



Threshold dynamics of a reaction-diffusion system in a cylinder with shifting effect [☆]

Qian Guo ^a, Taishan Yi ^a, Yurong Zhang ^b, Xingfu Zou ^{c,*}

^a School of Mathematics (Zhuhai), Sun Yat-Sen University, Zhuhai, Guangdong 519082, PR China

^b College of Mathematical Sciences, Chongqing Normal University, Chongqing, 401331, PR China

^c Department of Applied Mathematics, University of Western Ontario, London, ON, N6A 5B7, Canada

Received 12 July 2025; revised 26 December 2025; accepted 27 December 2025

Available online 6 January 2026

Abstract

In this paper, we study the threshold dynamics of a class of reaction-diffusion systems in a cylindrical domain with shifting effect. We first transform the reaction-diffusion system into a spatially inhomogeneous autonomous system using moving coordinates and analyze the fundamental properties of the solution to this new system. Then, we establish uniform asymptotic annihilation of the autonomous system by constructing an upper system sequence. Finally, employing the theory of asymptotic spectral radius, we investigate the threshold dynamics of the system, including existence/nonexistence and uniqueness of forced wave, as well as its global stability. Particularly, we establish a logarithmic relation between the asymptotic spectral radius and the standard generalized principal eigenvalue, thereby characterizing the influence of the climate shifting speed c on the asymptotic spectral radius.

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Keywords: Asymptotic annihilation; Cylindrical domain; Reaction-diffusion systems; Shifting effect; Threshold dynamics

[☆] Research was supported partially by the National Natural Science Foundation of China (11971494, 12231008)(Yi), the National Natural Science Foundation of China (12501196) and the Natural Science Foundation of Chongqing (CSTB2025NSCQ-GPX1001)(Zhang), and NSERC of Canada (RGPIN-2022-04744)(Zou).

* Corresponding author.

E-mail addresses: guoq55@mail2.sysu.edu.cn (Q. Guo), yitaishan@mail.sysu.edu.cn (T. Yi), cqzhangyurong@126.com (Y. Zhang), xzou@uwo.ca (X. Zou).

<https://doi.org/10.1016/j.jde.2025.114084>

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1. Introduction

Global climate change and industrialization have been a big concern in the scientific community. This has stimulated ecologists to explore how environmental shifts influence population dynamics (see, e.g., [18,33]). In this regard, mathematical modelling can play an important role and has actually provided some useful and helpful insight to explain and understand the impact of such shifting environments through reaction-diffusion. For example, in their pioneering work Berestycki et al. [1], the authors formulated a one-dimensional reaction-diffusion equation with a forced speed c :

$$\partial_t u(t, x) = d\partial_{xx}u(t, x) + f(x - ct, u(t, x)), \quad (t, x) \in (0, \infty) \times \mathbb{R}, \tag{1.1}$$

where $c > 0$ is the speed of climate change, d is the dispersal rate, and f represents the growth function in the shifting environment. Under the assumption that the favourable habitat is surrounded by unfavorable ones, they established the existence and uniqueness of travelling wave solutions with the speed c (called forced wave) and the long time dynamics of (1.1). Later, Berestycki and Rossi [4] extended the results of [1] to higher dimensional cases with more general assumptions on f . When f in (1.1) is logistic form, i.e., $f(x - ct, u) = u(r(x - ct) - u)$, Li et al. [21] investigated the persistence and extinction dynamics and established criteria to determine whether the population persists or becomes extinct. Furthermore, researchers have also investigated the spatial-temporal dynamics of other mathematical models in shifting habitats, such as integro-difference equation models [20,22,48], nonlocal diffusion equation models [23,34,35], and lattice differential equation models [13,25], delayed reaction diffusion equation models [7,16,37]. For more related studies, see [2,8–12,14,15,19,40,41,44,45] and references therein.

Among the various types of spatial domains are cylindrical domains. Reaction diffusion equations in cylinders have also received considerable attention in recent years, particularly regarding asymptotic propagation and traveling wave theory of reaction diffusion equations in such domains; see [3,24,27,28,31,46]. When there is a shifting effect accounting for the environmental shift, reaction-diffusion models in such cylindrical environments have also gained growing interest (see, e.g., [5,6,32,38]). Berestycki and Rossi [5] extended model (1.1) to an infinite cylinder with Neumann boundary condition:

$$\begin{cases} \partial_t u = \Delta u + f(x_1 - ct, y, u), & (t, x_1, y) \in (0, \infty) \times \mathbb{R} \times \Omega, \\ \partial_\nu u(t, x_1, y) = 0, & (t, x_1, y) \in (0, \infty) \times \mathbb{R} \times \partial\Omega. \end{cases} \tag{1.2}$$

In [5], assuming that the favourable habitat is bounded. They demonstrated that the long time dynamics of the solution are determined by the generalized principal eigenvalue λ of an associated linear elliptic operator: if $\lambda \geq 0$, the solution converges to zero; if $\lambda < 0$, the solution converges to the traveling wave $U(x_1 - ct, y)$, uniformly for $(x_1, y) \in \mathbb{R} \times \Omega$. Subsequent research by Vo [32] investigated reaction-diffusion models in shifting environments characterized by globally unfavorable mixed habitats with favourable pockets extending to infinity. The study established criteria for species persistence versus extinction in cylindrical and partially periodic domains. Bouhours and Giletti [6] analyzed the spreading and vanishing of monostable reaction-diffusion models in heterogeneous environments. They investigated the large time behaviour of solutions when the subdomain shifts or contracts at a given speed c with a positive reaction term. We point

out that the above works are all for scalar reaction diffusion equations. Despite these advances for scalar R-D equations, reaction-diffusion *systems* in cylinders with shifting effect are relatively less understood, compared to their one-dimensional counterparts. It is known that for R-D systems, the comparison theorem and maximum principle are generally no longer valid, and this brings in challenges in the study of the dynamics of R-D *systems* unless a system is cooperative.

Recent advances by Yi and Zhao [38] have significantly extended the theoretical framework for spreading speeds and traveling waves in monotone evolution systems without spatial translation invariance. Subsequently, Yi and Zhao [39] further generalized their theory to non-monotone evolution systems without spatial translation invariance, providing a powerful tool to analyze the global dynamics for certain evolution systems with their limiting systems being uniform asymptotic annihilation. In particular, they applied these abstract results to a reaction-diffusion system in a shifting environment. This development contrasts with traditional approaches based on generalized principal eigenvalues - a key methodology in propagation dynamics (see, e.g., [2,4,5,10,32]). As demonstrated in foundational works by Berestycki and collaborators [4,5], generalized principal eigenvalues serve as critical indicators for population persistence or extinction in *shifting habitats*. However, the introduction of generalized eigenvalues depends on the specific form of the equations, leading to distinct definitions for different classes of problems. Moreover, their applications usually require auxiliary tools such as regularity theory, comparison principle, and the construction of appropriate upper and lower solutions. In contrast, the asymptotic spectral radius theory developed by Yi and Zhao [39] transforms the above problems that require special treatment in the generalized eigenvalues into fundamental assumptions (asymptotic contraction, asymptotic translation, and strong positivity), and this can avoid repetitive arguments of the generalized eigenvalues and their derived procedures for certain systems.

Motivated by the aforementioned developments, in this paper, we investigate the following system

$$\begin{cases} \partial_t \mathbf{u} = D_1 \Delta_x \mathbf{u} + D_2 \Delta_y \mathbf{u} + \mathbf{f}(x - ct\xi, y, \mathbf{u}), & (t, x, y) \in (0, \infty) \times \mathbb{R}^m \times \Omega, \\ \partial_\nu \mathbf{u}(t, x, y) = \mathbf{0}, & (t, x, y) \in (0, \infty) \times \mathbb{R}^m \times \partial\Omega, \\ \mathbf{u}(0, x, y) = \phi(x, y), & (x, y) \in \mathbb{R}^m \times \bar{\Omega}, \end{cases} \quad (1.3)$$

where Ω is a bounded domain in \mathbb{R}^n with smooth boundary $\partial\Omega$, ν is the outer unit normal vector field to $\mathbb{R}^m \times \partial\Omega$, $\xi \in S^{m-1}$ is a given unit direction vector and $c \in \mathbb{R}$ is the speed of the climate change. Here,

$$\Delta_x = \sum_{i=1}^m \frac{\partial^2}{\partial x_i^2}, \quad \Delta_y = \sum_{j=1}^n \frac{\partial^2}{\partial y_j^2},$$

$D_j = \text{diag}(d_{j1}, \dots, d_{jN})$ is matrix of diffusivities, with $d_{ji} > 0$ for $i \in \{1, 2, \dots, N\}$, $j \in \{1, 2\}$. $\mathbf{f} = (f_1, f_2, \dots, f_N)^T$, $\mathbf{u} = (u_1, u_2, \dots, u_N)^T$, and $\phi = (\phi_1, \phi_2, \dots, \phi_N)^T$. We are concerned with the forced travelling wave solution to this system. Applying the abstract results in [39], we will establish the existence, nonexistence, and uniqueness of forced wave and its global stability. To rigorously prove the uniform asymptotic annihilation of solution to the reaction diffusion system (2.1) transformed from (1.3), we introduce a linear system and construct an upper system sequence. In addition, we define the truncation, amplification, and asymptotic spectral radius of the solution operator of the linear system to study the threshold dynamics of system (1.3). The

results reveal that when the asymptotic spectral radius is strictly greater than 1, the system admits a nontrivial traveling wave $W(x - ct\xi, y)$, while if it lies strictly between 0 and 1, the solution converges to $\mathbf{0}$.

The rest of this paper is organized as follows. In Section 2, we present some basic notations and assumptions. Then, we transform system (1.3) into an inhomogeneous reaction-diffusion-advection system using moving coordinates and establish the fundamental properties of its solution. In Section 3, we introduce an auxiliary linear system and analyze the asymptotic annihilation of the system (2.1) by constructing an upper system sequence. In Section 4, we first explore the threshold dynamics of system (1.3), then we establish the relation between the asymptotic spectral radius and the generalized principal eigenvalue defined in [5], thereby determining how the shifting speed c affects the threshold dynamics.

2. Preliminaries

In this section, we first give some basic notations and preliminary results that will be essential for the analysis in subsequent sections.

Let \mathbb{R}^m and \mathbb{R}^N be equipped with the Euclidean norm. Let $C = BC(\mathbb{R}^m \times \overline{\Omega}, \mathbb{R}^N)$ be the normed vector space of all bounded and continuous functions from $\mathbb{R}^m \times \overline{\Omega}$ to \mathbb{R}^N with the norm

$$\|\varphi\|_C = \sum_{k=1}^{\infty} 2^{-k} \sup\{\|\varphi(x, y)\|_{\mathbb{R}^N} : (x, y) \in \mathbb{R}^m \times \overline{\Omega}, \|x\|_{\mathbb{R}^m} \leq k\}.$$

In addition, let $C_+ = \{\varphi \in C : \varphi(x, y) \in \mathbb{R}_+^N, \forall (x, y) \in \mathbb{R}^m \times \overline{\Omega}\}$, and $C_+^\circ = \{\varphi \in C : \varphi(x, y) \in \text{Int}(\mathbb{R}_+^N), \forall (x, y) \in \mathbb{R}^m \times \overline{\Omega}\}$. It is clear that C_+ is a closed cone in the normed vector space C . For any $\xi, \eta \in C$, we write $\xi \geq \eta$ if $\xi - \eta \in C_+$; and $\xi > \eta$ if $\xi \geq \eta$ and $\xi \neq \eta$; $\xi \gg \eta$ if $\xi - \eta \in C_+^\circ$.

Let $K = \{1, 2, \dots, N\}$, $\mathbf{1} = (1, 1, \dots, 1) \in \mathbb{R}^N$, and $\check{\mathbf{1}} \in \mathbb{R}^{N \times N}$ be the matrix with $(\check{\mathbf{1}})_{ij} = 1$ for all $i, j \in K$. For any $\mathbf{r} \in \text{Int}(\mathbb{R}_+^N)$, we define

$$C_{\mathbf{r}} = \{\varphi \in C : \mathbf{0} \leq \varphi(x, y) \leq \mathbf{r}, \forall (x, y) \in \mathbb{R}^m \times \overline{\Omega}\}.$$

For any given $z \in \mathbb{R}^m$, we define the spatial translation operator \mathcal{T}_z on C by

$$\mathcal{T}_z[\varphi](x, y) = \varphi(x - z, y), \quad \forall (x, y) \in \mathbb{R}^m \times \overline{\Omega}, \varphi \in C.$$

Define $\|\varphi\|_\infty = \|\varphi\|_{L^\infty(\mathbb{R}^m \times \overline{\Omega}, \mathbb{R}^N)}$ for all $\varphi \in C$. Then $(C, C_+, \|\cdot\|_\infty)$ is an ordered Banach space.

An $N \times N$ matrix $A = (a_{ij})$ is said to be cooperative if $a_{ij} \geq 0$ for all $1 \leq i \neq j \leq N$, and the spectral bound of A is defined as $s(A) = \sup\{\Re \lambda : \det(\lambda I - A) = 0\}$. The norm of A is given by $\|A\|_{\mathbb{R}^{N \times N}} = \sqrt{\lambda_{\max}(AA^T)}$, where λ_{\max} is the largest eigenvalue of AA^T . A is irreducible if for any nonempty, proper subset I of the set K , there is an $i \in I$ and $j \in K \setminus I$ such that $a_{ij} \neq 0$. For simplicity, we denote $\|\cdot\|_*$ by $\|\cdot\|$, where $*$ represents $\mathbb{R}^m, \mathbb{R}^N, \mathbb{R}^{N \times N}, C(\overline{\Omega}, \mathbb{R}^N)$, or C .

Throughout this paper, we always assume that \mathbf{f} satisfies the following hypotheses.

- (H1) $\mathbf{f} \in C(\mathbb{R}^m \times \overline{\Omega} \times \mathbb{R}_+^N, \mathbb{R}^N)$, $\mathbf{f}(\cdot, \cdot, \mathbf{u}) \in L^\infty(\mathbb{R}^m \times \overline{\Omega}, \mathbb{R}^N)$, $\mathbf{f}(x, y, \cdot) \in C^1(\mathbb{R}_+^N, \mathbb{R}^N)$, $\mathbf{u} \mapsto \mathbf{f}(x, y, \mathbf{u})$ is locally Lipschitz continuous, uniformly for $(x, y) \in \mathbb{R}^m \times \overline{\Omega}$. In addition, we assume that $\mathbf{f}(x, y, \mathbf{0}) = \mathbf{0}$ for all $(x, y) \in \mathbb{R}^m \times \overline{\Omega}$.
- (H2) There exists $\mathbf{M} \in \text{Int}(\mathbb{R}_+^N)$ such that for any $(\alpha, x, y) \in [1, \infty) \times \mathbb{R}^m \times \overline{\Omega}$, $\mathbf{f}(x, y, \alpha \mathbf{M}) \ll \mathbf{0}$. Moreover, $\inf \left\{ \frac{\partial f_i(x, y, \mathbf{u})}{\partial u_i} : (x, y, \mathbf{u}, i) \in \mathbb{R}^m \times \overline{\Omega} \times [\mathbf{0}, \mathbf{M}]_{\mathbb{R}^N} \times K \right\} > -\infty$, and $\mathbf{f}(x, y, \cdot)$ is subhomogeneous on \mathbb{R}_+^N .
- (H3) The Jacobian matrix $D_{\mathbf{u}}\mathbf{f}(x, y, \mathbf{u}) := \left(\frac{\partial f_i(x, y, \mathbf{u})}{\partial u_j} \right)$ is continuous and cooperative for all $(x, y, \mathbf{u}) \in \mathbb{R}^m \times \overline{\Omega} \times \mathbb{R}_+^N$. Furthermore, there exists $(x_0, y_0) \in \mathbb{R}^m \times \overline{\Omega}$ such that $D_{\mathbf{u}}\mathbf{f}(x_0, y_0, \mathbf{0})$ is irreducible.
- (H4) The spectral bound of the matrix

$$s \left(\lim_{R \rightarrow \infty} \sup_{\substack{\|x\| > R \\ y \in \Omega}} D_{\mathbf{u}}\mathbf{f}(x, y, \mathbf{0}) \right) < 0,$$

where

$$\left(\lim_{R \rightarrow \infty} \sup_{\substack{\|x\| > R \\ y \in \Omega}} D_{\mathbf{u}}\mathbf{f}(x, y, \mathbf{0}) \right)_{ij} = \lim_{R \rightarrow \infty} \sup_{\substack{\|x\| > R \\ y \in \Omega}} \frac{\partial f_i}{\partial u_j}(x, y, \mathbf{0}).$$

- (H5) $\liminf_{(\|x\|, \mathbf{v}) \rightarrow (\infty, \mathbf{0})} \left[\min_{y \in \overline{\Omega}} (D_{\mathbf{u}}\mathbf{f}(x, \cdot, \mathbf{0}) - D_{\mathbf{u}}\mathbf{f}(x, \cdot, \mathbf{v})) \right] \geq \mathbf{0}$.
- (H6) There exists a \mathbf{f}_+ such that $\lim_{\|x\| \rightarrow \infty} \mathbf{f}(x, \cdot, \cdot) = \mathbf{f}_+$ in $C_{\text{loc}}^1(\overline{\Omega} \times \mathbb{R}_+^N, \mathbb{R}^N)$.

(H1) enumerates the continuity, boundedness, differentiability, and local Lipschitz continuity of \mathbf{f} . These properties ensure the existence and uniqueness of mild solutions. (H2) implies the boundedness and dissipativity of the system, which in population models typically corresponds to the fact that the environment has a carrying capacity. The subhomogeneity of \mathbf{f} ensures that the linearized system around the trivial solution can serve as an upper system for system (2.1). (H3) states that the system is cooperative, which implies monotonicity of solutions, while the irreducibility guarantees the strong positivity of solutions. Finally, (H4) and (H5), combined with subhomogeneity, establish the asymptotic annihilation of the system (2.1). (H6) states that both (1.3) and its transformed system (2.1) have limiting systems which will play a role in understanding the dynamics of (1.3).

To proceed, we transform system (1.3) into an autonomous system with spatial heterogeneity using moving coordinates:

$$\begin{cases} \partial_t \mathbf{u} = D_1 \Delta_x \mathbf{u} + D_2 \Delta_y \mathbf{u} + c \xi \cdot \nabla_x \mathbf{u} + \mathbf{f}(x, y, \mathbf{u}), & (t, x, y) \in (0, \infty) \times \mathbb{R}^m \times \Omega, \\ \partial_\nu \mathbf{u}(t, x, y) = \mathbf{0}, & (t, x, y) \in (0, \infty) \times \mathbb{R}^m \times \partial \Omega, \\ \mathbf{u}(0, x, y) = \phi(x, y), & (x, y) \in \mathbb{R}^m \times \overline{\Omega}. \end{cases} \quad (2.1)$$

Under (H6), (2.1) has the following limiting system:

$$\begin{cases} \partial_t \mathbf{u} = D_1 \Delta_x \mathbf{u} + D_2 \Delta_y \mathbf{u} + c\xi \cdot \nabla_x \mathbf{u} + \mathbf{f}_+(y, \mathbf{u}), & (t, x, y) \in (0, \infty) \times \mathbb{R}^m \times \Omega, \\ \partial_y \mathbf{u}(t, x, y) = \mathbf{0}, & (t, x, y) \in (0, \infty) \times \mathbb{R}^m \times \partial\Omega, \\ \mathbf{u}(0, x, y) = \phi(x, y), & (x, y) \in \mathbb{R}^m \times \overline{\Omega}. \end{cases} \quad (2.2)$$

According to Remark 5.2 in [43] and **(H1)**, for any $\phi \in C_+$, system (1.3) admits a unique mild solution $\mathbf{u}^\phi(t, x, y; \mathbf{f})$ with initial value $\mathbf{u}(0, x, y; \mathbf{f}) = \phi(x, y)$ on the maximal interval $[0, \infty)$, and $\mathbf{u}^\phi(t, \cdot, \cdot; \mathbf{f}) \in C_+$. Define $P : \mathbb{R}_+ \times C_+ \rightarrow C_+$ by $P[t, \phi; \mathbf{f}](x, y) = \mathbf{u}^\phi(t, x, y; \mathbf{f})$ for all $(t, x, y, \phi) \in \mathbb{R}_+ \times \mathbb{R}^m \times \overline{\Omega} \times C_+$. Let $Q[t, \phi; \mathbf{f}]$ and $Q_+[t, \phi; \mathbf{f}_+]$ be the mild solutions of systems (2.1) and (2.2) with $\mathbf{u}(0, \cdot, \cdot) = \phi \in C_+$, respectively. For convenience, we denote $P[t, \phi; \mathbf{f}]$, $Q[t, \phi; \mathbf{f}]$ and $Q_+[t, \phi; \mathbf{f}_+]$ by $P[t, \phi]$, $Q[t, \phi]$ and $Q_+[t, \phi]$, respectively. Based on the definitions of the mild solutions, we obtain the following expressions:

$$\begin{aligned} P[t, \phi](x, y) &= T(t)[\phi](x, y) + \int_0^t T(t-s)[\mathbf{f}(\cdot - cs\xi, \cdot, P[s, \phi](\cdot, \cdot))](x, y) \, ds, \\ Q[t, \phi](x, y) &= \overline{T}(t)[\phi](x, y) + \int_0^t \overline{T}(t-s)[\mathbf{f}(\cdot, \cdot, Q[s, \phi](\cdot, \cdot))](x, y) \, ds, \\ Q_+[t, \phi](x, y) &= \overline{T}(t)[\phi](x, y) + \int_0^t \overline{T}(t-s)[\mathbf{f}_+(\cdot, Q_+[s, \phi](\cdot, \cdot))](x, y) \, ds, \end{aligned}$$

where the semigroups $T(t)$ and $\overline{T}(t)$ are defined in the Appendix. Note that $Q[t, \phi] = \mathcal{T}_{-ct\xi} \circ P[t, \phi]$ for all $(t, \phi) \in \mathbb{R}_+ \times C_+$.

In view of Remark 5.2 in [43] and the assumptions on \mathbf{f} , we can easily verify that Q satisfies the properties of continuity, positive invariance, monotonicity, and asymptotic translation.

Proposition 2.1. *The following statements are valid:*

- (i) $Q : \mathbb{R}_+ \times C_{\mathbf{r}} \rightarrow C_+$ is continuous for all $\mathbf{r} \in \text{Int}(\mathbb{R}_+^N)$.
- (ii) $Q[\mathbb{R}_+ \times C_{\alpha\mathbf{M}}] \subseteq C_{\alpha\mathbf{M}}$ for all $\alpha \in [1, \infty)$.
- (iii) If $\phi, \psi \in C_+$ with $\phi \geq \psi$, then $Q[t, \phi] \geq Q[t, \psi]$ for all $t \in \mathbb{R}_+$.
- (iv) Let **(H6)** hold. Then $\lim_{\|z\| \rightarrow \infty} Q[t, \mathcal{T}_z[\phi]](\cdot + z, \cdot) = Q_+[t, \phi]$ in C for all $(t, \phi) \in \mathbb{R}_+ \times C_+$.
 Moreover, $\lim_{(k, \|z\|) \rightarrow (\infty, \infty)} Q[t, \mathcal{T}_z[\phi_k]](\cdot + z, \cdot) = Q_+[t, \phi_0]$ in C for all $t \in \mathbb{R}_+$ provided that $\{\phi_k\}_{k \in \mathbb{N}} \subset C_{\mathbf{r}}$ for some $\mathbf{r} \in \text{Int}(\mathbb{R}_+^N)$ with $\lim_{k \rightarrow \infty} \|\phi_k - \phi_0\| = 0$.

Next, under the assumptions on \mathbf{f} , we further establish the strong positivity, subhomogeneity, and compactness of Q .

Proposition 2.2. *The following statements are valid:*

- (i) $Q[t, \phi] \in C_+^\circ$ for all $(t, \phi) \in (0, \infty) \times (C_+ \setminus \{\mathbf{0}\})$.
- (ii) $Q[t, \gamma\phi] \geq \gamma Q[t, \phi]$ for all $(t, \gamma, \phi) \in \mathbb{R}_+ \times [0, 1] \times C_+$.
- (iii) $Q[t, C_{\mathbf{r}}]$ is precompact in C for all $(t, \mathbf{r}) \in (0, \infty) \times \text{Int}(\mathbb{R}_+^N)$.

Proof. (i) Without loss of generality, we assume that $\phi \in C_M \setminus \{0\}$ due to the monotonicity of $Q[t, \cdot]$. By (H2) and (H3), there exists a sufficiently large constant μ such that $\mu I + D_{\mathbf{u}}\mathbf{f}(x, y, \mathbf{u})$ is nonnegative for all $(x, y, \mathbf{u}) \in \mathbb{R}^m \times \overline{\Omega} \times [\mathbf{0}, \mathbf{M}]_{\mathbb{R}^N}$. Define $A(x, y, \mathbf{u}) = \mu I + \int_0^1 D_{\mathbf{u}}\mathbf{f}(x, y, \xi \mathbf{u}) d\xi$, $\mathbf{g}(x, y, \mathbf{u}) = A(x, y, \mathbf{u})\mathbf{u}$. Then $A(x, y, \mathbf{u})$ is nonnegative and $A(x_0, y_0, \mathbf{u})$ is irreducible for all $(x, y, \mathbf{u}) \in \mathbb{R}^m \times \overline{\Omega} \times [\mathbf{0}, \mathbf{M}]_{\mathbb{R}^N}$. Note that the solution of system (2.1) can also be expressed as

$$\mathbf{u}^\phi(t, x, y; \mathbf{f}) = e^{-\mu t} \overline{T}(t)[\phi](x, y) + \int_0^t e^{-\mu(t-s)} \overline{T}(t-s)[\mathbf{g}(\cdot, \cdot, \mathbf{u}^\phi(s, \cdot, \cdot; \mathbf{f}))](x, y) ds,$$

where the semigroup $\overline{T}(t)$ is defined in the Appendix. Since $\phi \in C_M \setminus \{0\}$, we may assume $\phi_1 \neq 0$. Then $\overline{T}_1(t)[\phi_1](x, y) > 0$ for all $(t, x, y) \in (0, \infty) \times \mathbb{R}^m \times \overline{\Omega}$, which implies $u_1^\phi(t, x, y; \mathbf{f}) \geq e^{-\mu t} \overline{T}_1(t)[\phi_1](x, y) > 0$. Define $I = \{i : i \in K, \exists (t, x, y) \in (0, \infty) \times \mathbb{R}^m \times \overline{\Omega} \text{ s.t. } u_i^\phi(t, x, y; \mathbf{f}) = 0\}$, $A(s) := A(x_0, y_0, \mathbf{u}^\phi(s, x_0, y_0; \mathbf{f}))$. Then $1 \notin I$. We now prove $I = \emptyset$; otherwise, suppose $I \neq \emptyset$. By the irreducibility of $A(s)$, there exist $i \in I, j \in I^c$ such that $a_{ij}(0) > 0$, and hence there exists $\delta > 0$ such that $a_{ij}(s) > 0, s \in (0, \delta)$. Thus,

$$(A(s)\mathbf{u}^\phi(s, x_0, y_0; \mathbf{f}))_i = \sum_{k=1}^N a_{ik}(s)u_k^\phi(s, x_0, y_0; \mathbf{f}) \geq a_{ij}(s)u_j^\phi(s, x_0, y_0; \mathbf{f}) > 0, \quad s \in (0, \delta),$$

which implies $\overline{T}_i(t-s)[g_i(\cdot, \cdot, \mathbf{u}^\phi(s, \cdot, \cdot; \mathbf{f}))](x, y) > 0$ for $\delta > t \geq s > 0, (x, y) \in \mathbb{R}^m \times \overline{\Omega}$. As a result,

$$u_i^\phi(t, x, y; \mathbf{f}) \geq \int_0^t e^{-\mu(t-s)} \overline{T}_i(t-s)[g_i(\cdot, \cdot, \mathbf{u}^\phi(s, \cdot, \cdot; \mathbf{f}))](x, y) ds > 0$$

for all $(t, x, y) \in (0, \infty) \times \mathbb{R}^m \times \overline{\Omega}$, which is in contradiction with the definition of I .

(ii) For any $\gamma \in [0, 1]$, let $\mathbf{z}(t, x, y) = Q[t, \gamma\phi](x, y) - \gamma Q[t, \phi](x, y)$. Then $\mathbf{z}(t, x, y)$ satisfies the following system:

$$\begin{cases} \partial_t \mathbf{z} \geq D\Delta_x \mathbf{z} + D\Delta_y \mathbf{z} + c\xi \cdot \nabla_x \mathbf{z} + \mathbf{g}(t, x, y)\mathbf{z}, & (t, x, y) \in (0, \infty) \times \mathbb{R}^m \times \Omega, \\ \partial_\nu \mathbf{z}(t, x, y) = \mathbf{0}, & (t, x, y) \in (0, \infty) \times \mathbb{R}^m \times \partial\Omega, \\ \mathbf{z}(0, x, y) = \mathbf{0}, & (x, y) \in \mathbb{R}^m \times \overline{\Omega}, \end{cases}$$

where

$$\mathbf{g}(t, x, y) := \int_0^1 D_{\mathbf{u}}\mathbf{f}(x, y, sQ[t, \gamma\phi](x, y) + (1-s)\gamma Q[t, \phi](x, y)) ds$$

is cooperative. It follows from the Phragmén-Lindelöf type maximum principle in [26] that $\mathbf{z}(t, x, y) \geq \mathbf{0}, \forall (t, x, y) \in \mathbb{R}_+ \times \mathbb{R}^m \times \overline{\Omega}$. Thus, we have $Q[t, \gamma\phi] \geq \gamma Q[t, \phi], \forall (t, \gamma, \phi) \in \mathbb{R}_+ \times [0, 1] \times C_+$.

(iii) We first claim that $Q[t, C_r]|_{\overline{B_R(0)} \times \overline{\Omega}}$ is an equicontinuous family in $C(\overline{B_R(0)} \times \overline{\Omega}, \mathbb{R}^N)$ for all $R > 0$. Let $D = \{\mathbf{f}(\cdot, \cdot, Q[s, \phi](\cdot, \cdot)) : s \in [0, t], \phi \in C_r\}$. Note that by **(H1)** and **(H2)**, there exists $d > 0$ such that $D \subset C_{d1} - C_{d1}$. For any $\varepsilon > 0$, and take $\delta = \frac{\varepsilon}{6d}$, by Lemma A.1-(ii) in the Appendix, $\overline{T}(t)[C_r]|_{\overline{B_R(0)} \times \overline{\Omega}}$ and $\{\overline{T}[s][\psi]|_{\overline{B_R(0)} \times \overline{\Omega}} : s \in [\delta, t], \psi \in D\}$ are families of equicontinuous functions in $C(\overline{B_R(0)} \times \overline{\Omega}, \mathbb{R}^N)$, it follows that for any $s \in [\delta, t], \phi \in C_r, \psi \in D, (x_1, y_1), (x_2, y_2) \in \overline{B_R(0)} \times \overline{\Omega}$ with $\|(x_1, y_1) - (x_2, y_2)\| \leq \delta$, we have

$$\|\overline{T}(t)[\phi](x_1, y_1) - \overline{T}(t)[\phi](x_2, y_2)\| < \frac{\varepsilon}{3}, \quad \|\overline{T}(t)[\psi](x_1, y_1) - \overline{T}(t)[\psi](x_2, y_2)\| < \frac{\varepsilon}{3t}.$$

Therefore,

$$\begin{aligned} & \|Q[t, \phi](x_1, y_1) - Q[t, \phi](x_2, y_2)\| \\ & \leq \|\overline{T}(t)[\phi](x_1, y_1) - \overline{T}(t)[\phi](x_2, y_2)\| \\ & + \left\| \int_0^{t-\frac{\varepsilon}{6d}} \overline{T}(t-s)[\mathbf{f}(\cdot, \cdot, Q[s, \phi](\cdot, \cdot))](x_1, y_1) - \overline{T}(t-s)[\mathbf{f}(\cdot, \cdot, Q[s, \phi](\cdot, \cdot))](x_2, y_2) \, ds \right\| \\ & + \left\| \int_{t-\frac{\varepsilon}{6d}}^t \overline{T}(t-s)[\mathbf{f}(\cdot, \cdot, Q[s, \phi](\cdot, \cdot))](x_1, y_1) - \overline{T}(t-s)[\mathbf{f}(\cdot, \cdot, Q[s, \phi](\cdot, \cdot))](x_2, y_2) \, ds \right\| \\ & \leq \frac{\varepsilon}{3} + \int_0^{t-\frac{\varepsilon}{6d}} \frac{\varepsilon}{3t} \, ds + \frac{\varepsilon}{3} < \varepsilon. \end{aligned}$$

This, combined with Proposition 2.1-(ii) and the Arzelà-Ascoli theorem, yields that $Q[t, C_r]$ is precompact in C . \square

3. Asymptotic annihilation

In this section, we will prove the asymptotic annihilation of system (2.1). We first introduce the following linear system:

$$\begin{cases} \partial_t \mathbf{u} = D_1 \Delta_x \mathbf{u} + D_2 \Delta_y \mathbf{u} + c\xi \cdot \nabla_x \mathbf{u} + \mathbf{h}(x, y)\mathbf{u}, & (t, x, y) \in (0, \infty) \times \mathbb{R}^m \times \Omega, \\ \partial_\nu \mathbf{u}(t, x, y) = 0, & (t, x, y) \in (0, \infty) \times \mathbb{R}^m \times \partial\Omega, \\ \mathbf{u}(0, x, y) = \phi(x, y), & (x, y) \in \mathbb{R}^m \times \overline{\Omega}, \end{cases} \quad (3.1)$$

where $\mathbf{h} \in BC(\mathbb{R}^m \times \overline{\Omega}, \mathbb{R}^{N \times N})$ and $\phi \in C_+$. Let $\mathbb{L}[t, \phi; \mathbf{h}]$ be the solution of the linear system (3.1). If $\lim_{\|x\| \rightarrow \infty} \mathbf{h}(x, \cdot) = \mathbf{h}_+$ in $C(\overline{\Omega}, \mathbb{R}^N)$, then we further define $\mathbb{L}[t, \cdot; \mathbf{h}_+]$ to be the solution map of the corresponding limiting system.

For the sake of convenience in our discussion. For any $\delta > 0$, we define

$$\mathbf{h}_\delta(x, y) = \max \left\{ \mathbf{h}(x, y), \delta \check{\mathbf{1}} + \mathbf{h}^\infty \right\}, \quad \forall (x, y) \in \mathbb{R}^m \times \overline{\Omega},$$

where $\mathbf{h}^\infty = \lim_{R \rightarrow \infty} \sup_{\substack{\|x\| > R \\ y \in \Omega}} \mathbf{h}(x, y)$. From the definition of $\mathbf{h}_\delta(x, y)$, it follows that $\lim_{\|x\| \rightarrow \infty} \mathbf{h}_\delta(x, \cdot) = \mathbf{h}^\infty$.

$$\mathbf{h}_\delta^+ := \delta \mathbf{1} + \mathbf{h}^\infty.$$

Based on Proposition 2.1 and 2.2, we can prove the following properties of $\mathbb{L}[t, \cdot; \mathbf{h}]$ and $\mathbb{L}[t, \cdot; \mathbf{h}_+]$.

Proposition 3.1. *Let $\mathbf{h} \in BC(\mathbb{R}^m \times \overline{\Omega}, \mathbb{R}^{N \times N}) \setminus \{\mathbf{0}\}$ with $\mathbf{h}(x, y)$ being cooperative for all $(x, y) \in \mathbb{R}^m \times \overline{\Omega}$. Then the following statements hold:*

- (i) *For any $t \in \mathbb{R}_+$, $\mathbb{L}[t, \cdot; \mathbf{h}], \mathbb{L}[t, \cdot; \mathbf{h}_+] : C \rightarrow C$ are bounded and positive linear operators on $(C, C_+, \|\cdot\|_\infty)$, and $\mathbb{L}[t, \cdot; \mathbf{h}], \mathbb{L}[t, \cdot; \mathbf{h}_+] : C_{\mathbf{r}} \rightarrow C$ are continuous and compact operators for all $(t, \mathbf{r}) \in \mathbb{R}_+ \times \text{Int}(\mathbb{R}_+^N)$.*
- (ii) *If $\lim_{\|x\| \rightarrow \infty} \mathbf{h}(x, \cdot) = \mathbf{h}_+$ in $C(\overline{\Omega}, \mathbb{R}^N)$, then $\lim_{\|z\| \rightarrow \infty} \mathcal{T}_{-z} [\mathbb{L}[t, \mathcal{T}_z[\phi]; \mathbf{h}]] = \mathbb{L}[t, \phi; \mathbf{h}_+]$ in C for all $(t, \phi) \in \mathbb{R}_+ \times C_+$. Moreover, for any $t \in \mathbb{R}_+$, $\lim_{(k, \|z\|) \rightarrow (\infty, \infty)} \mathcal{T}_{-z} [\mathbb{L}[t, \mathcal{T}_z[\phi_k]; \mathbf{h}]] = \mathbb{L}[t, \phi_0; \mathbf{h}_+]$ in C provided that $\{\phi_k\}_{k \in \mathbb{N}} \in C_{\mathbf{r}}$ for some $\mathbf{r} \in \text{Int}(\mathbb{R}_+^N)$ with $\lim_{k \rightarrow \infty} \|\phi_k - \phi_0\| = 0$.*
- (iii) *If there exists $(x_0, y_0) \in \mathbb{R}^m \times \overline{\Omega}$ such that $\mathbf{h}(x_0, y_0)$ is irreducible, then $\mathbb{L}[t, \phi; \mathbf{h}] \in C_+^o$ for all $(t, \phi) \in (0, \infty) \times (C_+ \setminus \{\mathbf{0}\})$.*

Next, we analyze the asymptotic annihilation of system (2.1) by constructing an upper system sequence for Q . We give the following results before presenting the main conclusion on asymptotic annihilation.

In the rest of this section, we assume that $\mathbf{h}(x, y) = D_{\mathbf{u}}\mathbf{f}(x, y, \mathbf{0})$. Then, (H4) implies $\exists \delta_0 > 0$ such that $s(\mathbf{h}_{\delta_0}^+) < 0$.

Lemma 3.1. *For any $\mathbf{r}^* \in \text{Int}(\mathbb{R}_+^N)$, the following statements are valid:*

- (i) $\lim_{t \rightarrow \infty} \|\mathbb{L}[t, \mathbf{r}^*; \mathbf{h}_{\delta_0}^+](0, \cdot)\| = 0$.
- (ii) $\liminf_{\|z\| \rightarrow \infty} \left[\inf\{\mathbb{L}[t, \mathbf{r}^*; \mathbf{h}_{\delta_0}^+](0, y) - Q[t, \mathbf{r}^*](z, y) : y \in \overline{\Omega}\} \right] \in \mathbb{R}_+^N$ for all $t > 0$.

Proof. (i) Note that $\mathbb{L}[t, \mathbf{r}^*; \mathbf{h}_{\delta_0}^+] = e^{\mathbf{h}_{\delta_0}^+ t} \overline{T}(t)[\mathbf{r}^*] = e^{\mathbf{h}_{\delta_0}^+ t} \mathbf{r}^*$. Thus, we have

$$\left\| \mathbb{L}[t, \mathbf{r}^*; \mathbf{h}_{\delta_0}^+](x, y) \right\| \leq \left\| e^{\mathbf{h}_{\delta_0}^+ t} \right\| \|\mathbf{r}^*\|, \quad \forall (x, y) \in \mathbb{R}^m \times \overline{\Omega}.$$

This, together with $s(\mathbf{h}_{\delta_0}^+) < 0$, implies that

$$\lim_{t \rightarrow \infty} \|\mathbb{L}[t, \mathbf{r}^*; \mathbf{h}_{\delta_0}^+](x, y)\| = 0.$$

In particular, $\lim_{t \rightarrow \infty} \|\mathbb{L}[t, \mathbf{r}^*; \mathbf{h}_{\delta_0}^+](0, \cdot)\| = 0$.

(ii) By the subhomogeneity of \mathbf{f} and Lemma 4.3 in [39], we have $\mathbf{f}(x, y, \mathbf{u}) \leq D_{\mathbf{u}}\mathbf{f}(x, y, \mathbf{0})\mathbf{u}$. This, combined with the definition of $\mathbf{h}_{\delta_0}(x, y)$ and the comparison principle, yields $Q[t, \mathbf{r}^*] \leq \mathbb{L}[t, \mathbf{r}^*; \mathbf{h}_{\delta_0}]$ for all $t > 0$. It follows from Proposition 3.1-(ii) that

$$\lim_{\|z\| \rightarrow \infty} \mathbb{L}[t, \mathbf{r}^*; \mathbf{h}_{\delta_0}](z, \cdot) = \mathbb{L}[t, \mathbf{r}^*; \mathbf{h}_{\delta_0}^+](0, \cdot), \quad \forall t > 0.$$

Thus, we obtain

$$\begin{aligned} & \liminf_{\|z\| \rightarrow \infty} \left[\inf\{\mathbb{L}[t, \mathbf{r}^*; \mathbf{h}_{\delta_0}^+](0, y) - Q[t, \mathbf{r}^*](z, y) : y \in \bar{\Omega}\} \right] \\ & \geq \liminf_{\|z\| \rightarrow \infty} \left[\inf\{\mathbb{L}[t, \mathbf{r}^*; \mathbf{h}_{\delta_0}^+](0, y) - \mathbb{L}[t, \mathbf{r}^*; \mathbf{h}_{\delta_0}](z, y) : y \in \bar{\Omega}\} \right] \\ & = \mathbf{0}. \quad \square \end{aligned}$$

Theorem 3.1. For any $\phi \in C_+$ and $\varepsilon > 0$, we have

$$\lim_{\alpha \rightarrow \infty} \left[\sup \left\{ \|Q[t, \phi](x, \cdot)\| : (t, \|x\|) \in [\alpha, \infty)^2 \right\} \right] = 0.$$

Consequently, $\lim_{t \rightarrow \infty} \left[\sup \{ \|Q[t, \phi](x, \cdot)\| : \|x\| \geq t\varepsilon \} \right] = 0$.

Proof. For any $\phi \in C_+$ and take $\alpha^* \in [1, \infty)$ such that $\mathbf{r}^{**} := \alpha^* \mathbf{M} \geq \phi$. Define

$$\tilde{Q}[t, \psi] = Q[t, \min\{\psi, \mathbf{r}^{**}\}], \quad \bar{Q}[\psi] = Q[1, \psi], \quad \bar{Q}^+[\psi] = \mathbb{L}[1, \psi; \mathbf{h}_{\delta_0}^+], \quad \forall t > 0, \psi \in C_+,$$

where \min denotes the componentwise pointwise minimum. By Propositions 2.1-(ii), (iii), we know that $\tilde{Q}[t, \psi], \bar{Q}[\psi] \in C_{\mathbf{r}^{**}}, \bar{Q}[1, \psi] \leq \bar{Q}[\psi], \bar{Q}[\varphi] \leq \bar{Q}[\psi]$ for all $\varphi \leq \psi \in C_{\mathbf{r}^{**}}$. These, together with Lemma 3.1 yield that $(\bar{Q}, \bar{Q}^+, \mathbf{r}^{**})$ satisfies uniform asymptotic annihilation (UAA) as stated in [39] and

$$\limsup_{t \rightarrow \infty} \left[\sup \left\{ \frac{\tilde{Q}[t, \psi](x, y)}{\mathbf{r}^{**} + 1} : (x, y) \in \mathbb{R}^m \times \bar{\Omega} \right\} \right] < 1, \quad \forall \psi \in C_+.$$

Note that $\tilde{Q}[t, \phi], t \geq 0$ is not a continuous-time semiflow on C_+ , but for any $\phi \in C_{\mathbf{r}^{**}}$, we have $\tilde{Q}[t, \phi] = Q[t, \phi]$. Thus, applying the proof of Theorem 3.3 in [39] to $(\bar{Q}, \bar{Q}, \bar{Q}^+)$, we obtain

$$\lim_{\alpha \rightarrow \infty} \left[\sup \left\{ \|\tilde{Q}[t, \phi](x, \cdot)\| : (t, \|x\|) \in [\alpha, \infty)^2 \right\} \right] = 0, \quad \forall \phi \in C_+,$$

and hence the desired conclusion follows. \square

4. Threshold dynamics

In this section, we study the threshold dynamics of system (1.3) based on the asymptotic spectral radius theory established in [39], including the existence, nonexistence and uniqueness of forced wave, as well as its global stability. Moreover, by deriving a relationship between the asymptotic spectral radius and the generalized principal eigenvalue defined in [5], we provide a framework to analyze how the asymptotic spectral radius depends on c . Before presenting the main results, we first provide a lower bound of \mathbf{f} , which is crucial for the proof of the subsequent theorem.

For any $\delta, r > 0$, define the function $\Gamma_{h,\delta,r} \in BC(\mathbb{R}^m \times \overline{\Omega}, \mathbb{R}_+)$ as follows:

$$\Gamma_{h,\delta,r}(x, y) = \begin{cases} \max\{0, h(x, y) - \delta\}, & x \in B_r, y \in \overline{\Omega}, \\ \max\{0, h(x, y) - \delta\}(1 + r - \|x\|), & x \in B_{r+1} \setminus B_r, y \in \overline{\Omega}, \\ 0, & x \notin B_{r+1}, y \in \overline{\Omega}, \end{cases}$$

where $h \in BC(\mathbb{R}^m \times \overline{\Omega}, \mathbb{R})$, $B_r = \{x \in \mathbb{R}^m : \|x\| \leq r\}$.

The following result is a extension of Lemma 4.12 in [39]. For the sake of completeness, we provide a proof here.

Lemma 4.1. For any $\delta, r > 0$, there exists $\tilde{\xi} := \tilde{\xi}_{\delta,r} \in \text{Int}(\mathbb{R}_+^N)$ such that

$$\mathbf{f}(x, y, \mathbf{u}) \geq (D_{\mathbf{u}}\mathbf{f}(x, y, \mathbf{0}) - \delta \check{\mathbf{1}})\mathbf{u}, \quad \forall (x, y, \mathbf{u}) \in B_{r+1} \times \overline{\Omega} \times [\mathbf{0}, \tilde{\xi}]_{\mathbb{R}^N}.$$

Let $\mu^* := -\inf \left\{ \frac{\partial f_i(x, y, \mathbf{u})}{\partial u_i} : (x, y, \mathbf{u}, i) \in \mathbb{R}^m \times \overline{\Omega} \times [\mathbf{0}, \mathbf{M}]_{\mathbb{R}^N} \times K \right\}$. Then

$$\mathbf{f}(x, y, \mathbf{u}) \geq \Lambda_{D_{\mathbf{u}}\mathbf{f}(x, y, \mathbf{0}), \delta, r}(x, y)\mathbf{u}, \quad \forall (x, y, \mathbf{u}) \in \mathbb{R}^m \times \overline{\Omega} \times [\mathbf{0}, \tilde{\xi}]_{\mathbb{R}^N},$$

where $\Lambda_{\mathbf{h}, \delta, r} \in BC(\mathbb{R}^m \times \overline{\Omega}, \mathbb{R}^{N \times N})$ is defined as

$$(\Lambda_{\mathbf{h}, \delta, r})_{ij}(x, y) = \begin{cases} -\mu^* + \Gamma_{\mu^* + \mathbf{h}_{ij}, \delta, r}(x, y), & i = j, \\ \Gamma_{\mathbf{h}_{ij}, \delta, r}(x, y), & i \neq j, \end{cases}$$

for all $(x, y, \mathbf{h}) \in \mathbb{R}^m \times \overline{\Omega} \times BC(\mathbb{R}^m \times \overline{\Omega}, \mathbb{R}^{N \times N})$.

Proof. Fix $\delta, r > 0$. Since $D_{\mathbf{u}}\mathbf{f}(x, y, \mathbf{u})$ is continuous in (x, y, \mathbf{u}) , there exists $\tilde{\xi} := \tilde{\xi}_{\delta,r} \in [\mathbf{0}, \mathbf{M}]_{\mathbb{R}^N} \cap \text{Int}(\mathbb{R}_+^N)$ such that

$$\|D_{\mathbf{u}}\mathbf{f}(x, y, \mathbf{u}) - D_{\mathbf{u}}\mathbf{f}(x, y, \mathbf{0})\| < \delta, \quad \forall (x, y, \mathbf{u}) \in B_{r+1} \times \overline{\Omega} \times [\mathbf{0}, \tilde{\xi}]_{\mathbb{R}^N}.$$

In particular, we have $\frac{\partial f_i(x, y, \mathbf{u})}{\partial u_j} > \frac{\partial f_i(x, y, \mathbf{0})}{\partial u_j} - \delta$, and hence $D_{\mathbf{u}}\mathbf{f}(x, y, \mathbf{u}) \geq D_{\mathbf{u}}\mathbf{f}(x, y, \mathbf{0}) - \delta \check{\mathbf{1}}$ for all $(x, y, \mathbf{u}) \in B_{r+1} \times \overline{\Omega} \times [\mathbf{0}, \tilde{\xi}]_{\mathbb{R}^N}$. It follows from $\mathbf{f}(x, y, \mathbf{0}) = \mathbf{0}$ that

$$\begin{aligned} \mathbf{f}(x, y, \mathbf{u}) &= \int_0^1 D_{\mathbf{u}}\mathbf{f}(x, y, s\mathbf{u}) ds \mathbf{u} \\ &\geq (D_{\mathbf{u}}\mathbf{f}(x, y, \mathbf{0}) - \delta \check{\mathbf{1}})\mathbf{u}, \quad \forall (x, y, \mathbf{u}) \in B_{r+1} \times \overline{\Omega} \times [\mathbf{0}, \tilde{\xi}]_{\mathbb{R}^N}. \end{aligned}$$

Let $(a_{ij}(x, y, \mathbf{u}))_{N \times N} := \int_0^1 D_{\mathbf{u}}\mathbf{f}(x, y, s\mathbf{u}) ds$. This, together with the definition of μ^* and the fact that $D_{\mathbf{u}}\mathbf{f}(x, y, \mathbf{u})$ is a cooperative matrix, yield that

$$\begin{cases} a_{ij}(x, y, \mathbf{u}) \geq 0, & \forall i \neq j, (x, y, \mathbf{u}) \in \mathbb{R}^m \times \overline{\Omega} \times [\mathbf{0}, \tilde{\xi}]_{\mathbb{R}^N}, \\ \mu^* + a_{ii}(x, y, \mathbf{u}) \geq 0, & \forall i, (x, y, \mathbf{u}) \in \mathbb{R}^m \times \overline{\Omega} \times [\mathbf{0}, \tilde{\xi}]_{\mathbb{R}^N}, \\ a_{ij}(x, y, \mathbf{u}) \geq \frac{\partial f_i(x, y, \mathbf{0})}{\partial u_j} - \delta, & \forall i, j, (x, y, \mathbf{u}) \in B_{r+1} \times \overline{\Omega} \times [\mathbf{0}, \tilde{\xi}]_{\mathbb{R}^N}. \end{cases}$$

By the definition of $\Gamma_{\frac{\partial f_i(\cdot, \cdot, \mathbf{0})}{\partial u_j}, \delta, r}(x, y)$, we obtain

$$\mu^* + a_{ii}(x, y, \mathbf{u}) \geq \Gamma_{\mu^* + \frac{\partial f_i(\cdot, \cdot, \mathbf{0})}{\partial u_i}, \delta, r}(x, y), \quad \forall (x, y, \mathbf{u}) \in \mathbb{R}^m \times \overline{\Omega} \times [\mathbf{0}, \tilde{\xi}]_{\mathbb{R}^N}, 1 \leq i \leq N,$$

$$a_{ij}(x, y, \mathbf{u}) \geq \Gamma_{\frac{\partial f_i(\cdot, \cdot, \mathbf{0})}{\partial u_j}, \delta, r}(x, y), \quad \forall (x, y, \mathbf{u}) \in \mathbb{R}^m \times \overline{\Omega} \times [\mathbf{0}, \tilde{\xi}]_{\mathbb{R}^N}, \forall i \neq j.$$

Hence, $\mathbf{f}(x, y, \mathbf{u}) \geq \Lambda_{\text{Duf}(\cdot, \cdot, \mathbf{0}), \delta, r}(x, y)\mathbf{u}$ for all $(x, y, \mathbf{u}) \in \mathbb{R}^m \times \overline{\Omega} \times [\mathbf{0}, \tilde{\xi}]_{\mathbb{R}^N}$. \square

Following the definitions in [39], we introduce the truncation, amplification, and asymptotic spectral radius of the operator \mathbb{L} , where $\mathbb{L} : C \rightarrow C$ is a bounded, linear and positive operator on $(C, C_+, \|\cdot\|_\infty)$. $\mathbb{L}|_{C_{\mathbf{r}}} : C_{\mathbf{r}} \rightarrow C$ is continuous for any $\mathbf{r} \in \text{Int}(\mathbb{R}^N)$, and $\mathbb{L}[C_1]$ is precompact in C .

For any given $\varrho \in (0, \infty]$, let

$$B_\varrho = \{x \in \mathbb{R}^m : \|x\| \leq \varrho\}, \quad \zeta_\varrho(x) := \max\{0, \min\{1, \varrho - \|x\|\}\}, \quad \forall x \in \mathbb{R}^m,$$

where B_∞ represents \mathbb{R}^m .

Define the truncation operator $\mathbb{L}_\varrho : C(B_\varrho \times \overline{\Omega}, \mathbb{R}^N) \rightarrow C(B_\varrho \times \overline{\Omega}, \mathbb{R}^N)$ by $\mathbb{L}_\varrho[\varphi] = \mathbb{L}[\zeta_\varrho\varphi]|_{B_\varrho \times \overline{\Omega}}$, where

$$(\zeta_\varrho\varphi)(x, y) = \begin{cases} \zeta_\varrho(x)\varphi(x, y), & (x, y) \in B_\varrho \times \overline{\Omega}, \\ 0, & x \notin B_\varrho, y \in \overline{\Omega}. \end{cases}$$

From the definitions of \mathbb{L}_ϱ and ζ_ϱ , it follows that

$$\mathbb{L}_\infty[\varphi] = \mathbb{L}[\zeta_\infty\varphi]|_{\mathbb{R}^m \times \overline{\Omega}} = \mathbb{L}[\varphi].$$

The asymptotic spectral radius of \mathbb{L} is defined as $\rho_\mathbb{L} := \lim_{\varrho \rightarrow \infty} \rho(\mathbb{L}_\varrho)$, where $\rho(\mathbb{L}_\varrho)$ denotes the spectral radius of the operator \mathbb{L}_ϱ .

For any given $(\sigma, d, \varrho) \in (0, \infty)^2 \times (0, \infty]$ and $(\varphi, \psi) \in C(B_\varrho \times \overline{\Omega}, \mathbb{R}^N) \times C$, define amplification operator $\widehat{\mathbb{L}}_{\sigma, d} : C \rightarrow C$ by

$$\widehat{\mathbb{L}}_{\sigma, d}[\psi](x, y) = (1 + \sigma\zeta_d(x))\mathbb{L}[\psi](x, y), \quad \forall (x, y) \in \mathbb{R}^m \times \overline{\Omega},$$

and the operator $\widehat{\mathbb{L}}_{\sigma, d, \varrho} : C(B_\varrho \times \overline{\Omega}, \mathbb{R}^N) \rightarrow C(B_\varrho \times \overline{\Omega}, \mathbb{R}^N)$ by $\widehat{\mathbb{L}}_{\sigma, d, \varrho}[\varphi] = \widehat{\mathbb{L}}_{\sigma, d}[\zeta_\varrho\varphi]|_{B_\varrho \times \overline{\Omega}}$.

Lemma 4.2. Let $\mathbf{h} \in BC(\mathbb{R}^m \times \overline{\Omega}, \mathbb{R}^{N \times N}) \setminus \{\mathbf{0}\}$ with $\mathbf{h}(x, y)$ being cooperative for all $(x, y) \in \mathbb{R}^m \times \overline{\Omega}$. Then the statements are valid:

- (i) If $t > 0$, $\rho_{\mathbb{L}[t, \cdot; \mathbf{h}]} > 1$, and there exists $(x_0, y_0) \in \mathbb{R}^m \times \overline{\Omega}$ such that $\mathbf{h}(x_0, y_0)$ is irreducible, then there exist $\delta_0, r_0 > 0$ such that $\rho_{\mathbb{L}[t, \cdot; \Lambda_{\mathbf{h}, \delta_0, r_0}]} > 1$.
- (ii) If $s(\mathbf{h}^\infty) < 0$, then for any $t > 0$, $\mathbb{L}[t, \cdot; \mathbf{h}]$ satisfies asymptotic contraction (AC) as defined in [39].

Proof. (i) Define $\mathbb{L} = \mathbb{L}[t, \cdot; \mathbf{h}]$. By the definition of $\rho_{\mathbb{L}}$, there exists $r_0 > 0$, such that $\rho(\mathbb{L}_{r_0}) > 1$. This, together with the compactness of \mathbb{L}_{r_0} and Proposition 7.26 in [42], implies that there exists $\psi_0 \in C(B_{r_0} \times \overline{\Omega}, \mathbb{R}^N) \setminus \{\mathbf{0}\}$ satisfying

$$\mathbb{L}_{r_0}[\psi_0] = \mathbb{L}[\zeta_{r_0} \psi_0]|_{B_{r_0} \times \overline{\Omega}} = \rho(\mathbb{L}_{r_0})\psi_0.$$

It follows from $\zeta_{r_0} \psi_0 \in C_+ \setminus \{\mathbf{0}\}$ and the irreducibility of $\mathbf{h}(x_0, y_0)$ that $\mathbb{L}[\zeta_{r_0} \psi_0] \in C_+^\circ$ and hence $\psi_0 \gg 0$. Set $\mathbb{L}^\delta := \mathbb{L}[t, \cdot; \Lambda_{\mathbf{h}, \delta, \frac{1}{\delta}}]$. By the definition of $\Lambda_{\mathbf{h}, \delta, \frac{1}{\delta}}$, we have $\lim_{\delta \rightarrow 0} \Lambda_{\mathbf{h}, \delta, \frac{1}{\delta}} = \mathbf{h}$ in $C_{\text{loc}}(\mathbb{R}^m \times \overline{\Omega})$. Due to the continuous dependence of solutions on the reaction terms (see [43, Remark 5.2]), we deduce that

$$\mathbb{L}^\delta[\zeta_{r_0} \psi_0]|_{B_{r_0} \times \overline{\Omega}} \rightarrow \mathbb{L}[\zeta_{r_0} \psi_0]|_{B_{r_0} \times \overline{\Omega}} = \rho(\mathbb{L}_{r_0})\psi_0$$

in $C(B_{r_0} \times \overline{\Omega}, \mathbb{R}^N)$ as $\delta \rightarrow 0$. Thus, there exists $\delta_0 > 0$ such that

$$(\mathbb{L}^{\delta_0})_{r_0}[\psi_0] = \mathbb{L}^{\delta_0}[\zeta_{r_0} \psi_0]|_{B_{r_0} \times \overline{\Omega}} \geq \frac{1 + \rho(\mathbb{L}_{r_0})}{2} \psi_0.$$

For any $r > r_0 + 1$,

$$\begin{aligned} (\mathbb{L}^{\delta_0})_r[\zeta_{r_0} \psi_0|_{B_r \times \overline{\Omega}}] &= \mathbb{L}^{\delta_0}[\zeta_r(\zeta_{r_0} \psi_0|_{B_r \times \overline{\Omega}})]|_{B_r \times \overline{\Omega}} \\ &= \mathbb{L}^{\delta_0}[\zeta_{r_0} \psi_0]|_{B_r \times \overline{\Omega}} \\ &\geq \begin{cases} \mathbb{L}^{\delta_0}[\zeta_{r_0} \psi_0]|_{B_{r_0} \times \overline{\Omega}}, & (x, y) \in B_{r_0} \times \overline{\Omega} \\ \mathbf{0}, & x \notin B_{r_0}, y \in \overline{\Omega} \end{cases} \\ &\geq \begin{cases} \frac{1 + \rho(\mathbb{L}_{r_0})}{2} \psi_0, & (x, y) \in B_{r_0} \times \overline{\Omega} \\ \mathbf{0}, & x \notin B_{r_0}, y \in \overline{\Omega} \end{cases} \\ &= \frac{1 + \rho(\mathbb{L}_{r_0})}{2} (\zeta_{r_0} \psi_0)|_{B_r \times \overline{\Omega}}. \end{aligned}$$

This, combined with the Gelfand’s formula, yields $\rho((\mathbb{L}^{\delta_0})_r) \geq \frac{1 + \rho(\mathbb{L}_{r_0})}{2}$ for all $r > r_0 + 1$. In view of the definition of $\rho_{\mathbb{L}^{\delta_0}}$, we have

$$\rho_{\mathbb{L}^{\delta_0}} \geq \frac{1 + \rho(\mathbb{L}_{r_0})}{2} > 1.$$

- (ii) Since $s(\mathbf{h}^\infty) < 0$, there exists $\varepsilon > 0$ such that $s(\varepsilon \check{\mathbf{1}} + \mathbf{h}^\infty) < 0$. Let

$$\mathbf{h}_\varepsilon(x, y) = \max \left\{ \mathbf{h}(x, y), \varepsilon \check{\mathbf{1}} + \mathbf{h}^\infty \right\}, \quad \forall (x, y) \in \mathbb{R}^m \times \overline{\Omega}.$$

Then $\mathbf{h}_\varepsilon^+ = \lim_{\|x\| \rightarrow \infty} \mathbf{h}_\varepsilon(x, \cdot) = \varepsilon \tilde{\mathbf{1}} + \mathbf{h}^\infty$ and $\mathbf{h}_\varepsilon(x, y) \geq \mathbf{h}(x, y)$ for all $(x, y) \in \mathbb{R}^m \times \overline{\Omega}$. By the comparison principle, we have

$$\mathbb{L}[t, \cdot; \mathbf{h}_\varepsilon] \geq \mathbb{L}[t, \cdot; \mathbf{h}].$$

It follows from $s(\mathbf{h}_\varepsilon^+) < 0$ that $\lim_{t \rightarrow \infty} \|\mathbb{L}[t, \mathbf{1}; \mathbf{h}_\varepsilon^+]\|_\infty = 0$. Moreover, by Proposition 3.1-(ii), $(\mathbb{L}[t, \cdot; \mathbf{h}_\varepsilon], \mathbb{L}[t, \cdot; \mathbf{h}_\varepsilon^+])$ satisfies asymptotic translation (AT) as introduced in [39]. This, together with Lemma 2.6 in [39], implies that $\mathbb{L}[t, \cdot; \mathbf{h}_\varepsilon]$ satisfies (AC) in [39]. As a result, $\mathbb{L}[t, \cdot; \mathbf{h}]$ also satisfies (AC) in [39]. \square

Next, we employ the theory developed in [39] to study the threshold dynamics of the system (1.3).

Theorem 4.1. *The following statements are valid:*

(i) *If $\phi \in C_+$, then for any $\varepsilon > 0$,*

$$\lim_{\alpha \rightarrow \infty} \left[\sup \{ \|P[t, \phi](x, \cdot)\| : t \geq \alpha \text{ and } \|x - ct\xi\| \geq \alpha \} \right] = 0,$$

which implies that $\lim_{t \rightarrow \infty} \left[\sup \{ \|P[t, \phi](x, \cdot)\| : \|x - ct\xi\| \geq t\varepsilon \} \right] = 0$.

(ii) *If $\rho_{\mathbb{L}[1, \cdot; D_{\mathbf{u}}\mathbf{f}(\cdot, \cdot, \mathbf{0})]} > 1$, then $P[t, \cdot]$ admits a nontrivial forced wave $W(x - ct\xi, \cdot)$ such that $W \in C_+^\circ$ and $\lim_{\|x\| \rightarrow \infty} W(x, \cdot) = \mathbf{0}$. Moreover, if there exists $(x_0, y_0) \in \mathbb{R}^m \times \overline{\Omega}$ such that $\mathbf{f}(\tilde{x}_0, \tilde{y}_0, \cdot)|_{\text{Int}(\mathbf{0}, \mathbf{M}]_{\mathbb{R}^N}}$ is strongly subhomogeneous, then*

$$\lim_{t \rightarrow \infty} \|P[t, \phi] - W(\cdot - ct\xi, \cdot)\|_{L^\infty(\mathbb{R}^m \times \overline{\Omega}, \mathbb{R}^N)} = 0, \quad \forall \phi \in C_+ \setminus \{\mathbf{0}\}.$$

(iii) *If $0 < \rho_{\mathbb{L}[1, \cdot; D_{\mathbf{u}}\mathbf{f}(\cdot, \cdot, \mathbf{0})]} < 1$, then for any $\phi \in C_+$,*

$$\lim_{t \rightarrow \infty} \|P[t, \phi]\|_{L^\infty(\mathbb{R}^m \times \overline{\Omega}, \mathbb{R}^N)} = 0,$$

which implies that $P[t, \cdot]$ admits no nontrivial forced wave $W(x - ct\xi, \cdot)$ in C_+ .

Proof. (i) follows from $Q[t, \cdot] = \mathcal{T}_{-ct\xi} \circ P[t, \cdot]$ and Theorem 3.1.

(ii) Since $\rho_{\mathbb{L}[1, \cdot; D_{\mathbf{u}}\mathbf{f}(\cdot, \cdot, \mathbf{0})]} > 1$, it follows from Lemma 4.2-(i) that there exist $\delta_0, r_0 > 0$ such that $\rho_{\mathbb{L}[1, \cdot; \Lambda_{D_{\mathbf{u}}\mathbf{f}(\cdot, \cdot, \mathbf{0}), \delta_0, r_0}]} > 1$. By the definition of Λ , we have

$$\lim_{\|x\| \rightarrow \infty} \Lambda_{D_{\mathbf{u}}\mathbf{f}(\cdot, \cdot, \mathbf{0}), \delta_0, r_0} = -\mu^* I,$$

where I denotes the $N \times N$ identity matrix. By (H4) and the Perron-Frobenius theorem, we may obtain $\mu^* > 0$, and hence $\lim_{t \rightarrow \infty} \|\mathbb{L}[t, \mathbf{1}; -\mu^* I]\|_\infty = 0$. By Proposition 3.1, we conclude that $\mathbb{L}[1, \cdot; \Lambda_{D_{\mathbf{u}}\mathbf{f}(\cdot, \cdot, \mathbf{0}), \delta_0, r_0}]$ satisfies all the conditions of Corollary 2.7 in [39].

It follows from Lemma 4.1 that there exists $\tilde{\xi} := \tilde{\xi}_{\delta_0, r_0} \in \text{Int}(\mathbb{R}_+^N)$ such that

$$\mathbf{f}(x, y, \mathbf{u}) \geq \Lambda_{D_{\mathbf{u}}\mathbf{f}(\cdot, \cdot, \mathbf{0}), \delta_0, r_0} \mathbf{u}, \quad \forall (x, y, \mathbf{u}) \in \mathbb{R}^m \times \overline{\Omega} \times [\mathbf{0}, \tilde{\xi}]_{\mathbb{R}^N}.$$

Take $\xi^* \in \text{Int}(\mathbb{R}_+^N) \cap [0, \tilde{\xi}]_{\mathbb{R}^N}$ such that $\mathbb{L}[t, C_{\xi^*}; \Lambda_{D_{\mathbf{u}}\mathbf{f}(\cdot, \cdot, \mathbf{0}), \delta_0, r_0}] \subseteq C_{\xi^*}$ for all $t \in [0, 1]$. By the comparison principle, we then obtain

$$\mathbf{M} \geq Q[t, \phi] \geq \mathbb{L}[t, \psi; \Lambda_{D_{\mathbf{u}}\mathbf{f}(\cdot, \cdot, \mathbf{0}), \delta_0, r_0}], \quad \forall (t, \phi, \psi) \in [0, 1] \times C_{\mathbf{M}} \times C_{\xi^*} \text{ with } \phi \geq \psi.$$

Hence, by the first statement of Theorem 3.9-(i) in [39], it follows that $P[t, \cdot]$ has a nontrivial forced wave $W(x - ct\xi, \cdot)$ in $C_{\mathbf{M}}$ with $\lim_{\|x\| \rightarrow \infty} W(x, \cdot) = \mathbf{0}$.

We now prove that Q has a unique nontrivial positive steady state $W \in C_+ \setminus \{\mathbf{0}\}$. Otherwise, suppose that Q has at least two nontrivial steady states $W_1, W_2 \in C_+ \setminus \{\mathbf{0}\}$ with $W_1 \neq W_2$. Take $\alpha^* \in [1, \infty)$ such that $\mathbf{r}^* := \alpha^* \mathbf{M} \geq \max\{W_1, W_2\}$. It then follows from Proposition 2.2-(i) that $W_1, W_2 \in C_{\mathbf{r}^*} \cap C_+^\circ$. Define

$$\mathcal{K} = \{\phi \in C_{\mathbf{r}^*} : \phi \geq W_2, \phi \geq W_1\}.$$

Then \mathcal{K} is a non-empty, closed, convex subset of $C_{\mathbf{r}^*}$, and $Q[t, \mathcal{K}] \subseteq \mathcal{K}$. This, integrated with the compactness of Q and Schauder's fixed point theorem, implies that Q has a fixed point $W^* \in \mathcal{K}$. Hence, without loss of generality, we assume $W_2 < W_1$.

For any $\varepsilon > 0$, define

$$A_\varepsilon = \varepsilon \check{\mathbf{1}} + \limsup_{R \rightarrow \infty} \sup_{\substack{\|x\| > R \\ y \in \Omega}} D_{\mathbf{u}}\mathbf{f}(x, y, \mathbf{0}).$$

By (H4) and (H5), there exist $\varepsilon_0, R_0 > 0$ such that

$$s_0 := s(A_{\varepsilon_0}) < 0, \quad D_{\mathbf{u}}\mathbf{f}(x, \cdot, \mathbf{v}) \leq \frac{\varepsilon_0}{3} \check{\mathbf{1}} + D_{\mathbf{u}}\mathbf{f}(x, \cdot, \mathbf{0}) \leq A_{\varepsilon_0}, \\ \forall (\|x\|, \mathbf{v}) \in [R_0, \infty) \times [0, \varepsilon_0 \mathbf{1}]_{\mathbb{R}^N}.$$

In view of (i), we may assume $W_1(x), W_2(x) \in [0, \varepsilon_0 \mathbf{1}]_{\mathbb{R}^N}$ when $\|x\| \geq R_0$. By the Perron-Frobenius theorem, there exists $\mathbf{v}_0 \in \text{Int}(\mathbb{R}_+^N)$ such that

$$A_{\varepsilon_0} \mathbf{v}_0 = s_0 \mathbf{v}_0 \ll \mathbf{0} \text{ and } \min\{(\mathbf{v}_0)_k : 1 \leq k \leq N\} = 1.$$

Define

$$a^* = \inf\{a \in \mathbb{R}_+ : aW_1(x, y) \leq W_2(x, y), \forall (x, y) \in B_{R_0} \times \overline{\Omega}\}.$$

Since $\mathbf{0} \ll W_2 < W_1$, we have $0 < a^* \leq 1$ and $a^*W_1|_{B_{R_0} \times \overline{\Omega}} \leq W_2|_{B_{R_0} \times \overline{\Omega}}$. We now divide the rest proof into two steps.

Step 1. Prove $a^*W_1 \leq W_2$.

Define

$$\mathbf{v}(t, x, y) = W_2(x, y) - a^*W_1(x, y) + \|\mathbf{r}^*\|e^{s_0 t} \mathbf{v}_0, \quad \forall (t, x, y) \in \mathbb{R}_+ \times \mathbb{R}^m \times \overline{\Omega}.$$

According to the definitions of a^* , R_0 , s_0 and \mathbf{v}_0 , \mathbf{v} satisfies the following system:

$$\begin{cases} \frac{\partial \mathbf{v}}{\partial t} \geq D_1 \Delta_x \mathbf{v} + D_2 \Delta_y \mathbf{v} + c\xi \cdot \nabla_x \mathbf{v} + \mathbf{g}(x, y)\mathbf{v}, & (t, x, y) \in (0, \infty) \times (\mathbb{R}^m \setminus B_{R_0}) \times \Omega, \\ \mathbf{v}(t, x, y) \geq \mathbf{0}, & (t, x, y) \in \mathbb{R}_+ \times \partial B_{R_0} \times \overline{\Omega}, \\ \mathbf{v}(0, x, y) \geq \mathbf{0}, & x \notin B_{R_0}, y \in \overline{\Omega}, \end{cases}$$

where

$$\mathbf{g}(x, y) := \int_0^1 D_{\mathbf{u}} \mathbf{f}(x, y, sW_2(x, y) + (1-s)a^*W_1(x, y)) ds$$

is cooperative. It follows from the Phragmén-Lindelöf maximum principle in [26] that $\mathbf{v}(t, x, y) \geq \mathbf{0}$ for all $(t, x, y) \in \mathbb{R}_+ \times (\mathbb{R}^m \setminus B_{R_0}) \times \overline{\Omega}$. Letting $t \rightarrow \infty$, we obtain $a^*W_1 \leq W_2$.

Step 2. Prove $a^* = 1$.

Assume, on the contrary, that $a^* < 1$. By **(H2)** and $\mathbf{f}(\tilde{x}_0, \tilde{y}_0, \cdot)|_{\text{Int}(\{0, \mathbf{M}\}_{\mathbb{R}^N})}$ is strongly sub-homogeneous, we have $\mathbf{f}(\cdot, \cdot, a^*W_1) \geq a^*\mathbf{f}(\cdot, \cdot, W_1)$ and $\mathbf{f}(\tilde{x}_0, \tilde{y}_0, a^*W_1(\tilde{x}_0, \tilde{y}_0)) \gg a^*\mathbf{f}(\tilde{x}_0, \tilde{y}_0, W_1(\tilde{x}_0, \tilde{y}_0))$. Define $\overline{W}(x, y) = W_2(x, y) - a^*W_1(x, y)$, $\forall (x, y) \in \mathbb{R}^m \times \overline{\Omega}$. Then \overline{W} satisfies the following system:

$$\begin{cases} D_1 \Delta_x \overline{W}(x, y) + D_2 \Delta_y \overline{W}(x, y) + c\xi \cdot \nabla_x \overline{W}(x, y) + \mathbf{g}(x, y)\overline{W}(x, y) \leq \mathbf{0}, & (x, y) \in \mathbb{R}^m \times \Omega, \\ D_1 \Delta_x \overline{W}(\tilde{x}_0, \tilde{y}_0) + D_2 \Delta_y \overline{W}(\tilde{x}_0, \tilde{y}_0) + c\xi \cdot \nabla_x \overline{W}(\tilde{x}_0, \tilde{y}_0) + \mathbf{g}(\tilde{x}_0, \tilde{y}_0)\overline{W}(\tilde{x}_0, \tilde{y}_0) \ll \mathbf{0}, \\ \overline{W}(x, y) \geq \mathbf{0}, & (x, y) \in \mathbb{R}^m \times \overline{\Omega}. \end{cases}$$

Applying the strong maximum principle in [26], we conclude that $\overline{W}(x, y) \gg \mathbf{0}$ for all $(x, y) \in \mathbb{R}^m \times \overline{\Omega}$. This implies that $W_2|_{B_{R_0} \times \overline{\Omega}} \gg a^*W_1|_{B_{R_0} \times \overline{\Omega}}$, which contradicts the choice of a^* . Thus, $W_1 = W_2$.

For any $\phi \in C_+$ and take $\alpha^{**} \in [1, \infty)$ such that $\mathbf{r}^{**} := \alpha^{**}\mathbf{M} \geq \phi$. According to the proof of Theorem 3.8-(i) in [39] and the choice of α^{**} , the convergence of $Q[t, \phi]$ does not require the following condition:

$$\limsup_{t \rightarrow \infty} \left[\sup \left\{ \frac{Q[t, \phi](x, y)}{\mathbf{r}^{**} + 1} : (x, y) \in \mathbb{R}^m \times \overline{\Omega} \right\} \right] < 1, \quad \forall \phi \in C_+.$$

As a result, we directly obtain

$$\lim_{t \rightarrow \infty} \|Q[t, \phi] - W\|_{L^\infty(\mathbb{R}^m \times \overline{\Omega}, \mathbb{R}^N)} = 0, \quad \forall \phi \in C_+ \setminus \{\mathbf{0}\},$$

and the desired conclusion follows.

(iii) It follows from $0 < \rho_{\mathbb{L}[1, \cdot; D_{\mathbf{u}} \mathbf{f}(\cdot, \cdot, \mathbf{0})]} < 1$, Proposition 3.1-(i), (ii), Lemma 4.2-(ii) and [39, Corollary 2.5-(ii)] that there exist $d_0, \sigma^* > 0, \rho^* \in (0, 1)$ and $\eta_\infty^* \in C_+^*$ such that

$$|\eta_\infty^*|_\infty = 1, \quad \mathbb{L}[1, \cdot; \widehat{D_{\mathbf{u}} \mathbf{f}(\cdot, \cdot, \mathbf{0})}]_{\sigma^*, 1+d_0}[\eta_\infty^*] = \rho^* \eta_\infty^* < \eta_\infty^*,$$

where $\mathbb{L}[1, \cdot; \widehat{D_{\mathbf{u}}\mathbf{f}}(\cdot, \cdot, \mathbf{0})]_{\sigma^*, 1+d_0}[\eta_\infty^*] = (1 + \sigma^* \zeta_{1+d_0}) \mathbb{L}[1, \cdot; D_{\mathbf{u}}\mathbf{f}(\cdot, \cdot, \mathbf{0})][\eta_\infty^*]$. According to Theorem 3.9-(ii) in [39], it suffices to show that for any $\phi \in C_+$, there exists $n \in \mathbb{N}$ such that

$$\omega(\phi) \subseteq C_{n\eta_\infty^*}.$$

By Lemma 4.2-(ii), there exists $(\delta_0, s_0, n_0) \in (0, \frac{1}{\sqrt{n}}) \times (0, \infty) \times \mathbb{N}$ such that

$$\mathbb{L}[n_0, \mathbf{1}; D_{\mathbf{u}}\mathbf{f}(\cdot, \cdot, \mathbf{0})](x, \cdot) \leq \delta_0 \mathbf{1}, \quad \forall \|x\| \geq s_0.$$

Define

$$b^* = \inf\{b \in \mathbb{R}_+ : \psi_k(x, y) \leq b(\eta_\infty^*)_k(x, y), \forall (k, x, y, \psi) \in K \times B_{s_0} \times \overline{\Omega} \times \omega(\phi)\}.$$

Then $b^* \in \mathbb{R}_+$ and $\omega(\phi)|_{B_{s_0} \times \overline{\Omega}} \leq b^* \eta_\infty^*|_{B_{s_0} \times \overline{\Omega}}$.

Now we claim $\omega(\phi) \leq b^* \eta_\infty^*$. Otherwise, we have

$$\gamma := \sup\{\psi_k(x, y) - b^*(\eta_\infty^*)_k(x, y) : (k, x, y, \psi) \in K \times \mathbb{R}^m \times \overline{\Omega} \times \omega(\phi)\} > 0.$$

It follows from Theorem 3.1 that $\lim_{\|x\| \rightarrow \infty} [\sup\{\|\psi(x, \cdot)\| : \psi \in \omega(\phi)\}] = 0$, and hence there exists $\alpha_0 > s_0$ such that

$$\psi_k(x, \cdot) - b^*(\eta_\infty^*)_k(x, \cdot) < \frac{\gamma}{2}, \quad \psi \in \omega(\phi), \quad \|x\| > \alpha_0.$$

Thus, by the definition of γ , we see that

$$\gamma = \sup\{\psi_k(x, y) - b^*(\eta_\infty^*)_k(x, y) : (k, x, y, \psi) \in K \times (B_{\alpha_0} \setminus B_{s_0}) \times \overline{\Omega} \times \omega(\phi)\}.$$

This, combined with the invariance of $\omega(\phi)$, implies that there exists $(k_0, x^*, y^*, \psi^*, \psi^{**}) \in K \times (B_{\alpha_0} \setminus B_{s_0}) \times \overline{\Omega} \times \omega(\phi) \times \omega(\phi)$ such that

$$\psi_{k_0}^*(x^*, y^*) - b^*(\eta_\infty^*)_{k_0}(x^*, y^*) = \gamma > 0 \text{ and } \psi^* = Q[n_0, \psi^{**}].$$

From the subhomogeneity of \mathbf{f} and Lemma 4.3 in [39], we have $\mathbf{f}(x, y, \mathbf{u}) \leq D_{\mathbf{u}}\mathbf{f}(x, y, \mathbf{0})\mathbf{u}$ for all $(x, y, \mathbf{u}) \in \mathbb{R}^m \times \overline{\Omega} \times \mathbb{R}_+^N$. It follows from the comparison principle that

$$Q[n_0, \psi^{**}] \leq \mathbb{L}[n_0, \psi^{**}; D_{\mathbf{u}}\mathbf{f}(\cdot, \cdot, \mathbf{0})].$$

According to the choice of $b^*, x^*, y^*, \rho^*, \psi^*, \psi^{**}, s_0$ and Lemma 2.1-(i) in [39], we obtain

$$\begin{aligned} & \psi^*(x^*, y^*) - b^* \eta_\infty^*(x^*, y^*) \\ &= Q[n_0, \psi^{**}](x^*, y^*) - b^* \eta_\infty^*(x^*, y^*) \\ &\leq \mathbb{L}[n_0, \psi^{**}; D_{\mathbf{u}}\mathbf{f}(\cdot, \cdot, \mathbf{0})](x^*, y^*) - \frac{b^*}{(\rho^*)^{n_0}} (\mathbb{L}[1, \cdot; \widehat{D_{\mathbf{u}}\mathbf{f}}(\cdot, \cdot, \mathbf{0})]_{\sigma^*, 1+d_0})^{n_0} [\eta_\infty^*](x^*, y^*) \\ &\leq \mathbb{L}[n_0, \psi^{**}; D_{\mathbf{u}}\mathbf{f}(\cdot, \cdot, \mathbf{0})](x^*, y^*) - \frac{b^*}{(\rho^*)^{n_0}} \mathbb{L}[n_0, \eta_\infty^*; D_{\mathbf{u}}\mathbf{f}(\cdot, \cdot, \mathbf{0})](x^*, y^*) \end{aligned}$$

$$\begin{aligned}
 &\leq \mathbb{L}[n_0, \psi^{**}; \mathbf{D}_u \mathbf{f}(\cdot, \cdot, \mathbf{0})](x^*, y^*) - b^* \mathbb{L}[n_0, \eta_\infty^*; \mathbf{D}_u \mathbf{f}(\cdot, \cdot, \mathbf{0})](x^*, y^*) \\
 &\leq \mathbb{L}[n_0, \psi^{**} - b^* \eta_\infty^*; \mathbf{D}_u \mathbf{f}(\cdot, \cdot, \mathbf{0})](x^*, y^*) \\
 &\leq [(\psi^*)_{k_0}(x^*, y^*) - b^*(\eta_\infty^*)_{k_0}(x^*, y^*)] \times \mathbb{L}[n_0, \mathbf{1}; \mathbf{D}_u \mathbf{f}(\cdot, \cdot, \mathbf{0})](x^*, y^*) \\
 &\leq \delta_0((\psi^*)_{k_0}(x^*, y^*) - b^*(\eta_\infty^*)_{k_0}(x^*, y^*)) \mathbf{1},
 \end{aligned}$$

which contradicts the fact that $(\psi^*)_{k_0}(x^*, y^*) - b^*(\eta_\infty^*)_{k_0}(x^*, y^*) > \gamma > 0$ and $0 < \delta_0 < \frac{1}{\sqrt{n}} \leq 1$. Hence, we conclude that $\omega(\phi) \leq b^* \eta_\infty^*$. This completes the proof of (iii). \square

For the remainder of this section, we investigate the relation between the asymptotic spectral radius and the generalized principal eigenvalue defined in [5]. We focus on the case where $N = m = 1$, for which the corresponding linearized equation is

$$\begin{cases} \partial_t u = d \Delta u + c \partial_x u + \partial_u f(x, y, 0)u, & (t, x, y) \in (0, \infty) \times \mathbb{R} \times \Omega, \\ \partial_\nu u(t, x, y) = 0, & (t, x, y) \in (0, \infty) \times \mathbb{R} \times \partial \Omega, \\ u(0, x, y) = \phi(x, y), & (x, y) \in \mathbb{R} \times \overline{\Omega}. \end{cases} \tag{4.1}$$

Define $h(x, y) = \partial_u f(x, y, 0)$ and the linear operator $L = d \Delta + c \partial_x + h(x, y)$. According to [5], the generalized Neumann principal eigenvalue of $-L$ in $\mathbb{R} \times \Omega$ is defined as

$$\lambda_1(-L, \mathbb{R} \times \Omega) = \sup \{ \lambda \in \mathbb{R} : \exists \psi > 0, (L + \lambda)\psi \leq 0 \text{ a.e. in } \mathbb{R} \times \Omega, \partial_\nu \psi \geq 0 \text{ on } \mathbb{R} \times \partial \Omega \},$$

where $\psi \in W^{2,p}((-r, r) \times \Omega), \forall r > 0, p > n + 1$. Let $\lambda_1 := \lambda_1(-L, \mathbb{R} \times \Omega)$. By Theorem 3.1 in [5], for any $r > 0$, there exists a principal eigenpair $(\lambda(r), \varphi_r) \in \mathbb{R}_+ \times W^{2,p}((-r, r) \times \Omega, \mathbb{R}_+)$ satisfying

$$\begin{cases} -L\varphi_r = \lambda(r)\varphi_r, & \text{a.e. in } (-r, r) \times \Omega, \\ \partial_\nu \varphi_r = 0, & \text{on } (-r, r) \times \partial \Omega, \\ \varphi_r = 0, & \text{on } \{\pm r\} \times \overline{\Omega}, \end{cases} \tag{4.2}$$

with the normalization condition $\varphi_r(0, y_0) = 1$, where $y_0 \in \Omega$ is fixed and independent of r . According to Proposition 1 in [5], $\lambda(r) \searrow \lambda_1, \varphi_r \rightarrow \varphi$ in $C_{loc}^1(\mathbb{R} \times \overline{\Omega})$ and weakly in $W_{loc}^{2,p}(\mathbb{R} \times \Omega)$ for any $p > 1$, as $r \rightarrow \infty$. Moreover, φ also satisfies $\varphi(0, y_0) = 1$ and

$$\begin{cases} -L\varphi = \lambda_1\varphi, & \text{a.e. in } \mathbb{R} \times \Omega, \\ \partial_\nu \varphi = 0, & \text{on } \mathbb{R} \times \partial \Omega. \end{cases} \tag{4.3}$$

Since Ω is a smooth bounded domain in \mathbb{R}^n , according to the proof in the appendix of [5], there exists $\varepsilon > 0$ such that the domain

$$\tilde{\Omega} := \{y \in \mathbb{R}^n : \text{dist}(y, \Omega) < \varepsilon\}$$

is smooth and every $y \in \tilde{\Omega} \setminus \Omega$ has a unique projection on $\overline{\Omega}$, denoted by $\pi(y)$. Let $\mathcal{R} : \mathbb{R} \times (\tilde{\Omega} \setminus \overline{\Omega}) \rightarrow \mathbb{R} \times \overline{\Omega}$ be the reflection with respect to $\mathbb{R} \times \partial \Omega$, i.e.,

$$\mathcal{R}(x, y) := (x, 2\pi(y) - y).$$

Define $\tilde{\beta} : \mathbb{R} \times \tilde{\Omega} \rightarrow \mathbb{R}$ and $\tilde{h} : \mathbb{R} \times \tilde{\Omega} \rightarrow \mathbb{R}$ by

$$\tilde{\beta}(x, y) := \begin{cases} 0, & \text{if } (x, y) \in \mathbb{R} \times \overline{\Omega}, \\ (J(x, y)^{-1})^T \operatorname{div} \left((J(x, y)^{-1})^T \right), & \text{if } (x, y) \in \mathbb{R} \times (\tilde{\Omega} \setminus \overline{\Omega}), \end{cases}$$

$$\tilde{h}(x, y) := \begin{cases} h(x, y), & \text{if } (x, y) \in \mathbb{R} \times \overline{\Omega}, \\ h(\mathcal{R}(x, y)), & \text{if } (x, y) \in \mathbb{R} \times (\tilde{\Omega} \setminus \overline{\Omega}), \end{cases}$$

and the matrix field $\tilde{A} : \mathbb{R} \times \tilde{\Omega} \rightarrow \mathcal{S}^{1+n}$ by

$$\tilde{A}(x, y) = \begin{cases} I, & \text{if } (x, y) \in \mathbb{R} \times \overline{\Omega}, \\ (J(x, y)J(x, y)^T)^{-1}, & \text{if } (x, y) \in \mathbb{R} \times (\tilde{\Omega} \setminus \overline{\Omega}), \end{cases}$$

where $J(x, y)$ denotes the Jacobian matrix of \mathcal{R} at (x, y) . By considering a smaller $\tilde{\Omega}$, one can conclude that \tilde{A} is uniformly Lipschitz continuous and elliptic in $\mathbb{R} \times \tilde{\Omega}$. Additionally, the functions $\tilde{A}(x, y)$, $\tilde{\beta}(x, y)$ and $\tilde{h}(x, y)$ are measurable and uniformly bounded.

Lemma 4.3. *There exists a constant $C > 1$ such that for any $r > 0$,*

$$\varphi_r(x, y) \leq e^{(|x|+2) \ln C}, \quad \forall (x, y) \in [-r, r] \times \overline{\Omega}.$$

Proof. In the appendix of [5], there exist a function $\tilde{\varphi}_r \in W^{2,p}((-r, r) \times \tilde{\Omega}) \cap C^1([-r, r] \times \overline{\Omega}, (0, \infty))$ such that $\tilde{\varphi}_r|_{[-r, r] \times \overline{\Omega}} = \varphi_r$ and $\tilde{\varphi}_r$ satisfies the following elliptic equation

$$\operatorname{div}(d\tilde{A}(x, y)\nabla\tilde{\varphi}_r) - \tilde{\beta}(x, y) \cdot \nabla\tilde{\varphi}_r + c\partial_x\tilde{\varphi}_r + (\tilde{h}(x, y) + \lambda(r))\tilde{\varphi}_r = 0, \quad (x, y) \in (-r, r) \times \tilde{\Omega}.$$

By repeatedly applying the Harnack inequality to the translation in x of $\tilde{\varphi}_r(x, y)$, that is, to $\tilde{\varphi}_r(x + \tau, y)$ for $|\tau| < r - 1$, one can show that there exists a constant $C > 1$ independent of r such that for any $r > 0$ and $\tau \in \mathbb{R}$ with $|\tau| < r - 1$, there holds

$$\sup_{(x, y) \in [\tau - 1, \tau + 1] \times \overline{\Omega}} \tilde{\varphi}_r(x, y) \leq C^{|\tau|+1}.$$

In particular, by the arbitrariness of τ and the choice of C and $\tilde{\varphi}_r$, it is easy to see that for any $r > 0$,

$$\varphi_r(x, y) \leq C^{(|x|+2)} = e^{(|x|+2) \ln C}, \quad \forall (x, y) \in [-r, r] \times \overline{\Omega}. \quad \square$$

Since $\varphi_r \rightarrow \varphi$ in $C^1_{\text{loc}}(\mathbb{R} \times \overline{\Omega})$, we directly obtain the following conclusion, which allows us to apply the Phragmén-Lindelöf maximum principle in the subsequent proof.

Proposition 4.1. *There exists a constant $C > 1$ such that*

$$\varphi(x, y) \leq e^{(|x|+2) \ln C}, \quad \forall (x, y) \in \mathbb{R} \times \overline{\Omega}.$$

Let $C = BC(\mathbb{R} \times \overline{\Omega}, \mathbb{R})$, and $\mathbb{L}[t, \cdot; h]$ be the solution map of equation (4.1) with initial data $\phi \in C$. Denote by $\rho_{\mathbb{L}[1, \cdot; h]}$ the asymptotic spectral radius of $\mathbb{L}[1, \cdot; h]$.

Next, we establish the following relation between the asymptotic spectral radius $\rho_{\mathbb{L}[1, \cdot; h]}$ and the generalized principal eigenvalue λ_1 . The inequality $\rho_{\mathbb{L}[1, \cdot; h]} \leq e^{-\lambda_1}$ follows from Proposition 4.1 together with the Phragmén-Lindelöf maximum principle. The reversed inequality can be proved by the arguments similar to those in [12, Proposition 4.2] and [16, Proposition 4.1]. For the sake of completeness, we provide a detailed derivation here.

Proposition 4.2. *There holds $\rho_{\mathbb{L}[1, \cdot; h]} = e^{-\lambda_1}$.*

Proof. We first prove $\rho_{\mathbb{L}[1, \cdot; h]} \leq e^{-\lambda_1}$. Although φ is generally not bounded, we still denote $\mathbb{L}[t, \varphi; h]$ as satisfying equation (4.1). By (4.3), we have $\mathbb{L}[t, \varphi; h] = e^{-\lambda_1 t} \varphi$. For any $\varrho > 0$, let $\mathbb{L}[t, \zeta_\varrho \varphi; h]$ be the solution of equation (4.1) with initial data $u(0, \cdot, \cdot) = \zeta_\varrho \varphi$, where $\varrho = r + 1$ and $\zeta_\varrho(x) := \max\{0, \min\{1, \varrho - |x|\}\}$ for all $x \in \mathbb{R}$. Then $\mathbb{L}_\varrho[t, \widehat{\varphi}_r; h] = \mathbb{L}[t, \zeta_\varrho \widehat{\varphi}_r; h]|_{[-\varrho, \varrho] \times \overline{\Omega}}$. Since $\zeta_\varrho(x)\varphi(x, y) \leq \varphi(x, y)$ for all $(x, y) \in \mathbb{R} \times \overline{\Omega}$, it then follows from Proposition 4.1 and the Phragmén-Lindelöf maximum principle that

$$\mathbb{L}[t, \zeta_\varrho \varphi; h](x, y) \leq \mathbb{L}[t, \varphi; h](x, y), \quad \forall (t, x, y) \in [0, \infty) \times \mathbb{R} \times \overline{\Omega}.$$

This, together with $\mathbb{L}[t, \varphi; h] = e^{-\lambda_1 t} \varphi$, implies that

$$\mathbb{L}[t, \zeta_\varrho \varphi; h]|_{[-\varrho, \varrho] \times \overline{\Omega}} \leq \mathbb{L}[t, \varphi; h]|_{[-\varrho, \varrho] \times \overline{\Omega}} = e^{-\lambda_1 t} \varphi|_{[-\varrho, \varrho] \times \overline{\Omega}}.$$

Particularly, $\mathbb{L}_\varrho[1, \varphi|_{[-\varrho, \varrho] \times \overline{\Omega}}; h] \leq e^{-\lambda_1} \varphi|_{[-\varrho, \varrho] \times \overline{\Omega}}$. This, combined with [17, Theorem 16.2] implies that $\rho(\mathbb{L}_\varrho[1, \cdot; h]) \leq e^{-\lambda_1}$. Letting $\varrho \rightarrow \infty$, yields that

$$\rho_{\mathbb{L}[1, \cdot; h]} = \lim_{\varrho \rightarrow \infty} \rho(\mathbb{L}_\varrho[1, \cdot; h]) \leq e^{-\lambda_1}.$$

Next, we prove $\rho_{\mathbb{L}[1, \cdot; h]} \geq e^{-\lambda_1}$ by appropriately adapting the proof of [12, Proposition 4.2] and [16, Proposition 4.1]. For any $r > 0$, let $\Phi_r(t)[\varphi_r]$ denote the solution of the initial-boundary value problem

$$\begin{cases} \partial_t u = Lu, & (t, x, y) \in (0, \infty) \times (-r, r) \times \Omega, \\ \partial_\nu u(t, x, y) = 0, & (t, x, y) \in (0, \infty) \times (-r, r) \times \partial\Omega, \\ u(t, x, y) = 0, & (t, x, y) \in (0, \infty) \times \{\pm r\} \times \overline{\Omega}, \\ u(0, x, y) = \varphi_r(x, y), & (x, y) \in [-r, r] \times \overline{\Omega}. \end{cases}$$

It follows directly from (4.2) that $\Phi_r(t)[\varphi_r] = e^{-\lambda(r)t} \varphi_r$. Define $\widehat{\varphi}_r : \mathbb{R} \times \overline{\Omega} \rightarrow \mathbb{R}_+$ by

$$\widehat{\varphi}_r(x, y) = \begin{cases} \varphi_r(x, y), & (x, y) \in [-r, r] \times \overline{\Omega}, \\ 0, & \text{otherwise.} \end{cases}$$

We also denote $\widehat{\varphi}_r|_{[-r, r] \times \overline{\Omega}}$ by $\widehat{\varphi}_r$ when there is no confusion. Let $\mathbb{L}[t, \zeta_\varrho \widehat{\varphi}_r; h]$ be the solution of equation (4.1) with initial data $u(0, \cdot, \cdot) = \zeta_\varrho \widehat{\varphi}_r = \widehat{\varphi}_r$. By the definition of ζ_ϱ and comparison principle, we obtain

$$\mathbb{L}[t, \zeta_\varrho \widehat{\varphi}_r; h](x, y) \geq \Phi_r(t)[\varphi_r](x, y), \quad \forall (t, x, y) \in \mathbb{R}_+ \times [-r, r] \times \overline{\Omega}.$$

In particular, $\mathbb{L}_\varrho[1, \widehat{\varphi}_r; h] \geq e^{-\lambda(r)} \widehat{\varphi}_r$. By virtue of [29, Proposition 3] (or [17, Theorem 16.1], [47, Lemma 3.1]), we have

$$\rho(\mathbb{L}_\varrho[1, \cdot; h]) \geq e^{-\lambda(r)}.$$

Combining with the definition of the asymptotic spectral radius and letting $r \rightarrow \infty$, we then obtain

$$\rho_{\mathbb{L}[1, \cdot; h]} = \lim_{\varrho \rightarrow \infty} \rho(\mathbb{L}_\varrho[1, \cdot; h]) \geq \lim_{r \rightarrow \infty} e^{-\lambda(r)} = e^{-\lambda_1}.$$

The proof is complete. \square

Remark 4.1. We make the following remarks.

- (i) For the case where $N > 1$ or $m > 1$, the theoretical framework can be established by extending the definition of the generalized principal eigenvalue as in [5] and following the proof techniques of Theorem 3.1 and Proposition 1 in [5]. Based on this, it can be further shown that Proposition 4.2 in this paper also holds for reaction-diffusion systems. A detailed analysis will be presented in future work.
- (ii) According to Section 3.2 in [5], we know that $\lambda_1 = \lambda_0 + \frac{c^2}{4d}$, where λ_0 is the generalized principal eigenvalue of the operator $-L_0 = -d\Delta - h(x, y)$ in $\mathbb{R} \times \Omega$, and can define the critical speed as

$$c_0 = \begin{cases} 2\sqrt{-d\lambda_0}, & \text{if } \lambda_0 < 0, \\ 0, & \text{otherwise.} \end{cases}$$

These, combined with the relation $\rho_{\mathbb{L}[1, \cdot; h]} = e^{-\lambda_1}$, lead to the conclusion that if $c > c_0$, $\rho_{\mathbb{L}[1, \cdot; h]} < 1$; if $c < c_0$, $\rho_{\mathbb{L}[1, \cdot; h]} > 1$.

Acknowledgments

This work was completed when Qian Guo was visiting the University of Western Ontario (UWO) supported by the International Program Fund (IPF) of Sun Yat-sen University for doctoral student. She would like to thank the IPF program at Sun Yat-sen University for its support, and is also grateful to the staff of the Department of Mathematics at UWO for their hospitality and help, and thank UWO for its excellent facilities and support during her visit.

Appendix A

In Sections 2 and 3, the operators $T(t)$ and $\overline{T}(t)$ play an crucial role. In this appendix, we confirm those properties of these two operators in the previous sections.

We begin by reviewing their definitions. Let $G_i(t, y, \bar{y})$ be the Green's function of the linear equation

$$\begin{cases} \partial_t u_i(t, y) = d_{2i} \Delta_y u_i(t, y), & (t, y) \in (0, \infty) \times \Omega, \\ \partial_\nu u_i(t, y) = 0, & (t, y) \in (0, \infty) \times \partial\Omega. \end{cases}$$

Consider the linear equation

$$\begin{cases} \partial_t u_i(t, x, y) = d_{1i} \Delta_x u_i(t, x, y) + d_{2i} \Delta_y u_i(t, x, y), & (t, x, y) \in (0, \infty) \times \mathbb{R}^m \times \Omega, \\ \partial_\nu u_i(t, x, y) = 0, & (t, x, y) \in (0, \infty) \times \mathbb{R}^m \times \partial\Omega, \\ u_i(0, x, y) = \phi_i(x, y), & (x, y) \in \mathbb{R}^m \times \bar{\Omega}. \end{cases} \quad (\text{A.1})$$

Then $T(t) = (T_1(t), T_2(t), \dots, T_N(t))$ is defined as the semi-flow of (A.1), that is,

$$T_i(t)[\phi_i](x, y) = \begin{cases} \phi_i(x, y), & t = 0, (x, y) \in \mathbb{R}^m \times \bar{\Omega}, \\ \frac{1}{(4\pi d_{1i} t)^{\frac{m}{2}}} \int_{\mathbb{R}^m \times \Omega} e^{-\frac{\|x-\bar{x}\|^2}{4d_{1i}t}} G_i(t, y, \bar{y}) \phi_i(\bar{x}, \bar{y}) d\bar{x} d\bar{y}, & t > 0, (x, y) \in \mathbb{R}^m \times \bar{\Omega}. \end{cases}$$

Similarly, $\bar{T}(t)[\phi] = (\bar{T}_1(t)[\phi_1], \dots, \bar{T}_N(t)[\phi_N])^T$ is the semi-flow of the linear system

$$\begin{cases} \partial_t u_i = d_{1i} \Delta_x u_i + d_{2i} \Delta_y u_i + c\xi \cdot \nabla_x u_i, & (t, x, y) \in (0, \infty) \times \mathbb{R}^m \times \Omega, \\ \partial_\nu u_i(t, x, y) = 0, & (t, x, y) \in (0, \infty) \times \mathbb{R}^m \times \partial\Omega, \\ u_i(0, x, y) = \phi_i(x, y), & (x, y) \in \mathbb{R}^m \times \bar{\Omega}, \end{cases}$$

that is,

$$\bar{T}_i(t)[\phi_i](x, y) = \begin{cases} \phi_i(x, y), & t = 0, (x, y) \in \mathbb{R}^m \times \bar{\Omega}, \\ \frac{1}{(4\pi d_{1i} t)^{\frac{m}{2}}} \int_{\mathbb{R}^m \times \Omega} e^{-\frac{\|x+c\xi t-\bar{x}\|^2}{4d_{1i}t}} G_i(t, y, \bar{y}) \phi_i(\bar{x}, \bar{y}) d\bar{x} d\bar{y}, & t > 0, (x, y) \in \mathbb{R}^m \times \bar{\Omega}. \end{cases}$$

From the definitions of $T(t)$ and $\bar{T}(t)$, and combining Corollary 7.2.4 in [30] and Lemma 5.9 in [36], we can confirm the following properties for $T(t)$.

Lemma A.1. *The following statements are valid:*

- (i) $T(t)[C_{\mathbf{r}}] \subseteq C_{\mathbf{r}}$ for all $(t, \mathbf{r}) \in (0, \infty) \times \text{Int}(\mathbb{R}_+^N)$.
- (ii) $T(t)[C_{\mathbf{r}}]$ is precompact in C for all $(t, \mathbf{r}) \in (0, \infty) \times \text{Int}(\mathbb{R}_+^N)$.
- (iii) $\mathbb{R}_+ \times C_{\mathbf{r}} \ni (t, \phi) \mapsto T(t)[\phi] \in C_+$ is continuous.

Proof. Define $S_t^1 : \mathbb{R}_+ \times C(\bar{\Omega}, \mathbb{R}_+^N) \rightarrow C(\bar{\Omega}, \mathbb{R}_+^N)$ and $S_t^2 : \mathbb{R}_+ \times BC(\mathbb{R}^m, \mathbb{R}_+^N) \rightarrow BC(\mathbb{R}^m, \mathbb{R}_+^N)$ by

$$S_t^1[\tilde{\phi}](y) = \int_{\Omega} G(t, y, \bar{y}) \tilde{\phi}(\bar{y}) d\bar{y}, \quad S_t^2[\tilde{\psi}](x) = \int_{\mathbb{R}^m} K(t, x - \bar{x}) \tilde{\psi}(\bar{x}) d\bar{x},$$

where $G(t, y, \bar{y}) = (G_1(t, y, \bar{y}), \dots, G_N(t, y, \bar{y}))^T$, $K(t, x - \bar{x}) = (K_1(t, x - \bar{x}), \dots, K_N(t, x - \bar{x}))^T$ with

$$K_i(t, x - \bar{x}) = \frac{1}{(4\pi d_{1i}t)^{\frac{m}{2}}} \exp\left(-\frac{\|x - \bar{x}\|^2}{4d_{1i}t}\right).$$

Let $\psi(t, x, y) = S_t^1[\phi(x, \cdot)](y)$ for $(t, \phi) \in \mathbb{R}_+ \times C_+$. Then we have $T(t)[\phi](x, y) = S_t^2[\psi(t, \cdot, y)](x)$.

(i) For any $(t, \mathbf{r}) \in (0, \infty) \times \text{Int}(\mathbb{R}_+^N)$, by applying Corollary 7.2.4 in [30] to S_t^1 , we obtain $S_t^1[C(\bar{\Omega}, [\mathbf{0}, \mathbf{r}]_{\mathbb{R}^N})] \subseteq C(\bar{\Omega}, [\mathbf{0}, \mathbf{r}]_{\mathbb{R}^N})$. Moreover, a slight extension of Lemma 5.9 in [36] yields $S_t^2[C(\mathbb{R}^m, [\mathbf{0}, \mathbf{r}]_{\mathbb{R}^N})] \subseteq C(\mathbb{R}^m, [\mathbf{0}, \mathbf{r}]_{\mathbb{R}^N})$. Since $T(t) = S_t^2 \circ S_t^1$, it follows that $T(t)[C_{\mathbf{r}}] \subseteq C_{\mathbf{r}}$.

(ii) Fix $(t, \mathbf{r}) \in (0, \infty) \times \text{Int}(\mathbb{R}_+^N)$. By (i) and the Arzelà-Ascoli theorem, we only need to show that for any $R > 0$, the set $\{T(t)[\phi] | \overline{B_R(0)} \times \bar{\Omega} : \phi \in C_{\mathbf{r}}\}$ is a family of equicontinuous functions in $C(\overline{B_R(0)} \times \bar{\Omega}, \mathbb{R}^N)$. Indeed, for any $\varepsilon > 0$, there exists $\tilde{R} = R(\varepsilon) > 0$ such that

$$\left\| \int_{\mathbb{R}^m \setminus B_{\tilde{R}}(0)} K(t, x - \bar{x}) \, d\bar{x} \right\| < \frac{\varepsilon}{8\|\mathbf{r}\| + 1}, \quad \forall x \in \mathbb{R}^m.$$

From the definition of $T(t)$, it follows that for any $\phi \in C_{\mathbf{r}}$, $(x_1, y_1), (x_2, y_2) \in \overline{B_R(0)} \times \bar{\Omega}$ satisfying $\|(x_1, y_1) - (x_2, y_2)\| < \delta$, we have

$$\begin{aligned} & \|T(t)[\phi](x_1, y_1) - T(t)[\phi](x_2, y_2)\| \\ &= \|S_t^2[\psi(t, \cdot, y_1)](x_1) - S_t^2[\psi(t, \cdot, y_2)](x_2)\| \\ &\leq \|S_t^2[\psi(t, \cdot, y_1)](x_1) - S_t^2[\psi(t, \cdot, y_1)](x_2)\| + \|S_t^2[\psi(t, \cdot, y_1)](x_2) - S_t^2[\psi(t, \cdot, y_2)](x_2)\|. \end{aligned}$$

By appropriately adapting the proof of Lemma 5.9 in [36], we see that

$$\|S_t^2[\psi(t, \cdot, y_1)](x_1) - S_t^2[\psi(t, \cdot, y_1)](x_2)\| < \frac{\varepsilon}{2}.$$

Set $R^* = \max\{R, \tilde{R}\}$. Then by the properties of $G(t, y, \bar{y})$, for any $x \in B_{R^*}(0)$, there holds

$$\|S_t^1[\phi(x, \cdot)](y_1) - S_t^1[\phi(x, \cdot)](y_2)\| < \frac{\varepsilon}{4\sqrt{N}}.$$

By direct computations we have

$$\begin{aligned} & \left\| S_t^2[\psi(t, \cdot, y_1)](x_2) - S_t^2[\psi(t, \cdot, y_2)](x_2) \right\| \\ &= \left\| \int_{\mathbb{R}^m} K(t, x_2 - \bar{x}) [\psi(t, \bar{x}, y_1) - \psi(t, \bar{x}, y_2)] \, d\bar{x} \right\| \\ &\leq \left\| \int_{B_{R^*}(0)} K(t, x_2 - \bar{x}) [\psi(t, \bar{x}, y_1) - \psi(t, \bar{x}, y_2)] \, d\bar{x} \right\| \end{aligned}$$

$$\begin{aligned}
 & + \left\| \int_{\mathbb{R}^m \setminus B_{R^*}(0)} K(t, x_2 - \bar{x}) [\psi(t, \bar{x}, y_1) - \psi(t, \bar{x}, y_2)] \, d\bar{x} \right\| \\
 & < \frac{\varepsilon}{4\sqrt{N}} \left\| \int_{B_{R^*}(0)} K(t, x_2 - \bar{x}) \, d\bar{x} \right\| + 2\|\mathbf{r}\| \left\| \int_{\mathbb{R}^m \setminus B_{R^*}(0)} K(t, x_2 - \bar{x}) \, d\bar{x} \right\| \\
 & \leq \frac{\varepsilon}{4\sqrt{N}} \cdot \sqrt{N} + 2\|\mathbf{r}\| \cdot \frac{\varepsilon}{8\|\mathbf{r}\| + 1} < \frac{\varepsilon}{2}.
 \end{aligned}$$

Thus, $\|T(t)[\phi](x_1, y_1) - T(t)[\phi](x_2, y_2)\| < \varepsilon$, which completes the proof of (ii).

(iii) Take $\{(t_k, \phi_k)\}_{k \in \mathbb{N}} \subset \mathbb{R}_+ \times C_{\mathbf{r}}$, and $(t, \phi) \in \mathbb{R}_+ \times C_{\mathbf{r}}$ such that $\lim_{k \rightarrow \infty} |t_k - t| = 0$, $\lim_{k \rightarrow \infty} \|\phi_k - \phi\|_C = 0$. We show that for any $R > 0$, there holds

$$\lim_{k \rightarrow \infty} \left(\sup\{\|T(t_k)[\phi_k](x, y) - T(t)[\phi](x, y)\| : (x, y) \in \overline{B_R(0)} \times \overline{\Omega}\} \right) = 0.$$

Without loss of generality, we may assume that $t_k \leq t + 1$ for all $k \in \mathbb{N}$. Note that for any $\varepsilon > 0$, there exists $\tilde{R} = R(\varepsilon) > R$ such that

$$\left\| \int_{\mathbb{R}^m \setminus B_{\tilde{R}}(0)} \sup_{t' \in (0, t+1)} K(t', x - \bar{x}) \, d\bar{x} \right\| < \frac{\varepsilon}{6\|\mathbf{r}\| + 1}, \quad \forall x \in \mathbb{R}^m.$$

By appropriately extending Lemma 5.9 in [36] and noting that $\{\psi(t, \cdot, y) : y \in \overline{\Omega}\}$ is precompact in $C(\mathbb{R}^m, \mathbb{R}^N)$, we know that there exists $k_0 > 0$ such that for any $k \geq k_0$, there holds

$$\|S_{t_k}^2[\psi(t, \cdot, y)](x) - S_t^2[\psi(t, \cdot, y)](x)\| < \frac{\varepsilon}{3}, \quad \forall (x, y) \in \overline{B_R(0)} \times \overline{\Omega}.$$

Let $\psi_k(t_k, x, y) = S_{t_k}^1[\phi_k(x, \cdot)](y)$ and $R^* = \max\{R, \tilde{R}\}$. It is easy to verify that there exists $k_1 > 0$ such that for any $k \geq k_1$, there holds

$$\|\psi_k(t_k, x, y) - \psi(t, x, y)\| < \frac{\varepsilon}{3\sqrt{N}}, \quad \forall (x, y) \in \overline{B_{R^*}(0)} \times \overline{\Omega}.$$

Thus, for any $k \geq \max\{k_0, k_1\}$, $(x, y) \in \overline{B_R(0)} \times \overline{\Omega}$, we have

$$\begin{aligned}
 & \|T(t_k)[\phi_k](x, y) - T(t)[\phi](x, y)\| \\
 & \leq \left\| S_{t_k}^2[\psi_k(t_k, \cdot, y)](x) - S_{t_k}^2[\psi(t, \cdot, y)](x) \right\| + \left\| S_{t_k}^2[\psi(t, \cdot, y)](x) - S_t^2[\psi(t, \cdot, y)](x) \right\| \\
 & < \frac{\varepsilon}{3\sqrt{N}} \left\| \int_{B_{R^*}(0)} K(t_k, x - \bar{x}) \, d\bar{x} \right\| + 2\|\mathbf{r}\| \left\| \int_{\mathbb{R}^m \setminus B_{R^*}(0)} K(t_k, x - \bar{x}) \, d\bar{x} \right\| + \frac{\varepsilon}{3} \\
 & < \frac{\varepsilon}{3\sqrt{N}} \cdot \sqrt{N} + 2\|\mathbf{r}\| \cdot \frac{\varepsilon}{6\|\mathbf{r}\| + 1} + \frac{\varepsilon}{3} < \varepsilon.
 \end{aligned}$$

This implies $\mathbb{R}_+ \times C_{\mathbf{r}} \ni (t, \phi) \mapsto T(t)[\phi] \in C_+$ is continuous. \square

Remark A.1. Note that $\bar{T}(t)[\phi] = \mathcal{T}_{-ct\xi} \circ T(t)[\phi]$ for all $(t, \phi) \in [0, \infty) \times C_+$. Thus, $\bar{T}(t)$ also satisfies properties (i)–(iii).

Data availability

No data was used for the research described in the article.

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