lambda_1 t}}{[1+\alpha_1e^{-\lambda_1 t}]^3[1+\alpha_1e^{-\lambda_1(t-c\tau_1)}]}\\ &\qquad\times\Big\{\alpha_1^2e^{-\lambda_1 t}e^{-\lambda_1(t-c\tau_1)}\big[D_1\lambda_1^2-c\lambda_1+a_1r_1k_1\big]\\ &\qquad+\alpha_1e^{-\lambda_1 t}\big[D_1\lambda_1^2-c\lambda_1+e^{\lambda_1c\tau_1}\big(2a_1r_1k_1-D_1\lambda_1^2-c\lambda_1\big)\big]\\ &\qquad+\big[-D_1\lambda_1^2-c\lambda_1+a_1r_1k_1e^{\lambda_1c\tau_1}\big]\Big\}\\ &\quad=\frac{k_1\alpha_1e^{-\lambda_1 t}}{[1+\alpha_1e^{-\lambda_1 t}]^3[1+\alpha_1e^{-\lambda_1(t-c\tau_1)}]}\\ &\qquad\times\Big\{\alpha_1e^{-\lambda_1 t}\big[D_1\lambda_1^2-c\lambda_1+e^{\lambda_1c\tau_1}\big(2a_1r_1k_1-D_1\lambda_1^2-c\lambda_1\big)\big]\\ &\qquad+\big[-D_1\lambda_1^2-c\lambda_1+a_1r_1k_1e^{\lambda_1c\tau_1}\big]\Big\}\end{aligned}$$

$$
\begin{aligned}
&= \frac{k_1\alpha_1 e^{-\lambda_1 t}}{[1+\alpha_1 e^{-\lambda_1 t}]^3[1+\alpha_1 e^{-\lambda_1(t-c\tau_1)}]} \\
&\quad \times \Big\{-\alpha_1 e^{-\lambda_1 t}\big[a_1r_1k_1 + e^{\lambda_1 c\tau_1}(2c\lambda_1 - 3a_1r_1k_1)\big] \\
&\qquad - \big[2c\lambda_1 - a_1r_1k_1 - a_1r_1k_1e^{\lambda_1 c\tau_1}\big]\Big\}.
\end{aligned}
$$

Note that λ_1 depends on on c and $c\lambda_1$ is decreasing since for any $c > \max\{2\times\sqrt{a_1r_1D_1k_1}, 2\sqrt{a_2r_2D_2k_2}\}$, since

$$
\frac{d}{dc}(c\lambda_1) = \frac{-2a_1r_1k_1[c-\sqrt{c^2-4a_1r_1k_1D_1}]}{c+\sqrt{c^2-4a_1r_1k_1D_1}} < 0.
$$

Furthermore, $\lim_{c\to+\infty}(c\lambda_1) = a_1r_1k_1$ and $\lim_{c\to 2\sqrt{a_1r_1k_1D_1}}(c\lambda_1) = 2a_1r_1k_1$. Thus, for any $c > \max\{2\sqrt{a_1r_1D_1k_1}, 2\sqrt{a_2r_2D_2k_2}\}$, $a_1r_1k_1 < c\lambda_1 < 2a_1r_1k_1$ and, therefore,

$$
\begin{aligned}
&\big[a_1r_1k_1 + e^{\lambda_1 c\tau_1}(2c\lambda_1 - 3a_1r_1k_1)\big]_{\tau_1=0} = 2c\lambda_1 - 2a_1r_1k_1 > 0,\\
&\big[2c\lambda_1 - a_1r_1k_1 - a_1r_1k_1e^{\lambda_1 c\tau_1}\big]_{\tau_1=0} = 2c\lambda_1 - 2a_1r_1k_1 > 0.
\end{aligned}
$$

So, for any $c > \max\{2\sqrt{a_1r_1D_1k_1}, 2\sqrt{a_2r_2D_2k_2}\}$, by continuity there exists $\tau_1^0(c) > 0$ such that when $0 \leqslant \tau_1 \leqslant \tau_1^0(c)$, we have

$$
\begin{cases}
a_1r_1k_1 + e^{\lambda_1 c\tau_1}[2c\lambda_1 - 3a_1r_1k_1] > 0,\\
2c\lambda_1 - a_1r_1k_1 - a_1r_1k_1e^{\lambda_1 c\tau_1} > 0.
\end{cases}
$$

This implies

$$
\begin{aligned}
&D_1\phi_1''(t) - c\phi_1'(t) + r_1\phi_1(t)\big[1 - a_1\phi_1(t-c\tau_1) + b_1\phi_2(t-c\tau_2)\big] \leqslant 0,\\
&\quad t\in R.
\end{aligned}
$$

Similarly, we can verify that when τ_2 is sufficiently small,

$$
\begin{aligned}
&D_2\phi_2''(t) - c\phi_2'(t) + r_2\phi_2(t)\big[1 + b_2\phi_1(t-c\tau_3) - a_2\phi_2(t-c\tau_4)\big] \leqslant 0,\\
&\quad t\in R.
\end{aligned}
$$

This completes the proof. □

Lemma 3.3. $\psi(t) = (\psi_1(t), \psi_2(t))$ *is a lower solution of* (3.1).

Proof. Obviously, $0 \leqslant \psi_1(t) \leqslant \varepsilon_1$ and $0 \leqslant \psi_2(t) \leqslant \varepsilon_2$. Hence,

$$
\begin{aligned}
&D_1\psi_1''(t) - c\psi_1'(t) + r_1\psi_1(t)\big[1 - a_1\psi_1(t-c\tau_1) + b_1\psi_2(t-c\tau_2)\big]\\
&\quad \geqslant D_1\psi_1''(t) - c\psi_1'(t) + r_1\psi_1(t)\big[1 - a_1\psi_1(t-c\tau_1)\big]\\
&\quad \geqslant D_1\psi_1''(t) - c\psi_1'(t) + r_1\psi_1(t)[1 - a_1\varepsilon_1]
\end{aligned}
$$

and

$$\begin{aligned}&D_2\psi_2''(t) - c\psi_2'(t) + r_2\psi_2(t)\big[1 + b_2\psi_1(t - c\tau_3) - a_2\psi_2(t - c\tau_4)\big]\\&\quad \geqslant D_2\psi_2''(t) - c\psi_2'(t) + r_2\psi_2(t)[1 - a_2\varepsilon_2].\end{aligned}$$

If $t > 0$, then $\psi_1(t) = \varepsilon_1$, and hence

$$D_1\psi_1''(t) - c\psi_1'(t) + r_1\psi_1(t)[1 - a_1\varepsilon_1] = r_1\varepsilon_1[1 - a_1\varepsilon_1] \geqslant 0.$$

If $t \leqslant 0$, then $\psi_1(t) = \varepsilon_1 e^{\lambda_3 t}$ and $\psi_1(t - c\tau_1) \leqslant \varepsilon_1 < 1/a_1 < k_1$, and thus

$$\begin{aligned}&D_1\psi_1''(t) - c\psi_1'(t) + r_1\psi_1(t)\big[1 - a_1\psi_1(t - c\tau_1)\big]\\&\quad \geqslant \varepsilon_1 e^{\lambda_3 t}\big[D_1\lambda_3^2 - c\lambda_3 + r_1(1 - a_1\varepsilon_1)\big]\\&\quad \geqslant \varepsilon_1 e^{\lambda_3 t}\big[D_1\lambda_3^2 - c\lambda_3 + r_1(1 - a_1k_1)\big] = 0.\end{aligned}$$

Therefore, we have

$$\begin{aligned}&D_1\psi_1''(t) - c\psi_1'(t) + r_1\psi_1(t)\big[1 - a_1\psi_1(t - c\tau_1) + b_1\psi_2(t - c\tau_2)\big] \geqslant 0,\\&\quad t \in R.\end{aligned}$$

Similarly, we can show that

$$\begin{aligned}&D_2\psi_2''(t) - c\psi_2'(t) + r_2\psi_2(t)\big[1 + b_2\psi_1(t - c\tau_3) - a_2\psi_2(t - c\tau_4)\big] \geqslant 0,\\&\quad t \in R,\end{aligned}$$

which means $\Psi(t) = (\psi_1(t), \psi_2(t))$ is a lower solution of (3.1). The proof is completed. $\square$

Lemma 3.4. *$e^{\beta t}[\Phi(t) - \Psi(t)]$ is nondecreasing in $t \in R$.*

Proof. We first verify that $e^{\beta_1 t}[\phi_1(t) - \psi_1(t)]$ is nondecreasing in $t \in R$.

(i) If $t > 0$, then $\psi_1(t) = \varepsilon_1$. Direct calculation shows that

$$\begin{aligned}\frac{d}{dt}\big\{e^{\beta_1 t}\big[\phi_1(t) - \psi_1(t)\big]\big\} &= \frac{d}{dt}\left\{e^{\beta_1 t}\left[\frac{k_1}{1 + \alpha_1 e^{-\lambda_1 t}} - \varepsilon_1\right]\right\}\\&= \frac{e^{\beta_1 t}}{[1 + \alpha_1 e^{-\lambda_1 t}]^2}\big[\beta_1(k_1 - \varepsilon_1) + (k_1\alpha_1\beta_1 + k_1\alpha_1\lambda_1 - 2\alpha_1\beta_1\varepsilon_1)e^{-\lambda_1 t}\\&\qquad - \alpha_1^2\varepsilon_1\beta_1 e^{-2\lambda_1 t}\big]\\&\geqslant \frac{e^{\beta_1 t}}{[1 + \alpha_1 e^{-\lambda_1 t}]^2}\big[\beta_1(k_1 - \varepsilon_1) + (k_1\alpha_1\beta_1 + k_1\alpha_1\lambda_1 - 2\alpha_1\beta_1\varepsilon_1\\&\qquad - \alpha_1^2\varepsilon_1\beta_1)e^{-\lambda_1 t}\big].\end{aligned}$$

Applying (3.5) and (3.7) to the right hand side of the above inequality, we get

$$\frac{d}{dt}\big\{e^{\beta_1 t}\big[\phi_1(t) - \psi_1(t)\big]\big\} > 0.$$

(ii) If $t \leqslant 0$, then $\psi_1(t) = \varepsilon_1 e^{\lambda_3 t}$. Since $\lambda_1 < \lambda_3$, then $e^{\lambda_1 t} \geqslant e^{\lambda_3 t}$. Straightforward calculation shows

$$\begin{aligned}
\frac{d}{dt}\bigl\{e^{\beta_1 t}\bigl[\phi_1(t) - \psi_1(t)\bigr]\bigr\} &= \frac{d}{dt}\left\{\frac{k_1 e^{\beta_1 t}}{1+\alpha_1 e^{-\lambda_1 t}} - \varepsilon e^{(\beta_1+\lambda_3)t}\right\} \\
&= \frac{e^{\beta_1 t}}{[1+\alpha_1 e^{-\lambda_1 t}]^2}\bigl[k_1\beta_1 + \alpha_1\beta_1 k_1 e^{-\lambda_1 t} + \alpha_1\lambda_1 k_1 e^{-\lambda_1 t} \\
&\qquad - \varepsilon_1(\beta_1+\lambda_3)e^{\lambda_3 t} - 2\alpha_1\varepsilon_1(\beta_1+\lambda_3)e^{(\lambda_3-\lambda_1)t} \\
&\qquad - \alpha_1^2\varepsilon_1(\beta_1+\lambda_3)e^{(\lambda_3-2\lambda_1)t}\bigr] \\
&\geqslant \frac{e^{\beta_1 t}}{[1+\alpha_1 e^{-\lambda_1 t}]^2}\bigl\{\bigl[k_1\alpha_1\beta_1 + \alpha_1\lambda_1 k_1 - \alpha_1^2\varepsilon_1(\beta_1+\lambda_3)\bigr]e^{-\lambda_1 t} + k_1\beta_1 \\
&\qquad - 2\alpha_1\varepsilon_1(\beta_1+\lambda_3) - \varepsilon_1(\beta_1+\lambda_3)\bigr\}.
\end{aligned}$$

Applying (3.8) and (3.9) to the above inequality, we see

$$\frac{d}{dt}\left\{\frac{k_1 e^{\beta_1 t}}{1+\alpha_1 e^{-\lambda_1 t}} - \varepsilon e^{(\beta_1+\lambda_3)t}\right\} \geqslant 0.$$

Therefore, $e^{\beta_1 t}[\phi_1(t) - \psi_1(t)]$ is nondecreasing in $t \in R$.

Similarly, $e^{\beta_2 t}[\phi_2(t) - \psi_2(t)]$ is nondecreasing in $t \in R$. The proof is completed. □

Finally, $\inf_{t\in R}\phi_1(t) = \inf_{t\in R}\phi_2(t) = 0$, $\sup_{t\in R}\psi_1(t) = \varepsilon_1 > 0$, $\sup_{t\in R}\psi_2(t) = \varepsilon_2 > 0$. Therefore $f(\tilde{u}) \neq 0$ for $u \in (0, \inf_{t\in R}\Phi(t)] \cup [\sup_{t\in R}\Psi(t), K) = [\varepsilon_1, k_1) \times [\varepsilon_2, k_2)$.

Now, applying Lemmas 3.1–3.4 and Theorem 2.1 to (3.1)–(3.2), we immediately obtain the following existence result.

Theorem 3.1. *Assume $a_1a_2 - b_1b_2 > 0$. Then, for every $c > \max\{2\sqrt{a_1r_1D_1k_1}, 2\sqrt{a_2r_2D_2k_2}\}$, (1.4) has a traveling wave front with speed c that connects $(0,0)$ and (k_1,k_2), provided that τ_1 and τ_4 are sufficiently small.*

Remark 3.1. Due to the cooperative feature, the delays in the interaction channel (i.e., τ_2 and τ_3) play no role in the above existence result.

Remark 3.2. When $b_1 = b_2 = 0$, system (1.4) is decoupled to (1.5). Note that in this case, $a_ik_1 = 1$, $i = 1, 2$, and thus Theorem 3.1 reproduces Theorem 5.1.5 in Wu and Zou [13].

Remark 3.3. In Tang and Fife [11] and van Vuuren [14], the authors were able to find the precise minimal wave speed. But due to the presence of the delays, (3.1) becomes an infinite dimensional system and thus is much hard to analyze than a non-delay system which would be finite dimensional. This fact seems to prevent one from being able to find the minimal speed, at least at the present.

Remark 3.4. In Theorem 3.1, existence of traveling wave fronts is established by constructing the appropriate upper and lower solutions of (3.1), and by using Theorem 2.1 which is from Wu and Zou [13]. Actually, Wu and Zou [13] also developed an iteration scheme which starts with the upper solution and generates a monotone sequence that converges to a profile function for the traveling wave front. Such an iteration brings convenience in approximating the wave front, and finding an appropriate upper solution and lower solution is crucial for the iteration. For details of such an interaction scheme, see Wu and Zou [13].

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