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Asymptotic profiles of steady states for a partially degenerate reaction-diffusion-advection model

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Abstract

This paper examines the steady states of a partially degenerate reaction-diffusion-advection system that models the dynamics of an aquatic species in a closed advective environment. The species is divided into two groups, drift and benthic groups, with the assumption that only individuals in the drift group are subject to advection and diffusion. We investigate the asymptotic profiles of positive steady states as the diffusion and advection rates approach zero or infinity. We find that, in regimes of large advection or small diffusion, the spatial distribution of the species exhibits a dichotomy: either (i) both drift and benthic species concentrate downstream, or (ii) drift individuals concentrate downstream, while benthic individuals are predominantly distributed in both downstream areas and regions where the net growth rate exceeds the benthic population release rate. Moreover, when the diffusion rate is sufficiently large, the drift species tends to distribute evenly, whereas the benthic species exhibits a spatially heterogeneous distribution. These findings provide valuable insights into population dynamics of both drift and benthic groups, and highlight the crucial influence of diffusion and advection rates in shaping the spatial distributions of species in advective environments.

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Keywords: Reaction-diffusion-advection; Benthic-drift model; Spatial heterogeneity; Asymptotic profiles; Closed advective environment

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1. Introduction

Streams and rivers are linear habitats of many aquatic species characterized by a predominantly unidirectional flow of water. This physical feature presents an ecological challenge for aquatic organisms: how to avoid being transported downstream into larger rivers, lakes, or marine environments, where physical or biological conditions may not support their growth and reproduction. This challenge is related to a vital issue in stream ecology, called "drift paradox" [3,12], which asks how stream-dwelling organisms can persist over many generations in a river or stream environment when continuously subjected to a unidirectional flow of water.

In efforts to address this paradox, a variety of hypotheses have been proposed. One early hypothesis, proposed by Müller [17,18], suggested that adult insects compensate for the downstream drift of their larvae by migrating upstream to oviposit. Extensive studies have been conducted to further investigate the validity of this hypothesis (see, e.g., [11,14,15,25]). Another hypothesis, proposed by Speirs and Gurney [22] based on a reaction-diffusion-advection model, suggested that random dispersal drives the persistence of aquatic species. This has stimulated extensive research on the impact of random dispersal on population persistence in advective environments (see, e.g., [8,9,23,24]). In addition, Waters [27] proposed that some insects primarily inhabit the benthos, with only the excess population beyond the local carrying capacity drifting downstream. Along this line, Pachepsky et al. [20] came up with the idea of dividing a river habitat into two zones: the drift zone and the benthic zone; accordingly, they categorized the populations in the river into two interacting subpopulations: individuals residing in the benthic zone and those dispersing in the drift zone, as was done in [1,2]. By assuming that advection and diffusion occur only in the drift zone and disregarding movement in the benthic zone, they introduced a *spatially homogeneous* benthic-drift model and derived conditions for persistence and spread of a population.

Given that natural and anthropogenic disturbances (e.g., floods, landslides, dam construction, water extraction, and land-use change) can alter the habitat conditions for aquatic species [10], it is essential to incorporate *spatial heterogeneity* into benthic-drift models. In this regard, some studies have been done to investigate the dynamics of spatially heterogeneous benthic-drift models. For example, Huang et al. [5] examined persistence metrics, such as the fundamental niche and net reproductive rate, for a benthic-drift model with logistic growth. Later, Jin et al. [6] extended the dynamical results for the benthic-drift model from one-dimensional rivers to two-dimensional depth-averaged rivers. In addition, Wang and Shi [26] analyzed the dynamics for a benthic-drift model with strong Allee effect growth. For further related research, see [7,13,28] and the references therein.

The above mentioned works on benthic-drift models do not *quantitatively* analyze the influence of key parameters, particularly the diffusion and advection rates, on species survival. To address this problem, Nie et al. [19] investigated the following benthic-drift system with logistic growth:

$$\begin{cases} \mathcal{U}_t = d\mathcal{U}_{xx} - \alpha\mathcal{U}_x - m_1(x)\mathcal{U} - \sigma\mathcal{U} + \mu\frac{A_2(x)}{A_1(x)}\mathcal{V}, & x \in (0, L),\ t > 0, \\ \mathcal{V}_t = g(x, \mathcal{V})\mathcal{V} - m_2(x)\mathcal{V} + \sigma\frac{A_1(x)}{A_2(x)}\mathcal{U} - \mu\mathcal{V}, & x \in [0, L],\ t > 0, \\ d\mathcal{U}_x(0, t) - \alpha\mathcal{U}(0, t) = d\mathcal{U}_x(L, t) - \alpha\mathcal{U}(L, t) = 0, & t > 0, \\ \mathcal{U}(x, 0) = u_0(x),\ \mathcal{V}(x, 0) = v_0(x), & x \in [0, L]. \end{cases} \tag{1.1}$$

Here, $\mathcal{U}(x,t)$ and $\mathcal{V}(x,t)$ represent the population densities in the drift and benthic zones, respectively, at location x and time t. The parameters $d>0$ and $\alpha>0$ are the diffusion rate and advection rate of drift species, respectively. The transfer rate of the drift population to the benthic zone (i.e., drift population release rate) is denoted by σ, and the transfer rate of the benthic population to the drift zone (i.e., benthic population release rate) is denoted by μ. The cross-sectional areas of the drift zone and the benthic zone are represented by $A_1(x)$ and $A_2(x)$, respectively. Additionally, $m_1(x)$ and $m_2(x)$ indicate the mortality rates of the drift and benthic populations, respectively, at location x. The function $g(x,v)$ represents the per capita growth rate of the benthic population at location x. The no-flux boundary conditions mean that no individual will pass through the boundaries at $x=0$ and $x=L$.

The main concern of [19] was the impact of the diffusion rate, advection rate, population transfer rates, and spatial heterogeneity on the global threshold dynamics of system (1.1), between persistence and extinction. In particular, it was shown that system (1.1) is persistent and admits a unique positive steady state that is globally asymptotically stable for all diffusion and advection rates in one of the following cases: (A) the maximum net growth rate of the benthic population exceeds its population release rate; (B) the maximum net growth rate is positive but lower than the benthic population release rate, while the drift population release rate is sufficiently large.

For the case when all parameters are constant except for the cross-section areas, (1.1) reduces to the model in Qu and Wang [21] where the authors analyzed the asymptotic profiles of the positive steady state in Case (B) as diffusion and advection rates vary. Their analysis focused on three limiting regimes: (i) $d>0$ is fixed and $\alpha\to+\infty$, or (ii) $\alpha>0$ is fixed and $d\to 0^+$, or (iii) $\alpha>0$ is fixed and $d\to+\infty$. We point out that the uniqueness of positive solutions to the limiting problem when $d\to+\infty$ *remains unsolved.* Thus, the impacts of the diffusion and advection rates on the spatial distributions of species for system (1.1) are not fully understood yet, particularly when there is certain spatial heterogeneity, e.g., location-dependent growth rate and mortality rates. Specifically, the three crucial and interesting questions arise and remain unaddressed:

(Q1) What are the asymptotic profiles of the positive steady state of system (1.1) when case (A) holds?

(Q2) Can the asymptotic behavior of the steady state be characterized when d and α tend to zero and infinity simultaneously, such as in the limits $(d,\alpha)\to(0,+\infty)$, $(d,\alpha)\to(0,0)$, $(d,\alpha)\to(+\infty,+\infty)$, or $(d,\alpha)\to(+\infty,0)$?

(Q3) Is it possible to establish the uniqueness of positive solutions to the limiting problem when $d\to+\infty$?

This paper aims to address these questions. To this end, we focus on the following steady-state system of (1.1):

$$\begin{cases} du_{xx}-\alpha u_x-(m_1(x)+\sigma)u+\mu\frac{A_2(x)}{A_1(x)}v=0, & x\in(0,L),\\ (g(x,v)-m_2(x)-\mu)v+\sigma\frac{A_1(x)}{A_2(x)}u=0, & x\in[0,L],\\ du_x(0)-\alpha u(0)=du_x(L)-\alpha u(L)=0. \end{cases} \tag{1.2}$$

Here, $u(x)$ and $v(x)$ represent the equilibrium densities of drift and benthic individuals, respectively, at location x. In this paper, we make the following assumptions:

(H1) $A_1(x)$, $A_2(x)$, $m_1(x)$, and $m_2(x)$ are positive and continuously differentiable functions on $[0, L]$.
(H2) $g(x, v)$ is continuously differentiable with respect to $x \in [0, L]$ and $v \in [0, +\infty)$. Moreover, $g(x, v)$ is strictly decreasing with respect to v and $\lim_{v \to +\infty} g(x, v) = -\infty$.

As stated above, the objective of this paper is to explore the asymptotic profiles of positive solutions to system (1.2) when d and α tend to zero or infinity. In other words, we aim to explore how the diffusion and advection rates of drift species influence the spatial distributions of drift and benthic populations when they persist. The main challenges and analytical approaches are summarized below, in terms of two cases that are complementary to [21]:

Case 1. Large advection rate or small diffusion rate. To analyze the asymptotic behavior of positive solutions to an equation of the form $du_{xx} - \alpha u_x + F(u) = 0$ as $\alpha \to +\infty$ or $d \to 0^+$, it is essential to first obtain an L^∞ estimate for $F(u)/u$. This estimate, together with a normalization technique, allows one to establish the uniform boundedness of the reaction term $F(u)$. Subsequently, the regularity theory for elliptic equations can be employed to characterize the asymptotic behavior of the positive solution (see, e.g., [4,16]). However, the difficulty we encounter is that the positive solution $(u(x), v(x))$ to system (1.2) may become unbounded as α and d vary. As a consequence, the reaction term $F(u) = -(m_1(x) + \sigma)u + \mu(A_2(x)/A_1(x))v$ may also become unbounded. To address this challenge, we proceed with two distinct cases as below. When $\mu > \max_{x \in [0,L]}\{g(x, 0) - m_2(x)\}$, we solve for v from the second equation in (1.2) and substitute it into the first equation. This enables us to establish the uniform boundedness of the normalized reaction term with respect to α and d. Consequently, by following the approach developed in [4,16], we are able to derive the asymptotic behavior of the positive solution $(u(x), v(x))$ as $\alpha \to +\infty$ or $d \to 0^+$ (see the proof of Theorem 2.4). When $\mu < \max_{x \in [0,L]}\{g(x, 0) - m_2(x)\}$, however, the approach in [4,16] cannot be applied directly. To proceed with this issue, through a detailed analysis of the algebraic equation in system (1.2), we establish the boundedness of the ratio $v(x)/u(x)$ as α or d vary (see Lemmas 3.1 and 3.2). This key finding enables us to obtain uniform boundedness of the reaction term. Then, by a similar strategy to that in [4,16], we succeed in determining the asymptotic profile of the positive solution as $\alpha \to +\infty$ or $d \to 0^+$ (see the proof of Theorem 2.2).

Case 2. Large diffusion rate. To investigate the asymptotic behavior of the positive solution $(u(x), v(x))$ to system (1.2) as the diffusion rate becomes sufficiently large, we follow two main steps (see the proof of Theorem 2.6). First, we derive the uniform boundedness of v, u, u_x and u_{xx} on the interval $[0, L]$ for large d. This enables us to demonstrate the convergence of $(u(x), v(x))$ as $d \to +\infty$. Next, we establish the uniqueness of positive solutions to the limiting problem as $d \to +\infty$, which was not addressed in [21].

The main results of this paper provide precise characterizations of the spatial distributions of both drift and benthic subpopulations when d and α are sufficiently large or small (see Theorems 2.2, 2.4, and 2.6). These findings constitute a significant complement to the results in [21] because they successfully address the three critical questions stated in (Q1) - (Q3); as such, they greatly enhance our understanding of the population dynamics of the benthic-drift model presented in [19] by further revealing the role of diffusion and advection rates in shaping the species' spatial distributions.

The remainder of the paper is organized as follows. In Section 2, we present the main results of this paper. In Section 3, we establish some preliminary results, which will be used in the subsequent analysis. Section 4 is devoted to the proofs of the main results on the asymptotic profiles of the unique positive solution to system (1.2). Finally, a brief discussion concludes the paper in Section 5.

2. Main results

In this section, we present the main results of this paper. We first recall some existing results regarding the existence and uniqueness of the positive steady state of system (1.1).

Theorem 2.1. *([19, Theorems 2.2, 4.3(i) and Proposition 4.13(ii)]) The following statements hold:*

(i) *If* $\max_{x\in[0,L]}\{g(x,0)-m_2(x)\}\le 0$*, then for any* $\mu,\sigma,d,\alpha>0$*,* $(0,0)$ *is globally asymptotically stable for system* (1.1) *among all nonnegative and nontrivial initial conditions.*
(ii) *If* $\max_{x\in[0,L]}\{g(x,0)-m_2(x)\}>\mu>0$*, then for any* $\sigma,d,\alpha>0$*, system* (1.1) *admits a unique positive steady state* $(u(x),v(x))$*, which is globally asymptotically stable among all nonnegative and nontrivial initial conditions.*
(iii) *If* $\mu>\max_{x\in[0,L]}\{g(x,0)-m_2(x)\}>0$ *and* $\sigma>\sigma_{\sup}$*, then for any* $d,\alpha>0$*, system* (1.1) *admits a unique positive steady state* $(u(x),v(x))$*, which is globally asymptotically stable among all nonnegative and nontrivial initial conditions. Here* $\sigma_{\sup}:=\sup\{\sigma_x : x\in[0,L]\}$ *with*

$$\sigma_x := \begin{cases} \dfrac{m_1(x)[\mu-(g(x,0)-m_2(x))]}{g(x,0)-m_2(x)}, & g(x,0)-m_2(x)>0, \\ +\infty, & g(x,0)-m_2(x)\le 0. \end{cases} \tag{2.1}$$

According to Theorem 2.1 (ii) and (iii), system (1.2) has a unique positive solution $(u(x),v(x))$ for all $d>0$ and $\alpha>0$ provided that one of the following alternatives holds:

(C1) $\max_{x\in[0,L]}\{g(x,0)-m_2(x)\}>\mu>0$ and $\sigma>0$;
(C2) $\mu>\max_{x\in[0,L]}\{g(x,0)-m_2(x)\}>0$ and $\sigma>\sigma_{\sup}$.

In the sequel, we present the main results on the asymptotic profiles of the unique positive solution $(u(x),v(x))$ to system (1.2) as the diffusion and advection rates vary. To this end, define

$$I_+ := \{x\in[0,L) : g(x,0)-m_2(x)>\mu\}.$$

From a biological perspective, we say that a point $x\in[0,L)$ is a *favorable benthic site* if $x\in I_+$, meaning that the net growth rate of the benthic species at x exceeds its population release rate. Thus, I_+ is indeed the collection of all favorable benthic sites.

Firstly, we describe the asymptotic profiles of $(u(x),v(x))$ under condition (C1) when $\alpha/d\to+\infty$ and $\alpha^2/d\to+\infty$ as below.

Theorem 2.2. *Assume that* (C1) *holds and let* $(u(x),v(x))$ *be the unique positive solution of system* (1.2)*. Then when*

$$\frac{\alpha}{d}\to+\infty \text{ and } \frac{\alpha^2}{d}\to+\infty,$$

(i) *the drift population density satisfies*

$$u(x) \to 0 \text{ for } x \in [0, L);$$

(ii) *the benthic population density satisfies*

$$v(x) \to \begin{cases} 0, & x \in [0, L) \setminus I_+, \\ v^*(x), & x \in I_+; \end{cases}$$

(iii) *the densities at the downstream end $x = L$ satisfy*

$$(u(L), v(L)) \to (+\infty, +\infty) \text{ and } \left(\frac{du(L)}{\alpha}, \frac{dv(L)}{\alpha} \right) \to \left(\frac{\mu \int_{I_+} \frac{A_2(x)}{A_1(x)} v^*(x) \mathrm{d}x}{m_1(L) + \sigma}, 0 \right).$$

Here $v^*(x)$ is a positive function defined on I_+ uniquely determined by the following equation:

$$g(x, v^*(x)) - m_2(x) - \mu = 0, \quad x \in I_+. \tag{2.2}$$

To further illustrate the results obtained in Theorem 2.2, we state five special cases of the limits $\alpha/d \to +\infty$ and $\alpha^2/d \to +\infty$ as the following corollary.

Corollary 2.3. *Assume that* (C1) *holds and let* $(u(x), v(x))$ *be the unique positive solution of system* (1.2). *If one of the following limiting cases holds:*

(a-1) $d > 0$ *is fixed and* $\alpha \to +\infty$*;*
(a-2) $\alpha > 0$ *is fixed and* $d \to 0^+$*;*
(a-3) $(d, \alpha) \to (0, +\infty)$*;*
(a-4) $(d, \alpha) \to (0, 0)$ *and* $\alpha^2/d \to +\infty$*;*
(a-5) $(d, \alpha) \to (+\infty, +\infty)$ *and* $\alpha/d \to +\infty$*,*

then the conclusions in Theorem 2.2 (i) - (iii) hold.

Secondly, we present the asymptotic profiles of $(u(x), v(x))$ under condition (C2) when $\alpha/d \to +\infty$ and $\alpha^2/d \to +\infty$ as below.

Theorem 2.4. *Assume that* (C2) *holds and let* $(u(x), v(x))$ *be the unique positive solution of system* (1.2). *Then when*

$$\frac{\alpha}{d} \to +\infty \text{ and } \frac{\alpha^2}{d} \to +\infty,$$

(i) *the population densities satisfy*

$$(u(x), v(x)) \to (0, 0) \text{ for } x \in [0, L);$$

(ii) *the densities at the downstream end $x = L$ satisfy*

$$(u(L), v(L)) \to (u^\infty, v^\infty),$$

where u^∞ and v^∞ are two positive constants uniquely determined by the following equations:

$$\begin{cases} -(m_1(L)+\sigma)u^\infty + \mu\frac{A_2(L)}{A_1(L)}v^\infty = 0, \\ (g(L, v^\infty)) - m_2(L) - \mu)v^\infty + \sigma\frac{A_1(L)}{A_2(L)}u^\infty = 0. \end{cases} \tag{2.3}$$

The following corollary is a direct result of Theorem 2.4.

Corollary 2.5. *Assume that* (C2) *holds and let* $(u(x), v(x))$ *be the unique positive solution of system* (1.2)*. If one of the cases (a-1) - (a-5) in Corollary 2.3 holds, then the conclusions in Theorem 2.4 (i) and (ii) hold.*

Thirdly, we present the following result concerning the asymptotic profiles of $(u(x), v(x))$ when $(d, \alpha) \to (+\infty, 0)$ or when $d \to +\infty$ with $\alpha > 0$ fixed.

Theorem 2.6. *Assume that either* (C1) *or* (C2) *holds, and let* $(u(x), v(x))$ *be the unique positive solution of system* (1.2)*. If*

$$\textit{either}\ \ (d, \alpha) \to (+\infty, 0) \quad \textit{or} \quad [d \to +\infty \textit{ and } \alpha > 0 \textit{ is fixed}],$$

then, there holds

$$(u(x), v(x)) \to (\bar{u}, \bar{v}(x)) \textit{ for } x \in [0, L],$$

where $\bar{u}$ is a positive constant and $\bar{v}(x)$ is a positive function uniquely determined by the following equations:

$$\begin{cases} \int_0^L [-(m_1(x)+\sigma)\bar{u} + \mu\frac{A_2(x)}{A_1(x)}\bar{v}(x)]\mathrm{d}x = 0, \\ \big(g(x, \bar{v}(x)) - m_2(x) - \mu\big)\bar{v}(x) + \sigma\frac{A_1(x)}{A_2(x)}\bar{u} = 0, \ x \in [0, L]. \end{cases} \tag{2.4}$$

Theorems 2.2, 2.4, and 2.6, together with Corollaries 2.3 and 2.5, characterize the asymptotic profiles of the positive solution to system (1.2) as the diffusion and advection rates become sufficiently large or sufficiently small. These findings indicate that the diffusion and advection rates significantly affect the spatial distributions of drift and benthic subpopulations along the river. More specifically, Theorems 2.2 and 2.4 reveal that, as $\alpha/d \to +\infty$ and $\alpha^2/d \to +\infty$, species' population distribution exhibits a **dichotomous pattern**, governed by conditions (C1) and (C2):

- Under condition (C1), there exist favorable benthic habitats. In this case, both species coexist: drift individuals concentrate near the downstream end, while benthic individuals are primarily distributed in both downstream areas and favorable benthic sites (see Theorem 2.2).

- Under condition (C2), there is no favorable benthic site. In this case, drift and benthic species coexist, and both species concentrating near the downstream end (see Theorem 2.4).

These observations indicate that favorable benthic sites play a crucial role in shaping species' spatial distributions in the co-existence scenario, particularly shaping the distribution of benthic species.

Theorem 2.6 indicates that when $(d, \alpha) \to (+\infty, 0)$ or when $d \to +\infty$ with $\alpha > 0$ fixed, both drift and benthic species coexist. In these regimes, a high diffusion rate of the drift species leads to an even distribution of drift subpopulation, while causing a spatially heterogeneous distribution of benthic subpopulation.

When the growth rate, mortality rates, and cross-sectional areas are spatially homogeneous, all positive solutions of system (1.2) are spatially constant, reflecting a uniform distribution of both drift and benthic subpopulations when they coexist. In contrast, Theorems 2.2, 2.4, and 2.6 suggest that spatial heterogeneity in these ecological parameters can destroy this uniform distribution, highlighting the crucial role of spatial variation in shaping species' distributions over the river in benthic-drift ecosystems.

3. Preliminaries

In this section, as preparation for the proofs of the main results, we present some preliminary results that will be used in the subsequent proofs.

Lemma 3.1. *Let $(u(x), v(x))$ be a positive solution of system (1.2). Then for any $x \in [0, L]$, when*

(i) *either $\alpha/d \to +\infty$ and $\alpha^2/d \to +\infty$,*
(ii) *or $(d, \alpha) \to (+\infty, 0)$,*
(iii) *or $\alpha > 0$ is fixed and $d \to +\infty$,*

there holds

$$u(x) \to +\infty \quad \text{if and only if} \quad v(x) \to +\infty. \tag{3.1}$$

Moreover, if $u(x) \to +\infty$, then $u(x)/v(x) \to +\infty$, equivalently $v(x)/u(x) \to 0$.

Proof. We only prove the case (iii) as the proofs of the other two cases are similar. To emphasize the dependence of $(u(x), v(x))$ on d, we rewrite it as $(u_d(x), v_d(x))$. Fix $x \in [0, L]$. Clearly,

$$\lim_{d\to+\infty} u_d(x) \geq 0 \text{ and } \lim_{d\to+\infty} v_d(x) \geq 0.$$

The second equation in (1.2) shows that

$$(g(x, v_d(x)) - m_2(x) - \mu)v_d(x) + \sigma \frac{A_1(x)}{A_2(x)} u_d(x) = 0. \tag{3.2}$$

Suppose $u_d(x) \to +\infty$ as $d \to +\infty$. For the sake of contradiction, we assume that there exists a sequence $\{d_n\}_{n=1}^{\infty}$ satisfying $d_n \to +\infty$ as $n \to +\infty$ such that $v_{d_n}(x)$ is bounded when

$n \to +\infty$. Then $u_{d_n}(x) \to +\infty$ as $n \to +\infty$ and $(g(x, v_{d_n}(x)) - m_2(x) - \mu)v_{d_n}(x)$ is bounded when $n \to +\infty$. This contradicts (3.2). Thus, there must be $v_d(x) \to +\infty$ as $d \to +\infty$.

Conversely, suppose $v_d(x) \to +\infty$ as $d \to +\infty$. It then follows from (3.2) and (H2) that

$$\lim_{d\to+\infty} u_d(x) = \lim_{d\to+\infty} \frac{A_2(x)}{\sigma A_1(x)}[\mu - (g(x, v_d(x)) - m_2(x))]v_d(x) = +\infty.$$

This proves (3.1).

Moreover, when $u_d(x) \to +\infty$ as $d \to +\infty$, it follows from (3.1) and (3.2) that $v_d(x) \to +\infty$ as $d \to +\infty$ and

$$\sigma \frac{A_1(x)}{A_2(x)} \cdot \frac{u_d(x)}{v_d(x)} = \mu - (g(x, v_d(x)) - m_2(x)). \tag{3.3}$$

Letting $d \to +\infty$ on both sides of (3.3) and using (H2), one then finds that $u_d(x)/v_d(x) \to +\infty$ as $d \to +\infty$, equivalently $v_d(x)/u_d(x) \to 0$ as $d \to +\infty$. This completes the proof. □

Next lemma passes the point-wise limits in the above lemma to limits in terms of $||\cdot||_\infty$ norm.

Lemma 3.2. *Let $(u(x), v(x))$ be a positive solution of system (1.2). Then, when*

(i) *either $\alpha/d \to +\infty$ and $\alpha^2/d \to +\infty$,*
(ii) *or $(d, \alpha) \to (+\infty, 0)$,*
(iii) *or $\alpha > 0$ is fixed and $d \to +\infty$,*

there holds

$$\|u\|_\infty \to +\infty \quad \textit{if and only if} \quad \|v\|_\infty \to +\infty. \tag{3.4}$$

Moreover, if $\|u\|_\infty \to +\infty$, then $\|u\|_\infty/\|v\|_\infty \to +\infty$, equivalently $\|v\|_\infty/\|u\|_\infty \to 0$.

Proof. We only prove the case (iii) since the proofs of the other cases follow similarly. To emphasize the dependence of $(u(x), v(x))$ on d, we rewrite it as $(u_d(x), v_d(x))$. Clearly, for any sequence $\{d_n\}_{n=1}^\infty$ satisfying $d_n \to +\infty$ as $n \to +\infty$, there exist sequences $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$ with $x_n, y_n \in [0, L]$ such that $v_{d_n}(x_n) = \|v_{d_n}\|_\infty$ and $u_{d_n}(y_n) = \|u_{d_n}\|_\infty$.

Suppose $\|u_d\|_\infty \to +\infty$ as $d \to +\infty$. Then $u_{d_n}(y_n) \to +\infty$ as $n \to +\infty$. It follows from (3.2) that for each $n \in \mathbb{N}$,

$$(g(y_n, v_{d_n}(y_n)) - m_2(y_n) - \mu)v_{d_n}(y_n) + \sigma \frac{A_1(y_n)}{A_2(y_n)} u_{d_n}(y_n) = 0. \tag{3.5}$$

If $\|v_{d_n}\|_\infty$ is bounded when $n \to +\infty$, then $(g(y_n, v_{d_n}(y_n)) - m_2(y_n) - \mu)v_{d_n}(y_n)$ is bounded when $n \to +\infty$. This contradicts (3.5). Thus, $\|v_{d_n}\|_\infty \to +\infty$ as $n \to +\infty$. By the arbitrariness of $\{d_n\}_{n=1}^\infty$, we obtain $\|v_d\|_\infty \to +\infty$ as $d \to +\infty$.

Conversely, suppose $\|v_d\|_\infty \to +\infty$ as $d \to +\infty$. Then $v_{d_n}(x_n) \to +\infty$ as $n \to +\infty$. In light of (3.2) and (H2), one finds that

$$\lim_{n\to+\infty} u_{d_n}(y_n) \geq \lim_{n\to+\infty} u_{d_n}(x_n) = \lim_{n\to+\infty} \frac{A_2(x_n)}{\sigma A_1(x_n)}\big[\mu - (g(x_n, v_{d_n}(x_n)) - m_2(x_n))\big] v_{d_n}(x_n)$$
$$= +\infty,$$

which leads to $\|u_{d_n}\|_\infty = u_{d_n}(y_n) \to +\infty$ as $n \to +\infty$. By the arbitrariness of $\{d_n\}_{n=1}^{\infty}$, we conclude that $\|u_d\|_\infty \to +\infty$ as $d \to +\infty$. This proves (3.4).

Moreover, when $\|u_d\|_\infty \to +\infty$ as $d \to +\infty$, it follows from (3.4) that $\|v_d\|_\infty \to +\infty$ as $d \to +\infty$. In particular, we have $v_{d_n}(x_n) = \|v_{d_n}\|_\infty \to +\infty$ as $n \to +\infty$. By (3.2) and (H2), one can derive that

$$\lim_{n\to+\infty} \frac{\|u_{d_n}\|_\infty}{\|v_{d_n}\|_\infty} \geq \lim_{n\to+\infty} \frac{u_{d_n}(x_n)}{v_{d_n}(x_n)} = \lim_{n\to+\infty} \frac{A_2(x_n)}{\sigma A_1(x_n)}\big[\mu - (g(x_n, v_{d_n}(x_n)) - m_2(x_n))\big] = +\infty,$$

which implies that $\|u_{d_n}\|_\infty / \|v_{d_n}\|_\infty \to +\infty$ as $n \to +\infty$. By the arbitrariness of $\{d_n\}_{n=1}^{\infty}$, we obtain $\|u_d\|_\infty / \|v_d\|_\infty \to +\infty$, equivalently $\|v_d\|_\infty / \|u_d\|_\infty \to 0$ as $d \to +\infty$. This completes the proof. □

We next derive some *a priori* estimates for the positive solution of system (1.2).

Lemma 3.3. *Suppose $(u(x), v(x))$ is a positive solution of system (1.2). Then there exists a positive constant M that is independent of $d > 0$ and $\alpha > 0$, such that*

(i) $\int_0^L u(x)\mathrm{d}x \leq M$ *and* $\int_0^L v(x)\mathrm{d}x \leq M$*;*
(ii) $-\frac{(K_1+\sigma)M}{\alpha}\left(1 - e^{-\frac{\alpha}{d}(L-x)}\right) \leq u(x) - u(L)e^{-\frac{\alpha}{d}(L-x)} \leq \frac{\mu K_2 M}{\alpha}\left(1 - e^{-\frac{\alpha}{d}(L-x)}\right)$ *for* $x \in [0, L]$*;*
(iii) $u(L) < \frac{M[\alpha+(K_1+\sigma)L]}{d(1-e^{-\frac{\alpha}{d}L})}$,

where $K_1 := \max_{x\in[0,L]} m_1(x)$ *and* $K_2 := \max_{x\in[0,L]}\{A_2(x)/A_1(x)\}$.

Proof. (i) Multiplying the second equation in (1.2) by $A_2(x)/A_1(x)$ and adding the result to the first equation in (1.2), one then finds that

$$\begin{cases} du_{xx} - \alpha u_x - m_1(x)u + (g(x, v) - m_2(x))\frac{A_2(x)}{A_1(x)}v = 0, & x \in (0, L), \\ du_x(0) - \alpha u(0) = du_x(L) - \alpha u(L) = 0. \end{cases} \tag{3.6}$$

Integrating the first equation in (3.6) over $[0, L]$, we obtain

$$\int_0^L m_1(x)u(x)\mathrm{d}x + \int_0^L m_2(x)\frac{A_2(x)}{A_1(x)}v(x)\mathrm{d}x = \int_0^L \frac{A_2(x)}{A_1(x)}g(x, v(x))v(x)\mathrm{d}x. \tag{3.7}$$

It follows from (H2) that for any $x \in [0, L]$, $g(x, v)v$ is continuously differentiable with respect to $v \in [0, +\infty)$ and $\lim_{v\to+\infty} g(x, v)v = -\infty$. Thus, one can find a constant $M_0 > 0$ large enough such that $\max_{x\in[0,L]} g(x, v(x))v(x) \leq M_0$. Define $C_1 := \min_{x\in[0,L]} m_1(x)$ and $C_2 := \min_{x\in[0,L]}\{m_2(x)A_2(x)/A_1(x)\}$. Then we deduce from (3.7) that

$$C_1\int_0^L u(x)\mathrm{d}x + C_2\int_0^L v(x)\mathrm{d}x \le K_2M_0L.$$

This implies that

$$\int_0^L u(x)\mathrm{d}x \le M \text{ and } \int_0^L v(x)\mathrm{d}x \le M,$$

where $M := \max\{K_2M_0L/C_1, K_2M_0L/C_2\}$ is a positive constant independent of d and α.

(ii) Integrating the first equation in (1.2) over $[0, x]$, we obtain

$$du_x(x) - \alpha u(x) = \int_0^x (m_1(y)+\sigma)u(y)\mathrm{d}y - \mu\int_0^x \frac{A_2(y)}{A_1(y)}v(y)\mathrm{d}y. \tag{3.8}$$

It then follows from (3.8) and statement (i) that

$$du_x(x) - \alpha u(x) \le \int_0^L (m_1(y)+\sigma)u(y)\mathrm{d}y \le (K_1+\sigma)M \tag{3.9}$$

and

$$du_x(x) - \alpha u(x) \ge -\mu\int_0^L \frac{A_2(y)}{A_1(y)}v(y)\mathrm{d}y \ge -\mu K_2 M. \tag{3.10}$$

Multiplying (3.9) by $e^{-\frac{\alpha}{d}x}$ and then integrating over $[x, L]$, we have

$$du(L)e^{-\frac{\alpha}{d}L} - du(x)e^{-\frac{\alpha}{d}x} \le \frac{d(K_1+\sigma)M}{\alpha}(e^{-\frac{\alpha}{d}x} - e^{-\frac{\alpha}{d}L}),$$

which implies that

$$u(x) - u(L)e^{-\frac{\alpha}{d}(L-x)} \ge -\frac{(K_1+\sigma)M}{\alpha}(1 - e^{-\frac{\alpha}{d}(L-x)}). \tag{3.11}$$

Similarly, multiplying (3.10) by $e^{-\frac{\alpha}{d}x}$ and then integrating over $[x, L]$ yields

$$du(L)e^{-\frac{\alpha}{d}L} - du(x)e^{-\frac{\alpha}{d}x} \ge -\frac{d\mu K_2M}{\alpha}(e^{-\frac{\alpha}{d}x} - e^{-\frac{\alpha}{d}L}),$$

which leads to

$$u(x) - u(L)e^{-\frac{\alpha}{d}(L-x)} \le \frac{\mu K_2 M}{\alpha}(1 - e^{-\frac{\alpha}{d}(L-x)}).$$

This, together with (3.11), proves statement (ii).

(iii) It follows from (3.11) that

$$u(L)e^{-\frac{\alpha}{d}(L-x)} \le u(x) + \frac{(K_1+\sigma)M}{\alpha}(1-e^{-\frac{\alpha}{d}(L-x)}). \tag{3.12}$$

Integrating (3.12) over $[0, L]$ and using statement (i), one then finds that

$$u(L)\frac{d(1-e^{-\frac{\alpha}{d}L})}{\alpha} \le M + \frac{(K_1+\sigma)M}{\alpha}\left[L - \frac{d(1-e^{-\frac{\alpha}{d}L})}{\alpha}\right] < \frac{M[\alpha+(K_1+\sigma)L]}{\alpha},$$

which implies that

$$u(L) < \frac{M[\alpha+(K_1+\sigma)L]}{d(1-e^{-\frac{\alpha}{d}L})},$$

completing the proof. □

Lemma 3.4. *Suppose $(u(x), v(x))$ is a positive solution of (1.2). If $\mu > \max_{x\in[0,L]}\{g(x,0) - m_2(x)\} > 0$, then there exist positive constants A and B, independent of d and α, such that whenever*

$$\frac{\alpha^2}{d} > \frac{4\mu\sigma}{\mu - \max\limits_{x\in[0,L]}\{g(x,0)-m_2(x)\}}, \tag{3.13}$$

there holds

$$e^{-\frac{\alpha}{d}(1+\frac{d}{\alpha^2}B)(L-x)} \le \frac{u(x)}{u(L)} \le e^{-\frac{\alpha}{d}(1-\frac{d}{\alpha^2}A)(L-x)} \text{ for } x \in [0, L]. \tag{3.14}$$

Proof. When $\mu > \max_{x\in[0,L]}\{g(x,0) - m_2(x)\}$, we derive from (H2) that

$$\mu > g(x,0) - m_2(x) > g(x, v(x)) - m_2(x) \text{ for } x \in [0, L].$$

Then the second equation in (1.2) yields

$$v(x) = \frac{A_1(x)}{A_2(x)} \cdot \frac{\sigma}{\mu - (g(x, v(x)) - m_2(x))} u(x) \text{ for } x \in [0, L]. \tag{3.15}$$

Thus, the $u(x)$ component is governed by

$$\begin{cases} du_{xx} - \alpha u_x - \left[(m_1(x)+\sigma) - \frac{\mu\sigma}{\mu-(g(x,v(x))-m_2(x))}\right]u = 0, & x \in (0, L), \\ du_x(0) - \alpha u(0) = du_x(L) - \alpha u(L) = 0. \end{cases} \tag{3.16}$$

Set $w(x) = e^{-\frac{\alpha}{d}\eta x}u(x)$, where η is a constant that will be chosen differently for different purposes. By a direct calculation, we find that w satisfies

$$\begin{cases} dw_{xx} + \alpha(2\eta - 1)w_x - \left[\frac{\alpha^2\eta(1-\eta)}{d} + m_1(x) + \sigma - \frac{\mu\sigma}{\mu-(g(x,v)-m_2(x))}\right]w = 0, & x \in (0, L), \\ dw_x(0) = (1-\eta)\alpha w(0), \quad dw_x(L) = (1-\eta)\alpha w(L). \end{cases} \tag{3.17}$$

On the one hand, if we let $\eta = 1 - dA/\alpha^2$, where A is a positive constant to be chosen later, then

$$\begin{cases} dw_{xx}+\alpha(1 - \frac{2d}{\alpha^2}A)w_x - \left[A(1 - \frac{d}{\alpha^2}A)+m_1(x)+\sigma - \frac{\mu\sigma}{\mu-(g(x,v)-m_2(x))}\right]w = 0, & x \in (0, L), \\ w_x(0) = \frac{A}{\alpha}w(0), \quad w_x(L) = \frac{A}{\alpha}w(L). \end{cases} \tag{3.18}$$

Let $x^* \in [0, L]$ be such that $w(x^*) = \max_{x\in[0,L]} w(x)$. Then $w(x^*) > 0$. From $w_x(0) = Aw(0)/\alpha > 0$, we conclude that $x^* \neq 0$. If $x^* \in (0, L)$, then $w_{xx}(x^*) \le 0$ and $w_x(x^*) = 0$. Choose

$$A = \frac{2\mu\sigma}{\mu - \max\limits_{x\in[0,L]}\{g(x,0) - m_2(x)\}}.$$

Then by (3.18) and (H2), we see that whenever (3.13) is satisfied, there holds

$$\begin{aligned} 0 \ge dw_{xx}(x^*) &= \left[A(1 - \frac{d}{\alpha^2}A) + m_1(x^*) + \sigma - \frac{\mu\sigma}{\mu - (g(x^*, v(x^*)) - m_2(x^*))}\right]w(x^*) \\ &\ge \left[A(1 - \frac{d}{\alpha^2}A) - \frac{\mu\sigma}{\mu - \max\limits_{x\in[0,L]}\{g(x,0) - m_2(x)\}}\right]w(x^*) \\ &= \frac{\mu\sigma}{\mu- \max\limits_{x\in[0,L]}\{g(x,0)-m_2(x)\}}\left(1-\frac{d}{\alpha^2}\cdot\frac{4\mu\sigma}{\mu- \max\limits_{x\in[0,L]}\{g(x,0)-m_2(x)\}}\right)w(x^*) \\ &> 0, \end{aligned}$$

a contradiction. Thus, we must have $x^* = L$, meaning that

$$e^{-\frac{\alpha}{d}(1-\frac{d}{\alpha^2}A)x}u(x) \le e^{-\frac{\alpha}{d}(1-\frac{d}{\alpha^2}A)L}u(L) \text{ for } x \in [0, L]$$

whenever (3.13) holds. It follows that

$$\frac{u(x)}{u(L)} \le e^{-\frac{\alpha}{d}(1-\frac{d}{\alpha^2}A)(L-x)} \text{ for } x \in [0, L] \tag{3.19}$$

whenever (3.13) holds.

On the other hand, if we let $\eta = 1 + dB/\alpha^2$, where B is a positive constant to be determined later, then

$$\begin{cases} dw_{xx}+\alpha(1+\frac{2d}{\alpha^2}B)w_x - \left[-B(1+\frac{d}{\alpha^2}B)+m_1(x)+\sigma - \frac{\mu\sigma}{\mu-(g(x,v)-m_2(x))}\right]w=0, & x \in (0, L), \\ w_x(0) = -\frac{B}{\alpha}w(0), \quad w_x(L) = -\frac{B}{\alpha}w(L). \end{cases} \tag{3.20}$$

Let $x_* \in [0, L]$ be such that $w(x_*) = \min_{x\in[0,L]} w(x)$. Then $w(x_*) > 0$. From $w_x(0) = -Bw(0)/\alpha < 0$, we conclude that $x_* \neq 0$. If $x_* \in (0, L)$, then $w_{xx}(x_*) \geq 0$ and $w_x(x_*) = 0$. Choose

$$B = 2\left(\max_{x\in[0,L]} m_1(x) + \sigma\right).$$

Then it follows from (3.20) and (H2) that for any $d > 0$ and $\alpha > 0$,

$$\begin{aligned}0 \leq d w_{xx}(x_*) &= \left[-B(1+\frac{d}{\alpha^2}B) + m_1(x_*) + \sigma - \frac{\mu\sigma}{\mu - (g(x_*, v(x_*)) - m_2(x_*))}\right] w(x_*)\\ &\leq [-B + \max_{x\in[0,L]} m_1(x) + \sigma] w(x_*)\\ &< 0,\end{aligned}$$

a contradiction. Thus, we must have $x_* = L$, which means that

$$e^{-\frac{\alpha}{d}(1+\frac{d}{\alpha^2}B)x} u(x) \geq e^{-\frac{\alpha}{d}(1+\frac{d}{\alpha^2}B)L} u(L) \text{ for } x \in [0, L].$$

It follows that

$$\frac{u(x)}{u(L)} \geq e^{-\frac{\alpha}{d}(1+\frac{d}{\alpha^2}B)(L-x)} \text{ for } x \in [0, L].$$

This, together with (3.19), implies that (3.14) holds whenever (3.13) holds, completing the proof. □

4. Proofs of the main results

In this section, we complete the proofs of Theorems 2.2, 2.4, and 2.6. To emphasize the dependence of d and α on $(u(x), v(x))$, we rewrite it as $(u_{d,\alpha}(x), v_{d,\alpha}(x))$.

4.1. *Proof of Theorem 2.2*

This subsection is devoted to proving Theorem 2.2, which describes the asymptotic profile, as $\alpha/d \to +\infty$ and $\alpha^2/d \to +\infty$, of the positive solution $(u_{d,\alpha}(x), v_{d,\alpha}(x))$ to system (1.2) under condition (C1). For clarity, we divide the proof into the following *four steps*.

Step 1. We show that as $\alpha/d \to +\infty$ and $\alpha^2/d \to +\infty$, if $u_{d,\alpha}(x) \to 0$ for $x \in [0, L)$, then

$$v_{d,\alpha}(x) \to \begin{cases} 0, & x \in [0, L) \setminus I_+, \\ v^*(x), & x \in I_+, \end{cases} \tag{4.1}$$

where $v^*(x)$ is a positive function defined on I_+ and uniquely determined by (2.2).

Indeed, since $u_{d,\alpha}(x) \to 0$ for $x \in [0, L)$, it follows from Lemma 3.1 that $v_{d,\alpha}(x)$ is bounded on $[0, L)$ when $\alpha/d \to +\infty$ and $\alpha^2/d \to +\infty$. Hence for any sequences $\{d_n\}_{n=1}^{\infty}$ and $\{\alpha_n\}_{n=1}^{\infty}$ satisfying $\alpha_n/d_n \to +\infty$ and $\alpha_n^2/d_n \to +\infty$ as $n \to +\infty$, there exist subsequences (still denoted by $\{d_n\}_{n=1}^{\infty}$ and $\{\alpha_n\}_{n=1}^{\infty}$) such that for any $x \in [0, L)$, the limit $\tilde{v}(x) := \lim_{n\to+\infty} v_{d_n,\alpha_n}(x)$ exists. Clearly, $\tilde{v}(x) \geq 0$ for $x \in [0, L)$. Moreover, it follows from (3.3) that for $x \in [0, L]$ and $n \in \mathbb{N}$,

$$\sigma \frac{A_1(x)}{A_2(x)} \cdot \frac{u_{d_n,\alpha_n}(x)}{v_{d_n,\alpha_n}(x)} = \mu - (g(x, v_{d_n,\alpha_n}(x)) - m_2(x)). \tag{4.2}$$

For $x \in [0, L) \setminus I_+$, we have $g(x, 0) - m_2(x) \leq \mu$. If $\tilde{v}(x) > 0$, then letting $n \to +\infty$ on both sides of (4.2) and using (H2), we obtain

$$0 = \lim_{n\to+\infty} \sigma \frac{A_1(x)}{A_2(x)} \cdot \frac{u_{d_n,\alpha_n}(x)}{v_{d_n,\alpha_n}(x)} = \mu - (g(x, \tilde{v}(x)) - m_2(x)) > \mu - (g(x, 0) - m_2(x)) \geq 0.$$

This contradiction implies that $\tilde{v}(x) = 0$ when $x \in [0, L) \setminus I_+$. Thus, by the arbitrariness of $\{d_n\}_{n=1}^{\infty}$ and $\{\alpha_n\}_{n=1}^{\infty}$, we conclude that $v_{d,\alpha}(x) \to 0$ for $x \in [0, L) \setminus I_+$ when $\alpha/d \to +\infty$ and $\alpha^2/d \to +\infty$.

For $x \in I_+$, we have $g(x, 0) - m_2(x) > \mu$. If $\tilde{v}(x) = 0$, then one can let $n \to +\infty$ on both sides of (4.2) to deduce that

$$0 \leq \lim_{n\to+\infty} \sigma \frac{A_1(x)}{A_2(x)} \cdot \frac{u_{d_n,\alpha_n}(x)}{v_{d_n,\alpha_n}(x)} = \mu - (g(x, 0) - m_2(x)) < 0.$$

This contradiction yields that $\tilde{v}(x) > 0$ when $x \in I_+$. We next prove that $\tilde{v}(x) = v^*(x)$, where $v^*(x)$ is determined by (2.2). Since $\lim_{n\to+\infty} u_{d_n,\alpha_n}(x) = 0$ and $\lim_{n\to+\infty} v_{d_n,\alpha_n}(x) = \tilde{v}(x) > 0$, one can let $n \to +\infty$ on both sides of (4.2) to derive that $g(x, \tilde{v}(x)) - m_2(x) - \mu = 0$, namely $\tilde{v}(x)$ satisfies (2.2). Define

$$F_1(v) := g(x, v) - m_2(x) - \mu \text{ for } v \in [0, +\infty).$$

Then $F_1(\tilde{v}) = F_1(v^*) = 0$. It follows from (H2) that $F_1(v)$ is strictly decreasing with respect to $v \in [0, +\infty)$ and $\lim_{v\to+\infty} F_1(v) = -\infty$. Moreover, for any $x \in I_+$, we have $F_1(0) = g(x, 0) - m_2(x) - \mu > 0$. Therefore, there exists a unique $v = v(x) > 0$ such that $F_1(v) = 0$, which implies that $\tilde{v}(x) = v^*(x)$. By the uniqueness, we can conclude that $v_{d,\alpha}(x) \to v^*(x)$ for $x \in I_+$ as $\alpha/d \to +\infty$ and $\alpha^2/d \to +\infty$. This proves Step 1.

Step 2. We now prove that $\|u_{d,\alpha}\|_\infty \to +\infty$ and $\|v_{d,\alpha}\|_\infty \to +\infty$ as $\alpha/d \to +\infty$ and $\alpha^2/d \to +\infty$.

By Lemma 3.2, it suffices to prove that $\|u_{d,\alpha}\|_\infty \to +\infty$ as $\alpha/d \to +\infty$ and $\alpha^2/d \to +\infty$. For the sake of contradiction, suppose $\|u_{d,\alpha}\|_\infty$ is bounded when $\alpha/d \to +\infty$ and $\alpha^2/d \to +\infty$. Then Lemma 3.2 implies that $\|v_{d,\alpha}\|_\infty$ is also bounded when $\alpha/d \to +\infty$ and $\alpha^2/d \to +\infty$.

Let $y = \alpha(L - x)/d$ and define

$$(U(y), V(y)) := \left(u_{d,\alpha}(L - \frac{d}{\alpha} y), v_{d,\alpha}(L - \frac{d}{\alpha} y) \right). \tag{4.3}$$

Then (U, V) satisfies the following equations:

$$\begin{cases} U_{yy} + U_y - \frac{d}{\alpha^2} \left[(m_1(L - \frac{d}{\alpha} y) + \sigma)U - \mu \frac{A_2(L - \frac{d}{\alpha} y)}{A_1(L - \frac{d}{\alpha} y)} V \right] = 0, \quad y \in (0, \frac{\alpha L}{d}), \\ U_y(0) + U(0) = U_y(\frac{\alpha L}{d}) + U(\frac{\alpha L}{d}) = 0. \end{cases} \tag{4.4}$$

Moreover, both $\|U\|_\infty$ and $\|V\|_\infty$ are bounded when $\alpha/d \to +\infty$ and $\alpha^2/d \to +\infty$. Thus, passing to a subsequence if necessary, we may assume that $U(y) \to U^\infty(y)$ in $C^1_{\text{loc}}([0,+\infty))$ as $\alpha/d \to +\infty$ and $\alpha^2/d \to +\infty$, where U^∞ satisfies the following equations in the weak sense:

$$\begin{cases} U^\infty_{yy} + U^\infty_y = 0, & y \in (0,\infty), \\ U^\infty_y(0) + U^\infty(0) = 0. \end{cases}$$

A direct calculation yields $U^\infty(y) = U^\infty(0)e^{-y}$. From $U(0) \to U^\infty(0)$, we obtain

$$|U(y) - U(0)e^{-y}| \to 0 \text{ in } C^1_{\text{loc}}([0,+\infty)) \text{ as } \frac{\alpha}{d} \to +\infty \text{ and } \frac{\alpha^2}{d} \to +\infty,$$

implying that

$$\left\| u_{d,\alpha}(x) - u_{d,\alpha}(L)e^{-\frac{\alpha(L-x)}{d}} \right\|_\infty \to 0 \text{ as } \frac{\alpha}{d} \to +\infty \text{ and } \frac{\alpha^2}{d} \to +\infty. \tag{4.5}$$

Since $\|u_{d,\alpha}\|_\infty$ is bounded, it follows that $u_{d,\alpha}(L)$ is bounded when $\alpha/d \to +\infty$ and $\alpha^2/d \to +\infty$. Then one derives from (4.5) that $u_{d,\alpha}(x) \to 0$ for all $x \in [0,L)$ as $\alpha/d \to +\infty$ and $\alpha^2/d \to +\infty$. Integrating the first equation in (1.2) over $[0,L]$ yields

$$\int_0^L (m_1(x)+\sigma)u_{d,\alpha}(x)\mathrm{d}x = \mu \int_0^L \frac{A_2(x)}{A_1(x)} v_{d,\alpha}(x)\mathrm{d}x. \tag{4.6}$$

Thus, it follows that

$$\lim_{\substack{\alpha/d\to+\infty \\ \alpha^2/d\to+\infty}} \mu \int_0^L \frac{A_2(x)}{A_1(x)} v_{d,\alpha}(x)\mathrm{d}x = \lim_{\substack{\alpha/d\to+\infty \\ \alpha^2/d\to+\infty}} \int_0^L (m_1(x)+\sigma)u_{d,\alpha}(x)\mathrm{d}x = 0. \tag{4.7}$$

We claim that $I_+ \neq \emptyset$ when (C1) holds. Indeed, if $I_+ = \emptyset$, then $g(x,0) - m_2(x) \le \mu$ for $x \in [0,L)$. Hence (C1) implies that $g(L,0) - m_2(L) > \mu$. According to (H1) and (H2), one can find $\epsilon > 0$ small enough such that $g(x,0) - m_2(x) > \mu$ for $x \in [L-\epsilon, L]$. This means that $L - \epsilon \in I_+$, a contradiction. Thus, $I_+ \neq \emptyset$.

Therefore, one can derive from Step 1 and the fact $I_+ \neq \emptyset$ that

$$\lim_{\substack{\alpha/d\to+\infty \\ \alpha^2/d\to+\infty}} \mu \int_0^L \frac{A_2(x)}{A_1(x)} v_{d,\alpha}(x)\mathrm{d}x = \mu \int_{I_+} \frac{A_2(x)}{A_1(x)} v^*(x)\mathrm{d}x > 0,$$

which contradicts (4.7). Therefore, there must be $\|u_{d,\alpha}\|_\infty \to +\infty$ and $\|v_{d,\alpha}\|_\infty \to +\infty$ when $\alpha/d \to +\infty$ and $\alpha^2/d \to +\infty$. This proves Step 2.

Step 3. We show that $u_{d,\alpha}(x) \to 0$ for $x \in [0,L)$ as $\alpha/d \to +\infty$ and $\alpha^2/d \to +\infty$.

For the sake of contradiction, suppose that there exist constants $x_0 \in [0,L)$ and $\delta_0 > 0$ such that

$$\lim_{\substack{\alpha/d\to+\infty\\ \alpha^2/d\to+\infty}} u_{d,\alpha}(x_0)\geq\delta_0.$$

Hence it follows from Lemma 3.3 (i) that there exists a constant $M>0$ such that

$$\lim_{\substack{\alpha/d\to+\infty\\ \alpha^2/d\to+\infty}}\int_0^L \frac{u_{d,\alpha}(x)}{u_{d,\alpha}(x_0)}\mathrm{d}x\leq\frac{M}{\delta_0}. \tag{4.8}$$

For (U,V) given in (4.3), we define

$$(\hat{U}(y),\hat{V}(y)):=\left(\frac{U(y)}{\|U\|_\infty},\frac{V(y)}{\|V\|_\infty}\right).$$

Then $0<\hat{U}(y),\hat{V}(y)\leq 1$ for $y\in[0,\alpha L/d]$. Moreover, $(\hat{U},\hat{V})$ satisfies the following problem:

$$\begin{cases}\hat{U}_{yy}+\hat{U}_y-\frac{d}{\alpha^2}\left[(m_1(L-\frac{d}{\alpha}y)+\sigma)\hat{U}-\mu\frac{A_2(L-\frac{d}{\alpha}y)}{A_1(L-\frac{d}{\alpha}y)}\cdot\frac{\|V\|_\infty}{\|U\|_\infty}\cdot\hat{V}\right]=0, & y\in(0,\frac{\alpha L}{d}),\\ \hat{U}_y(0)+\hat{U}(0)=\hat{U}_y(\frac{\alpha L}{d})+\hat{U}(\frac{\alpha L}{d})=0. \end{cases} \tag{4.9}$$

It follows from Step 2 that $\|U\|_\infty\to+\infty$ and $\|V\|_\infty\to+\infty$ as $\alpha/d\to+\infty$ and $\alpha^2/d\to+\infty$. Then Lemma 3.2 shows that $\|V\|_\infty/\|U\|_\infty\to 0$ as $\alpha/d\to+\infty$ and $\alpha^2/d\to+\infty$. Hence

$$\left(m_1(L-\frac{d}{\alpha}y)+\sigma\right)\hat{U}-\mu\frac{A_2(L-\frac{d}{\alpha}y)}{A_1(L-\frac{d}{\alpha}y)}\cdot\frac{\|V\|_\infty}{\|U\|_\infty}\cdot\hat{V}$$

is bounded when $\alpha/d\to+\infty$ and $\alpha^2/d\to+\infty$. Thus, passing to a subsequence if necessary, we can assume that $\hat{U}(y)\to\hat{U}^\infty(y)$ in $C^1_{\mathrm{loc}}([0,+\infty))$ as $\alpha/d\to+\infty$ and $\alpha^2/d\to+\infty$, where $\hat{U}^\infty$ satisfies the following equation in the weak sense:

$$\begin{cases}\hat{U}^\infty_{yy}+\hat{U}^\infty_y=0, & y\in(0,\infty),\\ \hat{U}^\infty_y(0)+\hat{U}^\infty(0)=0,\ \|\hat{U}\|_\infty=1.\end{cases}$$

A standard calculation shows that $\hat{U}^\infty(y)=e^{-y}$ is the unique solution of this equation. By the uniqueness, $\hat{U}(y)\to e^{-y}$ in $C^1_{\mathrm{loc}}([0,+\infty))$ as $\alpha/d\to+\infty$ and $\alpha^2/d\to+\infty$. In particular, we have $\hat{U}(0)=U(0)/\|U\|_\infty\to 1$ as $\alpha/d\to+\infty$ and $\alpha^2/d\to+\infty$. Thus, we obtain

$$\frac{U(y)}{U(0)}\to e^{-y}\text{ in }C^1_{\mathrm{loc}}([0,+\infty))\text{ as }\alpha/d\to+\infty\text{ and }\alpha^2/d\to+\infty.$$

This implies that

$$\left\|\frac{u_{d,\alpha}(x)}{u_{d,\alpha}(L)}-e^{-\frac{\alpha(L-x)}{d}}\right\|_\infty\to 0\text{ as }\frac{\alpha}{d}\to+\infty\text{ and }\frac{\alpha^2}{d}\to+\infty. \tag{4.10}$$

Then for any $\epsilon \in (0, \frac{1}{3})$, there exists $K > 0$ large enough such that when $\alpha/d \geq K$ and $\alpha^2/d \geq K$,

$$\frac{u_{d,\alpha}(x)}{u_{d,\alpha}(x_0)} = \frac{u_{d,\alpha}(x)}{u_{d,\alpha}(L)} \cdot \frac{u_{d,\alpha}(L)}{u_{d,\alpha}(x_0)} \geq \frac{e^{-\frac{\alpha(L-x)}{d}}(1-\epsilon)}{e^{-\frac{\alpha(L-x_0)}{d}}(1+\epsilon)} > \frac{e^{\frac{\alpha(x-x_0)}{d}}}{2} \text{ for } x \in [0, L].$$

It then follows that

$$\lim_{\substack{\alpha/d\to+\infty \\ \alpha^2/d\to+\infty}} \int_0^L \frac{u_{d,\alpha}(x)}{u_{d,\alpha}(x_0)} \mathrm{d}x \geq \lim_{\substack{\alpha/d\to+\infty \\ \alpha^2/d\to+\infty}} \int_0^L \frac{e^{\frac{\alpha(x-x_0)}{d}}}{2} \mathrm{d}x = \lim_{\substack{\alpha/d\to+\infty \\ \alpha^2/d\to+\infty}} \frac{d}{2\alpha}\left(e^{\frac{\alpha(L-x_0)}{d}} - e^{-\frac{\alpha x_0}{d}}\right) = +\infty,$$

which contradicts (4.8). Therefore, $u_{d,\alpha}(x) \to 0$ for $x \in [0, L)$ as $\alpha/d \to +\infty$ and $\alpha^2/d \to +\infty$.

Step 4. Completion of the proof.

Step 3 has confirmed that $u_{d,\alpha}(x) \to 0$ for $x \in [0, L)$ as $\alpha/d \to +\infty$ and $\alpha^2/d \to +\infty$. This, together with Step 1, further confirms that (4.1) holds. Thus, statement (i) and (ii) are proved.

We now prove statement (iii). By combining the results obtained in Steps 2 and 3, we have $u_{d,\alpha}(L) \to +\infty$ as $\alpha/d \to +\infty$ and $\alpha^2/d \to +\infty$. Hence by Lemma 3.1, $v_{d,\alpha}(L) \to +\infty$ as $\alpha/d \to +\infty$ and $\alpha^2/d \to +\infty$. It remains to calculate the following two limits:

$$\lim_{\substack{\alpha/d\to+\infty \\ \alpha^2/d\to+\infty}} \frac{du_{d,\alpha}(L)}{\alpha} \text{ and } \lim_{\substack{\alpha/d\to+\infty \\ \alpha^2/d\to+\infty}} \frac{dv_{d,\alpha}(L)}{\alpha}.$$

To this end, we first show that there exists $\epsilon \in (0, L)$ small enough such that

$$\int_{L-\epsilon}^{L} v_{d,\alpha}(x)\mathrm{d}x \to 0 \text{ as } \frac{\alpha}{d} \to +\infty \text{ and } \frac{\alpha^2}{d} \to +\infty. \tag{4.11}$$

For the sake of contradiction, suppose that for any $\epsilon > 0$, there exist sequences $\{d_n\}_{n=1}^{\infty}$ and $\{\alpha_n\}_{n=1}^{\infty}$, and a constant $\delta_1 > 0$ such that

$$\lim_{n\to+\infty} \int_{L-\epsilon}^{L} v_{d_n,\alpha_n}(x)\mathrm{d}x \geq \delta_1.$$

Since $u_{d,\alpha}(L) \to +\infty$ as $\alpha/d \to +\infty$ and $\alpha^2/d \to +\infty$, Lemma 3.1 shows that $u_{d,\alpha}(L)/v_{d,\alpha}(L) \to +\infty$ as $\alpha/d \to +\infty$ and $\alpha^2/d \to +\infty$. Thus, by taking ϵ sufficiently small, one can find a constant $\tilde{N} > 0$ large enough such that for $n > \tilde{N}$,

$$\frac{u_{d_n,\alpha_n}(x)}{v_{d_n,\alpha_n}(x)} \geq \frac{2M}{\delta_1} \text{ for } x \in [L-\epsilon, L],$$

where M is as in Lemma 3.3 (i). This, together with Lemma 3.3 (i), implies that

$$M \geq \lim_{n\to+\infty} \int_{L-\epsilon}^{L} u_{d_n,\alpha_n}(x)\mathrm{d}x \geq \frac{2M}{\delta_1} \lim_{n\to+\infty} \int_{L-\epsilon}^{L} v_{d_n,\alpha_n}(x)\mathrm{d}x \geq 2M.$$

This contradiction implies that (4.11) holds.

Thus, by (4.11), we derive from (4.1) that

$$\lim_{\substack{\alpha/d\to+\infty \\ \alpha^2/d\to+\infty}} \mu \int_0^L \frac{A_2(x)}{A_1(x)} v_{d,\alpha}(x)\mathrm{d}x = \mu \int_{I_+} \frac{A_2(x)}{A_1(x)} v^*(x)\mathrm{d}x. \tag{4.12}$$

Moreover, it follows from (4.10) that

$$\begin{aligned}
\lim_{\substack{\alpha/d\to+\infty \\ \alpha^2/d\to+\infty}} \int_0^L (m_1(x)+\sigma)u_{d,\alpha}(x)\mathrm{d}x &= \lim_{\substack{\alpha/d\to+\infty \\ \alpha^2/d\to+\infty}} \int_0^L (m_1(x)+\sigma)u_{d,\alpha}(L)\frac{u_{d,\alpha}(x)}{u_{d,\alpha}(L)}\mathrm{d}x \\
&= \lim_{\substack{\alpha/d\to+\infty \\ \alpha^2/d\to+\infty}} \int_0^L (m_1(x)+\sigma)u_{d,\alpha}(L)e^{-\frac{\alpha(L-x)}{d}}\mathrm{d}x \\
&= \lim_{\substack{\alpha/d\to+\infty \\ \alpha^2/d\to+\infty}} \frac{du_{d,\alpha}(L)}{\alpha} \int_0^{\frac{\alpha L}{d}} \left(m_1(L-\frac{d}{\alpha}y)+\sigma\right)e^{-y}\mathrm{d}y \\
&= (m_1(L)+\sigma) \lim_{\substack{\alpha/d\to+\infty \\ \alpha^2/d\to+\infty}} \frac{du_{d,\alpha}(L)}{\alpha}.
\end{aligned}$$

Combining this with (4.6) and (4.12) yields

$$\lim_{\substack{\alpha/d\to+\infty \\ \alpha^2/d\to+\infty}} \frac{du_{d,\alpha}(L)}{\alpha} = \frac{\mu \int_{I_+} \frac{A_2(x)}{A_1(x)} v^*(x)\mathrm{d}x}{m_1(L)+\sigma}.$$

Since $u_{d,\alpha}(L) \to +\infty$ as $\alpha/d \to +\infty$ and $\alpha^2/d \to +\infty$, Lemma 3.1 shows that $v_{d,\alpha}(L)/u_{d,\alpha}(L) \to 0$ as $\alpha/d \to +\infty$ and $\alpha^2/d \to +\infty$. Therefore, it follows that

$$\lim_{\substack{\alpha/d\to+\infty \\ \alpha^2/d\to+\infty}} \frac{dv_{d,\alpha}(L)}{\alpha} = \lim_{\substack{\alpha/d\to+\infty \\ \alpha^2/d\to+\infty}} \frac{du_{d,\alpha}(L)}{\alpha} \cdot \frac{v_{d,\alpha}(L)}{u_{d,\alpha}(L)} = 0.$$

This completes the proof of (iii), and hence, of the whole Theorem 2.2. □

4.2. Proof of Theorem 2.4

In this subsection, we present the proof of Theorem 2.4, which describes the asymptotic profile, as $\alpha/d \to +\infty$ and $\alpha^2/d \to +\infty$, of the positive solution $(u_{d,\alpha}(x), v_{d,\alpha}(x))$ to system (1.2) under condition (C2). Note that $(u_{d,\alpha}(x), v_{d,\alpha}(x))$ satisfies (3.15) and (3.16). For clarity, we split the proof into *two steps*.

Step 1. Proving that both $u_{d,\alpha}(L)$ and $v_{d,\alpha}(L)$ are bounded as $\alpha/d \to +\infty$ and $\alpha^2/d \to +\infty$.

To this end, we let $y = \alpha(L-x)/d$ and define

$$\tilde{U}_{d,\alpha}(y) = \frac{u_{d,\alpha}(L - \frac{d}{\alpha}y)}{\|u_{d,\alpha}\|_\infty}.$$

A direct calculation shows that $\tilde{U}_{d,\alpha}$ satisfies the following equations:

$$\begin{cases} (\tilde{U}_{d,\alpha})_{yy} + (\tilde{U}_{d,\alpha})_y + H_{d,\alpha}(y) = 0, & y \in (0, \frac{\alpha L}{d}), \\ (\tilde{U}_{d,\alpha})_y(0) + \tilde{U}_{d,\alpha}(0) = (\tilde{U}_{d,\alpha})_y(\frac{\alpha L}{d}) + \tilde{U}_{d,\alpha}(\frac{\alpha L}{d}) = 0, \end{cases} \tag{4.13}$$

where

$$H_{d,\alpha}(y) := -\frac{d}{\alpha^2}\left[m_1(L - \frac{d}{\alpha}y) + \sigma - \frac{\mu\sigma}{\mu - (g(L-\frac{d}{\alpha}y, v_{d,\alpha}(L-\frac{d}{\alpha}y)) - m_2(L-\frac{d}{\alpha}y))}\right]\tilde{U}_{d,\alpha}(y).$$

Integrating (4.13) over $[0, \alpha L/d]$ yields

$$\int_0^{\frac{\alpha L}{d}} \left[m_1(L - \frac{d}{\alpha}y) + \sigma - \frac{\mu\sigma}{\mu - (g(L - \frac{d}{\alpha}y, v_{d,\alpha}(L - \frac{d}{\alpha}y)) - m_2(L - \frac{d}{\alpha}y))}\right]\tilde{U}_{d,\alpha}(y)\mathrm{d}y = 0. \tag{4.14}$$

We first prove the boundedness of $v_{d,\alpha}(L)$. For the sake of contradiction, suppose there exist sequences $\{d_n\}_{n=1}^\infty$ and $\{\alpha_n\}_{n=1}^\infty$ satisfying $\alpha_n/d_n \to +\infty$ and $\alpha_n^2/d_n \to +\infty$ as $n \to +\infty$ such that $v_{d_n,\alpha_n}(L) \to +\infty$ as $n \to +\infty$. By the definition of $\tilde{U}_{d,\alpha}(y)$, it is obvious that $|\tilde{U}_{d,\alpha}(y)| \le 1$ on $[0, \alpha L/d]$ for $d > 0$ and $\alpha > 0$; accordingly $|\tilde{U}_{d_n,\alpha_n}(y)| \le 1$ on $[0, \alpha_n L/d_n]$ for $n = 1, 2, \cdots$. Thus, there exists a subsequence of $\{(d_n, \alpha_n)\}$, still denote it as $\{(d_n, \alpha_n)\}$ without loss of generality, with $\alpha_n/d_n \to +\infty$ and $\alpha_n^2/d_n \to +\infty$ such that

$$\tilde{U}_{d_n,\alpha_n}(y) \to \tilde{U}^\infty(y) \text{ in } C^1_{\text{loc}}([0, +\infty)) \text{ as } n \to +\infty.$$

Moreover, $\tilde{U}^\infty$ satisfies the following equation in the weak sense:

$$\begin{cases} \tilde{U}^\infty_{yy} + \tilde{U}^\infty_y = 0, & y \in (0, \infty), \\ \tilde{U}^\infty_y(0) + \tilde{U}^\infty(0) = 0, \ \|\tilde{U}^\infty\|_\infty = 1. \end{cases}$$

Clearly, $\tilde{U}^\infty(y) = e^{-y}$ is the unique solution of this equation. By the uniqueness, we then have

$$\tilde{U}_{d,\alpha}(y) \to e^{-y} \text{ in } C^1_{\text{loc}}([0,+\infty)) \text{ as } \frac{\alpha}{d} \to +\infty \text{ and } \frac{\alpha^2}{d} \to +\infty. \tag{4.15}$$

Now, taking $(d,\alpha) = (d_n, \alpha_n)$ in (4.14) and letting $n \to +\infty$, we derive from (4.15) and (H2) that

$$0 = \int_0^\infty (m_1(L) + \sigma)e^{-y}\mathrm{d}y = m_1(L) + \sigma > 0.$$

This contradiction implies that $v_{d,\alpha}(L)$ is bounded when $\alpha/d \to +\infty$ and $\alpha^2/d \to +\infty$. By Lemma 3.1, it follows that $u_{d,\alpha}(L)$ is also bounded when $\alpha/d \to +\infty$ and $\alpha^2/d \to +\infty$. This completes the proof of Step 1.

Step 2. Completing the proof of the theorem.

Part (i): From Lemma 3.4 and Step 1, we obtain $u_{d,\alpha}(x) \to 0$ for $x \in [0, L)$ when $\alpha/d \to +\infty$ and $\alpha^2/d \to +\infty$. In view of (H2), we derive from (3.15) that for $x \in [0, L]$,

$$\begin{aligned} v_{d,\alpha}(x) &= \frac{\sigma A_1(x)}{A_2(x)[\mu - (g(x, v_{d,\alpha}(x)) - m_2(x))]} u_{d,\alpha}(x) \\ &< \frac{\sigma A_1(x)}{[\mu - (g(x,0) - m_2(x))]A_2(x)} u_{d,\alpha}(x). \end{aligned}$$

Thus, letting $\alpha/d \to +\infty$ and $\alpha^2/d \to +\infty$ yields $v_{d,\alpha}(x) \to 0$ for $x \in [0, L)$, proving statement (i).

Part (ii): By Step 1, for any sequences $\{d_n\}_{n=1}^\infty$ and $\{\alpha_n\}_{n=1}^\infty$ satisfying $\alpha_n/d_n \to +\infty$ and $\alpha_n^2/d_n \to +\infty$ as $n \to +\infty$, there exist subsequences (still denoted by $\{d_n\}_{n=1}^\infty$ and $\{\alpha_n\}_{n=1}^\infty$) such that

$$\lim_{n\to+\infty} u_{d_n,\alpha_n}(L) =: u^\infty \text{ and } \lim_{n\to+\infty} v_{d_n,\alpha_n}(L) =: v^\infty.$$

Taking limit as $n \to \infty$ in

$$(g(L, v_{d_n,\alpha_n}(L)) - m_2(L) - \mu)v_{d_n,\alpha_n}(L) + \sigma \frac{A_1(L)}{A_2(L)} u_{d_n,\alpha_n}(L) = 0 \tag{4.16}$$

yields

$$(g(L, v^\infty) - m_2(L) - \mu)v^\infty + \sigma \frac{A_1(L)}{A_2(L)} u^\infty = 0. \tag{4.17}$$

Moreover, integrating the first equation in (1.2) with $(d,\alpha) = (d_n, \alpha_n)$ over $[0, L]$, we have

$$\int_0^L \left[(m_1(x) + \sigma)u_{d_n,\alpha_n}(x) - \mu \frac{A_2(x)}{A_1(x)} v_{d_n,\alpha_n}(x)\right] \mathrm{d}x = 0.$$

Setting $z = \alpha_n(L - x)/d_n$, it follows that

$$\int_0^{\frac{\alpha_n L}{d_n}} \left[(m_1(L - \frac{d_n}{\alpha_n} z) + \sigma) \cdot u_{d_n,\alpha_n}(L - \frac{d_n}{\alpha_n} z) - \mu \frac{A_2(L - \frac{d_n}{\alpha_n} z)}{A_1(L - \frac{d_n}{\alpha_n} z)} \cdot v_{d_n,\alpha_n}(L - \frac{d_n}{\alpha_n} z) \right] \mathrm{d}z = 0. \tag{4.18}$$

Letting $n \to +\infty$ in (4.18), one can derive that

$$-(m_1(L) + \sigma) u^\infty + \mu \frac{A_2(L)}{A_1(L)} v^\infty = 0. \tag{4.19}$$

Combining (4.17) and (4.19), it follows that (u^∞, v^∞) satisfies (2.3).

It remains to verify that $u^\infty, v^\infty > 0$ and (u^∞, v^∞) is unique. Indeed, it follows from (4.16) that

$$\lim_{n \to +\infty} \frac{u_{d_n,\alpha_n}(L)}{v_{d_n,\alpha_n}(L)} = \frac{A_2(L)[\mu - (g(L, v^\infty) - m_2(L))]}{\sigma A_1(L)}. \tag{4.20}$$

Dividing (4.18) by $v_{d_n,\alpha_n}(L)$ and letting $n \to +\infty$, we have

$$\lim_{n \to +\infty} \left[(m_1(L - \frac{d_n}{\alpha_n} z) + \sigma) \cdot \frac{u_{d_n,\alpha_n}(L - \frac{d_n}{\alpha_n} z)}{v_{d_n,\alpha_n}(L)} - \mu \frac{A_2(L - \frac{d_n}{\alpha_n} z)}{A_1(L - \frac{d_n}{\alpha_n} z)} \cdot \frac{v_{d_n,\alpha_n}(L - \frac{d_n}{\alpha_n} z)}{v_{d_n,\alpha_n}(L)} \right] = 0,$$

which implies that

$$\lim_{n \to +\infty} \frac{u_{d_n,\alpha_n}(L)}{v_{d_n,\alpha_n}(L)} = \frac{\mu A_2(L)}{A_1(L)(m_1(L) + \sigma)}.$$

Thus, combining this with (4.20), we obtain

$$\frac{\mu \sigma}{\mu - (g(L, v^\infty) - m_2(L))} - (m_1(L) + \sigma) = 0.$$

Define

$$F_2(v) := \frac{\mu \sigma}{\mu - (g(L, v) - m_2(L))} - (m_1(L) + \sigma) \text{ for } v \geq 0.$$

Then $F_2(v^\infty) = 0$. It follows from (H2) that $F_2(v)$ is strictly decreasing in $v \in [0, +\infty)$ and

$$\lim_{v \to +\infty} F_2(v) = -(m_1(L) + \sigma) < 0.$$

Moreover, condition (C2) shows that $\sigma > \sigma_{\sup} \geq \sigma_L$, where $\sigma_{\sup}$ and σ_L are defined in (2.1). A direct calculation yields

$$F_2(0) = \frac{\mu \sigma}{\mu - (g(L, 0) - m_2(L))} - (m_1(L) + \sigma) > 0.$$

Therefore, there exists a unique $v > 0$ such that $F_2(v) = 0$. This implies that $v^\infty > 0$ and v^∞ is uniquely determined by $F_2(v^\infty) = 0$. Moreover, (4.19) further uniquely determines u^∞ as

$$u^{\infty}=\frac{\mu A_2(L)}{A_1(L)(m_1(L)+\sigma)}v^{\infty}>0.$$

Summarizing the above, we proved that

$$\lim_{\substack{\alpha/d\to+\infty\\ \alpha^2/d\to+\infty}}(u_{d,\alpha}(L),v_{d,\alpha}(L))=(u^{\infty},v^{\infty}),$$

where u^{∞} and v^{∞} are two positive constants uniquely determined by (2.3). This completes the proof of Theorem 2.4. □

4.3. *Proof of Theorem 2.6*

In this subsection, we give the proof of Theorem 2.6, which describes the asymptotical behavior, as $(d,\alpha)\to(+\infty,0)$ or as $d\to+\infty$ with $\alpha>0$ fixed, of the positive solution $(u_{d,\alpha}(x),v_{d,\alpha}(x))$ to system (1.2). We focus on the case when $(d,\alpha)\to(+\infty,0)$, since the proof of the other case follows similarly. For clarity, we divide the proof into the following *two steps*.

Step 1. We first show that both $\|u_{d,\alpha}\|_\infty$ and $\|v_{d,\alpha}\|_\infty$ are bounded when $(d,\alpha)\to(+\infty,0)$.

According to Lemma 3.2, it suffices to prove that $\|u_{d,\alpha}\|_\infty$ is bounded when $(d,\alpha)\to(+\infty,0)$. For the sake of contradiction, suppose there exist sequences $\{d_n\}_{n=1}^{\infty}$ and $\{\alpha_n\}_{n=1}^{\infty}$ satisfying $(d_n,\alpha_n)\to(+\infty,0)$ as $n\to+\infty$ such that $\|u_{d_n,\alpha_n}\|_\infty\to+\infty$ as $n\to+\infty$. Define

$$p_n(x):=\frac{u_{d_n,\alpha_n}(x)}{\|u_{d_n,\alpha_n}\|_\infty}\ \text{and}\ w_n(x):=\frac{v_{d_n,\alpha_n}(x)}{\|u_{d_n,\alpha_n}\|_\infty}. \tag{4.21}$$

Then (p_n,w_n) satisfies the following problem:

$$\begin{cases} d_n(p_n)_{xx}-\alpha_n(p_n)_x-(m_1(x)+\sigma)p_n+\mu\frac{A_2(x)}{A_1(x)}w_n=0, & x\in(0,L),\\ (g(x,v_{d_n,\alpha_n}(x))-m_2(x)-\mu)w_n+\sigma\frac{A_1(x)}{A_2(x)}p_n=0, & x\in[0,L],\\ d_n(p_n)_x(0)-\alpha_n p_n(0)=d_n(p_n)_x(L)-\alpha_n p_n(L)=0. \end{cases} \tag{4.22}$$

Clearly, $p_n(x)$ is uniformly bounded on $[0,L]$ by its definition. Moreover, since $\|u_{d_n,\alpha_n}\|_\infty\to+\infty$ as $n\to+\infty$, it follows from Lemma 3.2 that $\|v_{d_n,\alpha_n}\|_\infty/\|u_{d_n,\alpha_n}\|_\infty\to0$ as $n\to+\infty$, which implies that $w_n(x)$ is uniformly bounded on $[0,L]$ when $n\to+\infty$. For any $x\in(0,L)$, integrating the first equation in (4.22) over $(0,x)$ yields

$$(p_n)_x(x)=\frac{1}{d_n}\left[\alpha_n p_n(x)+\int_0^x\Big((m_1(y)+\sigma)p_n(y)-\mu\frac{A_2(y)}{A_1(y)}w_n(y)\Big)\mathrm{d}y\right].$$

It follows that $(p_n)_x$ is uniformly bounded on $[0,L]$ when $n\to+\infty$. Then by the first equation in (4.22), $(p_n)_{xx}$ is uniformly bounded on $[0,L]$ when $n\to+\infty$. Therefore, passing to a subsequence if necessary, we may assume that

$$p_n(x)\to p^*(x)\ \text{in}\ C^{1+\beta}([0,L])$$

for some $\beta \in (0, 1)$. Dividing the first and third equations of (4.22) by d_n yields

$$\begin{cases} (p_n)_{xx} - \frac{\alpha_n}{d_n}(p_n)_x - \frac{1}{d_n}\left[(m_1(x)+\sigma)p_n - \mu\frac{A_2(x)}{A_1(x)}w_n\right] = 0, & x \in (0, L), \\ (p_n)_x(0) = \frac{\alpha_n}{d_n}p_n(0),\ (p_n)_x(L) = \frac{\alpha_n}{d_n}p_n(L). \end{cases}$$

It follows that $p^*(x)$ satisfies the following problem in the weak sense:

$$\begin{cases} p^*_{xx} = 0, & x \in (0, L), \\ p^*_x(0) = p^*_x(L) = 0,\ \|p^*\|_\infty = 1. \end{cases}$$

Clearly, $p^*(x) \equiv 1$ is the unique solution of this equation. By the uniqueness, we obtain

$$\lim_{n\to+\infty} \frac{u_{d_n,\alpha_n}(x)}{\|u_{d_n,\alpha_n}\|_\infty} = \lim_{n\to+\infty} p_n(x) = 1 \text{ in } C^{1+\beta}([0, L]). \tag{4.23}$$

Now, since $\|u_{d_n,\alpha_n}\|_\infty \to +\infty$ as $n \to +\infty$, we conclude from (4.23) that $u_{d_n,\alpha_n}(x) \to +\infty$ as $n \to +\infty$ for $x \in [0, L]$. Thus, it follows from Lemma 3.1 that $v_{d_n,\alpha_n}(x)/u_{d_n,\alpha_n}(x) \to 0$ as $n \to +\infty$ for $x \in [0, L]$, which implies that

$$\lim_{n\to+\infty} w_n(x) = 0 \text{ for } x \in [0, L].$$

Thus, one can integrate the first equation in (4.22) over $[0, L]$ and let $n \to +\infty$ to deduce that

$$0 = \mu \lim_{n\to+\infty} \int_0^L \frac{A_2(x)}{A_1(x)} w_n(x)\mathrm{d}x = \lim_{n\to+\infty} \int_0^L (m_1(x)+\sigma)p_n(x)\mathrm{d}x = \int_0^L (m_1(x)+\sigma)\mathrm{d}x > 0.$$

This contradiction implies that $\|u_{d,\alpha}\|_\infty$ is bounded when $(d, \alpha) \to (+\infty, 0)$, which proves Step 1.

Step 2. Completing the proof of Theorem 2.6.

From Step 1, $-(m_1(x)+\sigma)u_{d,\alpha} + \mu(A_2(x)/A_1(x))v_{d,\alpha}$ is uniformly bounded on $[0, L]$ when $(d, \alpha) \to (+\infty, 0)$. Thus, integrating the first equation in (1.2) over $[0, L]$, one finds that $(u_{d,\alpha})_x$ is uniformly bounded on $[0, L]$ when $(d, \alpha) \to (+\infty, 0)$. By the first equation in (1.2) again, $(u_{d,\alpha})_{xx}$ is uniformly bounded on $[0, L]$ when $(d, \alpha) \to (+\infty, 0)$. Therefore, for any sequences $\{d_n\}_{n=1}^\infty$ and $\{\alpha_n\}_{n=1}^\infty$ satisfying $(d_n, \alpha_n) \to (+\infty, 0)$ as $n \to +\infty$, there exist subsequences $\{\tilde{d}_n\}_{n=1}^\infty$ and $\{\tilde{\alpha}_n\}_{n=1}^\infty$ such that

$$u_{\tilde{d}_n,\tilde{\alpha}_n}(x) \to \bar{u}(x) \text{ in } C^{1+\gamma}([0, L]) \text{ as } n \to +\infty$$

for some $\gamma \in (0, 1)$. Dividing both sides of (1.2) by d, we obtain

$$\begin{cases} u_{xx} - \frac{1}{d}\left[\alpha u_x + (m_1(x)+\sigma)u - \mu\frac{A_2(x)}{A_1(x)}v\right] = 0, & x \in (0, L), \\ u_x(0) = \frac{\alpha}{d}u(0),\ u_x(L) = \frac{\alpha}{d}u(L). \end{cases}$$

By taking limit, we find that $\bar{u}(x)$ is a solution to the following problem in the weak sense:

$$\begin{cases} \bar{u}_{xx} = 0, & x \in (0, L), \\ \bar{u}_x(0) = \bar{u}_x(L) = 0, \end{cases}$$

which implies that $\bar{u} \equiv$ const.

By the boundedness of $\|v_{d,\alpha}\|_\infty$ (see Step 1), one can find subsequences of $\{\tilde{d}_n\}_{n=1}^\infty$ and $\{\tilde{\alpha}_n\}_{n=1}^\infty$ (still denoted by $\{\tilde{d}_n\}_{n=1}^\infty$ and $\{\tilde{\alpha}_n\}_{n=1}^\infty$) such that

$$v_{\tilde{d}_n,\tilde{\alpha}_n}(x) \to \bar{v}(x) \text{ for } x \in [0, L] \text{ as } n \to +\infty.$$

Moreover, integrating the first equation in (1.2) over $[0, L]$ and combining the resulting equation with the second equation in (1.2), one can verify that $(\bar{u}, \bar{v}(x))$ satisfies (2.4).

It remains to verify that $\bar{u} > 0$, $\bar{v}(x) > 0$ for $x \in [0, L]$, and $(\bar{u}, \bar{v}(x))$ is unique. Since the proof is quite lengthy, we divide the proof into the following *six claims.*

Claim 1. $\bar{u} > 0$ and $\bar{v}(x) > 0$ for $x \in [0, L]$.

We first prove that $\bar{u} > 0$. For the sake of contradiction, suppose $\bar{u} = 0$. Then the first equation in (2.4) shows that $\bar{v}(x) \equiv 0$ on $[0, L]$.

When (C1) holds, we have $\max_{x\in[0,L]}\{g(x, 0) - m_2(x)\} > \mu$. Then there exists $x_0 \in [0, L]$ such that

$$g(x_0, 0) - m_2(x_0) > \mu. \tag{4.24}$$

However, it follows from the fact $\bar{v}(x_0) = 0$ and the second equation in (1.2) that

$$\begin{aligned} 0 &\le \lim_{n\to+\infty} \frac{\sigma A_1(x_0)}{A_2(x_0)} \cdot \frac{u_{\tilde{d}_n,\tilde{\alpha}_n}(x_0)}{v_{\tilde{d}_n,\tilde{\alpha}_n}(x_0)} = \lim_{n\to+\infty} \left[\mu - (g(x_0, v_{\tilde{d}_n,\tilde{\alpha}_n}(x_0)) - m_2(x_0))\right] \\ &= \mu - (g(x_0, 0) - m_2(x_0)), \end{aligned}$$

which contradicts (4.24). Then we obtain $\bar{u} > 0$.

When (C2) holds, we know that (u, v) satisfies (3.16). Thus, $\tilde{p}_n(x) := u_{\tilde{d}_n,\tilde{\alpha}_n}(x)/\|u_{\tilde{d}_n,\tilde{\alpha}_n}\|_\infty$ satisfies

$$\begin{cases} \tilde{d}_n(\tilde{p}_n)_{xx} - \tilde{\alpha}_n(\tilde{p}_n)_x - \left[(m_1(x) + \sigma) - \frac{\mu\sigma}{\mu-(g(x,v_{\tilde{d}_n,\tilde{\alpha}_n}(x))-m_2(x))}\right]\tilde{p}_n = 0, & x \in (0, L), \\ \tilde{d}_n(\tilde{p}_n)_x(0) - \tilde{\alpha}_n\tilde{p}_n(0) = \tilde{d}_n(\tilde{p}_n)_x(L) - \tilde{\alpha}_n\tilde{p}_n(L) = 0. \end{cases} \tag{4.25}$$

By $\|\tilde{p}_n\|_\infty \le 1$, one can repeat the arguments as in Step 1 to derive that (4.23) holds. Integrating the first equation in (4.25) over $[0, L]$, we have

$$\int_0^L \left[\frac{\mu\sigma}{\mu - (g(x, v_{\tilde{d}_n,\tilde{\alpha}_n}(x)) - m_2(x))} - (m_1(x) + \sigma)\right]\tilde{p}_n(x)\mathrm{d}x = 0.$$

Since $\lim_{n\to+\infty} v_{\tilde{d}_n,\tilde{\alpha}_n}(x) = \bar{v}(x) \equiv 0$ on $[0, L]$, we can let $n \to +\infty$ and use (4.23) to deduce that

$$\int_0^L \left[\frac{\mu\sigma}{\mu - (g(x,0) - m_2(x))} - (m_1(x) + \sigma) \right] \mathrm{d}x = 0. \tag{4.26}$$

It follows from (C2) that $\sigma > \sigma_{\sup} \geq \sigma_x$ for $x \in [0, L]$. Thus, a direct calculation yields

$$\frac{\mu\sigma}{\mu - (g(x,0) - m_2(x))} - (m_1(x) + \sigma) > 0 \text{ for } x \in [0, L],$$

contradicting (4.26). Therefore, there must be $\bar{u} > 0$.

Moreover, if there exists $x_0 \in [0, L]$ such that $\bar{v}(x_0) = 0$, then the second equation in (2.4) shows

$$\bar{u} = \frac{A_2(x_0)}{\sigma A_1(x_0)} \cdot \left[\mu - (g(x_0, \bar{v}(x_0)) - m_2(x_0))\right]\bar{v}(x_0) = 0,$$

which contradicts the fact that $\bar{u} > 0$. Thus, $\bar{v}(x) > 0$ for $x \in [0, L]$. This proves Claim 1.

For any fixed $x \in [0, L]$ and $u > 0$, define

$$F_3(v) = g(x, v) - m_2(x) - \mu + \sigma \frac{A_1(x)}{A_2(x)} \cdot \frac{u}{v} \text{ for } v > 0. \tag{4.27}$$

Claim 2. For any $x \in [0, L]$ and $u > 0$, there exists a unique $v(u; x) > 0$ such that $F_3(v(u; x)) = 0$. Moreover, for any $x \in [0, L]$, $v(u; x)$ is continuously differentiable with respect to $u \in (0, +\infty)$ and satisfies $\lim_{u\to+\infty} v(u; x) = +\infty$,

$$\frac{\partial v(u;x)}{\partial u} = \frac{\sigma A_1(x)}{A_2(x)[\mu - (g(x, v(u;x)) - m_2(x)) - g_v(x, v(u;x))v(u;x)]}, \tag{4.28}$$

and

$$\lim_{u\to 0^+} v(u;x) = \begin{cases} 0, & x \in [0, L] \setminus \hat{I}_+, \\ v^*(x), & x \in \hat{I}_+, \end{cases} \tag{4.29}$$

where $\hat{I}_+ := \{x \in [0, L] : g(x, 0) - m_2(x) > \mu\}$, and $v^*(x)$ is a positive function defined on $\hat{I}_+$ and uniquely determined by $g(x, v^*(x)) - m_2(x) - \mu = 0$ for $x \in \hat{I}_+$.

Indeed, it follows from (H2) that for any $x \in [0, L]$ and $u > 0$, $F_3(v)$ is strictly decreasing with respect to $v \in (0, +\infty)$ satisfying $\lim_{v\to 0^+} F_3(v) = +\infty$ and $\lim_{v\to+\infty} F_3(v) = -\infty$. Thus, there exists a unique $v = v(u; x) > 0$ such that $F_3(v(u; x)) = 0$. By the implicit function theorem, it follows that for any $x \in [0, L]$, $v(u; x)$ is continuously differentiable with respect to $u \in (0, +\infty)$.

We now verify that $\lim_{u\to+\infty} v(u; x) = +\infty$ for $x \in [0, L]$. For the sake of contradiction, suppose there exists $x_0 \in [0, L]$ such that $v(u; x_0)$ is bounded when $u \to +\infty$. Then

$$\lim_{u\to+\infty} (g(x_0, v(u; x_0)) - m_2(x_0) - \mu)v(u; x_0)$$

is bounded. By $F_3(v(u; x)) = 0$, we have $F_3(v(u; x))v(u; x) = 0$; that is,

$$(g(x, v(u;x)) - m_2(x) - \mu)v(u;x) + \sigma \frac{A_1(x)}{A_2(x)} u = 0 \text{ for } x \in [0, L]. \tag{4.30}$$

Thus, letting $u \to +\infty$ on both sides of (4.30) with $x = x_0$, one can immediately derive a contradiction. Hence $\lim_{u\to+\infty} v(u;x) = +\infty$ for $x \in [0, L]$.

We next prove (4.28). Differentiating both sides of (4.30) with respect to u, one then finds that

$$-\big[\mu - (g(x, v(u;x)) - m_2(x)) - g_v(x, v(u;x))v(u;x)\big] \frac{\partial v(u;x)}{\partial u} + \sigma \frac{A_1(x)}{A_2(x)} = 0 \text{ for } x \in [0, L]. \tag{4.31}$$

From $F_3(v(u;x)) = 0$, we have

$$\mu - (g(x, v(u;x)) - m_2(x)) = \sigma \frac{A_1(x)}{A_2(x)} \cdot \frac{u}{v(u;x)} > 0 \text{ for } x \in [0, L].$$

This, together with (H2), implies that

$$\mu - (g(x, v(u;x)) - m_2(x)) - g_v(x, v(u;x))v(u;x) > 0 \text{ for } x \in [0, L].$$

Thus, it follows from (4.31) that (4.28) holds for $x \in [0, L]$.

Furthermore, by repeating the arguments used in Step 1 in the proof of Theorem 2.2, one can derive that (4.29) holds. This proves Claim 2.

For any $x \in [0, L]$ and $u > 0$, let $v(u;x)$ be uniquely determined by $F_3(v(u;x)) = 0$ and define

$$F_4(u) = \int_0^L \left[(m_1(x) + \sigma)u - \mu \frac{A_2(x)}{A_1(x)} v(u;x)\right] \mathrm{d}x.$$

Claim 3. $\lim_{u\to+\infty} F_4(u) = +\infty$ and

$$\lim_{u\to 0^+} F_4(u) = \begin{cases} -\mu \int_{\hat{I}_+} \frac{A_2(x)}{A_1(x)} v^*(x)\mathrm{d}x, & \text{when } (C1) \text{ holds}, \\ 0, & \text{when } (C2) \text{ holds}. \end{cases}$$

Indeed, by $F_3(v(u;x)) = 0$, it follows from Claim 2 and (H2) that

$$\lim_{u\to+\infty} \frac{u}{v(u;x)} = \lim_{u\to+\infty} \frac{A_2(x)}{\sigma A_1(x)} \big[\mu - (g(x, v(u;x)) - m_2(x))\big] = +\infty,$$

which yields

$$\lim_{u\to+\infty} \frac{F_4(u)}{u} = \lim_{u\to+\infty} \int_0^L \left[(m_1(x) + \sigma) - \mu \frac{A_2(x)}{A_1(x)} \cdot \frac{v(u;x)}{u}\right] \mathrm{d}x = \int_0^L (m_1(x) + \sigma)\mathrm{d}x.$$

This implies that $\lim_{u\to+\infty} F_4(u) = +\infty$.

Moreover, when (C1) holds, we have $\max_{x\in[0,L]}\{g(x, 0) - m_2(x)\} > \mu$. This implies that $\hat{I}_+ \neq \emptyset$. Furthermore, it follows from (4.29) that

$$\lim_{u\to 0^+} F_4(u) = \lim_{u\to 0^+} \int_0^L \left[(m_1(x)+\sigma)u - \mu \frac{A_2(x)}{A_1(x)} v(u;x) \right] \mathrm{d}x = -\mu \int_{\hat{I}_+} \frac{A_2(x)}{A_1(x)} v^*(x) \mathrm{d}x.$$

When (C2) holds, we have $\mu > \max_{x\in[0,L]}\{g(x,0) - m_2(x)\}$, which means that $\hat{I}_+ = \emptyset$. It then follows from (4.29) that

$$\lim_{u\to 0^+} F_4(u) = \lim_{u\to 0^+} \int_0^L \left[(m_1(x)+\sigma)u - \mu \frac{A_2(x)}{A_1(x)} v(u;x) \right] \mathrm{d}x = 0.$$

This proves Claim 3.

Claim 4. For any $u_0 > 0$ satisfying $F_4(u_0) = 0$, we have $F_4'(u_0) > 0$.

Indeed, by $u_0 > 0$, one can derive from $F_4(u_0) = 0$ that

$$\int_0^L (m_1(x)+\sigma)\mathrm{d}x = \int_0^L \mu \frac{A_2(x)}{A_1(x)} \cdot \frac{v(u_0;x)}{u_0} \mathrm{d}x.$$

Noticing that $F_3(v(u_0;x)) = 0$, we obtain

$$\frac{v(u_0;x)}{u_0} = \frac{A_1(x)}{A_2(x)} \cdot \frac{\sigma}{\mu - (g(x, v(u_0;x)) - m_2(x))} \quad \text{for } x \in [0, L],$$

which implies that

$$\int_0^L (m_1(x)+\sigma)\mathrm{d}x = \int_0^L \frac{\mu\sigma}{\mu - (g(x, v(u_0;x)) - m_2(x))} \mathrm{d}x. \tag{4.32}$$

Thus, by (4.28) and (4.32), we can conduct a direct calculation to deduce that

$$\begin{aligned} F_4'(u_0) &= \int_0^L \left[(m_1(x)+\sigma) - \mu \frac{A_2(x)}{A_1(x)} \cdot \frac{\partial v(u_0;x)}{\partial u} \right] \mathrm{d}x \\ &= \int_0^L \left[\frac{\mu\sigma}{\mu - (g(x, v(u_0;x)) - m_2(x)} - \frac{\mu\sigma}{\mu - (g(x, v(u_0;x)) - m_2(x)) - g_v(x, v(u_0;x)) v(u_0;x)} \right] \mathrm{d}x \\ &> 0, \end{aligned}$$

where the last inequality is due to $g_v(x, v(u_0;x)) < 0$ (see (H2)) and $v(u_0;x) > 0$. This proves Claim 4.

Claim 5. $\lim_{u\to 0^+} F_4'(u) < 0$ when (C2) holds.

Since $\mu > \max_{x\in[0,L]}\{g(x,0) - m_2(x)\}$ when (C2) holds, we have $\hat{I}_+ = \emptyset$. Thus, (4.29) implies that $\lim_{u\to 0^+} v(u;x) = 0$ for $x \in [0, L]$. It then follows from (4.28) and (4.29) that

$$\lim_{u\to 0^+} F_4'(u) = \int_0^L \left[(m_1(x)+\sigma) - \mu \frac{A_2(x)}{A_1(x)} \cdot \lim_{u\to 0^+} \frac{\partial v(u;x)}{\partial u} \right] \mathrm{d}x,$$
$$= \int_0^L \left[(m_1(x)+\sigma) - \frac{\mu\sigma}{\mu - (g(x,0) - m_2(x))} \right] \mathrm{d}x.$$

Condition (C2) shows that $\sigma > \sigma_{\sup} \ge \sigma_x$ for $x \in [0, L]$. Thus, a direct calculation yields

$$m_1(x) + \sigma - \frac{\mu\sigma}{\mu - (g(x,0) - m_2(x))} < 0 \text{ for } x \in [0, L].$$

Therefore, we obtain $\lim_{u\to 0^+} F_4'(u) < 0$. This proves Claim 5.

Claim 6. Problem (2.4) admits a unique positive solution.

Indeed, when (C1) holds, it follows from Claim 3 that

$$\lim_{u\to 0^+} F_4(u) < 0 \text{ and } \lim_{u\to +\infty} F_4(u) = +\infty.$$

This, together with Claim 4, implies that there exists a unique constant $u > 0$ such that $F_4(u) = 0$.

When (C2) holds, Claim 3 implies that

$$\lim_{u\to 0^+} F_4(u) = 0 \text{ and } \lim_{u\to +\infty} F_4(u) = +\infty.$$

Moreover, Claim 5 shows that $\lim_{u\to 0^+} F_4'(u) < 0$. Combining with Claim 4, there exists a unique constant $u > 0$ such that $F_4(u) = 0$.

Therefore, when either (C1) or (C2) holds, there exists a unique constant $u > 0$ such that $F_4(u) = 0$. This, together with Claim 2, implies that there exists a unique positive constant u and a unique positive function $v(u; x)$ such that

$$F_3(v(u;x)) = 0 \text{ and } F_4(u) = 0,$$

which means that (2.4) admits a unique positive solution $(u, v(u; x))$. Since $(\bar{u}, \bar{v}(x))$ is a positive solution of (2.4), it follows that $(\bar{u}, \bar{v}(x))$ is the unique positive solution of (2.4). By the uniqueness, we can conclude that

$$\lim_{(d,\alpha)\to(+\infty,0)} (u_{d,\alpha}(x), v_{d,\alpha}(x)) = (\bar{u}, \bar{v}(x)) \text{ for } x \in [0, L].$$

This completes the proof of Theorem 2.6. □

5. Discussions

Pachepsky et al. [20] divided a river into the drift zone and the benthic zone, and the species in the river are accordingly categorized into two interacting compartments: individuals dispersing in the drift zone and those residing in the benthic zone. They introduced a *spatially homogeneous* benthic-drift model to explore the mechanism for persistence or extinction of the species. Considering that variations in natural environments and human activities result in heterogeneous growth conditions for aquatic species, Nie et al. [19] investigated the global dynamics of a spatially heterogeneous benthic-drift system (i.e., system (1.1)), by which sufficient conditions for the uniqueness and global asymptotic stability of its positive steady state were identified. In this paper, we have further investigated the influence of diffusion and advection rates on the asymptotic profiles of the positive steady state of system (1.1), which is the unique positive solution $(u(x), v(x))$ of system (1.2). Our main findings indicate that diffusion and advection rates of the drift species can dramatically affect the spatial distributions of drift and benthic subpopulations. More specifically, we have studied the asymptotic profiles of $(u(x), v(x))$ in the following limit scenarios: (a) $\alpha/d \to +\infty$ and $\alpha^2/d \to +\infty$, (b) $\alpha > 0$ is fixed and $d \to +\infty$, and (c) $(d, \alpha) \to (+\infty, 0)$. Here, scenario (a) includes the following five cases:

(a-1) $d > 0$ is fixed and $\alpha \to +\infty$,
(a-2) $\alpha > 0$ is fixed and $d \to 0^+$,
(a-3) $(d, \alpha) \to (0, +\infty)$,
(a-4) $(d, \alpha) \to (0, 0)$ and $\alpha^2/d \to +\infty$,
(a-5) $(d, \alpha) \to (+\infty, +\infty)$ and $\alpha/d \to +\infty$.

For these scenarios, we have provided some comprehensive characterization of the asymptotic profiles of the positive solution $(u(x), v(x))$, as presented in Theorems 2.2, 2.4, and 2.6, and Corollaries 2.3 and 2.5.

We point out that scenario (a) (any of the five limit cases (a-1) - (a-5)) accounts for the situation of advection rate dominating the diffusion rate for the drifting subpopulation. In this scenario, the population distribution of species exhibits a *dichotomy*, determined by whether or not there exist favorable benthic sites (i.e., whether or not $I_+ := \{x \in [0, L) : g(x, 0) - m_2(x) > \mu\}$ is empty), in the following sense:

- When there is no favorable benthic site (i.e., (C2) holds), both drift and benthic species coexist. In this case, their subpopulations are concentrated near the downstream end (see Theorem 2.4 and Corollary 2.5).
- When there are favorable benthic sites (i.e., (C1) holds), co-existence persists. Here, the drift subpopulation remains concentrated near the downstream end, while the benthic subpopulation is distributed in both the downstream end and the aforementioned favorable benthic sites (see Theorem 2.2 and Corollary 2.3).

This indicates that favorable benthic sites play a crucial role in shaping the species' spatial distributions, particularly as they determine the primary spatial distribution of the *benthic species*.

On the other hand, in scenarios (b) and (c), the diffusion rate of the drift species approaches infinity (diffusion dominates). In this context, Theorem 2.6 demonstrates that drift and benthic species coexist, with the *drift species* tending to distribute *evenly* in the entire river, while the *benthic species* exhibit a spatially *heterogeneous* distribution.

Our main findings substantially complement and improve the results of [21] in the following three key aspects:

1. New insights under condition (C1). The spatial distributions of species under condition (C1) are not addressed in [21]. Theorems 2.2 and 2.4 provide a more thorough understanding of the population distributions by showing that when $\alpha/d \to +\infty$ and $\alpha^2/d \to +\infty$,

the asymptotic spatial distribution of species exhibits a dichotomous pattern, governed by whether condition (C1) or (C2) holds.

2. More complete coverage of the limit cases for the advection rate and diffusion rate. In [21], the authors only analyzed the spatial distributions of species when one of the limit cases of (a-1), (a-2) and (c). Our Theorem 2.4 generalizes their results to the cases where $\alpha/d \to +\infty$ and $\alpha^2/d \to +\infty$, which shows that both species will coexist and concentrate at the downstream end for each of the limit cases (a-1) - (a-5). Moreover, our Theorem 2.6 extends the asymptotic spatial distributions of species for sufficiently large diffusion rate to the limits (b) and (c).
3. Confirmation of the uniqueness of the positive steady state for large diffusion rate. The uniqueness of the positive steady state when $\alpha > 0$ and $d \to +\infty$ is not established in [21]. Our Theorem 2.6 demonstrates that the limiting problem (2.4) admits a unique positive solution (see Claims 2 - 6 in the proof). This indicates that for either (b) or (c), drift species distribute *evenly* in the entire river, while benthic species exhibit an *uneven* spatial distribution.

In conclusion, our main results in Theorems 2.2, 2.4, and 2.6 in this paper provide a more thorough and precise characterization of the spatial distributions of both drift and benthic species for the cases when d and α are sufficiently large or small. These results indicate that the diffusion and advection rates of the drift species significantly affect species' spatial distributions along the river. Moreover, they not only extend the dynamical insights of the benthic-drift model presented in [19], but also significantly enhance the conclusions of [21].

To conclude the paper, we point out that our discussions on the asymptotic profile of the steady states are for the cases of $\mu < \max_{x\in[0,L]}\{g(x,0) - m_2(x)\}$ and $\mu > \max_{x\in[0,L]}\{g(x,0) - m_2(x)\}$. For the critical case of $\mu = \max_{x\in[0,L]}\{g(x,0) - m_2(x)\}$, the asymptotic profile remains unclear. Although such a critical case is biologically less important (in reality, requiring all model parameters to satisfy an identity is too demanding and can hardly be achieved), it is mathematically interesting and yet challenging. This is because even the existence of the positive steady state for this critical case is still not guaranteed. This is an interesting mathematical problem for which, the methods in this paper fail to work (at least in our first try) and therefore, new ideas and/or tools need to be developed. We leave it as an open problem.

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Data availability

No data was used for the research described in the article.

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