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Propagation and heterogeneous steady states in a delayed nonlocal R-D equation without spatial translation-invariance ☆

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Abstract

We consider a delayed reaction-diffusion equation that models the population dynamics of a single species with the mature population living in the 1-D *whole* space $\mathbb{R}$ while the immature population only living in the *half* space $\mathbb{R}_+$, with homogeneous Dirichlet condition for the immatures at the boundary point. One of the important features of this model system is that it does have the translational-invariance. By linking the *non-translational-invariant* solution map for this equation to travelling wave maps for another related 1-D spatial homogeneous delay reaction-diffusion equation, we obtain some traveling-like *a priori* estimates for nontrivial solutions. We then establish the existence, uniqueness, and attractivity of heterogeneous steady states. As a result, we are able to describe the traveling-like asymptotic behaviours of nontrivial solutions in space-time region. These enable us to develop a new method for exploring the spreading speeds and asymptotic propagation phenomena for a class of *non-translation-invariant* delay reaction-diffusion equations on $\mathbb{R}$. As a corollary, we also recover some results on the asymptotic spreading speeds and traveling waves for monostable and spatial homogeneous delay reaction-diffusion equations in $\mathbb{R}$.

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1. Introduction

When studying the population dynamics of a species that has obvious age structure and is significantly mobile in the habitat, one is typically and naturally led to reaction diffusion equations of the form

$$\frac{\partial u}{\partial t}(t,x) = D_m \Delta u(t,x) - d_m u(t,x) + \varepsilon \int_{\Omega} \Gamma(\vartheta, x, y) b(u(t-\tau, y))\, dy, \ \ (t,x) \in (0,\infty) \times \Omega. \tag{1.1}$$

Here $u(t,x)$ is the mature population of the species at time t and location x, D_m and d_m are the diffusion rate and death rate of the mature population, τ is the maturation time for the species. The other two indirect parameters ε and ϑ are defined by $\varepsilon = exp(-\int_0^\tau d_I(a)da)$ $\vartheta = \int_0^\tau D_I(a)da$ where $D_I(a),\ d_I(a),\ a \in [0,\tau]$ are the age dependent diffusion rate and death rate of the immature individuals and hence, ε actually accounts for probability of surviving the immature period and ϑ measures the mobility of the immatures. Here, $b : \mathbb{R}_+ \to \mathbb{R}_+$ is the birth function of the species.

As for the kernel $\Gamma(\vartheta, x, y)$, it explains the probability that an individual born at location y will have dispersed to location x when becoming mature (τ time units later), provided that it can survive the immature period. Its form depends on the spatial domain Ω, and also on the conditions posed on the boundary of Ω when it has a boundary. For example, when Ω is a bounded 1-D domain (interval), [23] derived the forms of the kernel corresponding to various boundary conditions; when Ω is a general bounded domain in $\mathbb{R}^n$ and the homogeneous Robin boundary condition is posed, [51] obtained the kernel expressed as a Green function. When $\Omega = \mathbb{R}$, [39] derived $\Gamma(\vartheta, x, y)$ to be $\Gamma(\vartheta, x, y) = \Gamma_\vartheta(x-y)$ where $\Gamma_\vartheta(z)$ is the heat kernel (Gaussian function)

$$\Gamma_\vartheta(z) = \frac{1}{\sqrt{4\pi\vartheta}} e^{-z^2/4\vartheta}. \tag{1.2}$$

For more details about the background of such spatially non-local dynamic models, see [23,39,51,62] or the survey by Gourley and Wu [16].

We would particularly like to mention the case $\Omega = (0,\infty)$ in which, the spatial domain Ω is unbounded but is not the whole space, and hence, it has a boundary at $x = 0$. Such a spatial domain accounts for a scenario of species with both mature and immature individuals live and move, for example, in a big land that has a shore of ocean or lake on one side of the land, or species living in a large area of water (ocean or lake) with an obvious shore boundary on one side. For such a semi-infinite domain with a homogeneous Dirichlet boundary condition at $x = 0$ (hostile boundary), the kernel becomes $\Gamma(\vartheta, x, y) = \Gamma_\vartheta(x-y) - \Gamma_\vartheta(x+y)$. See [62] for more details about the model and the dynamics of the model.

We note that in all those spatially non-local reaction diffusion models mentioned above represented by (1.1), it is assumed that *both the matures and the immatures* live and diffuse in the *same spatial region*. However, in the real world there are biological species whose matures and immatures have different regions to live and diffuse. For instance, for an amphibious animal, its juveniles can only live in the water (lake or ocean), while its adults can live and diffuse both in the water (lake or ocean) and land (so that they can have more food resources). In [57], for the case of *bounded domain* Ω *for the immatures*, a general model has been derived to account for the differences of juvenile and adult habitat regions, assuming a hostile condition (homogeneous Dirichlet condition) on the boundary of the corresponding regions. By developing domain decomposition methods, Yi and Chen in [57] have obtained the existence and global attractivity of a positive steady-state solution, and the persistence of other solutions to the Dirichlet problem under the usual supremum norm, *regardless* of the mature population living in a *bounded or unbounded* environment. However, *when the living environment* Ω *of immature population is unbounded*, some completely different and difficult mathematical problems arise, and these motivate this paper.

In this paper, motivated by [57,62], we consider the case of adults individuals living in $\mathbb{R} = \mathbb{R}_- \cup \mathbb{R}_+$ while juveniles living in $\mathbb{R}_+$ with the *hostile* boundary characteristics at $x = 0$. Following the same procedure in deriving (1.1) in [39,57,62], but noting the difference in the spatial domain for immatures, we can obtain the following initial value problem (IVP) of nonlocal delayed reaction-diffusion equations on $\mathbb{R}$ for the mature population:

$$\begin{cases} \dfrac{\partial u}{\partial t} = D_m \dfrac{\partial^2 u}{\partial x^2} - d_m u + \begin{cases} \varepsilon \int_0^\infty b(u(t-\tau, y))[\Gamma_\vartheta(x-y) - \Gamma_\vartheta(x+y)]\mathrm{d}y, \\ \quad (t,x) \in (0,\infty) \times \mathbb{R}_+, \\ 0, \quad (t,x) \in (0,\infty) \times (-\infty, 0); \end{cases} \\ u_0 = \varphi \in C_+, \end{cases} \tag{1.3}$$

and $\varphi : [-\tau, 0] \times \mathbb{R} \to \mathbb{R}_+$ is a bounded and continuous function. Here the meanings of unknown $u(t,x)$ and all parameters remain the same as for (1.1).

To stimulate our study on (1.3), let us have a quick review of some works on (1.1). When the spatial domain Ω is *bounded*, various boundary conditions can be posed for (1.1), leading to different kernel $\Gamma(\theta, x, y)$ (see, e.g., [23,51]), among which are the homogeneous Neumann boundary value condition or zero flux condition (NBVC) and the homogeneous Dirichlet boundary value condition (DBVC) with the former accounting for an isolated domain and the latter explaining a scenario that the boundary is hostile for the species. The global dynamics of the semiflow generated by such models subject to either the Dirichlet or the Neumann boundary condition have been intensively and successfully studied (see, e.g., [6,13,17,20,38,40,49,52,53, 58–60,63]).

When the spatial domain Ω is *unbounded*, the existence and other qualitative properties of traveling wave solutions, as well as the theory of asymptotic spreading of (1.1) are the main concerns, and have been investigated by many authors under some particular forms of the kernel function $\Gamma(\tau, x, y)$ (see, e. g., [1,10,11,9,14,15,24,25,29,33,35,39,41,42,44,45,50,55,64] and the references therein). The study on traveling waves can be traced back to the celebrated papers of Fisher [12] and Kolmogorov *et al.* [21], while the study of asymptotic propagation was pioneered by Aronson and Weinberger [2]. Since solutions of initial value problems of reaction-diffusion equations can be considered as solutions to some discrete dynamical systems in appropriate spaces, Weinberger [46] and Lui [28] established the theory of spreading speeds and monostable traveling wavefronts for monotone *discrete* dynamical systems. This theory has been further developed recently in [8,24,47,61] for more general *monotone/non-monotone* semiflows so that it can be applied to a variety of discrete and continuous time evolution equations in homogeneous or periodic media. Moreover, Berestycki and Hamel [3] and Berestycki *et al.* [4] have obtained the existence and the minimal wave speed of pulsating fronts, and explored the notion of asymptotic spreading speed for a general *periodic* framework together with the Neumann boundary conditions on the boundary of a periodic domain.

However, very little attention has been paid to evolution equations for spatially heterogeneous but non-periodic cases. By applying some Hanack inequality, Berestycki *et al.* [5] have dealt with various notions of asymptotic spreading speeds for solutions with compactly supported initial data about the Neumann problem of Kolmogorov-Petrovsky-Piskunov type equations in non-periodic domains. By making links between travelling wave maps and solution maps, Yi and Chen [56] explored the spreading speed and asymptotic propagation phenomena for the Dirichlet boundary value problem of reaction-diffusion equations with Kolmogorov-Petrovsky-Piskunov type nonlinearities on $\mathbb{R}_+$.

In this paper, we consider equation (1.3) with *unimodal nonlinearity*. For unimodal nonlinear reaction terms, global stability is of great interest when the spatial domain is bounded; while for the case of *unbounded domain*, traveling wave solutions of various forms and asymptotic propagation characteristics are of great interest and significance, and are thus the main concerns. We point out that the existence of (periodic) traveling wave solutions heavily depends on the spatial (periodic) *translation invariance* of the equation and the domain. However, for equation (1.3), the kernel function $\Gamma(\vartheta, x, y) = \Gamma_\vartheta(x-y) - \Gamma_\vartheta(x+y)$ is *neither spatial periodic no translation invariant*. This implies that the solution map of (1.3) is *neither spatially periodic no translation invariant*, and therefore, the theory and method of Weinberger *et al.* cannot be applied directly to such Dirichlet problems. On the other hand, because the Harnack inequality cannot be extended to the boundary for Dirichlet problems and solutions for Dirichlet problems are zeros on the boundary, the methods of Berestycki *et al.* [5] are not feasible for Dirichlet problems in general domains. Fortunately, Yi and Chen [56] have provided a new class of methods to study asymptotic propagation for Dirichlet problem of KPP equations by using the iterative characteristics of travelling wave maps and integral operators associated with the Dirichlet diffusion kernel. It is natural to ask whether it is possible to, by developing the methods in [56], study the existences of heterogeneous steady states and the asymptotic spreading of other solutions of (1.3). This paper will seek answers to this question.

The rest of this paper is organized as follows. In Section 2, we investigate some basic properties of (1.3). In Section 3, as direct consequence of Theorem 3.6 in [61], we first establish some asymptotic properties of travelling wave maps with *non-symmetric* spatial kernels that are associated to the symmetric equation

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = D_m \Delta u(t,x) - d_m u(t,x) + \varepsilon \displaystyle\int_{\mathbb{R}} \Gamma_\vartheta(x-y) b(u(t-\tau,y))\,dy, & (t,x) \in (0,\infty)\times\mathbb{R} \\ u(t,x) = \varphi(t,x), & (t,x) \in [-\tau, 0]\times\mathbb{R}. \end{cases} \tag{1.4}$$

By linking travelling wave maps associated with (1.4) on $\mathbb{R}$ to another integro-difference equation with *non-symmetric* and *non-translation invariant* spatial kernels for (1.3), we obtain some iteration properties for the heat kernels and nonlocal kernels. In Section 4, with considerable modifications of the methods in [54,62], we give what will be very useful in proving traveling-like *a priori* estimates on nontrivial solutions to (1.3). With these estimates, in Section 5, we then establish the existence, uniqueness, and attractivity of heterogeneous steady states. In Section 6, we explore the traveling-like asymptotic behaviour of nontrivial solutions in space-time region.

This enables us to develop a new method for studying the spreading speeds and asymptotic propagation phenomena for (1.3). As a corollary of our main results, we easily re-establish the results on the travelling waves, spreading speeds and asymptotic propagation for (1.4) on $\mathbb{R}$ obtained in [9,24,39,55]. In Section 7, we apply our main results to two particular birth functions: the Ricker function and the Mackey-Glass hematopoiesis function, leading to some more specific results explicitly in terms of the parameters in this two functions.

2. Preliminary results

We first introduce some notations. Let $\mathbb{R}$, $\mathbb{R}_+$, and $\mathbb{N}$ be the sets of all reals, nonnegative reals, positive integers, respectively. Let $X = C(\mathbb{R}, \mathbb{R}) \cap L^\infty(\mathbb{R}, \mathbb{R})$ and $C = C([-\tau, 0], X)$. Equipped with the usual supremum norm $||\cdot||_X = ||\cdot||_{L^\infty(\mathbb{R},\mathbb{R})}$ and $||\cdot||_C = ||\cdot||_{L^\infty([-\tau,0]\times\mathbb{R},\mathbb{R})}$ respectively, X and C are Banach spaces. Let $X_+ = \{\phi \in X : \phi(x) \geq 0 \text{ for all } x \in \mathbb{R}\}$, $X_{++} = \{\phi \in X_+ : \phi(\mathbb{R}_+) \neq \{0\}\}$, $X_+^\circ = \{\phi \in X : \phi(x) > 0 \text{ for all } x \in \mathbb{R}\}$, $C_+ = \{\varphi \in C : \varphi(\theta, x) \geq 0 \text{ for all } (\theta, x) \in [-\tau, 0] \times \mathbb{R}\}$, $C_{++} = \{\varphi \in C_+ : \varphi([-\tau, 0) \times \mathbb{R}_+ \cup \{0\} \times \mathbb{R}) \neq \{0\}\}$ and $C_+^\circ = \{\varphi \in C : \varphi(\theta, x) > 0 \text{ for all } (\theta, x) \in [-\tau, 0] \times \mathbb{R}\}$. Clearly, X_+, C_+ are closed cones in X, C, respectively. For any $\xi, \eta \in X$ (resp. C), we write $\xi \geq \eta$ if $\xi - \eta \in X_+$ (resp. C_+), $\xi > \eta$ if $\xi \geq \eta$ and $\xi \neq \eta$, $\xi \gg \eta$ if $\xi - \eta \in X_+^\circ$ (resp. C_+°). Moreover, for $\gamma \in X_+$, $X_\gamma = \{\phi \in X_+ : \phi(x) \leq \gamma(x) \text{ for all } x \in \mathbb{R}\}$ and $C_\gamma = \{\varphi \in C_+ : \varphi(\theta, x) \leq \gamma(x) \text{ for all } (\theta, x) \in [-\tau, 0] \times \mathbb{R}\}$. Sometimes, we also write $BC(\mathcal{X}, \mathcal{Y})$ for $C(\mathcal{X}, \mathcal{Y}) \cap L^\infty(\mathcal{X}, \mathcal{Y})$, where $\mathcal{X}, \mathcal{Y}$ are topological spaces. For any $\zeta \in BC(\mathcal{X}, \mathcal{Y})$, we denote supremum norm of ζ by $||\zeta||_{L^\infty}$.

For convenience, we shall also treat an element $\varphi \in C$ as a function from $[-\tau, 0] \times \mathbb{R}$ into $\mathbb{R}$. For any $a \in \mathbb{R}$ or $\phi \in X$, we also use a, ϕ to denote the constant function taking constant value a, ϕ in the corresponding function space, when no confusion arises. So, we sometimes consider $\mathbb{R}, X$ as subsets of X, C, respectively, that is, $\mathbb{R} \subseteq X \subseteq C$.

For an interval $I \subseteq \mathbb{R}$, let $I + [-\tau, 0] = \{t + \theta : t \in I \text{ and } \theta \in [-\tau, 0]\}$. For $u : (I + [-\tau, 0]) \times \mathbb{R} \to \mathbb{R}$ and $t \in I$, we write $u_t(\cdot, \cdot)$ for the function defined by $u_t(\theta, x) = u(t + \theta, x)$ for $(\theta, x) \in [-\tau, 0] \times \mathbb{R}$.

For convenience, by letting $d = D_m$, $\mu = d_m$, and $f(u) = \frac{\varepsilon b(u)}{d_m}$ for all $u \in \mathbb{R}_+$, we transform system (1.3) to the following initial value problem (IVP) of nonlocal delayed reaction-diffusion equations in $\mathbb{R}$ with spatial switch:

$$\begin{cases} \dfrac{\partial u}{\partial t} = d\dfrac{\partial^2 u}{\partial x^2} - \mu u + \begin{cases} \mu \int_0^\infty f(u(t-\tau, y))[\Gamma_\vartheta(x-y) - \Gamma_\vartheta(x+y)]\mathrm{d}y, \\ \quad (t, x) \in (0, \infty) \times \mathbb{R}_+, \\ 0, \quad (t, x) \in (0, \infty) \times (-\infty, 0); \end{cases} \\ u_0 = \varphi \in C_+. \end{cases} \tag{2.1}$$

We will consider the mild solution of system (2.1) which solves the following integral equation with the given initial function,

$$\begin{cases} u(t, \cdot) = T(t)[\varphi(0, \cdot)] + \mu \int_0^t T(t-s)[K[f(u(s-\tau, \cdot))]]\mathrm{d}s, \qquad t \geq 0, \\ \quad u_0 = \varphi \in C_+. \end{cases} \tag{2.2}$$

Here $K : X \to X$ is defined by

$$K[\phi](x) = \begin{cases} \int_0^\infty \phi(y)[\Gamma_\vartheta(x-y) - \Gamma_\vartheta(x+y)]\mathrm{d}y, & x \in \mathbb{R}_+, \\ 0, & x \in (-\infty, 0), \end{cases}$$

and $T(t)$ is the semigroup generated by the linear system,

$$\begin{cases} \frac{\partial u}{\partial t} &= d\Delta u(t,x) - \mu u, & t>0, \\ u(0,x) &= \phi(x), & x \in \mathbb{R}, \end{cases}$$

that is, for $(x,\phi) \in \mathbb{R} \times X$,

$$\begin{cases} T(0)[\phi](x) &= \phi(x), \\ T(t)[\phi](x) &= \frac{\exp(-\mu t)}{\sqrt{4d\pi t}} \int_{-\infty}^{\infty} \phi(y) \exp\left(-\frac{(x-y)^2}{4dt}\right) \mathrm{d}y, \quad t>0. \end{cases} \tag{2.3}$$

For given $\varphi \in C_+$, by the method of steps and the definitions of $K[\cdot]$ and $T(t)$, it is easy to see (2.2) has a unique solution which exists for all $t \geq 0$. Denote by $u^\varphi(t,x)$ the unique solution of (2.2). Then it is clear that $(u^\varphi(\cdot,\cdot))_t \in C_+$ for all $t \geq 0$ and $\varphi \in C_+$ (see, e.g., Martin and Smith [31,32]). Thus, the solution map of (2.2) induces a continuous semiflow in C_+. Since the semigroup $T(t)$ is analytic, we know that a mild solution of (2.1) (i.e., solution of (2.2)) is also a classical solution of (2.1) for all $t > \tau$ when f is continuously differentiable (see, e.g., see [31, 32,43,48]). Therefore, in the sequel, we only need to consider solutions of (2.2).

Denote by Φ the solution semiflow of (2.2), that is, $\Phi : \mathbb{R}_+ \times C_+ \to C_+$ is defined by $\Phi(t,\varphi) = (u^\varphi)_t$ for all $(t,\varphi) \in \mathbb{R}_+ \times C_+$. Sometimes, we also write $\Phi(t,\varphi; f, K)$ for $\Phi(t,\varphi)$ to emphasize the dependence on the nonlinearity f and the nonlocal kernel K, if there is a need.

For any $0 < s < t < \infty$, let us define the mapping $\Phi^s(t,\cdot) : C_+ \ni \varphi \mapsto \Phi^s(t,\cdot)[\varphi] \in L^\infty([-s,0] \times \mathbb{R}, \mathbb{R})$ where $\Phi^s(t,\cdot)[\varphi](\theta,x) = \Phi(t+\theta,\varphi)(0,x)$ for all $(\theta,x) \in [-s,0] \times \mathbb{R}$.

In this section, we establish some basic results on the boundedness, compactness, and comparison principle of the solution maps to (2.1) or (2.2). We first collect some basic properties of $K[\cdot]$ and $T(t)$ established in [7,36,38,54].

Lemma 2.1. *$K[X] \subseteq X$, $K[X_+] \subseteq X_+$, $K[X_+ \setminus X_{++}] = \{0\}$ and $K[X_{++}] \subseteq \{\phi \in X_+ : \phi((-\infty,0]) = \{0\}$ and $\phi((0,\infty)) \subseteq (0,\infty)\} \subseteq X_{++}$.*

Lemma 2.2. *The following statements hold.*

- (i) *$T(t)[X] \subseteq UBC(\mathbb{R},\mathbb{R})$ for all $t > 0$, and $T(t)|_{UBC(\mathbb{R},\mathbb{R})}$ is an analytic and strongly continuous semigroup on X.*
- (ii) *$T(t)[X_+] \subseteq X_+$ for all $t \in \mathbb{R}_+$.*
- (iii) *$T(t)[X_+ \setminus \{0\}] \subseteq X_+^\circ$ and for $t > 0$.*
- (iv) *$||T(t)[\phi]||_X \leq ||\phi||_X e^{-\mu t}$ for all $t > 0$.*
- (v) *If $a, r > 0$, then $\{T(t)[\phi]|_{[-a,a]} : \phi \in X$ with $-r \leq \phi \leq r\}$ is pre-compact in $L^\infty([-a,a],\mathbb{R})$.*

Due to the non-compactness of the spatial domain, it is generally difficult and inconvenient to describe the asymptotic behaviour of solutions to (1.4) with respect to the L^∞-norm. To overcome this difficulty, we use the following norms on X and C, defined respectively by

$$||\phi|| \triangleq \sum_{n=1}^{\infty} 2^{-n} \sup\{|\phi(x)| : |x| \leq n\} \quad \text{for all } \phi \in X,$$

$$||\varphi|| \triangleq \sum_{n=1}^{\infty} 2^{-n} \sup\{|\varphi(\theta, x)| : (\theta, x) \in [-\tau, 0] \times [-n, n]\} \quad \text{for all} \quad \varphi \in C.$$

Similarly, for $a \leq b$, let $C_{a,b} = C([a, b] \times \mathbb{R}, \mathbb{R}) \cap L^{\infty}([a, b] \times \mathbb{R}, \mathbb{R})$, and $||\varphi|| \triangleq \sum_{n=1}^{\infty} 2^{-n} \sup\{|\varphi(\theta, x)| : (\theta, x) \in [a, b] \times [-n, n]\}$ for all $\varphi \in C_{a,b}$. Moreover, we shall still denote the corresponding topological vector spaces $(X, ||\cdot||)$, $(C, ||\cdot||)$, and $(C_{a,b}, ||\cdot||_{a,b})$ by X, C and $C_{a,b}$. Accordingly, we always assume that the tacit topologies on X, C and $C_{a,b}$ are induced by the new norms $||\cdot||$ and $||\cdot||_{a,b}$ in the sequel. In particular, $X = C_{0,0}$ and $C = C_{-\tau,0}$.

In what follows, we shall always assume the following for the nonlinearity f:

(H1) $f : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous with f being continuously differentiable in some right neighborhood of 0 with $f(0) = 0$ and $f'(0) > 1$; moreover, there exists $u^* \in (0, \infty)$ such that $f(u^*) = u^*$, $f(u) > u$ for all $u \in (0, u^*)$, and $0 < f(u) < u$ for all $u \in (u^*, \infty)$.

Under (H1), $\sup\{f(u) : u \in [0, u^*]\}$ exists and is denoted by $\mathcal{M}$ in the rest of this paper. Then, we have the following proposition which establishes the positivity, boundedness, and compactness for the solution semiflow of (2.2).

Proposition 2.1. *The following results hold.*

(i) $\Phi(\mathbb{R}_+ \times C_+) \subseteq C_+$, $\Phi(\mathbb{R}_+ \times (C_+ \setminus C_{++})) = \{0\}$, *and* $\Phi((\tau, \infty) \times C_{++}) \subseteq C_+^{\circ}$.
(ii) $\Phi(\mathbb{R}_+ \times C_{\delta+\mathcal{M}}) \subseteq C_{\delta+\mathcal{M}}$ *for any* $\delta > 0$.
(iii) $\overline{\lim_{t\to\infty}}(\sup\{||\Phi(t, \varphi)||_C : \varphi \in C_{\delta+\mathcal{M}}\}) \leq \mathcal{M}$ *for any* $\delta > 0$.
(iv) $\Phi(t, \cdot)$ *is a continuous semiflow on* $C_{\delta+\mathcal{M}}$ *for any* $\delta > 0$.
(v) *The mapping* $\Phi^s(t, \cdot) : C_{\delta+\mathcal{M}} \to C_{-s,0}$ *is precompact in* C *for any* $0 < s < t$ *and* $\delta > 0$. *In particular,* $\Phi(t, \cdot)|_{C_{\delta+\mathcal{M}}}$ *is precompact in* C *for* $t > \tau$ *and* $\delta > 0$.

Proof. The poofs of (i), (ii) and (iii) follow from (2.2), Lemma 2.1, Lemma 2.2 and (H1). Note that $\Phi(t, \cdot)$ is a semigroup on $C_{\delta+\mathcal{M}}$. By a similar proof to that of Theorem 2.8 in [54], we can show (iv) and (v). The proof is completed. □

By Proposition 2.1-(i), the solution is positive for all $t > 0$ if and only if the initial function is from $\varphi \in C_{++}$. The following result gives a comparison principle for the solution semiflow of (2.2).

Proposition 2.2. *Let* $\tilde{K} : C \to C$ *be a linear bounded operator satisfying* $(\tilde{K} - K)[X_+] \subseteq X_+$. *Let* $g \in C(\mathbb{R}_+, \mathbb{R}_+)$ *be nondecreasing on* $\mathbb{R}_+$ *satisfying* $f - g \in C(\mathbb{R}_+, \mathbb{R}_+)$. *Then, for any* $\varphi,\ \psi \in C_+$ *with* $\varphi \geq \psi$, *it hods that* $\Phi(t, \varphi; f, \tilde{K}) \geq \Phi(t, \psi; g, K)$ *for all* $t \geq 0$.

Proof. Suppose that $(\varphi, \psi) \in C_+ \times C_+$ and $\varphi \geq \psi$. It follows from Lemmas 2.1–2.2 and (2.2) that for any $t \in [0, \tau]$, we have

$$\Phi(t, \varphi; f, \tilde{K})(0, \cdot) = T(t)[\varphi(0, \cdot)] + \mu \int_0^t T(t - \varsigma)[\tilde{K}[f(\varphi(\varsigma - \tau, \cdot))]]\mathrm{d}\varsigma$$

$$
\begin{aligned}
&\geq T(t)[\varphi(0,\cdot)] + \mu \int_0^t T(t-\varsigma)[K[g(\varphi(\varsigma-\tau,\cdot))]]\mathrm{d}\varsigma \\
&\geq T(t)[\psi(0,\cdot)] + \mu \int_0^t T(t-\varsigma)[K[g(\psi(\varsigma-\tau,\cdot))]]\mathrm{d}\varsigma \\
&\geq \Phi(t,\psi; g, K)(0,\cdot).
\end{aligned}
$$

This, together with the semigroup properties of Φ, implies $\Phi(t,\varphi; f, \tilde{K}) \geq \Phi(t,\psi; g, K)$ for all $t \in \mathbb{R}_+$. The proof is completed. □

Before proceeding further, we collect some standard notions and notations.

Definition 2.1. An element $\varphi \in C_+$ is called an equilibrium of Φ if $\Phi(t,\varphi) = \varphi$ for all $t \in \mathbb{R}_+$. A subset $\mathcal{A}$ of C_+ is said to be positively invariant under Φ if $\Phi(t,\varphi) \in \mathcal{A}$ for all $\varphi \in \mathcal{A}$ and $t \in \mathbb{R}_+$.

We write $O(\varphi) = \{\Phi(t,\varphi) : t \in \mathbb{R}_+\}$ for the positive semi-orbit through the point $\varphi \in C_+$. The ω-limit set of $O(\varphi)$ is defined by $\omega(\varphi) = \bigcap_{t\in\mathbb{R}_+} \overline{O(\Phi(t,\varphi))}$, where $\overline{O(\Phi(t,\varphi))}$ represents the closure of $O(\Phi(t,\varphi))$. In what follows, we also write $O(f, K; \varphi)$ and $\omega(f, K; \varphi)$ for $O(\varphi)$ and $\omega(\varphi)$, respectively, to emphasize dependence on the nonlinearity f and nonlocal kernel K, if there is a need.

Definition 2.2. Let ϕ^* be an equilibrium of the semi-flow Φ and $\mathcal{A}$ be a positively invariant set of Φ. We say that ϕ^* is globally attractive in $\mathcal{A}$ if $\omega(\varphi) = \{\phi^*\}$ for all $\varphi \in \mathcal{A}$.

3. An integro-difference equation and its properties

Let $k(t,x) = \mu e^{-\mu t}\Gamma_{dt}(x)$, that is,

$$
k(t,x) = \frac{\mu e^{-\mu t}}{\sqrt{4d\pi t}} \exp\left(-\frac{x^2}{4dt}\right) \quad \text{for } t \in (0,\infty) \text{ and } x \in \mathbb{R}.
$$

Given $c \geq 0$, define $k_c : \mathbb{R} \to \mathbb{R}$ by

$$
k_c(x) = \frac{\mu}{\sqrt{4d\mu + c^2}} e^{-\frac{c}{2d}x - \sqrt{\frac{4d\mu+c^2}{4d^2}}|x|} \quad \text{for } x \in \mathbb{R}.
$$

Note that $\int_{\mathbb{R}_+} k(s, x+cs)\mathrm{d}s = k_c(x)$ for all $x \in \mathbb{R}$ by Lemma 2.1-(vi) in [54]. Clearly, $\int_{\mathbb{R}} k_c(y)\mathrm{d}y = 1$, and $\int_{\mathbb{R}} e^{\lambda y} k_c(y)\mathrm{d}y < \infty$ if and only if $\lambda \in (-\Delta_-, \Delta_+)$, where

$$
\Delta_\pm = \pm\frac{c}{2d} + \sqrt{\frac{4d\mu + c^2}{4d^2}} > 0.
$$

In our pursuing, we will need to relate our discussion to the following integro-difference equation

$$\begin{cases} u_{n+1}(x) &= \int_{\mathbb{R}^2} g(u_n(y+\tau c))\Gamma_\vartheta(z-y)k_c(x-z)\mathrm{d}y\mathrm{d}z \triangleq Q_c[g;u_n](x), \\ u_0 &\in X_+. \end{cases} \tag{3.1}$$

In other words, $u_n = (Q_c[g;\cdot])^n[u_0] \triangleq Q_c^n[g;u_0]$, for $u_0 \in X_+$. Here $g : \mathbb{R}_+ \to \mathbb{R}_+$ is assumed to satisfy (H1) and $g(u) \le g'(0)u$.

Let

$$l(c,\rho) = \frac{\mu g'(0)e^{-\frac{\rho\tau c}{d}}}{\rho c + \mu - d\rho^2} \int_{\mathbb{R}} e^{\rho y}\Gamma_\vartheta(y)\mathrm{d}y \text{ for all } c, \rho \in \mathbb{R}.$$

Then, for $c,\ \rho \in \mathbb{R}$, one can easily evaluate the integral to obtain

$$l(c,\rho) = \frac{\mu g'(0)e^{\vartheta\rho^2 - \frac{\rho\tau c}{d}}}{\rho c + \mu - d\rho^2} \text{ for all } c, \rho \in \mathbb{R}.$$

Let us define

$$l^{\pm}(c,\rho) = \begin{cases} l(c,\pm\rho) & \text{for } 0 < \rho < \dfrac{\pm c + \sqrt{c^2 + 4d\mu}}{2d}; \\ \infty & \text{for } \rho \ge \dfrac{\pm c + \sqrt{c^2+4d\mu}}{2d}. \end{cases} \tag{3.2}$$

Set

$$c_{\pm}^*(c) = \inf_{\rho>0} \frac{1}{\rho} \log l^{\pm}(c,\rho) \text{ for } c \in \mathbb{R}$$

and

$$c^* = \inf\{c \in \mathbb{R} : c_+^*(c) \le 0\}.$$

In what follows, we also write $c_{\pm}^*(g;c)$, $c^*(g)$ for $c_{\pm}^*(c)$, c^*, respectively, to emphasize the dependence on the nonlinearity g, if there is a need.

The following lemma follows from Lemma 4.4 in [62] by replacing (μ, c, k) with $(\frac{\mu}{d}, \frac{c}{d}, \Gamma_\vartheta)$.

Lemma 3.1. *Assume that $c_{\pm}^*(c), c^*$ are defined as above. If $c \ge 0$, then $\min\{c_-^*(c) > 0, c_+^*(c)\} > 0$ if and only if $c < c^*$.*

Lemma 3.1, together with Theorem 3.6 and Lemma 4.2 in [61] with (μ, c, k, f) replaced by $(\frac{\mu}{d}, \frac{c}{d}, \Gamma_\vartheta, g)$, gives the following results.

Proposition 3.1. *Assume that $c \ge 0$. If $g^2 = g \circ g$ has a unique positive fixed point u^*, then the following statements are valid.*

(i) *If* $c < c^*(g)$*, then* $\lim_{n\to\infty} ||Q_c^n[g;\phi] - u^*|| = 0$ *for any* $\phi \in BC(\mathbb{R}, \mathbb{R}_+) \setminus \{0\}$.
(ii) *If* $c \geq c^*(g)$*, then* (3.1) *has a decreasing steady state* $\phi(x)$ *with* $\phi(\infty) = 0$ *and* $\phi(-\infty) = u^*$.

Remark 3.1. It is easy to verify that the steady state ϕ in Proposition 3.1-(ii) satisfies the following equation,

$$-c\phi'(x) = d\phi''(x) - \mu\phi(x) + \mu \int_{\mathbb{R}} g(\phi(y + c\tau))\Gamma_\vartheta(x - y)\mathrm{d}y.$$

In other words, $\phi(x - ct)$ is indeed a travelling wave of the following delayed reaction-diffusion equation

$$\frac{\partial u}{\partial t}(t, x) = d\frac{\partial^2 u}{\partial x^2} - \mu u + \mu \int_{\mathbb{R}} f(u(t - \tau, y))\Gamma_\vartheta(x - y)\mathrm{d}y, \qquad (t, x) \in (0, \infty) \times \mathbb{R} \quad (3.3)$$

with $f = g$ such that $\phi(\infty) = 0$ and $\phi(-\infty) = u^*$. On the basis of this observation, we call Q_c a travelling wave map with wave speed c.

In the remainder of this paper, in addition to (H1) and $g(u) \leq g'(0)u$, we also further assume that g is *nondecreasing* on $\mathbb{R}_+$. This implies that the self-composition $g^2 = g \circ g$ has a unique positive fixed point u^*

Lemma 3.2. *Let* $c \in [0, c^*)$ *and let* $\phi \in C(\mathbb{R}, [0, u^*)) \setminus \{0\}$ *have a compact support. Then, for any* $\gamma \in (1, \frac{u^*}{||\phi||_{L^\infty}})$*, there is an* $n = n(c, \phi, \gamma) \in \mathbb{N}$ *such that* $Q_\sigma^n[g;\phi] \geq \gamma\phi$ *for all* $\sigma \in [0, c]$.

Proof. Clearly, by Proposition 3.1-(i), for any $\sigma \in [0, c]$ and $\gamma \in (1, \frac{u^*}{||\phi||_{L^\infty}})$, there is an $N = N(\sigma, \phi, \gamma) \in \mathbb{N}$ such that $Q_\sigma^N[g;\phi] \gg \gamma\phi$. Note that $Q_\sigma^N[g;\phi](x)$ is continuous at $(\sigma, x) \in [0, c] \times \mathbb{R}_+$ due to the definition of $Q_\sigma^N[g;\cdot]$. This, together with the compactness of $supp(\phi)$, implies that there is $\delta = \delta(\sigma, \phi, \gamma) > 0$ such that $Q_\varsigma^N[g;\phi] \gg \gamma\phi$ for all $\varsigma \in [0, c] \cap (\sigma - \delta, \sigma + \delta)$. Clearly, there exist $c_1, c_2, \ldots, c_l \in [0, c]$ such that $[0, c] \subseteq \bigcup_{i=1}^{l} (c_i - \delta(c_i, \phi, \gamma), c_i + \delta(c_i, \phi, \gamma))$. Let $n = n_{c,\phi,\gamma} \triangleq \prod_{i=1}^{l} N(c_i, \phi, \gamma)$. Then, by $\gamma > 1$, the choice of n, and the monotonicity of Q_σ, we have $Q_\sigma^n[g;\phi] \geq \gamma\phi$ for all $\sigma \in [0, c]$. $\square$

For any $c \geq 0$, and α, $\beta > 0$, we define linear operators

$$Q_{c,\alpha}[g;\cdot] : C([-\alpha, \alpha], \mathbb{R}) \to C([-\alpha, \alpha], \mathbb{R})$$

and

$$Q_{c,0,\infty}[g;\cdot],\ Q_{c,\beta}^\infty[g;\cdot] : BC(\mathbb{R}_+, \mathbb{R}) \triangleq C(\mathbb{R}_+, \mathbb{R}) \cap L^\infty(\mathbb{R}_+, \mathbb{R}) \to BC(\mathbb{R}_+, \mathbb{R})$$

respectively by

$$Q_{c,\alpha}[g;\phi](x)=\int_{-\alpha}^{\alpha}k_c(x+\tau c-y)\int_{-\alpha}^{\alpha}\Gamma_\vartheta(y-z)g(\phi(z))\mathrm{d}z\mathrm{d}y,\quad x\in[-\alpha,\alpha],$$

$$Q_{c,0,\infty}[g;\zeta](x)=\int_{0}^{\infty}k_c(x-y)\int_{0}^{\infty}\Gamma_\vartheta(y-z)g(\zeta(z+c\tau))\mathrm{d}z\mathrm{d}y,\quad x\in\mathbb{R}_+,$$

$$Q^\infty_{c,\beta}[g;\zeta](x)=\int_0^\beta\int_0^\infty k(s,x+cs-y)\int_0^\infty[\Gamma_\vartheta(y-z)-\Gamma_\vartheta(y+z)]g(\zeta(z+c\tau))\mathrm{d}z\mathrm{d}y\mathrm{d}s,$$
$$x\in\mathbb{R}_+.$$

It is easy to verify that these operators are order preserving.

Lemma 3.3. *If $c\in[0,c^*)$ and $\phi,\psi\in BC(\mathbb{R},\mathbb{R}_+)\setminus\{0\}$ have compact supports, then there exist $n_{c,\phi,\psi}\in\mathbb{N}$ and $\alpha^*_{c,\phi,\psi}>0$ such that $Q^{n_{c,\phi,\psi}}_{\sigma,\alpha}[g;\phi](0)\geq\frac{2u^*}{3}$ and $Q^{n_{c,\phi,\psi}}_{\sigma,\alpha}[g;\psi](0)\geq\frac{2u^*}{3}$ for all $\sigma\in[0,c]$ and $\alpha\geq\alpha^*_{c,\phi,\psi}$.*

Proof. Take $\beta^*>0$ and $\varphi\in C(\mathbb{R},[0,u^*))\setminus\{0\}$ with compact support such that $||\varphi||_{L^\infty}=\varphi(0)$, $\varphi\leq Q_{\sigma,\beta^*}[\phi]$ and $\varphi\leq Q_{\sigma,\beta^*}[\psi]$ for all $\sigma\in[0,c]$.

By Lemma 3.2 with

$$\gamma=\frac{\max\{1,\frac{2u^*}{3\varphi(0)}\}+\frac{u^*}{\varphi(0)}}{2},$$

we know that there is $n_1\in\mathbb{N}$ such that $Q^{n_1}_\sigma[g;\varphi]\geq\gamma\varphi$ for all $\sigma\in[0,c]$. According to $\lim\limits_{\alpha\to\infty}Q^{n_1}_{\sigma,\alpha}[g;\varphi](0)=Q^{n_1}_\sigma[g;\varphi](0)>\frac{2u^*}{3}$ for all $\sigma\in[0,c]$, we know that for any $\sigma\in[0,c]$, there exist $\alpha_\sigma>0$ such that $Q^{n_1}_{\sigma,\alpha_\sigma}[g;\varphi](0)>\frac{2u^*}{3}$. Since $Q^{n_1}_{\rho,\alpha}[g;\varphi](0)$ is continuous at $\rho\in[0,c]$, there exist $l\in\mathbb{N}$ and (c_1,δ_1), (c_2,δ_2), …, $(c_l,\delta_l)\in[0,c]\times(0,1)$ with $[0,c]\subseteq\bigcup\limits_{i=1}^{l}(c_i-\delta_i,c_i+\delta_i)$ such that $Q^{n_1}_{\sigma,\alpha_{c_i}}[g;\varphi](0)\geq\frac{2u^*}{3}$ for all $\sigma\in(c_i-\delta_i,c_i+\delta_i)$ and $i\in[0,l]\cap\mathbb{N}$. Let $\alpha^*=\max\{\alpha_i:i\in[0,l]\cap\mathbb{N}\}$. Then $Q^{n_1}_{\sigma,\alpha}[g;\varphi](0)\geq\frac{2u^*}{3}$ for all $\sigma\in[0,c]$ and $\alpha\geq\alpha^*$ due to the monotonicity of $Q^{n_1}_{\sigma,\alpha}[g;\varphi](0)$ with respect to $\alpha\in[0,\infty)$.

Thus, by the choice of φ and the monotonicity of $Q_{\sigma,\alpha}[g;\cdot](0)$ with respect to α, we have $Q^{n_1+1}_{\sigma,\alpha}[g;\phi](0)\geq Q^{n_1}_{\sigma,\alpha}[g;\varphi](0)\geq\frac{2u^*}{3}$ and $Q^{n_1+1}_{\sigma,\alpha}[g;\psi](0)\geq Q^{n_1}_{\sigma,\alpha}[g;\varphi](0)\geq\frac{2u^*}{3}$ for all $\sigma\in[0,c]$ and $\alpha\geq\alpha^*_{c,\phi,\psi}:=\max\{\alpha^*,\beta^*\}$. □

Lemma 3.4. *Let $c\in[0,c^*)$, $\delta\geq\alpha+\tau c>\tau c$, $\phi\in C([-\alpha,\alpha],\mathbb{R}_+)\setminus\{0\}$ and $\psi\in BC(\mathbb{R}_+,\mathbb{R}_+)\setminus\{0\}$. If $\psi(x+\delta)\geq\phi(x)$ for all $x\in[-\alpha,\alpha]$, then $Q_{c,0,\infty}[g;\psi](x+\delta)\geq Q_{c,\alpha}[g;\phi](x)$ and hence $Q^n_{c,0,\infty}[g;\psi](x+\delta)\geq Q^n_{c,\alpha}[g;\phi](x)$ for all $x\in[-\alpha,\alpha]$ and $n\in\mathbb{N}$.*

Proof. By the definitions of $Q_{c,0,\infty}[g;\cdot]$ and $Q_{c,\alpha}[g;\cdot]$, we can conclude that for any $x\in[-\alpha,\alpha]$,

$$Q_{c,0,\infty}[g;\psi](x+\delta)$$

$$
\begin{aligned}
&= \int_0^\infty k_c(x+\delta-y)\int_0^\infty \Gamma_\vartheta(y-z)g(\psi(z+\tau c))\mathrm{d}z\mathrm{d}y \\
&= \int_{-\delta}^\infty k_c(x-y)\int_{-\delta}^\infty \Gamma_\vartheta(y-z)g(\psi(z+\delta+\tau c))\mathrm{d}z\mathrm{d}y \\
&\geq \int_{-\alpha-\tau c}^{\alpha-\tau c} k_c(x-y)\int_{-\alpha-\tau c}^{\alpha-\tau c} \Gamma_\vartheta(y-z)g(\psi(z+\delta+\tau c))\mathrm{d}z\mathrm{d}y \\
&= \int_{-\alpha}^{\alpha} k_c(x+\tau c-y)\int_{-\alpha}^{\alpha} \Gamma_\vartheta(y-z)g(\psi(z+\delta))\mathrm{d}z\mathrm{d}y \\
&\geq \int_{-\alpha}^{\alpha} k_c(x+\tau c-y)\int_{-\alpha}^{\alpha} \Gamma_\vartheta(y-z)g(\phi(z))\mathrm{d}z\mathrm{d}y \\
&= Q_{c,\alpha}[g;\phi](x).
\end{aligned}
$$

This completes the proof. $\square$

For any given $T > 1$, define the function $h^T : \mathbb{R} \to \mathbb{R}_+$ by

$$
h^T(x) = \begin{cases} 1, & x \in [T, 2T], \\ x - T + 1, & x \in [T-1, T), \\ 2T - x + 1, & x \in (2T, 2T+1], \\ 0, & x \in \mathbb{R} \setminus [T-1, 2T+1]. \end{cases}
$$

Proposition 3.2. *If $c \in [0, c^*)$ and $\varepsilon > 0$, then there exist $n_0 = n_0(c, \varepsilon) \in \mathbb{N}$ and $t_0 = t_0(c, \varepsilon) > 4$ such that $(Q^\infty_{\sigma,\beta})^{n_0}[g; \varepsilon h^T] \geq \frac{u^*}{2} h^T$ for all $\beta, T \geq t_0$, and $\sigma \in [0, c]$.*

Proof. Let $\xi(x) = \varepsilon \max\{0, 1 - |x|\}$ for all $x \in \mathbb{R}$. By Lemma 3.3 with $\phi = \xi(\cdot + 1)$ and $\psi = \xi(\cdot - 1)$, there exist $n_\xi \in \mathbb{N}$ and $\alpha_\xi > 1 + \tau c$ such that $Q^{n_\xi}_{\sigma,\alpha}[g; \xi(\cdot \pm 1)](0) \geq \frac{2u^*}{3}$ for all $\sigma \in [0, c]$ and $\alpha > \alpha_\xi$. In particular, $Q^{n_\xi}_{\sigma,1+\alpha_\xi}[g; \xi(\cdot \pm 1)](0) \geq \frac{2u^*}{3}$ for all $\sigma \in [0, c]$. This, together with Lemma 3.4 and the fact that

$$
\varepsilon h^T(x+\delta) \begin{cases} \geq \xi(x-1), & \delta \in [T-1, 1.5T] \\ \geq \xi(x+1), & \delta \in (1.5T, 2T+1] \end{cases}
$$

for all $T > 4 + 2\alpha_\xi$ and $x \in [-1 - \alpha_\xi, 1 + \alpha_\xi]$, implies that, for all $T > 4 + 2\alpha_\xi$, $\delta \in [T-1, 2T+1]$, and $\sigma \in [0, c]$, we have

$$
Q^{n_\xi}_{\sigma,0,\infty}[g; \varepsilon h^T](\delta) \geq \min\{Q^{n_\xi}_{\sigma,1+\alpha_\xi}[g; \xi(\cdot - 1)](0), Q^{n_\xi}_{\sigma,1+\alpha_\xi}[g; \xi(\cdot + 1)](0)\} \geq \frac{2u^*}{3}.
$$

To complete the proof, we define two new maps $R_{\sigma,0,\infty}, R^\infty_{\sigma,\beta} : BC(\mathbb{R}_+, \mathbb{R}) \to BC(\mathbb{R}_+, \mathbb{R})$ respectively by

$$R_{\sigma,0,\infty}[g;\zeta](x) = \int_0^\infty k_\sigma(x-y)\int_0^\infty \Gamma_\vartheta(y+z)g(\zeta(z+\sigma\tau))\mathrm{d}z\mathrm{d}y,$$

$$R^\infty_{\sigma,\beta}[g;\zeta](x) = \int_\beta^\infty\int_0^\infty k(s,x+\sigma s-y)\int_0^\infty [\Gamma_\vartheta(y-z)-\Gamma_\vartheta(y+z)]g(\zeta(z+\sigma\tau))\mathrm{d}z\mathrm{d}y\mathrm{d}s$$

for all $x\in\mathbb{R}_+$ and $\zeta\in BC(\mathbb{R}_+,\mathbb{R})$. Clearly,

$$||Q_{\sigma,0,\infty}[g;\zeta]||_{L^\infty} \le \max\{u^*, ||\zeta||_{L^\infty}\},$$
$$||R_{\sigma,0,\infty}[g;\zeta]||_{L^\infty} \le \max\{u^*, ||\zeta||_{L^\infty}\},$$
$$||R^\infty_{\sigma,\beta}[g;\zeta]||_{L^\infty} \le \max\{u^*, ||\zeta||_{L^\infty}\}$$

for all $\zeta\in BC(\mathbb{R}_+,\mathbb{R})$. It is also obvious that

$$|R^\infty_{\sigma,\beta}[g;\zeta](x)| \le \max\{u^*, ||\zeta||_{L^\infty}\}\int_\beta^\infty\int_{-\infty}^\infty k(s,x+\sigma s-y)\mathrm{d}y\mathrm{d}s \le \max\{u^*, ||\zeta||_{L^\infty}\}e^{-\mu\beta}$$

for all $x,\sigma,\beta\in\mathbb{R}_+$, and $\zeta\in BC(\mathbb{R}_+,\mathbb{R})$. It then follows that, for any $x,\sigma\in\mathbb{R}_+$, and $\zeta\in BC(\mathbb{R}_+,\mathbb{R})$, we have

$$\begin{aligned}
&|R_{\sigma,0,\infty}[g;\zeta](x)|\\
&\le \max\{u^*, ||\zeta||_{L^\infty}\}\int_0^\infty k_\sigma(x-y)\int_0^\infty \Gamma_\vartheta(y+z)\mathrm{d}z\mathrm{d}y\\
&\le \max\{u^*, ||\zeta||_{L^\infty}\}\int_0^\infty k_\sigma(x-y)\int_0^\infty e^{-\frac{y^2}{4\vartheta}}\Gamma_\vartheta(z)\mathrm{d}z\mathrm{d}y\\
&= \frac{\max\{u^*, ||\zeta||_{L^\infty}\}}{2}[\int_0^x k_\sigma(x-y)e^{-\frac{y^2}{4\vartheta}}\mathrm{d}y + \int_x^\infty k_\sigma(x-y)e^{-\frac{y^2}{4\vartheta}}\mathrm{d}y]\\
&\le \frac{\mu\max\{u^*, ||\zeta||_{L^\infty}\}}{2\sqrt{4d\mu+\sigma^2}}[e^{-[\frac{\sigma}{2d}+\sqrt{\frac{4d\mu+\sigma^2}{4d^2}}]x}\int_0^x e^{\frac{\sigma}{2d}y+\sqrt{\frac{4d\mu+\sigma^2}{4d^2}}y}e^{-\frac{y^2}{4\vartheta}}\mathrm{d}y + e^{-\frac{x^2}{4\vartheta}}\int_x^\infty k_\sigma(x-y)\mathrm{d}y]\\
&\le \frac{\mu\max\{u^*, ||\zeta||_{L^\infty}\}}{2\sqrt{4d\mu+\sigma^2}}[e^{-[\frac{\sigma}{2d}+\sqrt{\frac{4d\mu+\sigma^2}{4d^2}}]x}\int_{-\infty}^\infty e^{\frac{\sigma}{2d}y+\sqrt{\frac{4d\mu+\sigma^2}{4d^2}}y}e^{-\frac{y^2}{4\vartheta}}\mathrm{d}y + e^{-\frac{x^2}{4\vartheta}}]\\
&= \frac{\mu\max\{u^*, ||\zeta||_{L^\infty}\}}{2\sqrt{4d\mu+\sigma^2}}[2\sqrt{\vartheta\pi}e^{\vartheta[\frac{\sigma}{2d}+\sqrt{\frac{4d\mu+\sigma^2}{4d^2}}]^2}e^{-[\frac{\sigma}{2d}+\sqrt{\frac{4d\mu+\sigma^2}{4d^2}}]x} + e^{-\frac{x^2}{4\vartheta}}]
\end{aligned}$$

$$\leq \frac{\mu \max\{u^*, ||\zeta||_{L^\infty}\}}{4\sqrt{d\mu}}[2\sqrt{\vartheta\pi}e^{\vartheta[\frac{\sigma}{2d}+\sqrt{\frac{4d\mu+\sigma^2}{4d^2}}]^2}e^{-\sqrt{\frac{\mu}{d}}x}+e^{-\frac{x^2}{4\vartheta}}].$$

Take $n_0 \triangleq n_0(c,\varepsilon) = n_\xi \in \mathbb{N}$ and $t_0 = t_0(c,\varepsilon) > 4 + 2\alpha_\xi$ such that $3^{n_0}\max\{\varepsilon, u^*\}e^{-\mu t_0} \leq \frac{u^*}{12}$ and

$$\frac{3^{n_0}\mu \max\{u^*, \varepsilon\}}{4\sqrt{d\mu}}[2\sqrt{\vartheta\pi}e^{\vartheta[\frac{c}{2d}+\sqrt{\frac{4d\mu+c^2}{4d^2}}]^2}e^{-\sqrt{\frac{\mu}{d}}(t_0-1)}+e^{-\frac{(t_0-1)^2}{4\vartheta}}] \leq \frac{u^*}{12}.$$

It follows that, for any $\beta, T \geq t_0$, $\sigma \in [0,c]$, and $x \in [T-1, 2T+1]$, we have

$$\begin{aligned}
(Q^\infty_{\sigma,\beta})^{n_0}[g;\varepsilon h^T](x) &= [Q_{\sigma,0,\infty} - R_{\sigma,0,\infty} - R^\infty_{\sigma,\beta}]^{n_0}[g;\varepsilon h^T](x) \\
&= \sum_{l=0}^{n_0} C^l_{n_0}[Q_{\sigma,0,\infty}]^{n_0-l}[-R_{\sigma,0,\infty} - R^\infty_{\sigma,\beta}]^l[g;\varepsilon h^T](x) \\
&= [Q_{\sigma,0,\infty}]^{n_0}[g;\varepsilon h^T](x) \\
&\quad + \sum_{l=1}^{n_0} C^l_{n_0}[Q_{\sigma,0,\infty}]^{n_0-l}[-R_{\sigma,0,\infty} - R^\infty_{\sigma,\beta}]^l[g;\varepsilon h^T](x) \\
&\geq [Q_{\sigma,0,\infty}]^{n_0}[g;\varepsilon h^T](x) - [R_{\sigma,0,\infty} + R^\infty_{\sigma,\beta}][g;3^{n_0}\max\{\varepsilon,u^*\}](x) \\
&\geq [Q_{\sigma,0,\infty}]^{n_0}[g;\varepsilon h^T](x) - 3^{n_0}\max\{\varepsilon,u^*\}e^{-\mu\beta} \\
&\quad - \frac{3^{n_0}\mu\max\{u^*,\varepsilon\}}{4\sqrt{d\mu}}[2\sqrt{\vartheta\pi}e^{\vartheta[\frac{\sigma}{2d}+\sqrt{\frac{4d\mu+\sigma^2}{4d^2}}]^2}e^{-\sqrt{\frac{\mu}{d}}x}+e^{-\frac{x^2}{4\vartheta}}] \\
&\geq [Q_{\sigma,0,\infty}]^{n_0}[g;\varepsilon h^T](x) - 3^{n_0}\max\{\varepsilon,u^*\}e^{-\mu\beta} \\
&\quad - \frac{3^{n_0}\mu\max\{u^*,\varepsilon\}}{4\sqrt{d\mu}}[2\sqrt{\vartheta\pi}e^{\vartheta[\frac{\sigma}{2d}+\sqrt{\frac{4d\mu+\sigma^2}{4d^2}}]^2}e^{-\sqrt{\frac{\mu}{d}}(t_0-1)}+e^{-\frac{(t_0-1)^2}{4\vartheta}}] \\
&\geq \frac{u^*}{2},
\end{aligned}$$

completing the proof. □

4. *A priori* estimates

In this section, we present several iteration properties involving the diffusion, nonlocal kernels, and the maturation time τ, which will be very useful in proving *a priori* traveling-like estimates on nontrivial solutions to (2.2).

For any g satisfying (H1) and $g(u) \leq \min\{g'(0)u, f(u)\}$ for $u \in \mathbb{R}_+$, we define three new operators $H_+(g,\cdot), H_-(g,\cdot), H(g,\cdot) : BC([-\tau,\infty)\times\mathbb{R}_+,\mathbb{R}) := C([-\tau,\infty)\times\mathbb{R}_+,\mathbb{R}) \cap L^\infty([-\tau,\infty)\times\mathbb{R}_+,\mathbb{R}) \to BC([-\tau,\infty)\times\mathbb{R}_+,\mathbb{R})$ by

$$H_+[g;\zeta](t,x) = \int_0^t \int_0^\infty k(t-s,x-y) \int_0^\infty \Gamma_\vartheta(y-z) g(\zeta(s-\tau,z))\mathrm{d}z\mathrm{d}y\mathrm{d}s,$$

$$H_-[g;\zeta](t,x) = \int_0^t \int_0^\infty k(t-s,x-y) \int_0^\infty \Gamma_\vartheta(y+z) g(\zeta(s-\tau,z))\mathrm{d}z\mathrm{d}y\mathrm{d}s,$$

$$H[g;\zeta](t,x) = \int_0^t \int_0^\infty k(t-s,x-y) \int_0^\infty [\Gamma_\vartheta(y-z)-\Gamma_\vartheta(y+z)] g(\zeta(s-\tau,z))\mathrm{d}z\mathrm{d}y\mathrm{d}s,$$

for all $\zeta \in BC([-\tau,\infty)\times\mathbb{R}_+,\mathbb{R})$ and $t,x\in\mathbb{R}_+$. Clearly, $H[g;\zeta] = H_+[g;\zeta] - H_-[g;\zeta]$ for all $\zeta \in BC([-\tau,\infty)\times\mathbb{R}_+,\mathbb{R})$.

In the following, we also further assume that g is *nondecreasing* on $\mathbb{R}_+$.

Lemma 4.1. *Suppose that $c,\alpha \ge 0$, $\beta > 0$, $\zeta \in BC([-\tau,\infty)\times\mathbb{R}_+,\mathbb{R}_+) \triangleq C([-\tau,\infty)\times\mathbb{R}_+,\mathbb{R}_+)\cap L^\infty([-\tau,\infty)\times\mathbb{R}_+,\mathbb{R})$, $\eta \in BC([-\tau,\infty)\times\mathbb{R},\mathbb{R}_+) \triangleq C([-\tau,\infty)\times\mathbb{R},\mathbb{R}_+)\cap L^\infty([-\tau,\infty)\times\mathbb{R},\mathbb{R})$, and $\phi \in BC(\mathbb{R}_+,\mathbb{R}_+)$ such that $\eta(t,x)\ge\zeta(t,x)$ for all $(t,x)\in[-\tau,\beta]\times\mathbb{R}_+$ and $\zeta(t,x+\alpha+ct)\ge\phi(x)$ for all $(t,x)\in[-\tau,\beta]\times[c\tau,\infty)$. Then the following statements are valid.*

(i) $g(||\zeta||_{L^\infty([-\tau,\infty)\times\mathbb{R}_+,\mathbb{R})}) \ge H_\pm[g;\zeta](t,x)\ge 0$ *for all* $(t,x)\in\mathbb{R}_+^2$.
(ii) $\int_0^t T(t-s)[K[g(\eta(s-\tau,\cdot))]](x)\mathrm{d}s \ge H[g;\zeta](t,x)$ *for all* $(t,x)\in[0,\beta+\tau]\times\mathbb{R}_+$.
(iii) $H_-[g;\zeta](t,x)\le \frac{\mu g'(0)||\zeta||_{L^\infty}}{4\sqrt{d\mu}}[2\sqrt{\vartheta\pi}e^{\frac{\mu\vartheta}{d}-\sqrt{\frac{\mu}{d}}x}+e^{-\frac{x^2}{4\vartheta}}]$ *for all* $(t,x)\in\mathbb{R}_+^2$.
(iv) $H[g;\zeta](t,x+\alpha+ct)\ge Q_{c,t}^\infty[g;\phi](x)$ *for all* $(t,x)\in[0,\beta+\tau]\times\mathbb{R}_+$.

Proof. (i) and (ii) follow from the definitions of H and $H_\pm$. For (iii), according to the proof of Proposition 3.2, we easily see that for any $t,x\in\mathbb{R}_+$,

$$\begin{aligned} H_-[g;\zeta](t,x) &\le |R_{0,0,\infty}[g;||\zeta||_{L^\infty}](x)| \\ &\le g'(0)||\zeta||_{L^\infty}\int_0^\infty k_0(x-y)\int_0^\infty \Gamma_\vartheta(y+z)\mathrm{d}z\mathrm{d}y \\ &\le \frac{\mu g'(0)||\zeta||_{L^\infty}}{4\sqrt{d\mu}}[2\sqrt{\vartheta\pi}e^{\frac{\mu\vartheta}{d}-\sqrt{\frac{\mu}{d}}x}+e^{-\frac{x^2}{4\vartheta}}]. \end{aligned}$$

For (iv), by the definitions of Γ_ϑ, k and H, and the Fubini's theorem, we can see that, for any $(t,x)\in[0,\beta+\tau]\times\mathbb{R}_+$,

$$\begin{aligned} &H[g;\zeta](t,x+\alpha+ct) \\ &= \int_0^t\int_0^\infty k(t-s,x+\alpha+ct-y)\int_0^\infty [\Gamma_\vartheta(y-z)-\Gamma_\vartheta(y+z)]g(\zeta(s-\tau,z))\mathrm{d}z\mathrm{d}y\mathrm{d}s \end{aligned}$$

$$= \int_0^t \int_{-cs-\alpha}^{\infty} k(t-s, x+c(t-s)-y)$$

$$\times \int_{-cs-\alpha}^{\infty} [\Gamma_\vartheta(y-z) - \Gamma_\vartheta(y+z+2cs+2\alpha)] g(\zeta(s-\tau, z+cs+\alpha)) \mathrm{d}z\mathrm{d}y\mathrm{d}s$$

$$\geq \int_0^t \int_0^{\infty} k(t-s, x+c(t-s)-y)$$

$$\times \int_0^{\infty} [\Gamma_\vartheta(y-z) - \Gamma_\vartheta(y+z+2cs+2\alpha)] g(\zeta(s-\tau, z+cs+\alpha)) \mathrm{d}z\mathrm{d}y\mathrm{d}s$$

$$\geq \int_0^t \int_0^{\infty} k(t-s, x+c(t-s)-y) \int_0^{\infty} [\Gamma_\vartheta(y-z) - \Gamma_\vartheta(y+z)] g(\zeta(s-\tau, z+cs+\alpha)) \mathrm{d}z\mathrm{d}y\mathrm{d}s$$

$$\geq \int_0^t \int_0^{\infty} k(t-s, x+c(t-s)-y) \int_0^{\infty} [\Gamma_\vartheta(y-z) - \Gamma_\vartheta(y+z)] g(\phi(z+c\tau)) \mathrm{d}z\mathrm{d}y\mathrm{d}s$$

$$= \int_0^t \int_0^{\infty} k(s, x+cs-y) \int_0^{\infty} [\Gamma_\vartheta(y-z) - \Gamma_\vartheta(y+z)] g(\phi(z+c\tau)) \mathrm{d}z\mathrm{d}y\mathrm{d}s$$

$$= Q^{\infty}_{c,t}[g;\phi](x). \quad \square$$

Lemma 4.2. *Let $c \geq 0$, $\beta > 0$ and $\phi \in BC(\mathbb{R}_+, \mathbb{R}_+)$. If $u \in BC([-\tau, \infty) \times \mathbb{R}, \mathbb{R}_+)$ is a solution of (2.2) such that $u(t, x+ct) \geq \phi(x)$ for all $(t, x) \in [-\tau, \beta] \times \mathbb{R}_+$, then, for any $I \in \mathbb{N}$ and $i \in [1, I] \bigcap \mathbb{N}$, we have $u(t, x+ct) \geq (Q^{\infty}_{c, \frac{\beta}{1+I}})^i[g;\phi](x)$ for all $(t, x) \in [\frac{i\beta}{1+I} + (i-1)\tau, \beta + i\tau] \times \mathbb{R}_+$. In particular, $u(t, x+ct) \geq (Q^{\infty}_{c, \frac{\beta}{1+I}})^I[g;\phi](x)$ for all $(t, x) \in [\frac{I\beta}{1+I} + (I-1)\tau, \beta + I\tau] \times \mathbb{R}_+$.*

Proof. Given $I \in \mathbb{N}$, by (2.2) combined with Lemma 4.1 and the fact that $u(s, x+cs) \geq \phi(x)$ for all $(s, x) \in [-\tau, \beta] \times [c\tau, \infty)$, we have, for any $(t, x) \in [0, \beta+\tau] \times \mathbb{R}_+$,

$$u(t, x+ct) = T(t)[\phi](x+ct) + \mu \int_0^t T(t-s)[K[f(u(s-\tau, \cdot))]](x+ct)\mathrm{d}s$$

$$\geq T(t)[\phi](x+ct) + \mu \int_0^t T(t-s)[K[g(u(s-\tau, \cdot))]](x+ct)\mathrm{d}s$$

$$\begin{aligned}
&\geq \mu \int_0^t T(t-s)[K[g(u(s-\tau,\cdot))]](x+ct)\mathrm{d}s \\
&\geq H[g;u(\cdot,\cdot))](t,x+ct) \\
&\geq (Q^{\infty}_{c,t})[g;\phi](x).
\end{aligned}$$

Thus $u(t,x+ct) \geq Q^{\infty}_{c,\frac{\beta}{1+I}}[g;\phi](x)$ for any $(t,x) \in [\frac{\beta}{1+I}, \beta+\tau] \times \mathbb{R}_+$ due to the monotonicity of $Q^{\infty}_{c,t}[g;\phi](x)$ with $t \in \mathbb{R}_+$, leading to the conclusion for $i=1$. Let

$$I_{j,\beta} = [\frac{j\beta}{1+I} + (j-1)\tau, \beta + j\tau] \text{ for all } j \in [1, I]$$

and

$$\begin{aligned}
i^* = \sup\{ & i \in [1,I] \bigcap \mathbb{N} : u(t,x+ct) \geq (Q^{\infty}_{c,\frac{\beta}{1+I}})^j[g;\phi](x) \text{ for all } j \in [1,i] \\
& \text{and } (t,x) \in I_{j,\beta} \times \mathbb{R}_+\}.
\end{aligned}$$

Then $i^* \geq 1$ and $u(t,x+ct) \geq (Q^{\infty}_{c,\frac{\beta}{1+I}})^{i^*}[g;\phi](x)$ for all $(t,x) \in I_{i^*,\beta} \times \mathbb{R}_+$. If $i^* < I$, then by the semi-group property of the solution flow of (2.2), Lemma 4.1, and the choice of i^*, we easily see that, for any $(t,x) \in I_{i^*+1,\beta} \times \mathbb{R}_+$,

$$\begin{aligned}
& u(t,x+ct) \\
&= u_{\frac{i^*\beta}{1+I}+i^*\tau}\left(t - \frac{i^*\beta}{1+I} - i^*\tau, x + c\left(t - \frac{i^*\beta}{1+I} - i^*\tau\right) + \frac{i^*\beta c}{1+I} + i^*\tau c\right) \\
&= T\left(t - \frac{i^*\beta}{1+I} - i^*\tau\right)\left[u\left(\frac{i^*\beta}{1+I} + i^*\tau, \cdot\right)\right](x+ct) \\
&\quad + \mu \int_0^{t-\frac{i^*\beta}{1+I}-i^*\tau} T\left(t - \frac{i^*\beta}{1+I} - i^*\tau - s\right)\left[K\left[f(u(s + \frac{i^*\beta}{1+I} + (i^*-1)\tau, \cdot))\right]\right](x+ct)\mathrm{d}s \\
&\geq \mu \int_0^{t-\frac{i^*\beta}{1+I}-i^*\tau} T\left(t - \frac{i^*\beta}{1+I} - i^*\tau - s\right)\left[K[f(u(s + \frac{i^*\beta}{1+I} + (i^*-1)\tau, \cdot))]\right](x+ct)\mathrm{d}s \\
&\geq \mu \int_0^{t-\frac{i^*\beta}{1+I}-i^*\tau} T\left(t - \frac{i^*\beta}{1+I} - i^*\tau - s\right)\left[K[g(u(s + \frac{i^*\beta}{1+I} + (i^*-1)\tau, \cdot))]\right](x+ct)\mathrm{d}s \\
&\geq H\left[g; u(\cdot + \frac{i^*\beta}{1+I} + i^*\tau, \cdot)\right]\left(t - \frac{i^*\beta}{1+I} - i^*\tau, x + c(t - \frac{i^*\beta}{1+I} - i^*\tau) + \frac{i^*\beta c}{1+I} + i^*\tau c\right)
\end{aligned}$$

$$\geq (Q^{\infty}_{c,t-\frac{i^*\beta}{1+I}-i^*\tau})[g;(Q^{\infty}_{c,\frac{\beta}{1+I}})^{i^*}[g;\phi]](x)$$

$$\geq (Q^{\infty}_{c,\frac{\beta}{1+I}})^{1+i^*}[g;\phi](x),$$

which yields a contradiction with the choice of i^*. Hence $i^* = I$. This completes the proof. □

The following result produces some *a priori* traveling-like estimates on nontrivial solutions to (2.2), which play a key role in the proof of the repellency of the trivial equilibrium, the global attractivity of the nontrivial equilibrium, as well as the *traveling-like* asymptotic behavior of nontrivial solutions to (2.2).

Proposition 4.1. *Suppose that $c \in [0, c^*(f))$. Then there exist $\varepsilon_0 > 0$, $T_0 > 4$, and $T^* > T_0$ such that, for all $\varepsilon \in (0, \varepsilon_0]$, $T \in [T_0, \infty)$, $\sigma \in [0, c]$, and solutions $u : [-\tau, \infty) \times \mathbb{R} \to [0, \mathcal{M} + 1]$ of (2.2) satisfying $u(t, \sigma t + x) \geq \varepsilon h^T(x)$ for all $(t, x) \in [-\tau, T^*] \times \mathbb{R}$, we have $u(t, \sigma t + \cdot) \geq \varepsilon h^T(x)$ for all $t \in \mathbb{R}_+$ and $u(t, \sigma t + \cdot) \gg \varepsilon h^T$ for all $t \in (T^*, \infty)$.*

Proof. Obviously, there exists $\delta \in (0, 1)$ and $\delta^* \in (0, f'(0) - 1)$ such that $(1 + \delta^*)\delta < u^*$, $c < c^*(g) \leq c^*(f)$ and $f(u) \geq g(u)$ for all $u \in [0, \mathcal{M}+1]$, where $g : \mathbb{R}_+ \to \mathbb{R}_+$ is defined by $g(u) = (1 + \delta^*) \min\{u, \delta\}$ for all $u \in \mathbb{R}_+$. Then for this g, clearly $g(\varepsilon u) \geq \varepsilon g(u)$ for all $u \in [0, 1 + \mathcal{M}]$ and $\varepsilon \in [0, 1]$. By applying Proposition 3.2, we know that there exist $n_0 \in \mathbb{N}$ and $t_0 > 4$ such that $(Q^{\infty}_{\sigma,\beta})^{n_0}[g; \varepsilon^* h_T] \geq 2\varepsilon^* h^T$ with $\varepsilon^* = \frac{(1+\delta^*)\delta}{5}$ for all $\beta, T \geq t_0$ and $\sigma \in [0, c]$.

Let $\varepsilon_0 = \frac{\varepsilon^*}{(g'(0))^{n_0+2}}$, $T_0 = t_0$, and $T^* = (1 + n_0)t_0 + n_0\tau$. Suppose that $\varepsilon \in (0, \varepsilon_0]$, $T \in [T_0, \infty)$, $\sigma \in [0, c]$, and a solution $u : [-\tau, \infty) \times \mathbb{R}_+ \to [0, 1 + \mathcal{M}]$ of (2.2) with $u(t, \sigma t + \cdot) \geq \varepsilon h^T$ for all $t \in [-\tau, T^*]$. It follows from Lemma 4.2 that, for any $t \in [T^*, T^* + n_0\tau]$, we have $u(t, \sigma t + \cdot) \geq (Q^{\infty}_{\sigma,\frac{T^*}{n_0+1}})^{n_0}[g; \varepsilon h_T] \geq (Q^{\infty}_{\sigma,T_0})^{n_0}[g; \varepsilon h_T] = (Q^{\infty}_{\sigma,T_0})^{n_0}[g; \frac{\varepsilon}{\varepsilon^*}\varepsilon^* h_T] \geq \frac{\varepsilon}{\varepsilon^*}(Q^{\infty}_{\sigma,T_0})^{n_0}[g; \varepsilon^* h_T] \geq 2\varepsilon h^T$. Let $T^{**} = \sup\{t \geq 0 : u(s, \sigma s + \cdot) \geq \varepsilon h^T$ for all $s \in [0, t]\}$. Then $T^{**} \geq T^* + n_0\tau > T^*$. We claim that $T^{**} = \infty$. Otherwise, $T^{**} < \infty$. By applying the above discussions, we have $u(t, \sigma t + \cdot) \geq (Q^{\infty}_{\sigma,\frac{T^{**}-n_0\tau}{n_0+1}})^{n_0}[g; \varepsilon h_T] \geq \frac{\varepsilon}{\varepsilon^*}(Q^{\infty}_{\sigma,T_0})^{n_0}[g; \varepsilon^* h_T] \geq 2\varepsilon h^T$ for all $t \in [T^{**}, T^{**} + n_0\tau]$. Hence, $u(t, \sigma t + \cdot) \geq \varepsilon h^T$ for all $t \in [-\tau, T^{**} + n_0\tau]$, a contradiction with the choice of T^{**}. Hence, $T^{**} = \infty$. Note that the previous discussions, together with Proposition 2.1(i) and the definition of h^T, also produce $u(t, \sigma t + \cdot) \gg \varepsilon h^T$ for all $t \in [T^*, \infty)$. □

Define $\Omega_{\sigma,\alpha} = \{(t, x) \in \mathbb{R}^2_+ : t \geq \alpha$ and $\alpha \leq x \leq 2\alpha + \sigma t\}$ for all $\alpha > 0, \sigma \in \mathbb{R}_+$. The following shows that the positive limit set of any nontrivial solution of (2.2) is far away from the trivial equilibrium.

Proposition 4.2. *Suppose that $c \in [0, c^*(f))$. If $\varphi \in C_{++}$, then there exist $\varepsilon_{c,\varphi} > 0$ and $\alpha_{c,\varphi} > 0$ such that $u^{\varphi}(t, x) \geq \varepsilon_{c,\varphi}$ for all $(t, x) \in \Omega_{c,\alpha_{c,\varphi}}$ and thus $\omega(\varphi) \geq \varepsilon_{c,\varphi} h^{\alpha_{c,\varphi},\infty}$, where $h^{\alpha_{c,\varphi},\infty}(x) = 1$ for all $x \in [\alpha_{c,\varphi}, \infty)$ and $h^{\alpha_{c,\varphi},\infty}(x) = 0$ for all $x \in [0, \alpha_{c,\varphi})$.*

Proof. Choose T_0, T^*, and ε_0 as in Proposition 4.1. By Proposition 2.1, we may assume that $1 + \mathcal{M} \geq u^{\varphi}(t, x) > 0$ for all $(t, x) \in [-\tau, \infty) \times \mathbb{R}$. Let $\varepsilon_1 = \inf\{u^{\varphi}(t, x) : t \in [-\tau, T^*]$ and $T_0 - 1 \leq x \leq 1 + 4T_0 + ct\}$ and $\varepsilon_{c,\varphi} = \min\{\varepsilon_0, \varepsilon_1\}$. Then $\varepsilon_1 > 0$, $\varepsilon_{c,\varphi} > 0$, and $u^{\varphi}(t, \sigma t + \cdot) \geq \varepsilon_{c,\varphi} h^T$ for all $t \in [-\tau, T^*]$, $T \in [T_0, 2T_0]$, and $\sigma \in [0, c]$. It follows from Proposition 4.1 and the choices of T_0, T^*, and ε_0 that $u^\varphi(t, \sigma t + \cdot) \geq \varepsilon_{c,\varphi} h^T$ for all $t \in \mathbb{R}_+$, $T \in [T_0, 2T_0]$, and $\sigma \in [0, c]$. In particular, $u^\varphi(t, x) \geq \varepsilon_{c,\varphi}$ for all $(t, x) \in \Omega_{c,\alpha_{c,\varphi}}$ with $\alpha_{c,\varphi} = T_0$. This, combined with the definition of $\omega(\varphi)$, implies $\xi \geq \varepsilon_{c,\varphi} h^{\alpha_{c,\varphi},\infty}$ for all $\xi \in \omega(\varphi)$. □

5. Unique positive heterogeneous steady state of (2.2) and its properties

In this section, we shall establish the existence, limit at $\pm\infty$, uniqueness, and attractivity of the heterogeneous steady state of (2.2).

Proposition 5.1. *Let $\mathcal{E}$ be the set of all nontrivial steady states of* (2.2)*. Then the following statements are valid.*

(i) $\emptyset \neq \mathcal{E} \subseteq X_+^\circ \cap C_{\mathcal{M}}$.

(ii) *For any $u \in \mathcal{E}$, $u(x) = u(0) e^{\sqrt{\frac{\mu}{d}} x}$ for all $x \in (-\infty, 0]$ and hence $\lim_{x \to -\infty} u(x) = 0$.*

(iii) *If $f^2 = f \circ f$ has a unique positive fixed point u^*, then $\lim_{x \to \infty} u(x) = u^*$.*

Proof. (i) Clearly, $\mathcal{E} \subseteq X_+^\circ \cap C_{\mathcal{M}}$ due to Proposition 2.1 if $\mathcal{E} \neq \emptyset$. Next, it suffices to prove $\mathcal{E} \neq \emptyset$. Note that by (H1), there is an $\epsilon_{\mathcal{M}} \in (0, \mathcal{M})$ such that $f'(x) > 1$ for all $x \in [0, \epsilon_{\mathcal{M}}]$ and $f(\epsilon_{\mathcal{M}}) = \min f([\epsilon_{\mathcal{M}}, \mathcal{M}])$. Define $\tilde{f} : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$\tilde{f}(u) = \begin{cases} f(u), & u \in [0, \epsilon_{\mathcal{M}}), \\ f(\epsilon_{\mathcal{M}}), & u \in [\epsilon_{\mathcal{M}}, \infty). \end{cases}$$

Then by Proposition 2.2, we have $\Phi(t, \psi; \tilde{f}) \leq \Phi(t, \varphi; \tilde{f}) \leq \Phi(t, \varphi)$ for all $(t, \varphi) \in \mathbb{R}_+ \times C_{\mathcal{M}}$ and $\psi \in C_{\mathcal{M}}$ with $\psi \leq \varphi$. Choose $\varphi^* \in C_{++}$ with $\varphi^* \leq \mathcal{M}$ and let ε_{φ^*}, T_{φ^*} defined as in Proposition 4.2 with f replaced by $\tilde{f}$. Then $\omega(\varphi^*; \tilde{f}) \geq \varepsilon_{\varphi^*} h^{T_{\varphi^*},\infty}$, and hence $\omega(\varphi^*; \tilde{f}) \subseteq C_+^\circ \cap C_{\mathcal{M}}$. Let $\mathcal{A} = \{\varphi \in C_+ : \omega(\varphi^*; \tilde{f}) \leq \varphi \leq \mathcal{M}\}$. Clearly, $\mathcal{A}$ is a nonempty, closed, and convex subset in C such that $\varepsilon_{\varphi^*} h^{T_{\varphi^*},\infty} \leq \mathcal{A} \subseteq C_+^\circ$ and $\Phi(t, \mathcal{A}) \subseteq \mathcal{A}$ for all $t \geq 0$.

Let $g_T(\mathcal{B}) \triangleq \overline{co}(\Phi(T, \mathcal{B}))$ for any $\mathcal{B} \subseteq \mathcal{A}$ and $T > 0$. Then by the induction, we have $\Phi(T, (g_T)^{k-1}(\mathcal{A})) \subseteq (g_T)^k(\mathcal{A}) \subseteq (g_T)^{k-1}(\mathcal{A})$ for any positive integers k and $T > 0$. We claim that there exists a compact convex subset $\mathcal{A}_T$ in $\mathcal{A}$ such that $\Phi(T, \mathcal{A}_T) \subseteq \mathcal{A}_T$ for any $T \in I \triangleq \{\frac{2\tau}{3^i} : i = 1, 2, \cdots\}$. Indeed, by Proposition 2.1-(v) and the fact that $\Phi(t, \varphi)(\theta, x) = u^\varphi(t + \theta, x)$ for all $(t, \theta, x) \in [0, \infty) \times [-\tau, 0] \times \mathbb{R}$, we know that $g_T(\mathcal{A})|_{[-\frac{T}{2}, 0] \times \mathbb{R}}$ is precompact in $C_{-\frac{T}{2}, 0}$ for any $T \in I$. By applying Proposition 2.1-(v) and the fact that $\Phi(t, \varphi)(\theta, x) = u^\varphi(t + \theta, x)$ for all $(t, \theta, x) \in [0, \infty) \times [-\tau, 0] \times \mathbb{R}$ repeatedly, for any $T > 0$ and positive integers i, k, l with $l \leq k \leq 3^i$ and $T = \frac{2\tau}{3^i}$, we may conclude that $(g_T)^k(\mathcal{A})|_{[(\frac{1}{2} - l)T, (1-l)T] \times \mathbb{R}}$ is precompact in $C_{(\frac{1}{2} - l)T, (1-l)T}$ for all $l \leq 1 + \frac{3^i}{2}$ and $(g_T)^k(\mathcal{A})|_{[(\frac{1}{2} - l)T + \tau, (1-l)T + \tau] \times \mathbb{R}}$ is precompact in $C_{(\frac{1}{2} - l)T + \tau, (1-l)T + \tau}$ for all $l > 1 + \frac{3^i}{2}$. These, combined with some simple computations, imply that $(g_T)^{3^i}(\mathcal{A})$ is precompact in C. Let $\mathcal{A}_T = (g_T)^{3^i}(\mathcal{A})$. Then $\mathcal{A}_T$ is a compact convex subset in C such that $\Phi(T, \mathcal{A}_T) \subseteq \mathcal{A}_T$. By the Schauder fixed point theorem, there is $\psi_T \in \mathcal{A}_T$ such that $\Phi(T, \psi_T) = \psi_T$. According to Proposition 2.1-(v) and the fact that $\{\psi_T : T \in I\} = \Phi(2\tau, \{\psi_T : T \in I\}) \subseteq \Phi(2\tau, \mathcal{A})$, we know that $\{\psi_T : T \in I\}$ is pre-compact in C, and thus there exist $\psi \in \mathcal{A}$ and a sequence $\{T_k\}$ in I such that $\lim_{T_k \to 0} \psi_{T_k} = \psi$. For any $t \in (0, \infty)$, there exist $r_k \in [0, T_k)$ and a nonnegative integer N_k such that $t = N_k T_k + r_k$. Obviously, $\lim_{k\to\infty} r_k = 0$. Hence, $\Phi(t,\psi) = \lim_{k\to\infty} \Phi(t, \psi_{T_k}) = \lim_{k\to\infty} \Phi(r_k, \psi_{T_k}) = \psi$, which implies that $u_+ \triangleq \psi \in \mathcal{E}$ is a positive steady state, located in $C_+^\circ \cap C_{\mathcal{M}}$ of (2.2).

(ii) Suppose that $u \in \mathcal{E}$. Then by (2.1) and the definition of K, we easily see that $||u||_{L^\infty(\mathbb{R})} < \infty$ and $du''(x) = \mu u(x)$ for all $x \in (-\infty, 0]$. Thus, $u(x) = u(0)e^{\sqrt{\frac{\mu}{d}}x}$ for all $x \in (-\infty, 0]$, which implies $\lim_{x\to-\infty} u(x) = 0$.

(iii) Let $\overline{u} = \limsup_{x\to\infty} u(x)$, $\underline{u} = \liminf_{x\to\infty} u(x)$ and $I = [\underline{u}, \overline{u}]$. Then by $u \geq \varepsilon h^{T,\infty}$ for some ε and $T \in (0,\infty)$ due to Proposition 4.2, we have $\underline{u} > 0$ and thus $I \subseteq (0,\infty)$.

Define $\underline{\phi} = \liminf_{x\to\infty} \phi(x)$, $\overline{\phi} = \limsup_{x\to\infty} \phi(x)$ and $P[\phi](x) = \int_0^\infty \phi(y)[p(x-y) - \lambda p(x+y)]\mathrm{d}y$ for all $x \in \mathbb{R}_+$ and $\phi \in X$, where $\lambda \geq 0$, $p : \mathbb{R} \to \mathbb{R}_+$ is a continuous and even function on $\mathbb{R}$ such that $\int_{\mathbb{R}} p(y) = 1$ and p is decreasing on $\mathbb{R}_+$. According to the proof of Proposition 3.3 in [62], we may obtain that $\underline{\phi} \leq \underline{P[\phi]} \leq \overline{P[\phi]} \leq \overline{\phi}$ for all $\phi \in X_+$.

If u_+ is a positive steady state of (2.2), then $u_+ = T(t)[u_+] + \mu \int_0^t T(t-s)[K[f(u_+)]]\mathrm{d}s$ for any $t \in \mathbb{R}_+$. This, together with the above claim, implies that

$$
\begin{aligned}
\underline{u_+} &\geq e^{-\mu t}\underline{S(t)[u_+]} + \mu \int_0^t e^{-\mu(t-s)}\underline{S(t-s)[K[f(u_+)]]}\mathrm{d}s \\
&\geq e^{-\mu t}\underline{u_+} + \mu \int_0^t e^{-\mu(t-s)}\underline{K[f(u_+)]}\mathrm{d}s \\
&\geq e^{-\mu t}\underline{u_+} + \mu \int_0^t e^{-\mu(t-s)}\underline{f(u_+)}\mathrm{d}s \\
&= e^{-\mu t}\underline{u_+} + (1 - e^{-\mu t})\underline{f(u_+)}.
\end{aligned}
$$

Thus, $\underline{u_+} \geq \underline{f(u_+)}$, and a similar argument yields $\overline{u_+} \leq \overline{f(u_+)}$. Consequently, $I \subseteq f(I)$. But, by (H1) and the assumption that $f^2 = f \circ f$ has a unique positive fixed point u^*, we know that there is a closed integer J such that $I \subseteq J \subseteq (0,\infty)$, $f(J) \subseteq J$, and $I \subseteq \bigcap_{n=0}^{\infty} f^n(J) = \{u^*\}$. So, $I = \{u^*\}$. In other word, $\liminf_{x\to\infty} u(x) = \limsup_{x\to\infty} u(x) = u^*$. The proof is completed. $\square$

In the following, we denote by u_+, the positive steady state of (2.2) obtained in Proposition 5.1-(i), and let $u_+^* = ||u_+||$. To address the attractiveness of u_+, we further need the following conditions on the nonlinear function f (in addition to (H1)), see [60,62]:

(H2) $f(u) < f'(0)u$ for all $u \in (0,\infty)$.

(H3) For any closed interval $[a,b] \neq \{1\}$ with $0 < a \leq b < \infty$, either (i) $G((0,u_+^*] \times [a,b]) \subseteq (a,\infty)$ or (ii) $G((0,u_+^*] \times [a,b]) \subseteq (0,b)$, where $G : (0,u_+^*] \times (0,\infty) \to (0,\infty)$ is defined by $G(k,u) = \frac{f(ku)}{f(k)}$.

Theorem 5.1. *Assume that (H2) and (H3) hold. Then (2.2) has a unique positive steady state u_+ which attracts all solutions of (2.2) with the initial value $\psi \in C_{++}$, in the sense that $\lim_{t\to\infty} |||(u^{\psi})_t - u_+||| = 0$ for all $\psi \in C_{++}$, where $|||\varphi||| \triangleq \sum_{n=1}^{\infty} 2^{-n} \sup\{|\varphi(\theta, x)| : (\theta, x) \in [-\tau, 0] \times (-\infty, n]\}$ for all $\varphi \in C$.*

Proof. The existence of u_+ is already established in Proposition 5.1-(i), and the uniqueness will be a consequence of the global attractiveness of u_+ in C_{++}. So, we firstly need to show that u_+ attracts all solutions of (2.2) with the initial value $\psi \in C_{++}$.

Suppose $\psi \in C_{++}$. Let $a^* = \sup\{a > 0 : \varphi(0, x) \ge a u_+(x)$ for all $(\varphi, x) \in \omega(\psi) \times \mathbb{R}_+\}$ and $b^* = \inf\{b > 0 : b u_+(x) \ge \varphi(0, x)$ for all $(\varphi, x) \in \omega(\psi) \times \mathbb{R}_+\}$. By the choices of a^*, b^*, Proposition 2.1, Proposition 4.2, and Proposition 5.1-(iii) give $0 < a^* u_+(x) \le \varphi(0, x) \le b^* u_+(x) < \infty$ for all $(\varphi, x) \in \omega(\psi) \times \mathbb{R}_+$. Hence, we have $a^* u_+(0) \le \varliminf_{t\to\infty} u^{\psi}(t, 0) \le \varlimsup_{t\to\infty} u^{\psi}(t, 0) \le b^* u_+(0)$, which implies that for any $\epsilon > 0$, there exists $t_\epsilon > 0$ such that $(a^* - \epsilon) u_+(0) \le u^{\psi}(t, 0) \le (b^* + \epsilon) u_+(0)$ for all $t \ge t_\epsilon$. Let $M = \max\{\mathcal{M}, ||\psi||_{L^\infty}, (b^* + \epsilon) u_+^*\}$ and $v(t, x) = (b^* + \epsilon) u_+(x) - u^{\psi}(t + t_\epsilon, x) + M e^{-\mu t}$ for all $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$. Then $v(t, x)$ satisfies the following equation:

$$\begin{cases} \frac{\partial v}{\partial t}(t, x) = d v_{xx}(t, x) - \mu v(t, x), & (t, x) \in (0, \infty) \times (-\infty, 0), \\ v(t, 0) \ge 0, & t \in \mathbb{R}_+, \\ v(0, x) \ge 0, & x \in (-\infty, 0]. \end{cases} \tag{5.1}$$

Then by the Phragmén-Lindelöf type maximum principle in [37], we have $v(t, x) \ge 0$ for all $(t, x) \in \mathbb{R}_+ \times (-\infty, 0]$. These imply that $\lim_{t\to\infty} (\inf\{(b^* + \epsilon) u_+(x) - u^{\psi}(t + t_\epsilon, x) : x \in (-\infty, 0]\}) \ge 0$. This, together with the definition of $\omega(\psi)$ and arbitrariness of ϵ, implies $\omega(\psi) \le b^* u_+$. Similarly, we have $\omega(\psi) \ge a^* u_+$.

Now, we shall prove that $a^* = b^* = 1$. Otherwise, $a^* \ne 1$ or $b^* \ne 1$. We shall show that this is impossible. By the assumption (H3) with $[a, b] = [a^*, b^*]$, we know that either (I) $f(ku) > a^* f(u)$ for all $(k, u) \in [a^*, b^*] \times (0, u_+^*]$ or (II) $f(ku) < b^* f(u)$ for all $(k, u) \in [a^*, b^*] \times (0, u_+^*]$.

We only consider (I) since we are similarly led to a contradiction for (II). By (I), there exists $\varepsilon > 0$ such that $f(ku) > a^* f(k) + \varepsilon$ for all $(k, u) \in [\frac{u^*}{2}, u_+^*] \times [a^*, b^*]$. Proposition 5.1-(iii) shows that there is $T_1 > 0$ such that $u_+(x) \ge \frac{u^*}{2}$ for all $x \ge T_1$. This, together with $\omega(\psi) \ge a^* u_+$ and the invariance of $\omega(\psi)$, implies that $b^* u_+^* \ge \varphi(-\tau, x) \ge \frac{a^* u^*}{2}$ for all $(x, \varphi) \in [T_1, \infty) \times \omega(\psi)$ and thus $f(\varphi(-\tau, \cdot)) - a^* f(u_+) \ge \varepsilon h^{T_1, \infty}$ for all $\varphi \in \omega(\psi)$. Note that $K[f(\varphi(-\tau, \cdot)) - a^* f(u_+)](x) \ge \varepsilon \int_{T_1}^{\infty} [\Gamma_\vartheta(x - y) - \Gamma_\vartheta(x + y)] dy$ for all $(x, \varphi) \in \mathbb{R}_+ \times \omega(\psi)$. Thus, there is $T_2 > T_1$ such that $K[f(\varphi(-\tau, \cdot)) - a^* f(u_+)](x) \ge \frac{\varepsilon}{2}$ for all $(x, \varphi) \in [T_2, \infty) \times \omega(\psi)$. Let us define $\eta : \mathbb{R}_+^2 \times \omega(\psi) \times \mathbb{R} \to \mathbb{R}$ by $\eta(t, x, \varphi, \alpha) = u^{\varphi}(t, x + T_2) - a^* u_+(x + T_2) - \frac{\varepsilon}{2} + \alpha e^{-\sqrt{\frac{\mu}{d}} x} + \alpha e^{-\mu t}$ for all $(t, x, \varphi) \in \mathbb{R}_+^2 \times \omega(\psi)$. Then $\eta(\cdot, \cdot, \varphi, \frac{\varepsilon}{2} + a^* u_+^*) \in B(\mathbb{R}_+^2, \mathbb{R}) \cap L^\infty(\mathbb{R}_+^2, \mathbb{R})$ for all $\varphi \in \omega(\psi)\}$. Let

$$\zeta \in \mathcal{D} \triangleq \{\eta(\cdot, \cdot, \varphi, \frac{\varepsilon}{2} + a^* u_+^*) \in BC(\mathbb{R}_+^2, \mathbb{R}) : \varphi \in \omega(\psi)\}.$$

It follows from the definition of $\mathcal{D}$ and (2.2) that $\zeta(t, x)$ satisfies the following equation:

$$\begin{cases} \frac{\partial \zeta}{\partial t}(t,x) & \geq & d\zeta_{xx}(t,x) - \mu\zeta(t,x), \qquad (t,x) \in (0,\infty)\times[0,\infty), \\ \zeta(t,0) & \geq & 0, \qquad t \in \mathbb{R}_+, \\ \zeta(0,x) & \geq & 0, \qquad x \in \mathbb{R}_+. \end{cases} \tag{5.2}$$

Then $\zeta(t,x) \geq 0$ for all $(t,x,\zeta) \in \mathbb{R}_+^2 \times \mathcal{D}$ due to the Phragmén-Lindelöf type maximum principle in [37]. Hence, there is $T_3 > T_2$ such that $u^\varphi(t,x) - a^* u_+(x) > \frac{\varepsilon}{3}$ for all $(t,x,\varphi) \in [T_3,\infty)^2 \times \omega(\psi)$ Hence the invariance of $\omega(\psi)$ forces that $\varphi(0,\cdot) - a^* u_+ \geq \frac{\varepsilon}{3} h^{T_3,\infty}$ for all $\varphi \in \omega(\psi)$. From (2.2), we have, for all $\varphi \in \omega(\psi)$ and $t > 0$,

$$\begin{aligned} u^\varphi(t,\cdot) - a^* u_+ &= T(t)[u^\varphi(0,\cdot)] + \mu \int_0^t T(t-s)[K[f(u^\varphi(s-\tau,\cdot))]]\mathrm{d}s - a^* u_+ \\ &= T(t)[u^\varphi(0,\cdot) - a^* u_+] \\ &\quad + \mu \int_0^t T(t-s)[K[f(u^\varphi(s-\tau,\cdot))] - a^* K[f(u_+)]]\mathrm{d}s \\ &\geq T(t)[u^\varphi(0,\cdot) - a^* u_+], \end{aligned}$$

and thus $u^\varphi(t,\cdot) - a^* u_+ \in X_+^\circ$ for all $t > 0$ and $\varphi \in \omega(\psi)$. Again the invariance of $\omega(\psi)$ forces that $\varphi(0,\cdot) - a^* u_+ \in X_+^\circ$ for all $\varphi \in \omega(\psi)$, which together with the compactness of $\omega(\psi)$, implies that there is $\delta > 0$ such that $\varphi(0,x) - a^* u_+(x) > \delta$ for all $(x,\varphi) \in [0,T_3] \times \omega(\psi)$. So, $\varphi(0,x) - a^* u_+(x) \geq \min\{\delta, \frac{\varepsilon}{3}\} > 0$ for all $(x,\varphi) \in \mathbb{R}_+ \times \omega(\psi)$, a contradiction to the choice of a^*.

For (II), we are similarly led to a contradiction. Consequently we see that $a^* = b^* = 1$ and hence $\omega(\psi) = \{u_+\}$.

To prove $\lim_{t\to\infty} |||(u^\psi)_t - u_+||| = 0$, it suffices to prove $\lim_{t\to\infty} \sup\{|u^\psi(t+\theta,x) - u_+(x)| : (\theta,x) \in [-\tau,0]\times(-\infty,0]\} = 0$. Indeed, for any $\epsilon > 0$, by Proposition 5.1, there is $\sigma = \sigma(\epsilon) > 0$ such that $u_+(x) < \frac{\epsilon}{3}$ for all $x \in (-\infty,-\sigma]$. In view of the previous discussions, there is $t^* = t^*(\epsilon,\psi) > 0$ such that $|u^\psi(t,x) - u_+(x)| < \frac{\epsilon}{3}$ for all $(t,x) \in [t^*,\infty) \times [-\sigma,0]$. Let $v(t,x) = u^\psi(t+t^*, x-\sigma) - \frac{2\epsilon}{3} e^{\sqrt{\frac{\mu}{d}}x} - \max\{\mathcal{M}, ||\psi||_{L^\infty}\} e^{-\mu t}$ for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$. Then by (2.2), $v(t,x)$ satisfies

$$\begin{cases} \frac{\partial v}{\partial t}(t,x) & = & dv_{xx}(t,x) - \mu v(t,x), \qquad (t,x) \in (0,\infty)\times(-\infty,0), \\ v(t,0) & \leq & 0, \qquad t \in \mathbb{R}_+, \\ v(0,x) & \leq & 0, \qquad x \in (-\infty,0]. \end{cases} \tag{5.3}$$

By the Phragmén-Lindelöf type maximum principle in [37], we easily see that $v(t,x) \leq 0$ for all $(t,x) \in \mathbb{R}_+ \times (-\infty,0]$. Hence, $u^\psi(t,x) \leq \frac{2\epsilon}{3} + \max\{\mathcal{M}, ||\psi||_{L^\infty}\} e^{-\mu(t-t^*)}$ for all $(t,x) \in [t^*,\infty) \times (-\infty,-\sigma]$. It follows that there is $t^{**} = t^{**}(\epsilon,\psi) > t^*$ such that $u^\psi(t,x) \leq \epsilon$ for all $(t,x) \in [t^{**},\infty) \times (-\infty,0]$. This completes the proof. □

Note that verifying (H3) is the key for applying Theorem 5.1. However since $u_+^* = ||u_+||_X$ cannot be explicitly obtained in general, verifying (H3) becomes impractical in applications. Motivated by [62], we seek a similar alternative condition that is given in terms of f only, as below:

(H4) $\liminf_{k\to 0+} G(k,u;f) \equiv u$ and $\frac{\partial G(k,u;f)}{\partial k}(1-u) > 0$ in $(0, f^*] \times ((0,\infty)\setminus\{1\})$, where $f^* \triangleq \max\{f(x) : x \in [0,u^*]\}$ and $G(k,u;f) = \frac{f(ku)}{f(k)}$.

As a direct corollary of Theorem 5.1, applying Lemma 3.7 in [62], we have the following theorem.

Theorem 5.2. *Assume that (H2) and (H4) hold. If $F^2 = F \circ F$ has a unique positive fixed point u^* with $F(\cdot) = G(f^*, \cdot; f)$, then u_+ is a globally attractive positive steady state of (2.2) in C_{++} in the sense that $\lim_{t\to\infty} |||(u^\psi)_t - u_+||| = 0$ for all $\psi \in C_{++}$.*

6. Asymptotic propagation and spreading speed

In this section, we explore the traveling-like asymptotic behaviour of nontrivial solutions in space-time region. This enables us to develop a unified method for studying spreading speeds and asymptotic propagation phenomena for (2.2) and (3.3).

The following lemma gives the space-time decision region for large (t_0, x_0).

Lemma 6.1. *For any $\epsilon > 0$, $\gamma \geq \mathcal{M}$, and $0 < a \leq b < \infty$, there exists $\varrho = \varrho(\epsilon, \gamma, a, b) > 0$ such that $(1+\epsilon)\max f([a,b]) \geq u^\varphi(t_0, x_0) \geq (1-\epsilon)\min f([a,b])$ for any $(\varphi, t_0, x_0) \in C_\gamma \times (\varrho, \infty)^2$ whenever $a \leq u^\varphi(t,x) \leq b$ for all $(t,x) \in (t_0,x_0) + [-\varrho-\tau, -\tau] \times [-\varrho, \varrho]$.*

Proof. Note that

$$\lim_{x\to\infty} \int_0^x \int_{-\frac{x}{2}}^{\frac{x}{2}} k(s,y)[\int_{-\frac{x}{2}}^{\frac{x}{2}} \Gamma_\vartheta(z)\mathrm{d}z - \int_{\frac{x}{2}}^{\infty} \Gamma_\vartheta(z)\mathrm{d}z]\mathrm{d}y\mathrm{d}s = 1$$

and

$$\lim_{x\to\infty} (\max f([a,b]) + \gamma e^{-\mu x} + \max f([0,\gamma])[2\int_{|y|\geq\frac{x}{2}} k_0(y)\mathrm{d}y + \int_{|z|\geq\frac{x}{2}} \Gamma_\vartheta(z)\mathrm{d}z]) = \max f([a,b]).$$

Suppose that $\epsilon > 0$, $\gamma \geq \mathcal{M}$, and $0 < a \leq b < \infty$. Then, there exists $\varrho = \varrho(\epsilon, \gamma, a, b) > 0$ such that

$$\int_0^\varrho \int_{-\frac{\varrho}{2}}^{\frac{\varrho}{2}} k(s,y)[\int_{-\frac{\varrho}{2}}^{\frac{\varrho}{2}} \Gamma_\vartheta(z)\mathrm{d}z - \int_{\frac{\varrho}{2}}^{\infty} \Gamma_\vartheta(z)\mathrm{d}z]\mathrm{d}y\mathrm{d}s > 1-\epsilon$$

and

$$\max f([a,b]) + \gamma e^{-\mu\varrho} + \max f([0,\gamma])[2\int_{|y|\geq\frac{\varrho}{2}} k_0(y)\mathrm{d}y + \int_{|z|\geq\frac{\varrho}{2}} \Gamma_\vartheta(z)\mathrm{d}z] < (1+\epsilon)\max f([a,b]).$$

Let $\varphi \in C_\gamma$ and $(t_0, x_0) \in (\varrho, \infty)^2$ be such that $a \le u(t,x) \triangleq u^\varphi(t,x) \le b$ for all $(t,x) \in (t_0, x_0) + [-\varrho - \tau, -\tau] \times [-\varrho, \varrho]$. By (2.2) and the Fubini's theorem,

$$
\begin{aligned}
& u(t_0, x_0) = u^{u_{t_0-\varrho}}(\varrho, x_0) \\
& \ge T(\varrho)[u(t_0-\varrho,\cdot)](x_0) + \mu \int_0^\varrho T(\varrho - s)[K[f(u(s+t_0-\varrho-\tau,\cdot))]](x_0)\mathrm{d}s \\
& \ge \mu \int_0^\varrho T(\varrho - s)[K[f(u(s+t_0-\varrho-\tau,\cdot))]](x_0)\mathrm{d}s \\
& \ge \int_0^\varrho \int_0^\infty k(\varrho - s, x_0 - y)[\int_0^\infty [\Gamma_\vartheta(y-z) - \Gamma_\vartheta(y+z)] f(u(s+t_0-\varrho-\tau, z))\mathrm{d}z]\mathrm{d}y\mathrm{d}s \\
& = \int_0^\varrho \int_{-x_0}^\infty k(\varrho - s, y)[\int_{-x_0}^\infty [\Gamma_\vartheta(y-z) - \Gamma_\vartheta(y+z+2x_0)] f(u(s+t_0-\varrho-\tau, z+x_0))\mathrm{d}z]\mathrm{d}y\mathrm{d}s \\
& \ge \min f([a,b]) \int_0^\varrho \int_{-\varrho}^\infty k(\varrho - s, y)[\int_{-\varrho}^\varrho [\Gamma_\vartheta(y-z) - \Gamma_\vartheta(y+z+2x_0)]\mathrm{d}z]\mathrm{d}y\mathrm{d}s \\
& \ge \min f([a,b]) \int_0^\varrho \int_{-\frac{\varrho}{2}}^{\frac{\varrho}{2}} k(s, y)[\int_{-\varrho}^\varrho [\Gamma_\vartheta(y-z) - \Gamma_\vartheta(y+z+2x_0)]\mathrm{d}z]\mathrm{d}y\mathrm{d}s \\
& \ge \min f([a,b]) \int_0^\varrho \int_{-\frac{\varrho}{2}}^{\frac{\varrho}{2}} k(s, y)[\int_{-\varrho}^\varrho \Gamma_\vartheta(y-z)\mathrm{d}z - \int_\varrho^\infty \Gamma_\vartheta(y+z)\mathrm{d}z]\mathrm{d}y\mathrm{d}s \\
& \ge \min f([a,b]) \int_0^\varrho \int_{-\frac{\varrho}{2}}^{\frac{\varrho}{2}} k(s, y)[\int_{-\frac{\varrho}{2}}^{\frac{\varrho}{2}} \Gamma_\vartheta(z)\mathrm{d}z - \int_{\frac{\varrho}{2}}^\infty \Gamma_\vartheta(z)\mathrm{d}z]\mathrm{d}y\mathrm{d}s \\
& \ge (1-\varepsilon)\min f([a,b]).
\end{aligned}
$$

On the other hand, again by (2.2) and Fubini's theorem, we have

$$
\begin{aligned}
& u(t_0, x_0) = u^{u_{t_0-\varrho}}(\varrho, x_0) \\
& = T(\varrho)[u(t_0-\varrho,\cdot)](x_0) + \mu \int_0^\varrho T(\varrho - s)[K[f(u(s+t_0-\varrho-\tau,\cdot))]](x_0)\mathrm{d}s
\end{aligned}
$$

$$
\begin{aligned}
&\le \gamma e^{-\mu\varrho} + \mu \int_0^{\varrho} T(\varrho - s)[K[f(u(s+t_0-\varrho-\tau,\cdot))]](x_0)\mathrm{d}s \\
&= \gamma e^{-\mu\varrho} + \int_0^{\varrho}\int_0^{\infty} k(\varrho - s, x_0 - y)[\int_0^{\infty}[\Gamma_\vartheta(y-z) - \Gamma_\vartheta(y+z)] f(u(s+t_0-\varrho-\tau,z))\mathrm{d}z]\mathrm{d}y\mathrm{d}s \\
&= \gamma e^{-\mu\varrho} + \int_0^{\varrho}\int_{-x_0}^{\infty} k(s,y)[\int_{-x_0}^{\infty}[\Gamma_\vartheta(y-z) - \Gamma_\vartheta(y+z+2x_0)] f(u(t_0-s-\tau,z+x_0))\mathrm{d}z]\mathrm{d}y\mathrm{d}s \\
&\le \gamma e^{-\mu\varrho} + \int_0^{\varrho}\int_{-x_0}^{\infty} k(s,y)[\int_{-x_0}^{\infty}\Gamma_\vartheta(y-z) f(u(t_0-s-\tau,z+x_0))\mathrm{d}z]\mathrm{d}y\mathrm{d}s \\
&\le \gamma e^{-\mu\varrho} + \int_0^{\varrho}\int_{-x_0}^{\infty} k(s,y)[\int_{-\varrho}^{\varrho}\Gamma_\vartheta(y-z) f(u(t_0-s-\tau,z+x_0))\mathrm{d}z]\mathrm{d}y\mathrm{d}s \\
&\quad + \int_0^{\varrho}\int_{-x_0}^{\infty} k(s,y)[\int_{|z|\ge\varrho}\Gamma_\vartheta(y-z) f(u(t_0-s-\tau,z+x_0))\mathrm{d}z]\mathrm{d}y\mathrm{d}s \\
&\le \gamma e^{-\mu\varrho} + \int_0^{\varrho}\int_{-\frac{\varrho}{2}}^{\frac{\varrho}{2}} k(s,y)[\int_{-\varrho}^{\varrho}\Gamma_\vartheta(y-z) f(u(t_0-s-\tau,z+x_0))\mathrm{d}z]\mathrm{d}y\mathrm{d}s \\
&\quad + \int_0^{\varrho}\int_{|y|\ge\frac{\varrho}{2}} k(s,y)[\int_{-\varrho}^{\varrho}\Gamma_\vartheta(y-z) f(u(t_0-s-\tau,z+x_0))\mathrm{d}z]\mathrm{d}y\mathrm{d}s \\
&\quad + \int_0^{\varrho}\int_{-\infty}^{\infty} k(s,y)[\int_{|z|\ge\varrho}\Gamma_\vartheta(y-z) f(u(t_0-s-\tau,z+x_0))\mathrm{d}z]\mathrm{d}y\mathrm{d}s \\
&\le \gamma e^{-\mu\varrho} + \max f([a,b]) \int_0^{\varrho}\int_{-\frac{\varrho}{2}}^{\frac{\varrho}{2}} k(s,y)[\int_{-\varrho}^{\varrho}\Gamma_\vartheta(y-z)\mathrm{d}z]\mathrm{d}y\mathrm{d}s \\
&\quad + \max f([0,\gamma]) \int_0^{\varrho}[\int_{|y|\ge\frac{\varrho}{2}} k(s,y)\int_{-\varrho}^{\varrho}\Gamma_\vartheta(y-z)\mathrm{d}z\mathrm{d}y + \int_{-\infty}^{\infty} k(s,y)\int_{|z|\ge\varrho}\Gamma_\vartheta(y-z)\mathrm{d}z\mathrm{d}y]\mathrm{d}s \\
&\le \gamma e^{-\mu\varrho} + \max f([a,b]) \\
&\quad + \max f([0,\gamma]) \int_0^{\varrho}[\int_{|y|\ge\frac{\varrho}{2}} k(s,y)\int_{-\varrho}^{\varrho}\Gamma_\vartheta(y-z)\mathrm{d}z\mathrm{d}y + \int_{-\infty}^{\infty} k(s,y)\int_{|z|\ge\varrho}\Gamma_\vartheta(y-z)\mathrm{d}z\mathrm{d}y]\mathrm{d}s
\end{aligned}
$$

$$\leq \gamma e^{-\mu\varrho} + \max f([a,b])$$

$$+ \max f([0,\gamma]) \int_0^{\varrho} [2 \int_{|y|\geq \frac{\varrho}{2}} k(s,y)\mathrm{d}y + \int_{|y|\leq \frac{\varrho}{2}} k(s,y) \int_{|z|\geq \frac{\varrho}{2}} \Gamma_\vartheta(z)\mathrm{d}z\mathrm{d}y]\mathrm{d}s$$

$$\leq \max f([a,b]) + \gamma e^{-\mu\varrho} + \max f([0,\gamma])[2 \int_{|y|\geq \frac{\varrho}{2}} k_0(y)\mathrm{d}y + \int_{|z|\geq \frac{\varrho}{2}} \Gamma_\vartheta(z)\mathrm{d}z]$$

$$\leq (1+\varepsilon)\max f([a,b]). \quad \square$$

Now, we are ready to derive the traveling-like asymptotic behavior of nontrivial solutions of (2.2) in space-time region.

Proposition 6.1. *Assume that $f^2 = f \circ f$ has a unique positive fixed point u^*. Let $c \in [0, c^*(f))$ and $\varphi \in C_{++}$. Then*

$$\lim_{\alpha\to\infty}[\inf\{u^\varphi(t,x) : (t,x) \in \Omega^*_{\alpha,c}\}] = \lim_{\alpha\to\infty}[\sup\{u^\varphi(t,x) : (t,x) \in \Omega^*_{\alpha,c}\}] = u^*,$$

where $\Omega^*_{c,\alpha} = \{(t,x) \in \mathbb{R}^2_+ : t \geq \alpha$ and $\alpha \leq x \leq ct\}$ for all $\alpha \in \mathbb{R}_+$.

Proof. In view of Proposition 2.1, we may assume that $\varphi \in C([-\tau,0]\times\mathbb{R}, (0, \frac{1}{2}+\mathcal{M}])$ and hence $0 < u^\varphi(t,x) \leq \frac{1}{2}+\mathcal{M}$ for all $(t,x) \in [-\tau,\infty)\times\mathbb{R}$. By Proposition 4.2, there exist $\alpha^* > 0$ and $\varepsilon^* > 0$ such that $u(t,x) \geq \varepsilon^*$ for all $t \geq \alpha^*$ and $\alpha^* \leq x \leq 2\alpha^* + \frac{c+c^*(f)}{2}t$. For any $\varepsilon \in [0, \frac{c^*(f)-c}{2}]$, define

$$U_-(\varepsilon) = \lim_{\alpha\to\infty}[\inf\{u^\varphi(t,x) : (t,x) \in \Omega^*_{c+\varepsilon,\alpha}\}]$$

and

$$U_+(\varepsilon) = \lim_{\alpha\to\infty}[\sup\{u^\varphi(t,x) : (t,x) \in \Omega^*_{c+\varepsilon,\alpha}\}].$$

Then $\varepsilon^* \leq U_-(\varepsilon) \leq U_+(\varepsilon) \leq \frac{1}{2}+\mathcal{M}$ for all $\varepsilon \in [0, \frac{c^*(f)-c}{2}]$. Note that $U_\pm(\varepsilon)$ are monotone in $\varepsilon \in [0, \frac{c^*(f)-c}{2}]$. Due to the monotonicity of $U_\pm$, we easily see that $U_\pm(\varepsilon)$ are continuous in $\varepsilon \in [0, \frac{c^*(f)-c}{2}]$ except possibly for ε from a countable set of $[0, \frac{c^*(f)-c}{2}]$. Therefore, we may assume, without loss of generality, that U_- and U_+ are continuous at some $\varepsilon_1 \in [0, \frac{c^*(f)-c}{2}]$.

We claim $U_-(\varepsilon) = U_+(\varepsilon) = u^*$ for some $\varepsilon \in (0, \frac{c^*(f)-c}{2}]$. Otherwise, $\{U_-(\varepsilon), U_+(\varepsilon)\} \neq \{u^*\}$ for all $\varepsilon \in (0, \frac{c^*(f)-c}{2}]$. In particular, $U_-(\varepsilon_1) < U_+(\varepsilon_1)$ or $U_-(\varepsilon_1) = U_+(\varepsilon_1) \neq u^*$. Since f^2 has a unique positive fixed point u^*, by Lemma 5.3 in [55], we have $\max f([U_-(\varepsilon_1), U_+(\varepsilon_1)]) < U_+(\varepsilon_1)$ or $\min f([U_-(\varepsilon_1), U_+(\varepsilon_1)]) > U_-(\varepsilon_1)$. Thus, there is $\delta_1 \in (0, \min\{1, \frac{U_-(\varepsilon_1)}{3}\})$ such that $(1+\delta_1)\max f([U_-(\varepsilon_1)-\delta_1, U_+(\varepsilon_1)+\delta_1]) < U_+(\varepsilon_1)$ or $(1-\delta_1)\min f([U_-(\varepsilon_1)-\delta_1, U_+(\varepsilon_1)+\delta_1]) > U_-(\varepsilon_1)$.

It suffices to consider the case of $(1+\delta_1)\max f([U_-(\varepsilon_1)-\delta_1, U_+(\varepsilon_1)+\delta_1]) < U_+(\varepsilon_1)$ since similarly we may deal with the case of $(1-\delta_1)\min f([U_-(\varepsilon_1)-\delta_1, U_+(\varepsilon_1)+\delta_1]) >$

$U_-(\varepsilon_1)$. In view of the definitions of $U_\pm(\varepsilon_1)$, there is $\alpha_1 > 0$ such that $U_+(\varepsilon_1) + \delta_1 > u^\varphi(t,x) \geq U_-(\varepsilon_1) - \delta_1$ for all $(t,x) \in \Omega^*_{c+\varepsilon_1,\alpha_1}$. Applying Lemma 6.1 with $a = U_-(\varepsilon_1) - \delta_1$, $b = U_+(\varepsilon_1) + \delta_1$, $\epsilon = \delta_1$, and $\gamma = \frac{1}{2} + \mathcal{M}$, there is $\varrho = \varrho(\epsilon, \gamma, a, b) > 0$ such that $u^\varphi(t,x) \leq (1+\delta_1) \max f([U_-(\varepsilon_1) - \delta_1, U_+(\varepsilon_1) + \delta_1])$ when $t, x \in [\varrho, \infty)$ with $(t,x) + [-\varrho - \tau, -\tau] \times [-\varrho, \varrho] \subseteq \Omega^*_{c+\varepsilon_1,\alpha_1}$. Suppose $\varsigma \in (0, \varepsilon_1)$. According to the definition of $\Omega^*_{c+\varepsilon,\alpha}$, there exists $\alpha_2 > \alpha_1$ such that $(t,x) + [-\varrho - \tau, -\tau] \times [-\varrho, \varrho] \subseteq \Omega^*_{c+\varepsilon_1,\alpha_1}$ for all $(t,x) \in \Omega^*_{c+\varsigma,\alpha_2}$. So, $u^\varphi(t,x) \leq (1+\delta_1) \max f([U_-(\varepsilon_1) - \delta_1, U_+(\varepsilon_1) + \delta_1])$ for all $(t,x) \in \Omega^*_{c+\varsigma,\alpha_2}$, and hence $U_+(\varsigma) \leq (1+\delta_1) \max f([U_-(\varepsilon_1) - \delta_1, U_+(\varepsilon_1) + \delta_1])$. By the continuity of U_+ at ε_1 and letting $\tau \to \varepsilon_1$, we have $U_+(\varepsilon_1) \leq (1+\delta_1) \max f([U_-(\varepsilon_1) - \delta_1, U_+(\varepsilon_1) + \delta_1]) < U_+(\varepsilon_1)$, a contradiction. Therefore, $U_-(\varepsilon_2) = U_+(\varepsilon_2) = u^*$ for some $\varepsilon_2 \in (0, \frac{c^*(f)-c}{2}]$. This, together with the monotonicity of $U_\pm$, yields that $U_-(\varepsilon) = U_+(\varepsilon) = U_-(0) = U_+(0) = u^*$ for all $\varepsilon \in [0, \varepsilon_2]$ and hence the conclusion follows. This completes the proof. $\square$

The following gives the spreading speeds and asymptotic propagation phenomena for (2.2).

Theorem 6.1. *Assume that (H2) and either (H3) or (H4) hold. Then the following statements hold.*

(i) (2.2) *has a unique positive steady state u_+ in X_+° with $\lim_{x\to-\infty} u(x) = 0$ and $\lim_{x\to\infty} u(x) = u^*$.*

(ii) *For any $c > c^*(f)$, if $\varphi \in C_+$ has a compact support then*

$$\lim_{t\to\infty} (\sup\{u^\varphi(t,x) : |x| \geq tc\}) = 0.$$

(iii) *For any $0 \leq c < c^*(f)$ and $\varphi \in C_{++}$,*

$$\lim_{t\to\infty} (\sup\{|u^\varphi(t,x) - u_+(x)| : -\infty < x \leq tc\}) = 0.$$

Proof. (i) follows from Proposition 5.1 and Theorem 5.1. For (ii), fix $c > c^*(f)$ and $\varphi \in C_+$ with a compact support. Define $\tilde{f} : \mathbb{R}_+ \to \mathbb{R}_+$ by $\tilde{f}(x) = \min\{f'(0)x, 1 + ||\varphi||_{L^\infty} + \mathcal{M}\}$ for all $x \in \mathbb{R}_+$. Let $\tilde{c} = \frac{c+c^*(f)}{2}$. Then by the remark after Proposition 3.1, there is $\phi \in X_+^\circ$ such that $\phi(x - \tilde{c}t)$ is a travelling wave of (3.3) with $f = \tilde{f}$ such that $\phi(\infty) = 0$ and $\phi(-\infty) = 1 + ||\phi||_{L^\infty} + \mathcal{M}$. Thus, by the compactness of $supp(\varphi)$, without loss of generality, we may assume that $\varphi \leq \phi$. It follows from $\tilde{c} < c$, Proposition 2.1 and the definition of $\tilde{f}$ that $u^\varphi(t,x) \leq \phi(x - \tilde{c}t)$ for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$, which implies

$$\lim_{t\to\infty} \max\{u^\varphi(t,x) : x \geq tc\} \leq \lim_{t\to\infty} \max\{\phi(x - \tilde{c}t) : x \geq tc\} = \phi(\infty) = 0.$$

Therefore, the conclusion holds.

For (iii), suppose that $c < c^*(f)$ and $\varphi \in C_{++}$. For any $\varepsilon > 0$, by Proposition 5.1 and Proposition 6.1, there exists $\alpha_1 > 0$ such that $|u_+(x) - u^*| < \frac{\varepsilon}{3}$ for all $x \in [\alpha_1, \infty)$ and $|u^\varphi(t,x) - u^*| < \frac{\varepsilon}{3}$ for all $(t,x) \in \mathbb{R}_+^2$ with $t \geq \alpha_1$ and $\alpha_1 \leq x \leq ct$. It follows that $|u^\varphi(t,x) - u_+(x)| < \varepsilon$ for all $(t,x) \in \mathbb{R}_+^2$ with $t \geq \alpha_1$ and $\alpha_1 \leq x \leq ct$. Again by Theorems 5.1 and 5.2, we have $\lim_{t\to\infty} ||u^\varphi(t,\cdot) - u_+||_{L^\infty((-\infty,\alpha_1],\mathbb{R})} = 0$ and hence there exists $\alpha_2 > \alpha_1$ such that $|u^\varphi(t,x) - u_+(x)| < \varepsilon$ for all $(t,x) \in [\alpha_2, \infty) \times (-\infty, \alpha_1]$. In other words, $|u^\varphi(t,x) - u_+(x)| < \varepsilon$ for all $t \geq \alpha_2$ and $ct \geq x > -\infty$. So, the conclusion follows. This completes the proof. $\square$

In terms of the description after Theorem 2.1 in [47], Theorem 6.1 states that if $\phi(x)$ is zero for all large values of x, then an observer who moves toward either of the two direction with a speed above $c^*(f)$ will see the solution go down to 0, while an observer who moves toward to the right at a speed below $c^*(f)$ will see the solution approach u_+.

Finally, we would like to apply our approach to equation (3.3) on $\mathbb{R}$ with the initial value $\varphi \in C_+$. This allows us to easily re-establish those results on spreading speed and traveling wave fronts for (3.3) with monostable nonlinearity.

To begin with, denote the solution of (3.3) on $[-\tau, \infty)$ by $\tilde{u}^\varphi(t,x)$. Then, by a similar proof in Lemma 6.1, we may obtain the following lemma.

Lemma 6.2. *For any $\epsilon > 0$, $\gamma \geq \mathcal{M}$, and $0 < a < b < \infty$, there exists $\varrho = \varrho(a,b,\epsilon,\gamma) > 0$ such that $(1+\epsilon)\max f([a,b]) \geq \tilde{u}^\varphi(t_0,x_0) \geq (1-\epsilon)\min f([a,b])$ for any $\varphi \in C_\gamma$ and $(t_0,x_0) \in (\varrho,\infty)\times\mathbb{R}$ whenever $a \leq \tilde{u}^\varphi(t,x) \leq b$ for all $(t,x) \in (t_0,x_0) + [-\varrho-\tau, -\tau] \times [-\varrho,\varrho]$.*

The following theorem gives the basic results for (3.3) on the spreading speed and travelling waves for monostable reaction-diffusion equations $\mathbb{R}$ in the line [2,21].

Theorem 6.2. *Suppose that (H2) holds and $f^2 = f \circ f$ has a unique positive fixed point u^*. Then the following statements are true.*

(i) *For any $c \geq c^*(f)$, there is $\phi \in X_+^\circ$ such that $\phi(x-ct)$ is a travelling wave of (3.3) with $\phi(\infty) = 0$ and $\phi(-\infty) = u^*$.*

(ii) *For any $c > c^*(f)$, if $\varphi \in C_+$ has a compact support then*

$$\lim_{t\to\infty}(\sup\{\tilde{u}^\varphi(t,x) : |x| \geq tc\}) = 0.$$

(iii) *For any $0 \leq c < c^*(f)$ and $\varphi \in C_+$,*

$$\lim_{t\to\infty}(\sup\{|\tilde{u}^\varphi(t,x) - u^*| : |x| \leq tc\}) = 0.$$

Proof. (i) follows from Remark 3.1 and (ii) follows from a similar discussion in the proof of Theorem 6.1-(ii).

(iii) Fix $\varphi \in C_+$ and $c < c^*(f)$. We *claim* that there exist $\kappa = \kappa(\varphi) > 0$, $a^* = a^*(\varphi) > 0$, and $b^* = b^*(\varphi) > 0$ such that $b^* \geq \tilde{u}^\varphi(t,x) \geq a^*$ for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$ with $t \geq \kappa$ and $|x| \leq ct$. Indeed, letting $b^* = b^*(\varphi) = \max\{\mathcal{M}, ||\varphi||_{L^\infty}\}$, we know that, by Proposition 2.1, there exists $\kappa_1 = \kappa_1(\varphi) > 0$ such that $b^* \geq \tilde{u}^\varphi(t,x)$ for all $(t,x) \in [\kappa_1,\infty)\times\mathbb{R}$. Note that there is $\epsilon_0 = \epsilon_0(\varphi) \in (0,1)$ such that $f'(u) > f'(0) - \epsilon_0 > 1$ for all $u \in [0,\epsilon_0]$, $f(\epsilon_0) = \min f([\epsilon_0, b^*])$, and $c < c^*(\tilde{f})$ where $\tilde{f} : \mathbb{R}_+ \to \mathbb{R}_+$ is defined by

$$\tilde{f}(u) = (f'(0) - \epsilon_0)\min\{u, \epsilon_0\} \text{ for all } u \in \mathbb{R}_+.$$

By Theorem 6.1 with $f = \tilde{f}$, there is $\phi^* \in X_+^\circ$ such that $\phi^*(\infty) = \epsilon_0(f'(0) - \epsilon_0)$ and

$$\lim_{t\to\infty}(\sup\{|u^\varphi(t,x) - \phi^*(x)| : x \leq tc\}) = 0.$$

Hence, by letting $a^* = a^*(\varphi) = \frac{1}{2}\min\{\phi^*(x) : x \in \mathbb{R}_+\}$, there exists $\kappa = \kappa(\varphi) > \kappa_1$ such that $a^* \le u^\varphi(t,x) \le \tilde{u}^\varphi(t,x)$ for all $(t,x) \in [\kappa_1, \infty) \times \mathbb{R}_+$ with $|x| \le ct$. These, together with the fact that $\tilde{u}^\varphi(t,-\cdot) = \tilde{u}^{\varphi(-\cdot)}(t,\cdot)$ satisfies (3.3), yield the above claim.

Let $\overline{\varphi}(\varepsilon) = \overline{\lim_{t\to\infty}} \max\{\tilde{u}^\varphi(t,x) : |x| \le t(c+\varepsilon)\}$ and $\underline{\varphi}(\varepsilon) = \underline{\lim_{t\to\infty}} \min\{\tilde{u}^\varphi(t,x) : |x| \le t(c+\varepsilon)\}$ for any $\varepsilon \in [0, \frac{c^*(f)-c}{2}]$. Then by the above claim, we have $\infty > b^* \ge \overline{\varphi}(\varepsilon) \ge \underline{\varphi}(\varepsilon) \ge a^* > 0$ for any $\varepsilon \in [0, \frac{c^*(f)-c}{2}]$. By using similar discussions in the proof of Proposition 6.1 with Lemma 6.1 replaced by Lemma 6.2, we easily see that there exists an $\varepsilon_1 \in [0, \frac{c^*(f)-c}{2}]$ such that $\underline{\varphi}(\varepsilon) = \overline{\varphi}(\varepsilon) = \underline{\varphi}(0) = \overline{\varphi}(0) = u^*$ for all $\varepsilon \in [0, \varepsilon_1]$ and hence the conclusion of (iii) follows. This completes the proof. $\square$

7. Examples

In this section, we illustrate the results of Theorems 6.1 and 6.2 by considering two concrete examples, that is, the non-local diffusive Nicholson's blowflies equation and the non-local diffusive Mackey-Glass equation.

Example 7.1. Consider the following birth function for (1.3) and (1.4):

$$b(u) = \frac{pu}{1+u^n} \tag{7.1}$$

where p and n are all positive constants. This function was initially used by Mackey and Glass [30] to model the blood cell production in an ordinary differential equation model. Since then various modified models have been proposed and studied by many researchers. One of the main topics on these models is the stability of a positive equilibrium, accounting for a long term stable blood concentration level. See, for example, [22,59] and the references therein.

By applying Lemma 4.1 in [62] and Theorem 6.1, we then immediately obtain the following results for (1.3) with $b(u)$ given by (7.1).

Theorem 7.1. *If $p\varepsilon > d_m$ and*

$$n \le \max\left\{\frac{p\varepsilon}{p\varepsilon - d_m}, 2\frac{(p\varepsilon)^n(n-1)^{n-1} + n^n d_m^n}{(p\varepsilon)^n(n-1)^{n-1}}\right\},$$

then there is $c^{**} > 0$ such that the following statements hold for (1.3) with $b(u)$ given by (7.1).

(i) (1.3)–(7.1) *has a unique positive steady state u_+ in X_+° satisfying $\lim_{x\to-\infty} u(x) = 0$ and $\lim_{x\to\infty} u(x) = [\frac{p\varepsilon - d_m}{d_m}]^{1/n}$.*

(ii) *For any $c > c^{**}$, if $\varphi \in C_+$ has a compact support then*

$$\lim_{t\to\infty} \sup\{u^\varphi(t,x) : |x| \ge tc\} = 0.$$

(iii) *For any $0 \le c < c^{**}$ and $\varphi \in C_{++}$,*

$$\lim_{t\to\infty} (\sup\{|u^\varphi(t,x) - u_+(x)| : -\infty < x \le tc\}) = 0.$$

Applying Theorem 6.2 and taking advantage of the proof of Theorem 4.2 and Remark 4.3 in [59], we then obtain the following results for (1.4)–(7.1).

Theorem 7.2. *If $p\varepsilon > d_m$ and $n \le \max\{2, \frac{2p\varepsilon}{p\varepsilon - d_m}\}$, then there is $c^{**} > 0$ such that the following statements are valid for* (1.4)–(7.1)

(i) *For any $c \ge c^{**}$, there is $\phi \in X_+^\circ$ such that $\phi(x-ct)$ is a travelling wavefront of* (1.4)–(7.1) *with $\phi(\infty) = 0$ and $\phi(-\infty) = [\frac{p\varepsilon - d_m}{d_m}]^{1/n}$.*

(ii) *For any $c > c^{**}$, if $\varphi \in C_+$ has a compact support then*

$$\lim_{t\to\infty} (\sup\{\tilde{u}^\varphi(t,x) : |x| \ge tc\}) = 0.$$

(iii) *For any $0 \le c < c^{**}$ and $\varphi \in C_+$,*

$$\lim_{t\to\infty} \sup\left\{\left|\tilde{u}^\varphi(t,x) - \left[\frac{p\varepsilon - d_m}{d_m}\right]^{1/n}\right| : |x| \le tc\right\} = 0.$$

Example 7.2. Consider the following so-called Ricker birth function for (1.3) and (1.4)

$$b(u) = pue^{-u}, \tag{7.2}$$

where p is a positive constant. This birth function has been used in the Nicholson's blowfly equation and its variations as well as in many other models for, e.g., fish population dynamics. See for example, [18,19,9,26,27,24,34,38,39,58,60,62] and the references therein.

Applying Lemma 4.2 in [62] and Theorem 6.1, we obtain the following results for (1.3) with the birth function given by (7.2).

Theorem 7.3. *If $2ed_m \ge p\varepsilon > d_m$ then there is $c^{**} > 0$ such that the following statements hold for* (1.3)–(7.2).

(i) (1.3)–(7.2) *has a unique positive steady state u_+ in X_+° with $\lim_{x\to-\infty} u(x) = 0$ and $\lim_{x\to\infty} u(x) = \ln \frac{p\varepsilon}{d_m}$.*

(ii) *For any $c > c^{**}$, if $\varphi \in C_+$ has a compact support then*

$$\lim_{t\to\infty} \sup\{u^\varphi(t,x) : |x| \ge tc\} = 0.$$

(iii) *For any $0 \le c < c^{**}$ and $\varphi \in C_{++}$,*

$$\lim_{t\to\infty} \sup\{|u^\varphi(t,x) - u_+(x)| : -\infty < x \le tc\} = 0.$$

Applying Theorem 6.2 and making use of the proof of Theorem 4.1 and Remark 4.3 in [59], we then obtain the following results for (1.4) with the birth function given by (7.2).

Theorem 7.4. *If $e^2 d_m \ge p\varepsilon > d_m$, then there is $c^{**} > 0$ such that the following statements are valid for* (1.4)–(7.2).

(i) *For any* $c \geq c^{**}$, *there is* $\phi \in X_+^\circ$ *such that* $\phi(x-ct)$ *is a travelling wave of* (1.4)–(7.2) *with* $\phi(\infty)=0$ *and* $\phi(-\infty)=\ln\frac{p\varepsilon}{d_m}$.

(ii) *For any* $c > c^{**}$, *if* $\varphi \in C_+$ *has a compact support then*

$$\lim_{t\to\infty} \sup\{\tilde{u}^{\varphi}(t,x) : |x| \geq tc\} = 0.$$

(iii) *For any* $0 \leq c < c^{**}$ *and* $\varphi \in C_+$,

$$\lim_{t\to\infty} \sup\left\{\left|\tilde{u}^{\varphi}(t,x) - \ln\frac{p\varepsilon}{d_m}\right| : |x| \leq tc\right\} = 0.$$

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