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Existence, multiplicity, shape and attractivity of heterogeneous steady states for bistable reaction-diffusion equations in the plane ☆

Taishan Yi [a], Xingfu Zou [b,*]

[a] School of Mathematics (Zhuhai), Sun Yat-Sen University, Zhuhai, Guangdong 519082, PR China
[b] Department of Applied Mathematics, University of Western Ontario, London, Ontario, Canada, N6A 5B7

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Abstract

We consider a class of bistable reaction-diffusion equations in the plane. First we introduce a partition of the plane into infinitely many sectors and consider Dirichlet problems in these sectors. By establish some *a priori* estimates for nontrivial solutions to these sub-systems, we obtain the existence and attractivity of a heterogeneous steady state of the Dirichlet problem in each of the sectors and prove the existence of a maximum positive steady state and describe the asymptotic behaviours of positive steady states at the infinities. We also estimate $\omega-$limit sets at the vicinities of the boundaries of the sectors near origin and at infinities. Further assuming the sub-linearity for the reaction term, we obtain the uniqueness and attractivity of a heterogeneous steady state by applying the dynamical and sliding methods. These results help us describe the multiplicity, shape and attractivity of the heterogeneous steady states for the equation.

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Keywords: Reaction diffusion equation; Steady state; Heterogeneity; Attractivity; Bistable; Multiplicity

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* Corresponding author.
E-mail addresses: yitaishan@mail.sysu.edu.cn (T. Yi), xzou@uwo.ca (X. Zou).

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1. Introduction

Consider the following reaction-diffusion equation

$$\frac{\partial u}{\partial t}(t,x) = \Delta u(t,x) + f(u(t,x)), \quad t > 0, \quad x \in \Omega \subset \mathbb{R}^N \tag{1.1}$$

where Δ is the Laplacian in $\mathbb{R}^N$ and $f : \mathbb{R} \to \mathbb{R}$ is a continuously differentiable function. Naturally, one would like to understand the global structure of the trajectories of equation (1.1) and their asymptotic behaviours as $t \to \infty$. However, with very few exceptions of special cases, this problem is far too complicated to deal with as a whole. A basic question for (1.1) is the following: do globally defined and bounded solutions converge to a steady state as $t \to \infty$? This question is, in general, mathematically challenging and still largely remains open. Depending on the situations of spatial domains Ω and reaction terms f, different problems may arise.

When Ω is a bounded domain in $\mathbb{R}^N$, various boundary conditions can be posed depending on the practical scenario of the problem, and accordingly, various topics would be the focus. Among these boundary conditions are the typical homogeneous Dirichlet boundary value condition and Neumann boundary value condition, and corresponding major concerns are convergence or quasi-convergence of bounded solutions. For example, under the homogeneous Neumann boundary condition, (1.1) is gradient like with an energy function, and thus, by standard parabolic regularity estimates, one can easily conclude that each bounded solution approaches a set of equilibria (see e.g., [3,28] for details). Moreover, by applying the zero number method, convergence to equilibria has been proven on a bounded interval ($N = 1$) in [6,27,37] or in a circle in [5]. By the Lojasiewicz-Simon inequality, Jendoubil [23] showed and Simon [34] also confirmed that each bounded solution converges to an equilibrium when the nonlinearity f is analytic. By posing some strong restrictions on the linearization at any equilibrium or assuming a special structure of the set of equilibria, other convergence results have also been obtained in [21,22,25]. A reader is referred to the nice survey [29] for more details on the methods and results in case of bounded domains. On the other hand, it is known that if the equation is not spatially homogeneous, then on a *multidimensional domain* there may exist non-convergent bounded trajectories (see, e.g., [31,32]).

When the spatial domain is unbounded, the asymptotic behaviours of the bounded solutions of equations (1.1) become extremely complicated. It turns out that one usually has to impose more restrictions on initial functions and reaction terms, and sometimes on the spatial domains to obtain results for more elaborate systems. For systems with the domain being the whole space, traveling wave solutions and asymptotic propagation are two important topics [1,18,24]. Traveling wave solutions may quite often determine the long term behaviour of other solutions while asymptotic propagation describes how other solutions converge to an equilibrium as $t \to \infty$. The study on traveling waves for R-D equation with monostable reaction terms can be traced back to the celebrated papers of Fisher [18] and Kolmogorov et al. [24] while exploration of the asymptotic spreading speeds was mathematically started by Aronson and Weinberger [1]. Recently, Berestycki et al. [2] have outlined a theory of various asymptotic spreading speeds for (1.1) in general non-periodic spatially unbounded domains under Neumann boundary conditions and exhibited that spatial domains may affect the spreading speeds.

For bistable or more complicated reaction terms, traveling waves was initially studied in [17]. There are some essential differences between bistable and monostable nonlinearities. For example, by the results in [9], it is seen that every bounded steady state of (1.1) is constant when f

is monostable and Ω is the full Euclidean space. However, this shall not be case for a bistable or general f since a bounded steady states of (1.1) may be a ground steady solution or other heterogeneous steady state, see [4,30] and the references therein. As pointed out in [30], both quasiconvergent and non-quasiconvergent solutions are possible even for $\Omega = \mathbb{R}$, pending on the initial functions. Moreover, there are examples of spatially non-homogeneous equations on $\mathbb{R}^{\mathbb{N}}$ with $N \geq 3$, which possess nontrivial recurrent orbits (see, e.g., [20]). The existence of heterogeneous steady states and recurrent orbits impede us to explore the idea of general solutions tending to constant steady states in some sense. On the other hand, it is obvious that a traveling front with the speed $c \neq 0$ does not approach globally any equilibrium although it can locally approach a constant. Therefore, the choice of the underlying topology becomes a trickier and more important issue than in bounded domains.

Luckly, with some proper assumptions on the reaction term f, the results in [7,8,10–16,19, 30,38], roughly speaking, indicate that each bounded positive solution converges to a single steady state solution if the initial value u_0 is a nonnegative function with compact support or $\lim_{|x|\to\infty} u_0(x)$ exists, together with some extra assumption(s). Moreover, an involving steady state solution is either a *constant*, or *radially symmetric and radially decreasing* about some $x_0 \in \mathbb{R}^N$ with $\lim_{|x-x_0|\to\infty} u_0(x) =$ Constant. These restrictions on the initial functions and involving steady states do not seem to allow us to obtain information about other types of steady states and their asymptotic behaviours. For example, when $N = 1$, there can be spatially periodic steady states and spatially strictly increasing or decreasing steady states. For $N \geq 2$, more complicated steady state solutions are also possible. As such, it is worthwhile to explore the existence and asymptotic behaviour of other types of steady state solutions and this constitutes the goal of this paper.

In this paper, we focus on the following semilinear reaction-diffusion equation in $\mathbb{R}^2$, i.e. $N = 2$,

$$\begin{cases} \frac{\partial u}{\partial t}(t,x,y) = \Delta_{x,y} u(t,x,y) + f(u(t,x,y)), & t > 0,\ (x,y) \in \mathbb{R}^2, \\ u(0,x,y) = \phi(x,y), & (x,y) \in \mathbb{R}^2, \end{cases} \tag{1.2}$$

where $\phi \in L^\infty(\mathbb{R}^2)$ and $f \in C^1(\mathbb{R},\mathbb{R})$ with $f(u) = -f(-u)$ for all $u \in \mathbb{R}$ (implying $f(0) = 0$).

It is well-known that for a given $\phi \in L^\infty(\mathbb{R}^2)$, (1.2) has a unique mild solution on the maximal time interval $[0, \eta_\phi)$ for some $\eta_\phi \in (0, \infty]$ in the sense of Lunardi [26]. Denote this solution by $u^\phi(t,x,y)$. If the solution $u^\phi(t,x,y)$ is bounded, that is, $u^\phi \in L^\infty([0,\eta_\phi) \times \mathbb{R}^2)$, then its existence is global: $\eta_\phi = \infty$. Moreover, if ϕ is also continuous in $\mathbb{R}^2$, then $u^\phi \in C^{1,2}((0,\eta_\phi) \times \mathbb{R}^2))$ and $u^\phi(t,\cdot,\cdot) \in L^\infty(\mathbb{R}^2)$ for all $t \in (0,\eta_\phi)$, and thus, the solution $u^\phi(t,x,y)$ is also a classical solution of (1.2). This suggests that we only need to consider the initial functions belonging to $C(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$.

The remainder of the paper is organized as follows. In Section 2, we give some basic properties of (1.2). In Section 3, to further describe the complicated dynamics of this type of equations, we shall identify some positively invariant sets (sectors) that can not be mutually transformed under the translations and orthogonal transformations. These sets play a key role in obtaining the multiplicity of steady states in this work. More precisely, in each of these sectors, we consider the Dirichlet problem for (1.2) with (x,y) restricted to the sector, treating it as a sub-system of (1.2). We show that some dynamical properties of these sub-systems remain valid for (1.2) as well. Making use of these properties, we can obtain the multiplicity, shape and attractivity of heterogeneous steady states for the bistable reaction-diffusion equations in the plane. In Section 4, by developing some methods similar to those in [35,36] but with considerable modifications, we establish some *a priori* estimates for the sub-systems. More specifically, we give some iteration property of the diffusion, which turns out to be very useful for establishing *a priori* estimates for nontrivial solutions. We establish such an *a priori* estimate for nontrivial solutions after describing the delicate asymptotic properties of the diffusion operator. These allow us to show that the positive limit set of a positive solution is far away from zero at infinity locations away from the boundary of the sector domain. This finding plays a key role in the proof of existence and attractivity of heterogeneous steady states of (1.2). In Section 5, we shall establish the existence of heterogeneous steady state of Dirichlet boundary problem by using the *a priori* estimate obtained in the previous section, and we shall also establish the existence of a maximum positive steady state and explore the asymptotic behaviours of the positive steady state at infinities. In Section 6, by further assuming the sublinearity for the nonlinear term, we address the uniqueness and attractivity of the maximal positive steady state obtained in Section 5, by using the dynamical and sliding methods. Finally, Section 7 summarizes the main results about the multiplicity, shape and attractivity of the heterogeneous steady states for the bistable reaction-diffusion equation in $\mathbb{R}^2$ from the previous sections.

2. Preliminary results

In this section, we shall introduce some notations and present some preliminary results on the problem.

Let $\mathbb{N}$, $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{R}^2$ be the sets of all positive integers, reals, nonnegative reals, and 2-dimension vectors, respectively. Denote the Euclidean norm of $\mathbb{R}^2$ by $||\cdot||$.

Equip $X = C(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ with the usual supremum norm $||\cdot||_X = ||\cdot||_{L^\infty(\mathbb{R}^2)}$. Then X is a Banach space. Let $X_+ = \{\phi \in X : \phi(x, y) \geq 0 \text{ for all } x, y \in \mathbb{R}\}$. For a given number $r > 0$, define $X_r = \{\phi \in X : ||\phi||_X \leq r\}$.

For any functions $\xi : \mathbb{R}^2 \supseteq Dom(\xi) \to \mathbb{R}$, $\eta : \mathbb{R}^2 \supseteq Dom(\eta) \to \mathbb{R}$, we write $\xi \geq \eta$ if $\xi(x, y) \geq \eta(x, y)$ for all $(x, y) \in Dom(\xi) \cap Dom(\eta)$, $\xi > \eta$ if $\xi \geq \eta$ and $\xi \neq \eta$, $\xi \gg \eta$ if $\xi(x, y) > \eta(x, y)$ for all $(x, y) \in Int(Dom(\xi) \cap Dom(\eta))$.

We will consider the mild solution of system (1.2), which solves the following integral equation with the given initial function,

$$\begin{cases} u(t,\cdot) = T(t)[\phi] + \int_0^t T(t-s)[F(u(s,\cdot))]\mathrm{d}s, & t \geq 0, \\ u_0 = \phi \in X, \end{cases} \tag{2.1}$$

where

$$\begin{cases} T(0)[\phi](x, y) = \phi(x, y) \\ T(t)[\phi](x, y) = \dfrac{1}{4\pi t} \displaystyle\int_{\mathbb{R}^2} \phi(\tilde{x}, \tilde{y})[\exp(-\frac{(x-\tilde{x})^2}{4t})\exp(-\frac{(y-\tilde{y})^2}{4t})]\mathrm{d}\tilde{x}\mathrm{d}\tilde{y} \quad \text{for } t > 0 \end{cases} \tag{2.2}$$

for $(x, y, \phi) \in \mathbb{R}^2 \times X$ and $F : X \to X$ is defined by

$$F(\phi)(x, y) = f(\phi(x, y)) \qquad \text{for } (x, y) \in \mathbb{R}^2 \text{ and } \phi \in X.$$

As stated in the introduction, for any given $\phi \in X$, (2.1) has a unique solution on a maximal interval $[0, \eta_\phi)$, denoted by $u^\phi(t, x, y)$, which is also the classical solution of (1.2) on $(0, \eta_\phi)$ with $[0, \eta_\phi) \ni t \mapsto u^\phi(t, \cdot, \cdot) \in X$ being continuous and $\limsup_{t \to \eta_\phi^+} ||u^\phi(t, x, y)||_X = \infty$ when $\eta_\phi < \infty$.

Due to the non-compactness of the spatial domain, it is generally difficult and inconvenient to describe the global asymptotic behaviour of solutions with respect to the L^∞ norm. To overcome this difficulty, we shall introduce a weaker topology induced by the local L^∞-norm for $\phi \in X$. For simplicity of notation, when there is no confusion about the spaces involved, we will just write $|| \cdot ||_{L^\infty_{loc}}$ for the local L^∞-norm. Moreover, we also denote the normed vector space $(X, || \cdot ||_{L^\infty_{loc}})$ still by X.

The following proposition establishes the monotonicity and *global* existence of the solution to (2.1) (hence, (1.2)). It involves the following hypothesis:

(H1) There exists $u^* \geq 0$ such that $f(\pm u^*) = 0$ and $uf(u) \leq 0$ for all $|u| \geq u^*$.

Proposition 2.1. *Let $\psi, \phi \in X$ with $\phi \leq \psi$. Then $u^\phi(t, x, y) \leq u^\psi(t, x, y)$ for all $(t, x, y) \in [0, \min\{\eta_\phi, \eta_\psi\}) \times \mathbb{R}^2$. Moreover, if (H1) holds, then $\eta_\phi = \infty$ for all $\phi \in X$.*

Proof. The monotonicity follows from the Phragmén-Lindelöf type maximum principle in [33].

When ψ is a constant function in $\mathbb{R}^2$, by (H1), we easily see that $-\max\{||\psi||_X, u^*\} \leq u^\psi(t, x, y) \leq \max\{||\psi||_X, u^*\}$ for all $(t, x, y) \in \mathbb{R}_+ \times \mathbb{R}^2$. For any $\phi \in X$, applying the monotonicity to ϕ and $\psi = ||\phi||_X$, one conclude that $\eta_\phi = \infty$. □

In the rest of the paper, we shall always assume (H1) holds so that $\eta_\phi = \infty$ for all $\phi \in X$. Thus, we may define $\Phi : \mathbb{R}_+ \times X \to X$ by $\Phi(t, \phi) = u^\phi(t, \cdot)$ for all $(t, \phi) \in \mathbb{R}_+ \times X$. To emphasize the dependence on the nonlinearity f, we sometimes write $\Phi(t, \phi, f)$ for $\Phi(t, \phi)$. It is obvious that for any $r \geq u^*$, $\Phi|_{\mathbb{R}_+ \times X_r}$ is a continuous, compact and monotone semiflow on X_r. Here, the topology of X_r is induced by the norm $|| \cdot ||_{L^\infty_{loc}}$. Accordingly, we always assume that the tacit topology of X is induced by the L^∞_{loc}-norm in the sequel.

Definition 2.1. An element $\phi \in X$ is called an equilibrium of Φ if $\Phi(t, \phi) = \phi$ for all $t \in \mathbb{R}_+$. A subset $\mathcal{A}$ of X is said to be positively invariant under Φ if $\Phi(t, \phi) \in \mathcal{A}$ for every $\phi \in \mathcal{A}$ and $t \in \mathbb{R}_+$.

We write $O(\phi) = \{\Phi(t, \phi) : t \in \mathbb{R}_+\}$ for the positive semi-orbit through the point ϕ. The ω-limit set of $O(\phi)$ (or of ϕ) is defined by $\omega(\phi) = \bigcap_{t \in \mathbb{R}_+} Cl(O(\Phi(t, \phi)))$, where $Cl(O(\Phi(t, \phi)))$ represents the closure of $O(\Phi(t, \phi))$ with respect to the L^∞_{loc}-norm.

Definition 2.2. Let ϕ be an equilibrium and $\mathcal{A}$ be a positively invariant set of the semiflow Φ. We say that ϕ is globally attractive in $\mathcal{A}$ if $\omega(\psi) = \{\phi\}$ for all $\psi \in \mathcal{A}$.

By the local parabolic estimates and the related discussions in [11], we may obtain a relation between the topologies defined by the $L^\infty_{loc}(\mathbb{R}^2)$-norm and the $C^2_{loc}(\mathbb{R}^2)$-norm for the positive semi-orbits of uniformly bounded subsets, which shall be very useful for proving the existence and attractivity of heterogeneous steady states for (1.2).

Proposition 2.2. *If $t, \gamma > 0$ and $K \subseteq X_\gamma$, then $\Phi(t, K)$ is precompact with $C^2_{loc}(\mathbb{R}^2)$-norm, and hence the $L^\infty_{loc}(\mathbb{R}^2)$-norm and $C^2_{loc}(\mathbb{R}^2)$-norm define the same topology (in the sense of equivalence) on $\Phi(t, K)$. In particular, for any $\phi \in X$, $O(\phi)$ and $\omega(\phi)$ are precompact with $C^2_{loc}(\mathbb{R}^2)$-norm, and hence the $L^\infty_{loc}(\mathbb{R}^2)$-norm and $C^2_{loc}(\mathbb{R}^2)$-norm define the same topology in $Cl(O(\phi))$ as well as in $\omega(\phi)$.*

From this proposition, if ψ is globally attractive in $\mathcal{A}$ then $\lim_{t\to\infty} ||u^\phi(t,\cdot) - \psi||_{L^\infty_{loc}} = 0$ and hence $\lim_{t\to\infty} ||u^\phi(t,\cdot) - \psi||_{C^2_{loc}(\mathbb{R}^2)} = 0$ for all $\phi \in \mathcal{A}$.

Definition 2.3. We say that ψ is globally attractive in X with respect to the usual supremum norm if $\lim_{t\to\infty} ||u^\phi(t,\cdot) - \psi||_X = 0$ for all $\phi \in X$.

For the trivial steady state $\psi = 0$, we have the following result.

Theorem 2.1. *If $uf(u) < 0$ for all $u \neq 0$, then 0 is a globally attractive equilibrium of (2.1) in X with respect to the $L^\infty(\mathbb{R}^2)$-norm.*

Proof. Note that each solution $u(t)$ of $u'(t) = f(u(t))$ tends to 0 as $t \to \infty$. By the comparison principal, we know that for any $\phi \in X$, $u^{-||\phi||_X}(t,\cdot) \leq u^\phi(t,\cdot) \leq u^{||\phi||_X}(t,\cdot)$ for all $t \in [0,\infty)$. Thus, $||u^\phi(t,\cdot)||_{L^\infty(\mathbb{R}^2)} \to 0$ as $t \to \infty$, and the proof is complete. □

The condition on f in Theorem 2.1 implies that $f'(0) \leq 0$. If $f'(0) > 0$, then 0 is not a locally attractive equilibrium. In the next sections, we tackle the global dynamics of the bistable form for (2.1) under the condition $f'(0) > 0$ together with the following assumption for a *bistable scenario*:

(H2) $f'(0) > 0$ and there is $u^* > 0$ such that $f(\pm u^*) = 0$ and $(u - u^*)f(u) < 0$ for all $u \in (0,\infty) \setminus \{u^*\}$.

It is clear that (H2) implies (H1). The following result shows that every solution of (1.2) is attracted to X_{u^*} under the above bistable condition.

Theorem 2.2. *If (H2) holds, then $\limsup_{t\to\infty} ||u^\phi(t,\cdot)||_X \leq u^*$ for all $\phi \in X$.*

Proof. Proposition 2.1 implies that $\eta_\phi = \infty$ and $u^{\phi_-}(t,\cdot) \leq u^\phi(t,\cdot) \leq u^{\phi_+}(t,\cdot)$ for all $(t,\phi) \in \mathbb{R}_+ \times X$, where $\phi_+(x,y) = \max\{u^*, \phi(x,y)\}$ and $\phi_-(x,y) = \min\{-u^*, \phi(x,y)\}$ for all $(x,y) \in \mathbb{R}^2$. Hence, it suffices to prove $\limsup_{t\to\infty} ||u^\phi(t,\cdot)||_X = u^*$ for all $\phi \geq u^*$. Now, let us suppose $\phi \geq u^*$. Again, Proposition 2.1 implies $u^\phi(t,\cdot) \geq u^*$ for all $t \in \mathbb{R}_+$. Let $u(t,x,y) = u^\phi(t,x,y) - u^*$ and let $\tilde{f}(u) = sign(u)f(|u| + u^*)$ for all $u \in (-\infty,\infty)$. Then $u(t,x,y) \geq 0$ satisfies (2.1) with $f = \tilde{f}$. By applying Theorem 2.1 with $f = \tilde{f}$, we have $\limsup_{t\to\infty} ||u(t,\cdot)||_X = 0$. Thus, $\limsup_{t\to\infty} ||u^\phi(t,\cdot)||_X = u^*$, completing the proof. □

3. Invariant sub-systems

In this section, we construct a sequence of domains in $\mathbb{R}^2$ with which a sequence of positive invariant sub-systems of (2.1) is associated. By studying the invariance of (2.1) under *translations and orthogonal transformations*, we identify those subsystems that cannot be mutually transformed and use these subsystems to establish the multiplicity of steady states for (2.1).

Denote the orthogonal transformation group on $\mathbb{R}^2$ by $O(2)$. One can easily verify that the solutions of (2.1) are invariant under translations and orthogonal transformations, that is, for any $\phi \in X$, $u^{\phi}(t, a_{11}x + a_{12}y + b_1, a_{12}x + a_{22}y + b_2)$ satisfies the integral equation of (2.1) where $(a_{ij})_{2\times 2} \in O(2)$ and $b = (b_1, b_2) \in \mathbb{R}^2$.

For a given non-negative integer m, set

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad B_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B_m = \begin{pmatrix} \cos\frac{2\pi}{m} & \sin\frac{2\pi}{m} \\ -\sin\frac{2\pi}{m} & \cos\frac{2\pi}{m} \end{pmatrix} \quad \text{for } m \geq 1,$$

and accordingly define the following sets in $\mathbb{R}^2$ depending on m:

$$\Omega_m = \begin{cases} \mathbb{R}^2, & m = 0, \\ \mathbb{R} \times \mathbb{R}_+, & m = 1, \\ \mathbb{R}^2_+, & m = 2, \\ \{(x, y) \in \mathbb{R}^2_+ : y \leq x \tan(\frac{\pi}{m})\}. & m \geq 3. \end{cases}$$

Denote by $\frac{\partial}{\partial \nu}$ the derivative in the outward normal direction of $\partial\Omega_m \setminus \{(0, 0)\}$. Let

$$\begin{aligned} \mathcal{X}_m &= \{\phi \in C(\Omega_m, \mathbb{R}) \cap L^\infty(\Omega_m, \mathbb{R}) : \phi|_{\partial\Omega_m} = 0\}, \\ \mathcal{X}_m^+ &= \{\phi \in \mathcal{X}_m : \phi(x, y) \geq 0 \text{ for all } (x, y) \in \Omega_m\}, \\ \mathcal{X}_m^r &= \{\phi \in \mathcal{X}_m^+ : ||\phi||_X \leq r\}, \quad \text{for } r > 0. \end{aligned}$$

For any ϕ, $\psi \in \mathcal{X}_m$, we write $\phi \geq \psi$ if $\phi - \psi \in X_+$, $\phi > \psi$ if $\phi \geq \psi$ and $\phi \neq \psi$, $\phi \gg \psi$ if $\phi(x, y) > \psi(x, y)$ for all $(x, y) \in Int(\Omega_m)$.

Define the operator $P_m : \mathcal{X}_m \longrightarrow X$ such that

$$P_m[\phi]|_{\Omega_m} = \phi, \;\; (P_m[\phi]) \circ A|_{\Omega_m} = -\phi \text{ and } (P_m[\phi]) \circ B_m = P_m[\phi] \text{ for all } \phi \in \mathcal{X}_m$$

where $(\phi \circ B)(x, y) = \phi((x, y)B)$ for all $(x, y) \in \mathbb{R}^2$, $\phi \in X$ and $B \in O(2)$. Also define

$$k_{t,m}(x, y, \tilde{x}, \tilde{y}) = \begin{cases} \frac{1}{4\pi t} exp(-\frac{||(x,y)-(\tilde{x},\tilde{y})||^2}{4t}), & m = 0, \\ \sum_{l=0}^{m-1} \frac{1}{4\pi t}[exp(-\frac{||(x,y)B_m^{-l}-(\tilde{x},\tilde{y})||^2}{4t}) - exp(-\frac{||(x,y)B_m^{-l}A^{-1}-(\tilde{x},\tilde{y})||^2}{4t})] & m \geq 1 \end{cases}$$

for $(x, y), (\tilde{x}, \tilde{y}) \in \Omega_m$. For $m \geq 0$ define

$$\begin{cases} \mathcal{T}_m(0)[\phi](x, y) = \phi(x, y), \\ \mathcal{T}_m(t)[\phi](x, y) = \int_{\Omega_m} k_{t,m}(x, y, \tilde{x}, \tilde{y})\phi(\tilde{x}, \tilde{y})\mathrm{d}\tilde{x}\mathrm{d}\tilde{y} \end{cases}$$

for all $(t, \phi) \in \mathbb{R}_+ \times \mathcal{X}_m$ and $(x, y), (\tilde{x}, \tilde{y}) \in \Omega_m$.

The next lemma shows that for every $m \geq 0$, $P_m[\mathcal{X}_m]$ is a positively invariant set with respect to Φ.

Proposition 3.1. *For any positive integer m, $P_m[\mathcal{X}_m]$ is a positively invariant subset for Φ, that is, $\Phi(t, P_m[\mathcal{X}_m]) \subseteq P_m[\mathcal{X}_m]$ for all $t \in \mathbb{R}_+$.*

Proof. It follows from the expression of $T(t)$ and (2.1) that for any $(t, \phi) \in \mathbb{R}_+ \times X$ and $B \in O(2)$, we have

$$\begin{aligned} u^{\phi}(t, \cdot) \circ B &= T(t)[\phi] \circ B + \mu \int_0^t T(t-s)[f(u^{\phi}(s, \cdot) \circ B)]\mathrm{d}s \\ &= T(t)[\phi \circ B] + \mu \int_0^t T(t-s)[f(u^{\phi}(s, \cdot) \circ B)]\mathrm{d}s, \end{aligned}$$

which implies that $u^{\phi}(t, \cdot) \circ B$ satisfies (2.1) with the initial value $\phi \circ B$. Note that $\phi \circ A = -\phi$ and $\phi \circ B_m = \phi$ for all $\phi \in P_m[\mathcal{X}_m]$. Thus, by the uniqueness of solution to (2.1), we conclude that $u^{\phi}(t, \cdot) = -u^{\phi}(t, \cdot) \circ A$ and $u^{\phi}(t, \cdot) = u^{\phi}(t, \cdot) \circ B_m$ for all $(t, \phi) \in \mathbb{R}_+ \times P_m[\mathcal{X}_m]$. In other words, $u^{\phi}(t, \cdot) \in P_m[\mathcal{X}_m]$ for any $(t, \phi) \in \mathbb{R}_+ \times P_m[\mathcal{X}_m]$. This completes the proof. □

Now, we consider the following auxiliary problem for the nonlinear reaction-diffusion equation in Ω_m:

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u + f(u(t, x, y)), & t > 0, \\ u(t, x, y) = 0, & (t, x, y) \in (0, \infty) \times \partial\Omega_m, \\ u(0, x, y) = \phi(x, y), & \phi \in \Omega_m. \end{cases} \tag{3.1}$$

A mild solution of system (3.1) solves the following integral equation with the given initial function,

$$\begin{cases} u(t, \cdot) = \exp(-\mu t)\mathcal{T}_m(t)[\phi] + \int_0^t \exp(-\mu(t-s))\mathcal{T}_m(t-s)[\mathcal{F}_m(u(s, \cdot))]\mathrm{d}s, & t \geq 0, \\ u_0 = \phi \in \mathcal{X}_m, \end{cases} \tag{3.2}$$

where $\mu \geq 0$, and $\mathcal{F}_m : \mathcal{X}_m^+ \to \mathcal{X}_m$ is defined by

$$\mathcal{F}_m(\phi)(x, y) = \mu\phi(x, y) + f(\phi(x, y)), \ \ (x, y) \in \Omega_m.$$

In what follows, we sometimes need to emphasize the parameter $\mu \geq 0$ and thus, will refer (3.2) as $(3.2)_\mu$, and simply denote $(3.2)_0$ by (3.2).

Proposition 3.2. *For any positive integer m and $\phi \in \mathcal{X}_m$, $u^{P_m[\phi]}(t, \cdot)|_{\Omega_m}$ satisfies $(3.2)_\mu$ for $\mu \geq 0$, and thus, satisfies (3.1). Moreover, $u^{P_m[\phi]}(t, \cdot)|_{\Omega_m} \in \mathcal{X}_m^+$ for all $(t, \phi) \in \mathbb{R}_+ \times \mathcal{X}_m^+$.*

Proof. By Proposition 3.1, we easily see that $u^{P_m[\phi]}(t,\cdot)|_{\Omega_m}$ satisfies $(3.2)_0$. Thus, a standard argument shows that $u^{P_m[\phi]}(t,\cdot)|_{\Omega_m}$ satisfies $(3.2)_\mu$ for all $\mu > 0$.

Now, we prove $u^{P_m[\phi]}(t,\cdot)|_{\Omega_m} \in \mathcal{X}_m^+$ for all $(t,\phi) \in \mathbb{R}_+ \times \mathcal{X}_m^+$. Indeed, for any $\phi \in \mathcal{X}_m^+$, let us denote $v(t,x,y) = u^{P_m[\phi]}(t,x,y)$ for all $(t,x,y) \in \mathbb{R}_+ \times \Omega_m$. It follows from (3.1) that v satisfies the following equation

$$\begin{cases} \frac{\partial v}{\partial t}(t,x,y) = \Delta v + c(t,x,y)v(t,x,y), & (t,x,y) \in (0,\infty)\times\Omega_m, \\ v(0,x,y) = 0, & (x,y) \in \Omega_m, \\ v(t,x,y) \geq 0, & (t,x,y) \in [0,\infty)\times\partial\Omega_m. \end{cases} \tag{3.3}$$

Here,

$$c(t,x,y) = \begin{cases} \frac{f(v(t,x,y))}{v(t,x,y)}, & v(t,x,y) \neq 0, \\ f'(0) & v(t,x,y) = 0. \end{cases}$$

By the Phragmén-Lindelöf type maximum principle in [33], we have $v(t,x,y) \geq 0$ for all $(t,x,y) \in [0,\infty)\times\Omega_m$, completing the proof. □

In the following, if there is no confusion, we will abuse the notation $u^\phi(t,\cdot)$ for $u^{P_m[\phi]}(t,\cdot)|_{\Omega_m}$ for all $(t,\phi) \in \mathbb{R}_+ \times \mathcal{X}_m$. The actual meaning of $u^\phi(t,\cdot)$ depends on whether ϕ is in X or in $\mathcal{X}_m$ which will be clear from the context.

The following proposition reveals a key relation between solutions of (2.1) and (3.2).

Proposition 3.3. *For any non-negative integer m, $P_m[u^\phi(t,\cdot)] = u^{P_m[\phi]}(t,\cdot)$ for any $(t,\phi) \in \mathbb{R}_+ \times \mathcal{X}_m$.*

Proof. Let $m \geq 0$ be given and $\phi \in \mathcal{X}_m^+$. It follows from (3.2), the definition of P_m and Lemma A.2-(iii) that for any $t \in \mathbb{R}_+$, we have

$$\begin{aligned} P_m[u^\phi(t,\cdot)] &= P_m[\mathcal{T}_m(t)[\phi]] + P_m[\int_0^t \mathcal{T}_m(t-s)[f(u^\phi(s,\cdot))]\mathrm{d}s] \\ &= T(t)[P_m[\phi]] + \int_0^t T(t-s)[f(P_m[u^\phi(s,\cdot)])]\mathrm{d}s. \end{aligned}$$

This, together with the uniqueness of solutions of (2.1), implies $P_m[u^\phi(t,\cdot)] = u^{P_m[\phi]}(t,\cdot)$ for all $t \in \mathbb{R}_+$. This completes the proof. □

To explore the dynamics of (2.1), we only focus on (3.2). In what follows, we always assume that the tacit topology of $\mathcal{X}_m^+$ is induced by $L^\infty_{loc}(\Omega_m)$-norm.

According to Proposition 3.2, we may define $\Psi : \mathbb{R}_+ \times \mathcal{X}_m^+ \to \mathcal{X}_m^+$ by $\Psi(t,\phi) = u^\phi(t,\cdot)$ for all $(t,\phi) \in \mathbb{R}_+ \times \mathcal{X}_m^+$, where the topology and the partial ordering of $\mathcal{X}_m^+$ are induced by the norm $||\cdot|| \triangleq ||\cdot||_{L^\infty_{loc}(\Omega_m)}$ and $\mathcal{X}_m^+$ respectively. To emphasize the nonlinearity f, we sometimes write $\Psi(t,\phi; f)$ for $\Psi(t,\phi)$.

Definition 3.1. An element $\phi \in \mathcal{X}_m^+$ is called an equilibrium of Ψ if $\Psi(t, \phi) = \phi$ for all $t \in \mathbb{R}_+$. A subset $\mathcal{A}$ of $\mathcal{X}_m^+$ is said to be positively invariant under Ψ if $\Psi(t, \phi) \in \mathcal{A}$ for every $\phi \in \mathcal{A}$ and $t \in \mathbb{R}_+$.

We write $O(\phi) = \{\Psi(t, \phi) : t \in \mathbb{R}_+\}$ for the positive semi-orbit through the point ϕ. The ω-limit set of ϕ is defined by $\omega(\phi) = \bigcap_{t \in \mathbb{R}_+} Cl(O(\Psi(t, \phi)))$, where $Cl(O(\Psi(t, \phi)))$ represents the closure of $O(\Psi(t, \phi))$ with respect to the $L^\infty_{loc}(\Omega_m)$-norm.

Definition 3.2. Let ϕ be an equilibrium and $\mathcal{A}$ be a positively invariant set of the semiflow Ψ. We say that ϕ is globally attractive in $\mathcal{A}$ if $\omega(\phi) = \{\phi\}$ for all $\phi \in \mathcal{A}$.

To continue, we collect some basic properties of Ψ as follows.

Proposition 3.4. *For any $r \geq u^*$, $\Psi|_{\mathbb{R}_+ \times \mathcal{X}_m^r}$ is a continuous, compact and monotone semiflow on $\mathcal{X}_m^r$ with respect to the topology of $\mathcal{X}_m^r$ induced by the L^∞_{loc}-norm and the partial ordering induced by $\mathcal{X}_m^+$.*

Proof. By Propositions 2.1 and 3.3, we easily see that $\Psi|_{\mathbb{R}_+ \times \mathcal{X}_m^r}$ is a continuous and compact semiflow on $\mathcal{X}_m^r$. But these two Propositions can not directly give the monotonicity of Ψ due to the fact that $P_m[\mathcal{X}_m^+] \setminus X_+ \neq \varnothing$. Indeed, for any $\phi, \psi \in \mathcal{X}_m^+$ with $\psi - \phi \in \mathcal{X}_m^+$, by letting $v(t, x, y) = u^\psi(t, x, y) - u^\phi(t, x, y)$ and

$$c(t, x, y) = \begin{cases} \frac{f(u^\psi(t,x,y)) - f(u^\phi(t,x,y))}{v(t,x,y)}, & v(t, x, y) \neq 0, \\ f'(0) & v(t, x, y) = 0, \end{cases}$$

we easily check that $v(t, x, y)$ satisfies (3.3). By the Phragmén-Lindelöf type maximum principle in [33], we have $v(t, x, y) = u^\psi(t, x, y) - u^\phi(t, x, y) \geq 0$ for all $(t, x, y) \in [0, \infty) \times \Omega_m$. In other words, Ψ is monotone with respect to the ordering induced by $\mathcal{X}_m^+$. This completes the proof. □

Moreover, by the strong maximum principle and the Hopf boundary lemma, we can obtain some further information about the monotonicity and boundary property of Ψ, as stated in the following proposition.

Proposition 3.5. *If $\psi, \phi \in \mathcal{X}_m^+$ with $\psi < \phi$, then $\Psi(t, \psi)(x, y) < \Psi(t, \phi)(x, y)$ for all $(t, x, y) \in (0, \infty) \times Int(\Omega_m)$ and $\frac{\partial \Psi(t,\phi)}{\partial \nu}(x, y) < 0$ for all $(t, x, y) \in (0, \infty) \times (\partial \Omega_m \setminus \{(0, 0)\})$.*

By Theorem 2.2 and Proposition 3.3, we have the following result.

Proposition 3.6. $\limsup_{t \to \infty} ||P_m[u^\phi(t, \cdot)]||_X \leq u^*$ *for all* $\phi \in \mathcal{X}_m^+$.

In the following sections, we shall explore the existence, shape and attractivity of the heterogeneous steady states for (3.2) with the initial function $\phi \in \mathcal{X}_m^+$.

4. Some priori estimates

To overcome the difficulty in showing that the trivial equilibrium expels nontrivial solutions due to the lack of compactness and smoothness of the spatial domain Ω_m, we establish an *a* *priori* estimate for nontrivial solutions after describing the delicate asymptotic properties of the diffusion operator.

Let $\mu = 1 + \max\{|f'(u)| : u \in [-1-u^*, 1+u^*]\}$. Define $f_\mu : \mathbb{R} \longrightarrow \mathbb{R}$ by $f_\mu(u) = u + \frac{f(u)}{\mu}$ for all $u \in \mathbb{R}$. Then $\mu u + f(u) = \mu f_\mu(u) > 0$ and $\mu + f'(u) \geq 1$ for all $u \in (0, 1+u^*]$. It suffices to study $(3.2)_\mu$ with the initial function $\phi \in \mathcal{X}_m^{1+u^*}$, according to Proposition 3.6.

Let $l(x) = \frac{\sqrt{\mu}}{2}\exp(-\sqrt{\mu x^2})$ and $l(t,x) = \frac{1}{\sqrt{4\pi t}}\exp(-\frac{x^2}{4t})$ for all $t \in (0,\infty)$ and $x \in \mathbb{R}$. To proceed further, we define some new linear operators $L^t,\ H[t,\cdot],\ H[\cdot],\ H_t : \mathcal{X}_m \to \mathcal{X}_m$ as below:

$$\begin{aligned}
L^t[\zeta](x,y) &= \mathcal{T}_m(t)[\zeta](x,y),\\
H[t,\zeta](x,y) &= \int_0^t \mu\exp(-\mu s)L^s[\zeta](x,y)\mathrm{d}s,\\
H[\zeta](x,y) &= \int_{\mathbb{R}_+} \mu\exp(-\mu s)L^s[\zeta](x,y)\mathrm{d}s,\\
H_t[\zeta] &= H[\zeta] - H[t,\zeta],
\end{aligned}$$

for all $\zeta \in \mathcal{X}_m$, $t \in [0,\infty)$ and $(x,y) \in \Omega_m$.

It is easily seen that these operators can be extended from $\mathcal{X}_m$ to $L^\infty(\Omega_m, \mathbb{R})$ and the extensions are also order preserving in the sense of pointwise ordering due to the nonnegativity of $k_{t,m}$, see Lemma A.2-(v) in the Appendix.

For given $\tilde{T} > 2T > 0$, let

$$\begin{aligned}
\Omega_0^{T,\tilde{T}} &= [-\tilde{T}, \tilde{T}]^2,\\
\Omega_1^{T,\tilde{T}} &= \{(x,y) \in \Omega_1 : x \in [-\tilde{T}, \tilde{T}] \text{ and } y \in [T, \tilde{T}]\},\\
\Omega_2^{T,\tilde{T}} &= \{(x,y) \in \Omega_2 : x, y \in [T, \tilde{T}]\},\\
\Omega_m^{T,\tilde{T}} &= (T, T\tan(\tfrac{\pi}{2m})) + \{(x,y), (x,y)AB_{2m} : (x,y) \in \Omega_{2m} \text{ with } T \leq x \leq \tilde{T} - T\},\ m \geq 3.
\end{aligned}$$

Note that for $m \geq 3$, $\Omega_m^{T,\tilde{T}}$ is a result of sliding the region $D_1 \cup D_2$ along the direction of the line $y = x\tan(\pi/2m)$ by $T\sec(\pi/2m)$, where $D_1 = \{(x,y) : (x,y) \in \Omega_{2m}$ with $T \leq x \leq \tilde{T} - T\}$, $D_2 = \{(x,y)AB_{2m} : (x,y) \in \Omega_{2m}$ with $T \leq x \leq \tilde{T} - T\}$. It is easy to see that $\Omega_m^{T,\tilde{T}}$ is a hexagon with each angle $\geq \frac{\pi}{2}$. See Fig. 1 for an illustration. Define the function $h_m^{T,\tilde{T}} : \Omega_m \to \mathbb{R}_+$ by $h_m^{T,\tilde{T}}(x,y) = 1$ for all $(x,y) \in \Omega_m^{T,\tilde{T}}$ and $h_m^{T,\tilde{T}}(x,y) = 0$ for all $(x,y) \in \Omega_m \setminus \Omega_m^{T,\tilde{T}}$. Let $h^{T,\tilde{T}} : \mathbb{R}^2 \to \mathbb{R}$ be the extension of $h_m^{T,\tilde{T}}$ through P_m in the sense that $h^{T,\tilde{T}}(x,y) = P_m[h_m^{T,\tilde{T}}](x,y)$ for all $(x,y) \in \mathbb{R}^2$. Let $A^{T,\tilde{T}} = \{\phi \in \mathcal{X}_m^+ : \phi(x,y) \geq h_m^{T,\tilde{T}}(x,y)$ for all $(x,y) \in \Omega_m\}$.

For any $n \in \mathbb{N}$, $(x,y) \in \mathbb{R}^2$, $B \in O(2)$ and $D \subseteq \mathbb{R}^2$, let us define $B[(x,y)] = (x,y) \circ B = (x,y)B$, $B[D] = \{d \circ B : d \in D\}$ and

$$I_{x,y}(n; D) = \{\mathbf{y} = (\tilde{x}_1, \tilde{y}_1, \ldots, \tilde{x}_n, \tilde{y}_n) \in \mathbb{R}^{2n} : (x + \sum_{i=1}^n \tilde{x}_i, y + \sum_{i=1}^n \tilde{y}_i) \in D\}.$$

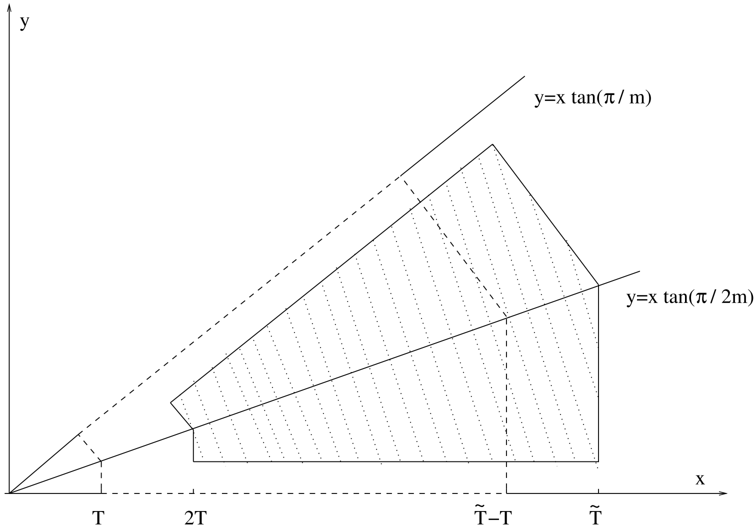

Fig. 1. Illustration of the set $\Omega_m^{T,\tilde{T}}$: the shaded region in $\mathbb{R}^2$ represents this hexagon.

Lemma 4.1. *If $T > 0$, $\tilde{T} \geq 4T$, $n \in \mathbb{N}$ and $(x, y) \in \Omega_m^{T,\tilde{T}}$, then $I_{x,y}(n; \Omega_m^{T,\tilde{T}}) \supseteq I_{0,0}(n; B[\Omega_{m,T_m}])$ for some $B \in O(2)$ where $\Omega_{m,T_m} = [0, \frac{T_m}{2}]^2$ with $T_m = T$ for all $m \leq 2$ and $T_m = T\tan(\frac{\pi}{2m})$ for all $m > 2$.*

Proof. Fix $T > 0$, $\tilde{T} \geq 4T$, $n \in \mathbb{N}$ and $(x, y) \in \Omega_m^{T,\tilde{T}}$. Note that for $m \leq 2$, $\Omega_m^{T,\tilde{T}}$ is a rectangle with lengths of the sides $\geq T$; and for $m > 2$, $\Omega_m^{T,\tilde{T}}$ is a hexagon with each angle $\geq \frac{\pi}{2}$ and length of a side $\geq T\tan(\frac{\pi}{2m})$. Thus, $\Omega_m^{T,\tilde{T}} - (x, y)$ contains a square with $(0, 0)$ being one of the four vertices and length of the sides equaling to $\frac{T_m}{2}$. In other words, there exists $B \in O(2)$ such that $I_{x,y}(n; \Omega_m^{T,\tilde{T}}) \supseteq I_{0,0}(n; B[\Omega_{m,T_m}])$. This completes the proof. □

Lemma 4.2. *If $T > 0$, $\tilde{T} \geq 4T$, $n \in \mathbb{N}$ and $(x, y) \in \Omega_m^{T,\tilde{T}}$, then $I_{x,y}(n; B_m^l A^j[\Omega_m^{T,\tilde{T}}]) \subseteq D_{m,n}^T$ for all nonnegative integers $l \leq m - 1$ and $j \leq 1$ with $(j, l) \neq (0, 0)$, where $D_{m,n}^T \triangleq \{(\tilde{x}_1, \tilde{y}_1, \dots, \tilde{x}_n, \tilde{y}_n) \in \mathbb{R}^{2n} : \sqrt{\sum_{i=1}^{n}[\tilde{x}_i^2 + \tilde{y}_i^2]} \geq \frac{T_m}{\sqrt{n}}\}$ with $T_m = T$ for all $m \leq 2$ and $T_m = T\tan(\frac{\pi}{2m})$ for all $m > 2$.*

Proof. Suppose that $T > 0$, $\tilde{T} \geq 4T$, $n \in \mathbb{N}$, $l \leq m - 1$, $j \leq 1$ with $(j, l) \neq (0, 0)$, and $(x, y) \in \Omega_m^{T,\tilde{T}}$. Let $T_m = T$ for all $m \leq 2$ and $T_m = T\tan(\frac{\pi}{2m})$ for all $m > 2$. Note that $dist(\Omega_m^{T,\tilde{T}}, B_m^l A^j[\Omega_m^{T,\tilde{T}}]) \geq T_m$, and hence $dist((x, y), B_m^l A^j[\Omega_m^{T,\tilde{T}}]) \geq T_m$. By the definition of $I_{x,y}(n; B_m^l A^j[\Omega_m^{T,\tilde{T}}])$, we have

$$\sqrt{[\sum_{i=1}^{n}\tilde{x}_i]^2 + [\sum_{i=1}^{n}\tilde{y}_i]^2} \geq dist((x, y), B_m^l A^j[\Omega_m^{T,\tilde{T}}]) \geq T_m$$

for all $(\tilde{x}_1, \tilde{y}_1, \dots, \tilde{x}_n, \tilde{y}_n) \in I_{x,y}(n; B_m^l A^j[\Omega_m^{T,\tilde{T}}])$. This gives

$$\sqrt{\sum_{i=1}^{n}[\tilde{x}_i^2 + \tilde{y}_i^2]} \geq \frac{T_m}{\sqrt{n}}$$

for all $(\tilde{x}_1, \tilde{y}_1, \dots, \tilde{x}_n, \tilde{y}_n) \in I_{x,y}(n; B_m^l A^j[\Omega_m^{T,\tilde{T}}])$, and thus $I_{x,y}(n; B_m^l A^j[\Omega_m^{T,\tilde{T}}]) \subseteq D_{m,n}^T$. □

Lemma 4.3. *For any $n \in \mathbb{N}$ and $\delta \in (0, \frac{1}{4})$, there exists $T_{n,\delta} > 0$ such that $H^n[h_m^{T,\tilde{T}}] \geq (\frac{1}{4} - \delta)h_m^{T,\tilde{T}}$ for all $T \geq T_{n,\delta}$ and $\tilde{T} \geq 4T$, and hence $H^n[A^{T,\tilde{T}}] \subseteq (\frac{1}{4} - \delta)A^{T,\tilde{T}}$ for all $T \geq T_{n,\delta}$ and $\tilde{T} \geq 4T$, where H^n represents the nth-composition of H.*

Proof. Fix $n \in \mathbb{N}$ and $\delta \in (0, \frac{1}{4})$. It suffices to prove that there exists $T_{n,\delta} > 0$ such that $H^n[h_m^{T,\tilde{T}}] \geq (\frac{1}{4} - \delta)h_m^{T,\tilde{T}}$ for all $T \geq T_{n,\delta}$ and $\tilde{T} \geq 4T$ due to the monotonicity of H.

Define $g_{n,\mu} : \mathbb{R}^{2n} \to \mathbb{R}$ by

$$g_{n,\mu}(\mathbf{y}) = \prod_{i=1}^{n}(l(-\tilde{x}_i)(l(-\tilde{y}_i)) = (\frac{\sqrt{\mu}}{2})^n \exp(-\sqrt{\mu}\sum_{i=1}^{n}[|\tilde{x}_i| + |\tilde{y}_i|]),$$

where $\mathbf{y} = (\tilde{x}_1, \tilde{y}_1, \dots, \tilde{x}_n, \tilde{y}_n) \in \mathbb{R}^{2n}$. Actually, since $\int_{\mathbb{R}_+} l(t,x)\mathrm{d}t = l(x)$ for all $x \in \mathbb{R}$ due to Lemma 2.1-(vi) in [35], it follows from Fubini's Theorem and the linear transformations of variables that for any $\tilde{T} \geq 4T > 0$ and $(x, y) \in \Omega_m$, we have

$$H^n[h_m^{T,\tilde{T}}](x,y) = \int_{\mathbb{R}^{2n}} h^{T,\tilde{T}}\left(x + \sum_{i=1}^{n}\tilde{x}_i, y + \sum_{i=1}^{n}\tilde{y}_i\right) g_{n,\mu}(\mathbf{y})\mathrm{d}\mathbf{y}.$$

This together with Lemmas 4.1 and 4.2, shows that for any $T > 0$, $\tilde{T} \geq 4T$ and $(x, y) \in \Omega_m^{T,\tilde{T}}$, letting $J_{x,y}^{l,j} = I_{x,y}(n; B_m^l[A^j[\Omega_m^{T,\tilde{T}}]])$ with given nonnegative integers $j \leq 1$ and $l \leq m-1$, we know that there exists $B \in O(2)$ such that

$$\begin{aligned} H^n[h_m^{T,\tilde{T}}](x,y) &= \int_{J_{x,y}^{0,0}} g_{n,\mu}(\mathbf{y})\mathrm{d}\mathbf{y} + \sum_{(l,j)\neq(0,0)} (-1)^j \int_{J_{x,y}^{l,j}} g_{n,\mu}(\mathbf{y})\mathrm{d}\mathbf{y} \\ &\geq \int_{I_{0,0}(n;B[\Omega_{m,T_m}])} g_{n,\mu}(\mathbf{y})\mathrm{d}\mathbf{y} - (2m-1)\int_{D_{m,n}^T} g_{n,\mu}(\mathbf{y})\mathrm{d}\mathbf{y}. \end{aligned}$$

By the definition of $g_{n,\mu}$ and the fact that $\int_{\mathbb{R}} l(y)\mathrm{d}y = 1$, we easily see that

$$\int_{\mathbb{R}^n} g_{n,\mu}(\mathbf{y})\mathrm{d}\mathbf{y} = 1.$$

This together with the definition of $D^T_{m,n}$ implies that

$$\lim_{T\to\infty} \int\limits_{D^T_{m,n}} g_{n,\mu}(\mathbf{y})\mathrm{d}\mathbf{y} = 0.$$

Let $V_i^T = (B_4)^i[B[\Omega_{m,T_m}]]$ for $i = 0, 1, 2, 3$. Then $\lim\limits_{T\to\infty} \cup_{i=1}^4 V_i^T = \mathbb{R}^2$, which implies that

$$\sum_{i=1}^4 \lim_{T\to\infty} \int\limits_{I_{0,0}(n;V_i^T)} g_{n,\mu}(\mathbf{y})\mathrm{d}\mathbf{y} = 1.$$

It follows from the definitions of B_4 and $g_{n,\mu}$ that for any $i \in \{0, 1, 2, 3\}$, we have

$$\int\limits_{I_{0,0}(n;V_i^T)} g_{n,\mu}(\mathbf{y})\mathrm{d}\mathbf{y} = \int\limits_{I_{0,0}(n;V_0^T)} g_{n,\mu}(\mathbf{y})\mathrm{d}\mathbf{y}.$$

By letting $T \to \infty$, we obtain $\lim\limits_{T\to\infty} \int_{I_{0,0}(n;B[\Omega_{m,T_m}])} g_{n,\mu}(\mathbf{y})\mathrm{d}\mathbf{y} = \frac{1}{4}$. Therefore, there exists $T_{n,\delta} > 0$ such that for all $T \geq T_{n,\delta}$,

$$\int\limits_{I_{0,0}(n;B[\Omega_{m,T_m}])} g_{n,\mu}(\mathbf{y})\mathrm{d}\mathbf{y} - (2m-1) \int\limits_{D^T_{m,n}} g_{n,\mu}(\mathbf{y})\mathrm{d}\mathbf{y} \geq \frac{1}{4} - \delta.$$

So, for any $T \geq T_{n,\delta}$, $\tilde{T} \geq 4T$ and $(x, y) \in \Omega_m^{T,\tilde{T}}$, we have $H^n[h_m^{T,\tilde{T}}](x, y) \geq \frac{1}{4} - \delta$, that is, $H^n[h_m^{T,\tilde{T}}] \geq (\frac{1}{4} - \delta)h_m^{T,\tilde{T}}$ and hence the proof is completed. □

To continue our discussions, we give some iteration property of the diffusion, which shall be very useful to prove *a priori* estimate for nontrivial solutions for $(3.2)_\mu$.

Lemma 4.4. *For any $n \in \mathbb{N}$ and $\delta \in (0, \frac{1}{4})$, there exist $T_{n,\delta} > 0$ and $s_{n,\delta} > 0$ such that $H^n[h_m^{T,\tilde{T}}] \geq (\frac{1}{4} - \delta)h_m^{T,\tilde{T}}$ and $(H(s,\cdot))^n[h_m^{T,\tilde{T}}] \geq (\frac{1}{4} - \delta)h_m^{T,\tilde{T}}$ for all $T \geq T_{n,\delta}$, $\tilde{T} \geq 4T$ and $s \geq s_{n,\delta}$, where H^n and $(H(s,\cdot))^n$ represent the nth-composition of H and $H(s,\cdot)$, respectively.*

Proof. Fix $n \in \mathbb{N}$ and $\delta \in (0, \frac{1}{4})$. By Lemma 4.3, there exists $T_{n,\delta} > 0$ such that $H^n[h_m^{T,\tilde{T}}] \geq [\frac{1}{4} - \frac{\delta}{2}]h_m^{T,\tilde{T}}$ for all $T \geq T_{n,\delta}$ and $\tilde{T} \geq 4T$.

Let $s_{n,\delta} = \frac{1}{\mu}\ln(\frac{2^{1+n}}{\delta})$. Now fix $T \geq T_{n,\delta}$, $\tilde{T} \geq 4T$ and $s \geq s_{n,\delta}$. By the definitions of $H, H_s, H(s,\cdot)$, we have $H_s \circ H(s,\cdot) = H(s,\cdot) \circ H_s$, $H_s[1] \leq e^{-\mu s}$, $(H_s)^j[h_m^{T,\tilde{T}}] \leq 1$ and $(H(s,\cdot))^j[h_m^{T,\tilde{T}}] \leq 1$ for all $j \in \mathbb{N}$. It follows that

$$
\begin{aligned}
H^n[h_m^{T,\tilde{T}}] &= \sum_{j=0}^{n} C_n^j (H_s)^j (H(s,\cdot))^{n-j}[h_m^{T,\tilde{T}}] \\
&= (H(s,\cdot))^n[h_m^{T,\tilde{T}}] + H_s[\sum_{j=1}^{n} C_n^j (H_s)^{j-1}(H(s,\cdot))^{n-j}[h_m^{T,\tilde{T}}]] \\
&\le (H(s,\cdot))^n[h_m^{T,\tilde{T}}] + (2^n-1)H_s[1] \\
&\le (H(s,\cdot))^n[h_m^{T,\tilde{T}}] + 2^n e^{-\mu s} \\
&\le (H(s,\cdot))^n[h_m^{T,\tilde{T}}] + \frac{\delta}{2}.
\end{aligned}
$$

This, combined with the fact that $H^n[h_m^{T,\tilde{T}}] \ge [\frac{1}{4} - \frac{\delta}{2}]h_m^{T,\tilde{T}}$, implies that $(H(s,\cdot))^n[h_m^{T,\tilde{T}}] \ge (\frac{1}{4} - \delta)h_m^{T,\tilde{T}}$. This completes the proof. □

The following result gives *a priori* estimate for nontrivial solutions for $(3.2)_\mu$, which plays a key role in the proof of existence and attractivity for the heterogeneous steady states of $(3.2)_\mu$.

Proposition 4.1. *Suppose that $M \ge u^*$. Then there exist $\varepsilon_0 > 0$, $T_0 > 0$ and $T^* > 0$ such that for all $\varepsilon \in [0, \varepsilon_0]$, $T \in [T_0, \infty)$, $\tilde{T} \in [4T, \infty)$ and a solution $u : [0,\infty) \times \Omega_m \to [0, M]$ of $(3.2)_\mu$ with $u(t,\cdot) \ge \varepsilon h_m^{T,\tilde{T}}$ for all $t \in [0, T^*]$, we have $u(t,\cdot) \ge \varepsilon h_m^{T,\tilde{T}}$ for all $t \in [0,\infty)$ and $u(t,\cdot) \gg \varepsilon h_m^{T,\tilde{T}}$ for all $t \in [T^*, \infty)$.*

Proof. Obviously, there exist $n \in \mathbb{N}$ and $\beta \in (1, 1 + \frac{1}{\mu} f'(0))$ such that $\beta^n > 10$.

By the choices of β and μ, one can easily see that there exists a $\varepsilon_1 \in (0, M)$ such that $u + \frac{1}{\mu} f(u) \ge \beta u$ for all $u \in [0, \varepsilon_1]$ and $u + \frac{1}{\mu} f(u) \ge \beta\varepsilon_1$ for all $u \in [\varepsilon_1, M]$.

By applying Lemma 4.4 with $\delta = \frac{1}{8}$, we know that there exists $s_n > 0$ such that $H^n[h_m^{T,\tilde{T}}] \ge \frac{1}{8}h_m^{T,\tilde{T}}$ and $(H(s,\cdot))^n[h_m^{T,\tilde{T}}] \ge \frac{1}{8}h_m^{T,\tilde{T}}$ for all $\frac{\tilde{T}}{4} \ge T \ge s_n$ and $s \ge s_n$.

Let $\varepsilon_0 = \frac{\varepsilon_1}{\beta^{n+1}}$, $T_0 = s_n$ and $T^* = ns_n$. Suppose that $\varepsilon \in [0, \varepsilon_0]$, $T \in [T_0, \infty)$, $\tilde{T} \in [4T, \infty)$ and $u : [0, \infty) \times \Omega_m \to [0, M]$ is a solution of $(3.2)_\mu$ such that $u(t,\cdot) \ge \varepsilon h_m^{T,\tilde{T}}$ for all $t \in [0, T^*]$. Let $\phi = u(0,\cdot)$. Then $u(t,x,y) = u^\phi(t,x,y) = \Psi(t,\phi)(x,y)$ for all $(t,x,y) \in [0,\infty) \times \Omega_m$. Due to the choices of ε and β, one can easily obtain $\beta^j (H(t,\cdot))^j[\varepsilon h_m^{T,\tilde{T}}] < \varepsilon_1$ and $\beta^j (H(t,\cdot))^j[\varepsilon h_m^{T,\tilde{T}}] + \frac{1}{\mu} f(\beta^j (H(t,\cdot))^j[\varepsilon h_m^{T,\tilde{T}}]) \ge \beta^{j+1}(H(t,\cdot))^j[\varepsilon h_m^{T,\tilde{T}}]$ for all $t \ge 0$ and $j = 0, 1, \dots, n$.

Let $n^* = \sup\{j \in \{0, 1, 2, \cdots, n\} : u(t,\cdot) \ge \varepsilon\beta^j (H(s_n,\cdot))^j[h_m^{T,\tilde{T}}]$ for all $t \in [js_n, T^*]\}$. We claim $n^* = n$; otherwise, $n^* \in [0, n-1]$ and $u(t,\cdot) \ge \varepsilon\beta^{n^*}(H(s_n,\cdot))^{n^*}[h_m^{T,\tilde{T}}]$ for all $t \in [n^*s_n, T^*]$. These, combined with $(3.2)_\mu$ and Fubini's theorem imply that for any $t \in [(1 + n^*)s_n, T^*]$,

$$
\begin{aligned}
u(t,\cdot) &= u(s_n + (t - s_n), \cdot) \\
&= \Psi(s_n, \Psi(t - s_n, \phi))(\cdot) \\
&= e^{-\mu s_n}\mathcal{T}_m(s_n)[u(t - s_n, \cdot)]
\end{aligned}
$$

$$+\int_0^{s_n} e^{-\mu(s_n-s)}\mathcal{T}_m(s_n-s)[\mu u(s+t-s_n,\cdot)+f(u(s+t-s_n,\cdot))]\mathrm{d}s$$
$$\geq H(s_n,\cdot)[\varepsilon\beta^{n^*}(H(s_n,\cdot))^{n^*}[h_m^{T,\tilde{T}}],\cdot+\frac{1}{\mu}f(\varepsilon\beta^{n^*}(H(s_n,\cdot))^{n^*}[h_m^{T,\tilde{T}}])]$$
$$\geq \varepsilon\beta^{n^*+1}(H(s_n,\cdot))^{n^*+1}[h_m^{T,\tilde{T}}],$$

which yields a contradiction and thus means that $u(t,\cdot)\geq\varepsilon\beta^n(H(s_n,\cdot))^n[h_m^{T,\tilde{T}}]$ for all $t\in[ns_n,T^*]$. That is, $u(T^*,\cdot)\geq\varepsilon\beta^n(H(s_n,\cdot))^n[h_m^{T,\tilde{T}}])\geq\varepsilon\frac{\beta^n}{8}h_m^{T,\tilde{T}}$, and thus $u(T^*,\cdot)\gg\varepsilon h_m^{T,\tilde{T}}$. Let $T^{**}=\sup\{t\geq 0: u([0,t],\cdot)\geq\varepsilon h_m^{T,\tilde{T}}\}$. Then $T^{**}>T^*$ due to the continuity of $u(\cdot,\cdot)$ and the fact that $u(T^*,\cdot)\gg\varepsilon h_m^{T,\tilde{T}}$. We claim that $T^{**}=\infty$; otherwise, $T^{**}<\infty$. By applying the above discussions with $u^{\Psi(t^*,\phi)}(t,\cdot)$ with $t^*\in[0,T^{**}-T^*]$, we have $u(t,\cdot)=u^{\Psi(t-T^*,\phi)}(T^*,\cdot)\gg\varepsilon h_m^{T,\tilde{T}}$ for all $t\in[T^*,T^{**}]$. In particular, $u(T^{**},\cdot)\gg\varepsilon h_m^{T,\tilde{T}}$, a contradiction with choice of T^{**}. Hence, $T^{**}=\infty$, and the previous discussions also give $u(t,\cdot)\gg\varepsilon h_m^{T,\tilde{T}}$ for all $t\in[T^*,\infty)$. □

Given $T>0$, let us define $\Omega_m^{T,\infty}=\lim_{\tilde{T}\to\infty}\Omega_m^{T,\tilde{T}}$ and $h_m^{T,\infty}(x,y)=1$ for all $(x,y)\in\Omega_m^{T,\infty}$ and $h_m^{T,\infty}(x,y)=0$ for all $(x,y)\in\Omega_m\setminus\Omega_m^{T,\infty}$.

The following theorem shows that the positive limit set of a positive solution of (3.2) is far away from zero for locations away from the boundary of Ω_m.

Theorem 4.1. *Suppose that $f'(0)>0$. If $\phi\in\mathcal{X}_m^+\setminus\{\mathbf{0}\}$, then there exist $\varepsilon_\phi>0$ and $T_\phi>0$ such that $\omega(\phi)\geq\varepsilon_\phi h_m^{T,\tilde{T}}$ for all $T\geq T_\phi$ and $\tilde{T}\geq 4T$. In other words, $\omega(\phi)\geq\varepsilon_\phi h_m^{T_\phi,\infty}$.*

Proof. By Propositions 3.5 and 3.6, we may assume that $\phi\in\mathcal{X}_m^{1+u^*}$ with $\phi(x,y)>0$ for all $(x,y)\in Int(\Omega_m)$, and hence $u^\phi(t,x,y)\in(0,1+u^*]$ for all $(t,x,y)\in[0,\infty)\times Int(\Omega_m)$. Choose T_0, T^*, and ε_0 as in Proposition 4.1. Let $T_\phi=T_0$, $\varepsilon_1=\inf\{u^\phi(t,x,y):(t,x,y)\in[0,T^*]\times\Omega_m^{T_0,4T_0}\}$ and $\varepsilon_\phi=\min\{\varepsilon_0,\varepsilon_1\}$. Then $\varepsilon_1>0$ and $\varepsilon_\phi>0$. By Proposition 4.1 and the choices of T_0, T^* and ε_0, we get $u^\phi(t,\cdot)\geq\varepsilon_\phi h_m^{T_\phi,4T_\phi}$ for all $t\geq 0$. This, combined with the definition of $\omega(\phi)$, implies $\xi\geq\varepsilon_\phi h_m^{T_\phi,4T_\phi}$ for all $\xi\in\omega(\phi)$.

For any $\xi\in\omega(\phi)$, let $a_\xi=\sup\{a\geq T_\phi:\xi(x,y)\geq\varepsilon_\phi$ for all $(x,y)\in\Omega_m^{T_\phi,a}\}$. Then $a_\xi\geq 4T_\phi$ for all $\xi\in\omega(\phi)$. Let $T^\phi=\inf\{a_\xi:\xi\in\omega(\phi)\}$. Then $T^\phi\geq 4T_\phi$.

We claim that $T^\phi=\infty$. By way of contradiction, suppose that $T^\phi<\infty$. Take $\xi^*\in\omega(\phi)$. Then, the invariance of $\omega(\phi)$ implies that $u^{\xi^*}(t,\cdot)\geq\varepsilon_\phi h_m^{T_\phi,T^\phi}$ for all $t\in[0,\infty)$. Again, by Proposition 4.1 and the choices of T_0, T^* and ε_0, we have $u^{\xi^*}(t,\cdot)\gg\varepsilon_\phi h_m^{T_\phi,T^\phi}$ for all $t\in[T^*,\infty)$. In particular, there exists $\tilde{T}>T^\phi$ such that $u^{\xi^*}(t,\cdot)\gg\varepsilon_\phi h_m^{T_\phi,\tilde{T}}$ for all $t\in[T^*,2T^*]$. On the other hand, by the definition of $\omega(\phi)$, there exists a sequence $\{s_n\}_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty}||(u^\phi)_{s_n}-\xi^*||_{L^\infty_{loc}}=0$. It follows that $\lim_{n\to\infty}(\sup\{|u^\phi(s_n+t,x,y)-u^{\xi^*}(t,x,y)|:(t,x,y)\in[T^*,2T^*]\times\Omega_m^{T_\phi,\tilde{T}}\})=0$. Thus there exists $n^*>1$ such that $u^\phi(s_{n^*}+t,\cdot)\geq\varepsilon_\phi h_m^{T_\phi,\tilde{T}}$ for all $t\in[T^*,2T^*]$. It follows from Proposition 4.1 that $u^\phi(s_{n^*}+t,\cdot)=u^{\Psi(s_{n^*}+T^*,\phi)}(t-T^*,\cdot)\geq\varepsilon_\phi h_m^{T_\phi,\tilde{T}}$ for all $t\in[T^*,\infty)$. This and the definition of $\omega(\phi)$ produce $\xi\geq\varepsilon_\phi h_m^{T_\phi,\tilde{T}}$ for all $\xi \in \omega(\phi)$. In view of $\tilde{T} > T^{\phi}$, we have $a_{\xi} \geq \tilde{T} > T^{\phi}$ for all $\xi \in \omega(\phi)$. Then $T^{\phi} = \inf\{a_{\xi} : \xi \in \omega(\phi)\} \geq \tilde{T} > T^{\phi}$, a contradiction. This proves the claim, that is, $T^{\phi} = \infty$. By the definition of T^{ϕ}, we conclude that $\omega(\phi) \geq \varepsilon_{\phi} h_m^{T_{\phi},\infty}$, completing the proof. □

5. Existence and properties of heterogeneous steady states

In this section, we first establish the existence of heterogeneous steady state of (3.2) by using the *a priori* estimate established in the previous section. Then we will discuss the asymptotic behaviours of the positive steady state at infinities.

To proceed, we first explore some properties of the first derivatives of elements in the ω-limit set, which shall be very useful for proving the existence and attractivity of heterogeneous steady state for (3.2).

Lemma 5.1. *If $\psi \in \mathcal{X}_m^+$, then*

$$\sup\{|\frac{\partial \phi}{\partial x}(x, y)| + |\frac{\partial \phi}{\partial y}(x, y)| : (x, y) \in \Omega_m \text{ and } \phi \in \omega(\psi)\} \leq C_f,$$

where C_f is a constant depending on f only.

Proof. By Proposition 3.6, $\omega(\psi) \subseteq \mathcal{X}_m^{u^*}$. Thus, the conclusion follows from the standard parabolic estimates. □

Lemma 5.2. *Assume that D is a convex, closed and nonempty subset of $\mathcal{X}_m$ such that $0 \notin D \subseteq \mathcal{X}_m^{u^*}$ and $\Psi(t, D) \subseteq D$ for all $t \in \mathbb{R}_+$. Then (3.2) has a positive steady state, located in D.*

Proof. Let $I = \{\frac{1}{2^i} : i = 1, 2, \cdots\}$. Then for any $T \in I$, $\Psi(T, \cdot) : D \to D$ is compact due to Proposition 3.4. By the Schauder fixed point theorem, there is $\psi_T \in D$ such that $\Psi(T, \psi_T) = \psi_T$. According to the compactness of Ψ due to Proposition 3.4 and the fact that $\{\psi_T : T \in I\} \subseteq \Psi(1, D)$, we know that $\{\psi_T : T \in I\}$ is pre-compact in $\mathcal{X}_m$, and thus there exist $\psi \in D$ and a sequence $\{T_k\}$ in I such that $\lim_{T_k \to 0} \psi_{T_k} = \psi$. For any $t \in (0, \infty)$, there exist $r_k \in [0, T_k)$ and nonnegative integer N_k such that $t = N_k T_k + r_k$. Obviously, $\lim_{k\to\infty} r_k = 0$. Hence, for all $t \geq 0$, we have $\Psi(t, \psi) = \lim_{k\to\infty} \Psi(t, \psi_{T_k}) = \lim_{k\to\infty} \Psi(r_k, \psi_{T_k}) = \psi$, which implies that ψ is as required, completing the proof. □

Proposition 5.1. *Eq. (3.2) has at least on positive steady state, located in $\mathcal{X}_m^{u*}$.*

Proof. Take $M = u^*$ and $\phi^* \in \mathcal{X}_m^M \setminus \{0\}$. Let $\varepsilon_{\phi^*}, T_{\phi^*}$ be defined as in Theorem 4.1. Let $D = \{\phi \in \mathcal{X}_m^+ : \omega(\phi^*) \leq \phi \leq M\}$. Then $\varepsilon_{\phi^*} h^{T_{\phi^*},\infty} \leq \omega(\phi^*) \leq D$, and hence $D \subseteq \mathcal{X}_m^+$ and $\phi(x, y) > 0$ for all $(x, y, \phi) \in Int(\Omega_m) \times D$. Clearly, D is a convex and closed subset of $\mathcal{X}_m$ and $\Psi(t, D) \subseteq D$ for all $t \geq 0$. On the other hand, Lemma 5.1 ensures that there exists a $\gamma > 1$ such that $|\frac{\partial \phi(x,y)}{\partial x}|^2 + |\frac{\partial \phi(x,y)}{\partial y}|^2 \leq \gamma^2$ for all $(x, y, \phi) \in \Omega_m \times \omega(\phi^*)$. Let $\zeta(\cdot, x, y) = \min\{\gamma dist((x, y), \partial\Omega_m), M\}$ for all $(x, y) \in \Omega_m$. Then the choices of γ and ζ imply that $\zeta \geq \omega(\phi^*)$ and $\zeta \leq M$. Thus $\zeta \in D$ and $D \neq \emptyset$. Thus the result follows from Lemma 5.2, completing the proof. □

In what follows, we denote by $\mathcal{E}_m$ the set of all positive steady states of (3.2). The following result implies the existence of the maximum element of $\mathcal{E}_m$.

Proposition 5.2. $\sup \mathcal{E}_m \in \mathcal{E}_m \subseteq \mathcal{X}_m^{u^*} \setminus \{u^*\}$, *where* $\sup \mathcal{E}_m(x, y) = \sup\{\phi(x, y) : \phi \in \mathcal{E}_m\}$ *for all* $(x, y) \in \Omega_m$.

Proof. Clearly, Proposition 3.6 implies $\mathcal{E}_m \subseteq \mathcal{X}_m^{u^*} \setminus \{u^*\}$. By the Zorn's lemma and the compactness of Ψ, we easily see that $\mathcal{E}_m$ has a maximal element ζ. We shall prove $\zeta = \sup \mathcal{E}_m$; otherwise, there is $\eta \in \mathcal{E}_m$ such that $\zeta - \eta \notin \mathcal{X}_m^+$. Let $D = \{\phi \in \mathcal{X}_m^{u^*} : \phi - \zeta, \phi - \eta \in \mathcal{X}_m^+\}$. Then D is a convex, closed and nonempty subset of $\mathcal{X}_m$ such that $0 \notin D \subseteq \mathcal{X}_m^{u^*}$ and $\Psi(t, D) \subseteq D$ for all $t \in \mathbb{R}_+$. Thus Lemma 5.2 shows that (3.2) has a positive steady state ψ, located in D. Thus, $\psi \in \mathcal{E}_m$ and $\psi > \zeta$, a contradiction. Consequently, $\zeta = \sup \mathcal{E}_m$ and hence the proof is completed. □

For $\phi \in \mathcal{X}_m$, define $|\phi|_T = \inf\{\phi(x, y) : (x, y) \in \Omega_m^{T,\infty}\}$ and $|\phi|^T = \sup\{\phi(x, y) : (x, y) \in \Omega_m^{T,\infty}\}$. Clearly, $|\phi|_T$ ($|\phi|_T$) is nondecreasing (nonincreasing) in T. Let $\underline{\phi} = \lim_{T\to\infty} |\phi|_T$ and $\overline{\phi} = \lim_{T\to\infty} |\phi|^T$ for any $\phi \in \mathcal{X}_m$.

The next result gives some information about the asymptotic behaviour of positive steady states at infinity location.

Proposition 5.3. *If* $\phi \in \mathcal{E}_m$, *then* $\lim_{T\to\infty} |\phi|_T = \lim_{T\to\infty} |\phi|^T = u^*$ *and* $||P_m[\phi]||_X = u^*$*; in other words, for any* $\varepsilon > 0$ *there exists* $T_\varepsilon > 0$ *such that* $|\phi(x, y) - u^*| < \varepsilon$ *for all* $(x, y) \in \Omega_m^{T_\varepsilon,\infty}$.

Proof. Fix $\phi \in \mathcal{E}_m$. By Proposition 3.6 and Theorem 4.1, $u^* \geq \phi \geq \epsilon h_m^{T,\infty}$ for some ϵ and $T \in (0, \infty)$. Thus, we have $u^* \geq \overline{\phi} \geq \underline{\phi} > \epsilon$ and thus $I \triangleq [\underline{\phi}, \overline{\phi}] \subseteq [\epsilon, u^*]$.

We claim that $\underline{\phi} \leq \underline{\mathcal{T}_m(t)[\phi]} \leq \overline{\mathcal{T}_m(t)[\phi]} \leq \overline{\phi}$ for all $(t, \phi) \in \mathbb{R}_+ \times \mathcal{X}_m^+$. Indeed, it follows from the definitions of Ω_m and $\mathcal{T}_m(t)$ that, for any $(t, \phi) \in (0, \infty) \times \mathcal{X}_m^+$, we have

$$\begin{aligned}
&\overline{\mathcal{T}_m(t)[\phi]} \\
&= \lim_{T\to\infty} |\mathcal{T}_m(t)[\phi]|_T \\
&= \lim_{T\to\infty} |T(t)[P_m[\phi]]|_{\Omega_m}|_T \\
&\leq \frac{1}{4\pi t} \sum_{l=0}^{m-1} \lim_{T\to\infty} \{\sup\{ \int_{B_m^l[\Omega_m]} P_m[\phi](\tilde{x}, \tilde{y}) \exp(-\frac{(x-\tilde{x})^2 + (y-\tilde{y})^2}{4t}) \mathrm{d}\tilde{x}\mathrm{d}\tilde{y} : (x, y) \in \Omega_m^{T,\infty}\}\} \\
&\leq \frac{1}{4\pi t} \lim_{T\to\infty} \{\sup\{ \int_{\Omega_m} P_m[\phi](\tilde{x}, \tilde{y}) \exp(-\frac{(x-\tilde{x})^2 + (y-\tilde{y})^2}{4t}) \mathrm{d}\tilde{x}\mathrm{d}\tilde{y} : (x, y) \in \Omega_m^{T,\infty}\}\} \\
&\quad + \frac{(m-1)||P_m[\phi]||_X}{4\pi t} \lim_{T\to\infty} \int_{\sqrt{\tilde{x}^2+\tilde{y}^2}\geq T_m} \exp(-\frac{\tilde{x}^2+\tilde{y}^2}{4t}) \mathrm{d}\tilde{x}\mathrm{d}\tilde{y} \\
&= \frac{1}{4\pi t} \lim_{T\to\infty} \{\sup\{ \int_{\Omega_m} \phi(\tilde{x}, \tilde{y}) \exp(-\frac{(x-\tilde{x})^2 + (y-\tilde{y})^2}{4t}) \mathrm{d}\tilde{x}\mathrm{d}\tilde{y} : (x, y) \in \Omega_m^{T,\infty}\}\}
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{1}{4\pi t}\lim_{T\to\infty}\{\sup\{\int\limits_{\Omega_m\setminus\Omega_m^{\frac{T}{2},\infty}} \phi(\tilde{x},\tilde{y})\exp(-\frac{(x-\tilde{x})^2+(y-\tilde{y})^2}{4t})\mathrm{d}\tilde{x}\mathrm{d}\tilde{y} : (x,y)\in\Omega_m^{T,\infty}\}\} \\
&\quad + \frac{1}{4\pi t}\lim_{T\to\infty}\{\sup\{\int\limits_{\Omega_m^{\frac{T}{2},\infty}} \phi(\tilde{x},\tilde{y})\exp(-\frac{(x-\tilde{x})^2+(y-\tilde{y})^2}{4t})\mathrm{d}\tilde{x}\mathrm{d}\tilde{y} : (x,y)\in\Omega_m^{T,\infty}\}\} \\
&\le \frac{1}{4\pi t}||P_m[\phi]||_X \lim_{T\to\infty}\int\limits_{\sqrt{\tilde{x}^2+\tilde{y}^2}\ge\frac{T_m}{2}} \exp(-\frac{\tilde{x}^2+\tilde{y}^2}{4t})\mathrm{d}\tilde{x}\mathrm{d}\tilde{y} \\
&\quad + \frac{1}{4\pi t}\lim_{T\to\infty}\{\sup\{\int\limits_{\Omega_m^{\frac{T}{2},\infty}} \phi(\tilde{x},\tilde{y})\exp(-\frac{(x-\tilde{x})^2+(y-\tilde{y})^2}{4t})\mathrm{d}\tilde{x}\mathrm{d}\tilde{y} : (x,y)\in\Omega_m^{T,\infty}\}\} \\
&\le \frac{1}{4\pi t}\lim_{T\to\infty}\{|\phi|^{\frac{T}{2}}\sup\{\int\limits_{\Omega_m^{\frac{T}{2},\infty}} \exp(-\frac{(x-\tilde{x})^2+(y-\tilde{y})^2}{4t})\mathrm{d}\tilde{x}\mathrm{d}\tilde{y} : (x,y)\in\Omega_m^{T,\infty}\}\} \\
&\le \frac{1}{4\pi t}\int\limits_{\mathbb{R}^2}\exp(-\frac{\tilde{x}^2+\tilde{y}^2}{4t})\mathrm{d}\tilde{x}\mathrm{d}\tilde{y}\lim_{T\to\infty}|\phi|^{\frac{T}{2}} \\
&\le \overline{\phi}.
\end{aligned}
$$

Here, $T_m = T$ for all $m \le 2$ and $T_m = T\tan(\frac{\pi}{2m})$ for all $m > 2$. Similarly, $\underline{\mathcal{T}_m(t)[\phi]} \ge \underline{\phi}$ for all $(t,\phi)\in\mathbb{R}_+\times\mathcal{X}_m^+$.

Let $f_\mu(x) = x + \frac{1}{\mu}f(x)$ for all $x\in\mathbb{R}$. Since ϕ is a positive steady state of $(3.2)_\mu$, we obtain

$$
\phi = e^{-\mu t}\mathcal{T}_m(t)[\phi] + \int\limits_0^t e^{-\mu(t-s)}\mathcal{T}_m(t-s)[\mu f_\mu(\phi)]\mathrm{d}s \text{ for any } t\in\mathbb{R}_+.
$$

This, together with the above claim, implies that

$$
\begin{aligned}
\underline{\phi} &\ge e^{-\mu t}\underline{\mathcal{T}_m[\phi]} + \int\limits_0^t \mu e^{-\mu(t-s)}\underline{\mathcal{T}_m(t-s)[f_\mu(\phi)]}\mathrm{d}s \\
&\ge e^{-\mu t}\underline{\phi} + \int\limits_0^t \mu e^{-\mu(t-s)}\underline{f_\mu(\phi)}\mathrm{d}s \\
&\ge e^{-\mu t}\underline{\phi} + \mu\int\limits_0^t e^{-\mu(t-s)}\underline{f_\mu(\phi)}\mathrm{d}s \\
&= e^{-\mu t}\underline{\phi} + (1-e^{-\mu t})\underline{f_\mu(\phi)}.
\end{aligned}
$$

Thus, $\underline{\phi} \geq \underline{f_\mu(\phi)}$. A similar argument yields $\overline{\phi} \leq \overline{f_\mu(\phi)}$. Hence, $I \subseteq f(I)$, which together with the fact that $f_\mu(x) > x$ for all $x \in I \setminus \{u^*\}$, implies that $I = \{u^*\}$. Hence, $\liminf_{T\to\infty} |\phi|_T = \limsup_{T\to\infty} |\phi|^T = u^*$ and $||P_m[\phi]||_X = u^*$. This completes the proof. □

6. Global attractivity of heterogeneous steady state

In this section, we shall investigate the uniqueness and attractivity of heterogeneous steady states of (3.2). In the sequel, we denote by u^+ the maximal positive steady states obtained in Proposition 5.2. Then we have the following attractivity result for u^+.

Proposition 6.1. $\omega(\psi) = \{u^+\}$ *for all* $\psi \in \mathcal{X}_m^+ \setminus \{0\}$ *with* $\psi \geq u^+$.

Proof. Otherwise, there exists $\psi \in \mathcal{X}_m^+ \setminus \{0\}$ such that $\psi \geq u^+$ and $\omega(\psi) \neq \{u^+\}$. By the monotonicity of Ψ and Proposition 3.6, we have $u^+ \leq \omega(\psi) \leq u^*$. Let $D = \{\phi \in \mathcal{X}_m^+ : \omega(\psi) \leq \phi \leq u^*\}$. Clearly, by the definition of D and Lemma 5.1, we know that $u^+ < D$ and D is a convex, closed and nonempty set in $\mathcal{X}_m$. These together with $\Psi(\mathbb{R}_+ \times D) \subseteq D$ and Lemma 5.2, show that (3.2) has a steady state in D, a contraction with the choice of u^+. This completes the proof. □

To address the attractiveness of u^+ for (3.2), we further need the following *sublinear condition* on the function f:

(H3) $f(\alpha x) \geq \alpha f(x)$ for all $(\alpha, x) \in [0, 1] \times [0, u^*]$.

Clearly, (H3) holds if and only if $\frac{f(x)}{x}$ is nonincreasing on $[0, u^*]$. For example, (H1)-(H3) hold for $f(x) = px(q - x^2)$ for $p, q > 0$.

Theorem 6.1. *Assume that (H1) and (H3) hold. Then for any nonnegative integer m,* (3.2) *has a unique positive steady state* u^+ *which attracts all solutions of (3.2) with the initial value* $\psi \in \mathcal{X}_m^+ \setminus \{0\}$.

Proof. The existence of u^+ is already established in Proposition 5.1, and the uniqueness will be a consequence of the global attractiveness of u^+ in $\mathcal{X}_m^+ \setminus \{0\}$. So, we only need to show that u^+ attracts all solutions of (3.2) with the initial value $\psi \in \mathcal{X}_m^+ \setminus \{0\}$.

We claim that if $l > 0$ and lu^+ is a positive steady state of (3.2), then $l = 1$. Otherwise $l \neq 1$. By using Proposition 5.3, we have $||P_m[lu^+]||_X = u^*$ and thus $l = 1$, a contradiction.

Let $\psi \in \mathcal{X}_m^+ \setminus \{0\}$. To prove $\omega(\psi) = \{u^+\}$, we may assume that $\psi \leq u^+$ due to the monotonicity of Ψ and Proposition 6.1.

Suppose that $\omega(\psi) \neq \{u^+\}$. If $m = 0$, then by Proposition 3.6 and Theorem 4.1, we have $0 < a^* \triangleq \inf\{\phi(x, y) : (x, y, \phi) \in \mathbb{R}^2 \times \omega(\psi)\} \leq b^* \triangleq \sup\{\phi(x, y) : (x, y, \phi) \in \mathbb{R}^2 \times \omega(\psi)\} \leq u^*$. Clearly, $a^* < u^*$. Thus, there is $\delta > 0$ such that $f_\mu(\phi(x, y)) \geq a^* + \delta$ for all $(x, y, \phi) \in \mathbb{R}^2 \times \omega(\psi)$. It follows from $(3.2)_\mu$ that we know that for all $(x, y, \phi) \in \mathbb{R}^2 \times \omega(\psi)$,

$$u^\phi(1, x, y) = \exp(-\mu)T(1)[\phi](x, y) + \mu \int_0^1 \exp(-\mu(1-s))T(1-s)[f_\mu(u^\phi(s, \cdot))](x, y)\mathrm{d}s$$

$$\geq \exp(-\mu)T(t)[a^*](x,y) + \mu \int_0^1 \exp(-\mu(1-s))T(1-s)[a^*+\delta](x,y)\mathrm{d}s$$
$$= a^* + \delta(1-\exp(-\mu)).$$

Hence the invariance of $\omega(\psi)$ forces that $\phi(x,y) \geq a^* + \delta(1-\exp(-\mu)) > a^*$ for all $(x,y,\phi) \in \mathbb{R}^2 \times \omega(\psi)$, a contradiction to choice of a^*. So, $\omega(\psi) = \{u^+\}$.

Suppose that $m \geq 1$. We now use the famous sliding method to finish the proof. Define $(a_m, b_m) : \mathbb{R}_+ \to \mathbb{R}^2$ by

$$(a_m(T), b_m(T)) = \begin{cases} (0, T), & m = 1, \\ (T, T), & m = 2, \\ (T, T\tan(\frac{\pi}{2m})), & m \geq 3. \end{cases}$$

Let $\alpha(T;\phi) = \sup\{\alpha \geq 0 : \phi(t, x + a_m(T), y + b_m(T)) \geq \alpha u^+(x,y)$ for all $(x,y) \in \Omega_m\}$ for all $T \in \mathbb{R}_+$ and $\phi \in \omega(\psi)$. Then $0 \leq \alpha(T;\phi) \leq 1$ and $\phi(t, x + a_m(T), y + b_m(T)) \geq \alpha(T,\phi)u^+(x,y)$ for all $T \in \mathbb{R}_+$, $(x,y) \in \Omega_m$ and $\phi \in \omega(\psi)$.

Now we shall prove $\alpha(T;\phi) = 1$ for all $(T,\phi) \in (0,\infty) \times \omega(\psi)$ by two steps.

Step 1: Prove that for any $T > 0$, there is $\varepsilon = \varepsilon_{T,\psi} > 0$ such that $\alpha(T;\phi) \geq \varepsilon > 0$ for all $\phi \in \omega(\psi)$.

Fix $T > 0$. By Theorem 4.1, there exist $\varepsilon_1, T_1 > 0$ and $\zeta \in \mathcal{X}_m^1$ such that $\phi(\cdot + a_m(T), \cdot + b_m(T)) \geq \varepsilon_1 \zeta \geq \varepsilon_1 h^{T_1,\infty}$ for all $\phi \in \omega(\psi)$. This together with $u^+ \leq u^*$ implies that

$$\phi(t, x + a_m(T), y + b_m(T)) \geq \frac{\varepsilon_1}{u^*} u^+(x,y) \text{ for all } (x,y,\phi) \in \Omega_m^{T_1,\infty} \times \omega(\psi). \tag{6.1}$$

It follows from $(3.2)_\mu$ and Lemma A.3-(ii) and (iv) that for any $(t,\phi) \in (0,\infty) \times \omega(\psi)$ and $(x,y) \in \Omega_m \setminus \Omega_m^{T_1,\infty}$ with $\sqrt{x^2+y^2} \geq 1$, we have

$$u^\phi(t, x + a_m(T), y + b_m(T))$$
$$= \exp(-\mu t)\mathcal{T}_m(t)[\phi](x + a_m(T), y + b_m(T)) +$$
$$\mu \int_0^t \exp(-\mu(t-s))\mathcal{T}_m(t-s)[f_\mu(u^\phi(s,\cdot))](x + a_m(T), y + b_m(T))\mathrm{d}s$$
$$\geq \varepsilon_1 \exp(-\mu t)\mathcal{T}_m(t)[\zeta](x,y)$$
$$\geq \varepsilon_1 b(1,t,T_1)\{\exp(-\mu t)\mathcal{T}_m(t)[\frac{u^+}{u^*}](x,y) +$$
$$\mu \int_0^t \exp(-\mu(t-s))\mathcal{T}_m(t-s)[\frac{f_\mu(u^+)}{\max f_\mu([0,u^*])}](x,y)\mathrm{d}s\}$$
$$\geq \frac{\varepsilon_1 b(1,t,T_1)}{\max\{u^*, \max f_\mu([0,u^*])\}}\{\exp(-\mu t)\mathcal{T}_m(t)[u^+](x,y) +$$

$$\mu \int_0^t \exp(-\mu(t-s))\mathcal{T}_m(t-s)[f_\mu(u^+)](x,y)\mathrm{d}s\}$$

$$= \frac{\varepsilon_1 b(1,t,T_1)}{\max\{u^*, \max f_\mu([0,u^*])\}} u^+(x,y).$$

In particular, we have

$$u^\phi(1, x+a_m(T), y+b_m(T)) \geq \frac{\varepsilon_1 b(1,1,T_1)}{\max\{u^*, \max f_\mu([0,u^*])\}} u^+(x,y) \triangleq \varepsilon_2 u^+(x,y)$$

for all $\phi \in \omega(\psi)$ and $(x,y) \in \Omega_m \setminus \Omega_m^{T_1,\infty}$ with $\sqrt{x^2+y^2} \geq 1$. Note that $\varepsilon_2 > 0$. Hence by the invariance of $\omega(\psi)$, we may get that for all $\phi \in \omega(\psi)$ and $(x,y) \in \Omega_m \setminus \Omega_m^{T_1,\infty}$ with $\sqrt{x^2+y^2} \geq 1$,

$$\phi(x+a_m(T), y+b_m(T)) \geq \varepsilon_2 u^+(x,y). \tag{6.2}$$

Let

$$\varepsilon_3 = \frac{\min\{\phi(x+a_m(T), y+b_m(T)) : (x,y,\phi) \in \Omega_m \times \omega(\psi) \text{ with } x^2+y^2 \leq 1\}}{\max\{u^+(x,y) : (x,y) \in \Omega_m \text{ with } x^2+y^2 \leq 1\}}.$$

Then $\varepsilon_3 > 0$ and $\phi(x+a_m(T), y+b_m(T)) \geq \varepsilon_3 u^+(x,y)$ for all $\phi \in \omega(\psi)$ and $(x,y) \in \Omega_m$ with $x^2+y^2 \leq 1$. This together with (6.1) and (6.2), implies that $\phi(x+a_m(T), y+b_m(T)) \geq \min\{\frac{\varepsilon_1}{u^*}, \varepsilon_2, \varepsilon_3\} u^+(x,y)$ for all $(x,y,\phi) \in \Omega_m \times \omega(\psi)$. So, by the definition of $\alpha(T;\phi)$, we have $\alpha(T;\phi) \geq \varepsilon \equiv \varepsilon_{T,\psi} \triangleq \min\{\frac{\varepsilon_1}{u^*}, \varepsilon_2, \varepsilon_3\} > 0$ for all $\phi \in \omega(\psi)$.

Step 2: Prove that $\alpha(T;\phi) = 1$ for all $(T,\phi) \in (0,\infty) \times \omega(\psi)$.

Fix $T > 0$. Suppose that $0 < \alpha(T;\bar{\phi}) < 1$ for some $\bar{\phi} \in \omega(\psi)$. Let $\alpha^* = \inf\{\alpha(T;\phi) : \phi \in \omega(\psi)\}$. Then by Step 1 and the definition of α^*, we have $1 > \alpha^* > 0$ and $\phi(\cdot + a_m(T), \cdot + b_m(T)) \geq \alpha^* u^+$ for all $\phi \in \omega(\psi)$.

For any $\phi \in \omega(\psi)$, let us define $v^{T,\phi} : \mathbb{R}_+ \times \Omega_m \to \mathbb{R}$ by

$$v^{T,\phi}(t,x,y) = u^\phi(t, x+a_m(T), y+b_m(T)) - \alpha^* u^+(x,y).$$

It follows from (3.1) and (H3) that $v^{T,\phi}$ satisfies the following equation

$$\begin{cases} \frac{\partial v^{T,\phi}}{\partial t}(t,x,y) \geq \Delta v^{T,\phi} + c(t,x,y)v^{T,\phi}(t,x,y), & (t,x,y) \in (0,\infty) \times \Omega_m, \\ v^{T,\phi}(0,x,y) \geq 0, & (x,y) \in \Omega_m, \\ v^{T,\phi}(t,x,y) > 0, & (t,x,y) \in [0,\infty) \times \partial\Omega_m, \end{cases} \tag{6.3}$$

where

$$c(t,x,y) = \frac{f(u^\phi(t, x+a_m(T), y+b_m(T))) - f(\alpha^* u^+(x,y))}{v^{T,\phi}(t,x,y)}.$$

By the Phragmén-Lindelöf type maximum principle in [33] and the strong maximum principle, we know that $v^{T,\phi}(t,x,y) > 0$ for all $(t,x,y) \in (0,\infty) \times \Omega_m$. Thus, by the invariance of $\omega(\psi)$, we may obtain that

$$\phi(x + a_m(T), y + b_m(T)) > \alpha^* u^+(x,y) \text{ for all } (x,y,\phi) \in \Omega_m \times \omega(\psi). \tag{6.4}$$

By (6.4) and a contradict argument, we may show that there is $\gamma^* > 0$ such that

$$\phi(x + a_m(T), y + b_m(T)) \geq (\alpha^* + \gamma^*) u^+(x,y),\ (x,y,\phi) \in \Omega_m \times \omega(\psi) \text{ with } \sqrt{x^2+y^2} \leq 1. \tag{6.5}$$

Note that there exist $\epsilon > 0$ and $\delta \in (0, u^*)$ such that $f_\mu(\alpha^* u) > \alpha^* f_\mu(u) + \epsilon$ for all $u \in [u^* - \delta, u^*]$. By applying Theorem 4.1 and Proposition 5.3, we know that there exist $T^* > 0$ and $\varepsilon^* \in (0,\delta)$ such that $\phi(x,y) \geq \varepsilon^*$ and $u^+(x,y) \geq u^* - \varepsilon^* > u^* - \delta$ for all $(x,y,\phi) \in \Omega_m^{T^*,\infty} \times \omega(\psi)$. These, together with (H3) and the monotonicity of f_μ, give $f_\mu(\phi(\cdot + (a_m(T), b_m(T)))) - \alpha^* f_\mu(u^+(\cdot)) \geq \epsilon h_m^{T^*,\infty}$ for all $\phi \in \omega(\psi)$.

Applying Lemma 4.4 with $n = 1$ and $\delta = \frac{1}{8}$, we obtain that there exists $T^{**} > \max\{T, T^*\}$ such that $H(s, h_m^{\tilde{T},\infty}) \geq \frac{1}{8} h_m^{\tilde{T},\infty}$ for all $s, \tilde{T} \geq T^{**}$.

It follows from $(3.2)_\mu$ and Lemma A.3-(ii) that we know that for all $(x,y,\phi) \in \Omega_m \times \omega(\psi)$ and $t \geq T^{**}$,

$$\begin{aligned}
v^{T,\phi}(t,x,y) &= u^\phi(t, x + a_m(T), y + b_m(T)) - \alpha^* u^+(x,y) \\
&= \exp(-\mu t)\mathcal{T}_m(t)[\phi](x + a_m(T), y + b_m(T)) - \exp(-\mu t)\mathcal{T}_m(t)[\alpha^* u^+](x,y) \\
&\quad + \mu \int_0^t \exp(-\mu(t-s))\mathcal{T}_m(t-s)[f_\mu(u^\phi(s,\cdot))](x + a_m(T), y + b_m(T))\mathrm{d}s \\
&\quad - \mu \int_0^t \exp(-\mu(t-s))\mathcal{T}_m(t-s)[\alpha^* f_\mu(u^+) + \epsilon h^{T^*,\infty}](x,y)\mathrm{d}s \\
&\quad + \mu \int_0^t \exp(-\mu(t-s))\mathcal{T}_m(t-s)[\epsilon h^{T^*,\infty}](x,y)\mathrm{d}s \\
&\geq \mu \int_0^t \exp(-\mu(t-s))\mathcal{T}_m(t-s)[\epsilon h^{T^*,\infty}](x,y)\mathrm{d}s \\
&\geq \epsilon H(t,\cdot)[h^{T^{**},\infty}](x,y) \\
&\geq \frac{\epsilon}{8} h^{T^{**},\infty}(x,y).
\end{aligned}$$

Hence the invariance of $\omega(\psi)$ forces that

$$\phi(x + a_m(T), y + b_m(T)) \geq \alpha^* u^+(x,y) + \frac{\epsilon}{8} \text{ for all } (x,y,\phi) \in \Omega_m^{T^{**},\infty} \times \omega(\psi). \tag{6.6}$$

Again, from $(3.2)_\mu$ and Lemma A.3-(ii) and (iv) that we know that for all $(x, y, \phi) \in (\Omega_m \setminus \Omega_m^{T^{**},\infty}) \times \omega(\psi)$ with $\sqrt{x^2+y^2} \geq 1$,

$$\begin{aligned}
v^{T,\phi}(1,x,y) &= u^{\phi}(1, x+a_m(T), y+b_m(T)) - \alpha^* u^+(x,y) \\
&\geq \exp(-\mu)\mathcal{T}_m(1)[\phi](x+a_m(T), y+b_m(T)) - \exp(-\mu)\mathcal{T}_m(1)[\alpha^* u^+](x,y) \\
&\geq \exp(-\mu)\mathcal{T}_m(1)[\frac{\epsilon}{8}h^{T^{**},\infty}](x,y) \\
&\geq \frac{\epsilon b(1,1,T^{**})}{8}[\exp(-\mu)\mathcal{T}_m(1)[\frac{u^+}{u^*}](x,y) + \\
&\quad \mu\int_0^1 \exp(-\mu(1-s))\mathcal{T}_m(1-s)[\frac{f_\mu(u^+)}{\max f_\mu([0,u^*])}](x,y)\mathrm{d}s] \\
&\geq \frac{\epsilon b(1,1,T^{**})}{8\max\{u^*, \max f_\mu([0,u^*])\}}[\exp(-\mu)\mathcal{T}_m(1)[u^+](x,y) + \\
&\quad \mu\int_0^1 \exp(-\mu(1-s))\mathcal{T}_m(1-s)[f_\mu(u^+)](x,y)\mathrm{d}s] \\
&= \frac{\epsilon b(1,1,T^{**})}{8\max\{u^*, \max f_\mu([0,u^*])\}}u^+(x,y).
\end{aligned}$$

So, by the invariance of $\omega(\psi)$, we know that for all $(x, y, \phi) \in (\Omega_m \setminus \Omega_m^{T^{**},\infty}) \times \omega(\psi)$ with $\sqrt{x^2+y^2} \geq 1$,

$$\phi(x+a_m(T), y+b_m(T)) \geq \alpha^* u^+(x,y) + \frac{\epsilon b(1,1,T^{**})}{8\max\{u^*, \max f_\mu([0,u^*])\}}u^+(x,y). \tag{6.7}$$

It follows from (6.5)-(6.7) that there is $\delta^* > 0$ such that $\phi(x+a_m(T), y+b_m(T)) \geq [\alpha^* + \delta^*]u^+(x,y)$ for all $(x,y,\phi) \in \Omega_m \times \omega(\psi)$. Hence by the definition of $\alpha(T;\phi)$, we have $\alpha(T;\phi) \geq \alpha^* + \delta^* > \alpha^*$ for all $\phi \in \omega(\psi)$, a contradiction to choice of α^*. Therefore, $\alpha(T;\phi) = 1$ for all $(T,\phi) \in (0,\infty) \times \omega(\psi)$.

Thus, by Step 2, we easily see that $\omega(\psi) \geq u^+$. This together with $\omega(\psi) \leq u^+$, yields $\omega(\psi) = \{u^+\}$. This completes the proof. □

Corollary 6.1. *Assume that (H1) and (H3) hold. Then for any nonnegative integer m, we have the following results:*

(i) (3.2) *has a unique positive steady state u^+ in $\mathcal{X}_m^+ \setminus \{0\}$.*
(ii) $\lim_{T\to\infty} |u^+|_T = \lim_{T\to\infty} |u^+|^T = u^*$.
(iii) *u^+ is a symmetry function with respect to $\{y = \tan(\frac{\pi}{2m})x\}$. In other words, $u^+(x,y) = u^+((x,y)B_{4m}^{-1}AB_{4m})$ for all $(x,y) \in \Omega_m$.*
(iv) *For any $(T,x,y) \in (0,\infty) \times \Omega_m$, we have*

$$u^+(x+a_m(T), y+b_m(T)) > u^+(x,y), \tag{6.8}$$

where $(a_m(T), b_m(T))$ is defined in the proof of Theorem 6.1. Hence, $u^+(x, y)$ is strictly increasing from $(x_0, y_0) \in \partial\Omega_m$ along the ray parallel to $\{y = \tan(\frac{\pi}{2m})x\}$.

Proof. (i) and (ii) follow from Theorem 6.1 and Proposition 5.3, respectively.

(iii) Let $B = B_{2m} A B_{2m}^{-1}$. Then $B \in O(2)$, $B[\Omega_m] = \Omega_m$ and $u^+ \circ B \in \mathcal{X}_m^+ \setminus \{0\}$. Thus by the proof of Proposition 3.1, we have

$$P_m[u^+] \circ B = T(t)[P_m[u^+] \circ B] + \mu \int_0^t T(t-s)[f(P_m[u^+] \circ B)]\mathrm{d}s \ \text{ for any } t \in \mathbb{R}_+.$$

This, together with the fact that $P_m[u^+] \circ B = P_m[u^+ \circ B]$, implies that

$$P_m[u^+ \circ B] = T(t)[P_m[u^+ \circ B]] + \mu \int_0^t T(t-s)[f(P_m[u^+ \circ B])]\mathrm{d}s \ \text{ for any } t \in \mathbb{R}_+.$$

As a result, Lemma A.2 gives

$$u^+ \circ B = \mathcal{T}_m(t)[u^+ \circ B] + \mu \int_0^t \mathcal{T}_m(t-s)[f(u^+ \circ B)]\mathrm{d}s \ \text{ for any } t \in \mathbb{R}_+.$$

So, $u^+ \circ B$ is a positive steady state of (3.2) in $\mathcal{X}_m^+ \setminus \{0\}$. By the uniqueness of the positive steady states for (3.2), we have $u^+ \circ B = u^+$.

(iv) By the proof of Theorem 6.1, we know that for any $(T, x, y) \in (0, \infty) \times \Omega_m$,

$$u^+(x + a_m(T), y + b_m(T)) \geq u^+(x, y). \tag{6.9}$$

This, together with the strong maximum principle, implies that for any $(T, x, y) \in (0, \infty) \times \Omega_m$,

$$u^+(x + a_m(T), y + b_m(T)) > u^+(x, y). \tag{6.10}$$

So, $u^+(x, y)$ is strictly increasing from $(x_0, y_0) \in \partial\Omega_m$ along the ray parallel to $\{y = \tan(\frac{\pi}{2m})x\}$. This completes the proof. □

Without assuming the sublinearity (H3), we have the following results.

Theorem 6.2. *For any nonnegative integer m, we have the following results:*

(i) *If $m = 0$, then (3.2) has a unique positive steady state $u^+ \equiv u^*$, which attracts all solutions of (3.2) with the initial value $\psi \in X_+ \setminus \{0\}$.*
(ii) *There exist $u_+, u^+ \in \mathcal{E}_m$ such that $u_+ \leq u \leq u^+$ for every $u \in \mathcal{E}_m$; moreover u_+ (u^+) attracts all solutions of (3.2) with the initial value in $\{\phi \in \mathcal{X}_m^+ \setminus \{0\} : \phi \leq u_+\}$ ($\{\phi \in \mathcal{X}_m^+ \setminus \{0\} : \phi \geq u^+\}$).*
(iii) *(3.2) has a unique positive steady state u^+ if and only if u^+ attracts all solutions of (3.2) with the initial value $\psi \in \mathcal{X}_m^+ \setminus \{0\}$.*

Proof. (i) Clearly, $u^+ = u^*$. By checking the proof of Theorem 6.1 for $m = 0$, we easily see that (3.2) has a unique positive steady state $u^+ \equiv u^*$, which attracts all solutions of (3.2) with the initial value $\psi \in X_+ \setminus \{0\}$.

(ii) By Propositions 5.2 and 6.1, there exist $u^+ \in \mathcal{E}_m$ such that $\mathcal{E}_m \leq u^+$, and u^+ attracts all solutions of (3.2) with the initial value $\{\phi \in \mathcal{X}_m^+ \setminus \{0\} : \psi \geq u^+\}$.

Clearly, there is $g \in C^1(\mathbb{R}, \mathbb{R})$ such that $g|_{[0,u^*]} \leq f|_{[0,u^*]}$ satisfies (H1) and (H3). For example, there is a large $\alpha > 0$ such that $g_\alpha|_{[0,u^*]} \leq f|_{[0,u^*]}$ satisfies (H1) and (H3) where $g_\alpha(x) = \frac{f'(0)}{2}x[1 - \alpha x^2]$ for all $x \in \mathbb{R}$. By applying Theorem 6.1 with $f = g$, there is $u_g^+ \in \mathcal{X}_m^+ \setminus \{0\}$ such that $\omega(\psi; g) = \{u_g^+\}$ for all $\psi \in X_+ \setminus \{0\}$. By the choice of g and the monotonicity of Ψ, we have $\Psi(t, \psi) \geq \Psi(t, \psi; g)$ for all $t \in \mathbb{R}_+$ and $\psi \in X_+$. Thus, the definition of $\omega-$limit set, we have $u_g^+ \leq \omega(\psi)$ for all $\psi \in X_+ \setminus \{0\}$. In particular, $u_g^+ \leq \mathcal{E}_m$ and thus $\omega(u_g^+) \leq \mathcal{E}_m$. Let $D = \{\phi \in \mathcal{X}_m^{u^*} : u_g^+ \leq \phi \leq \omega(u_g^+)\}$. Then D is a convex, closed and nonempty subset of $\mathcal{X}_m$ such that $0 \notin D \subseteq \mathcal{X}_m^{u^*}$ and $\Psi(t, D) \subseteq D$ for all $t \in \mathbb{R}_+$. Thus Lemma 5.2 shows that (3.2) has a positive steady state u_+, located in D. So, $u_g^+ \leq u_+ = \inf \mathcal{E}_m \leq \omega(u_g^+)$. For any $\psi \in \{\phi \in \mathcal{X}_m^+ \setminus \{0\} : \phi \leq u_+\}$, we have $\omega(\psi) \leq u_+ \leq \omega(u_g^+)$, and $\omega(\psi) \geq \omega(u_g^+)$ due to $\omega(\psi) \geq u_g^+$, which imply $\omega(\psi) = \{u_+\}$.

(iii) The sufficiency is clear. Now suppose that (3.2) has a unique positive steady state u^+. Then $u_+ = u^+$. For any $\psi \in \mathcal{X}_m^+ \setminus \{0\}$, by statement (ii) we have $\omega(\psi_+) = \omega(\psi^+) = \{u^+\}$, where $\psi_+(x, y) = \min\{\psi(x, y), u_+(x, y)\}$ and $\psi^+(x, y) = \max\{\psi(x, y), u^+(x, y)\}$ for all $(x, y) \in \Omega_m$. By the monotonicity of Ψ, we easily see that $\omega(\psi) = \{u^+\}$ for all $\psi \in \mathcal{X}_m^+ \setminus \{0\}$, that is, u^+ attracts all solutions of (3.2) with the initial value $\psi \in \mathcal{X}_m^+ \setminus \{0\}$.

This completes the proof. □

7. Main results

In this section, the following theorem summarizes the main results about the bistable reaction-diffusion equation in $\mathbb{R}^2$ by using Proposition 3.3 and Theorem 6.2.

Theorem 7.1. *Assume that (H1) holds. Let $|\cdot|_T, |\cdot|^T, \Omega_m^{T,\infty}$ be defined as in Section 4. Then for every nonnegative integer m, there exist $u_{+,m}, u^{+,m} \in \mathcal{X}_m$ and $\mathcal{E}_m \subseteq \{\phi \in \mathcal{X}_m^+ : \phi(x, y) > 0$ for all $(x, y) \in Int(\Omega_m)\}$ with $u_{+,m} = \inf \mathcal{E}_m, u^{+,m} = \sup \mathcal{E}_m \in \mathcal{E}_m$ and $\lim_{T\to\infty} |u|_T = \lim_{T\to\infty} |u|^T = u^*$ for all $u \in \mathcal{E}_m$ such that the following hold:*

- (i) *$P_m[\mathcal{E}_m]$ is the set of all nontrivial steady states of (1.2) in $P_m[\mathcal{X}_m^+]$;*
- (ii) *$P_m[u_{m,+}]$ (resp. $P_m[u^{m,+}]$) attracts all solutions with initial functions in $P_m[\{\phi \in \mathcal{X}_m^+ \setminus \{0\} : \phi \leq u_{m,+}\}]$ (resp. $P_m[\{\phi \in \mathcal{X}_m^+ \setminus \{0\} : \phi \geq u^{m,+}\}]$);*
- (iii) *If $u_{m,+} = u^{m,+}$, then $P_m[u_{m,+}] = P_m[u^{m,+}]$ is a globally attractive steady state of (1.2) in $P_m[\mathcal{X}_m^+] \setminus \{0\}$; in other words, $\lim_{t\to\infty} ||\Phi(t, \phi) - P_m[u_{m,+}]||_{C^2_{loc}(\mathbb{R}^2)} = 0$ for all $\phi \in P_m[\mathcal{X}_m^+] \setminus \{0\}$.*

We also have the following results about the multiplicity, shape and attractivity of the heterogeneous steady states for the bistable reaction-diffusion equation in $\mathbb{R}^2$.

Theorem 7.2. *Assume that (H1) and (H3) hold. Then the following hold:*

(i) *-Multiplicity: for every positive integer m,* (1.2) *has a unique nontrivial steady state* u^m *in* $P_m[\mathcal{X}_m^+]$.

(ii) *-Shape: for the* u^m *in (i), we have* $u^m = P_m[u^m|_{\Omega_m}]$*; moreover,*

(ii)-1: there is the asymptotic property: $\lim_{T\to\infty} |u^m|_T = \lim_{T\to\infty} |u^m|^T = u^*$, *where* $|u|_T = \inf\{u(x,y) : (x,y) \in \Omega_m^{T,\infty}\}$ *and* $|u|^T = \sup\{u(x,y) : (x,y) \in \Omega_m^{T,\infty}\}$*;*

(ii)-2: there is the symmetry: u^m *is a symmetric function with respect to* $\{y = \tan(\frac{\pi}{2m})x\}$*;*

(ii)-3: there is the monotonicity:
$u^m(x,y)$ *is strictly increasing from* $(x_0, y_0) \in \partial\Omega_m$ *along the ray parallel to* $\{y = \tan(\frac{\pi}{2m})x\}$.

(iii) *-Attractivity:* u^m *is a globally attractive steady state of* (1.2) *in* $P_m[\mathcal{X}_m^+] \setminus \{0\}$*, in the sense that* $\lim_{t\to\infty} ||\Phi(t,\phi) - u^m||_{C^2_{loc}(\mathbb{R}^2)} = 0$ *for all* $\phi \in P_m[\mathcal{X}_m^+] \setminus \{0\}$.

Proof. (i) and (iii) follow from Proposition 3.3 and Theorem 6.1. (ii) follows from Proposition 3.3 and Corollary 6.1. This completes the proof. □

Appendix A. The linear operator semigroups

In this appendix, we give some basic properties of $T(t)$ and $\mathcal{T}_m(t)$.

By the explicit expression of $T(t)$, we easily get the following results.

Lemma A.1. *Let* $T(t)$ *be defined in Section 2. Then the following statements are true.*

(i) $T(t) : X \to X$ *is a linear operator such that* $||T(t)[\phi]||_X \le ||\phi||_X$ *for all* $(t,\phi) \in \mathbb{R}_+ \times X$ *and* $T|_{\mathbb{R}_+\times X_r} : \mathbb{R}_+ \times X_r \to X_r$ *is continuous, where* $t \in \mathbb{R}_+, r \in (0,\infty)$, $X_r \triangleq \{\phi \in X : ||\phi||_X \le r\}$ *and* $T|_{\mathbb{R}_+\times X_r}(t,\phi) = T(t)[\phi]$*;*

(ii) $T(0) = Id_X$, $T(t)[T(s)[\phi]] = T(t+s)[\phi]$ *for all* $t, s \in \mathbb{R}_+$ *and* $\phi \in X$.

Now, we give a lemma to establish some key properties of $k_{t,m}$, $T(t)$ and $\mathcal{T}_m(t)$ for all positive integers m.

Lemma A.2. *For any positive integer m, we have the following results:*

(i) $k_{t,m}(x,y,\tilde{x},\tilde{y}) = 0$ *for all* $(t,x,y,\tilde{x},\tilde{y}) \in \mathbb{R}_+ \times \partial\Omega_m \times \Omega_m$*, and hence* $\mathcal{T}_m(t)[\mathcal{X}_m] \subseteq \mathcal{X}_m$ *for all* $t \in \mathbb{R}_+$*;*

(ii) $\phi \circ A = -\phi$, $\phi \circ B_m = \phi$, $T(t)[\phi] \circ A = -T(t)[\phi]$ *and* $T(t)[\phi] \circ B_m = T(t)[\phi]$ *for all* $(t,\phi) \in \mathbb{R}_+ \times P_m[\mathcal{X}_m]$*. Hence,* $P_m[\mathcal{X}_m]$ *is a positively invariant subset of* $T(t)$*, that is,* $T(t)[P_m[\mathcal{X}_m]] \subseteq P_m[\mathcal{X}_m]$ *for all* $t \in \mathbb{R}_+$*;*

(iii) $\mathcal{T}_m(t)[\phi] = (P_m)^{-1}|_{P_m[\mathcal{X}_m]} \circ T(t) \circ P_m[\phi]$ *for all* $(t,\phi) \in \mathbb{R}_+ \times \mathcal{X}_m$*, in other words,* $P_m \circ \mathcal{T}_m(t)[\phi] = T(t) \circ P_m[\phi]$ *for all* $(t,\phi) \in \mathbb{R}_+ \times \mathcal{X}_m$*;*

(iv) $\mathcal{T}_m(t)(t \ge 0)$ *is a linear operator semigroup such that* $\mathcal{T}_m|_{\mathbb{R}_+\times\mathcal{X}_m^r} : \mathbb{R}_+ \times \mathcal{X}_m^r \to \mathcal{X}_m^r$ *is continuous, where* $t \in \mathbb{R}_+, r \in (0,\infty)$, $\mathcal{X}_m^r \triangleq \{\phi \in \mathcal{X}_m^+ : ||P_m[\phi]||_X \le r\}$ *and* $\mathcal{T}_m|_{\mathbb{R}_+\times\mathcal{X}_m^r}(t,\phi) = \mathcal{T}_m(t)[\phi]$*;*

(v) $\mathcal{T}_m(t)[\mathcal{X}_m^+] \subseteq \mathcal{X}_m^+$ *and hence* $k_{t,m}(x,y,\tilde{x},\tilde{y}) \ge 0$ *for all* $(t,x,y,\tilde{x},\tilde{y}) \in \mathbb{R}_+ \times \Omega_m \times \Omega_m$*. Moreover,* $\mathcal{T}_m(t)[\phi](x,y) > 0$ *for all* $(t,x,y) \in (0,\infty) \times Int(\Omega_m)$ *and* $\phi \in \mathcal{X}_m^+ \setminus \{0\}$ *and* $\frac{\partial \mathcal{T}_m(t)[\phi]}{\partial \nu}(x,y) < 0$ *for all* $(x,y) \in \partial\Omega_m \setminus \{(0,0)\}$ *and* $(t,\phi) \in (0,\infty) \times (\mathcal{X}_m^+ \setminus \{0\})$.

(vi) $T(t)[\phi] \in P_m[\mathcal{X}_m^+]$ *for all* $(t,\phi) \in \mathbb{R}_+ \times P_m[\mathcal{X}_m^+]$.

Proof. (i) Clearly, for any integers l, by some simple computations, we have

$$B_m^l = \begin{pmatrix} \cos\frac{2l\pi}{m} & \sin\frac{2l\pi}{m} \\ -\sin\frac{2l\pi}{m} & \cos\frac{2l\pi}{m} \end{pmatrix}.$$

By the choice of Ω_m, $(x, y) \in \partial\Omega_m$ implies $y = 0$ or $x = y$. Note that $(x, 0)B_m^{-l} = (x, 0)B_m^{l-m}A^{-1}$ and $(x, x)B_m^{-l} = (x, x)B_m^{l-m-1}A^{-1}$ for all integers $l \in [0, m-1]$. These together with the presentation of $k_{t,m}$ and $B_m^{-m} = B_m^0$, give (i).

(ii) Suppose that $(t, \phi) \in \mathbb{R}_+ \times P_m[\mathcal{X}_m]$. By the definitions of Ω_m and P_m, we easily verify $\phi \circ A = -\phi$ and $\phi \circ B_m = \phi$. Thus, $T(t)[\phi] \circ A = -T(t)[\phi]$ and $T(t)[\phi] \circ B_m = T(t)[\phi]$ when $t = 0$. If $t > 0$, then for any $(x, y) \in \mathbb{R}^2$, we have

$$\begin{aligned} T(t)[\phi] \circ A(x, y) &= \frac{1}{4\pi t}\int_{\mathbb{R}^2} \phi(\tilde{x}, \tilde{y}) \exp(-\frac{||(x, y)A - (\tilde{x}, \tilde{y})||^2}{4t})\mathrm{d}\tilde{x}\mathrm{d}\tilde{y} \\ &= \frac{1}{4\pi t}\int_{\mathbb{R}^2} \phi(\tilde{x}, \tilde{y}) \exp(-\frac{||(x, y) - (\tilde{x}, \tilde{y})A^{-1}||^2}{4t})\mathrm{d}\tilde{x}\mathrm{d}\tilde{y} \\ &= \frac{1}{4\pi t}\int_{\mathbb{R}^2} \phi((\tilde{x}, \tilde{y})A) \exp(-\frac{||(x, y) - (\tilde{x}, \tilde{y})||^2}{4t})\mathrm{d}\tilde{x}\mathrm{d}\tilde{y} \\ &= -\frac{1}{4\pi t}\int_{\mathbb{R}^2} \phi(\tilde{x}, \tilde{y}) \exp(-\frac{||(x, y) - (\tilde{x}, \tilde{y})||^2}{4t})\mathrm{d}\tilde{x}\mathrm{d}\tilde{y} \\ &= -T(t)[\phi](x, y). \end{aligned}$$

Similarly, by $\phi \circ B_m = \phi$ due to the definition of P_m, we also easily see $T(t)[\phi] \circ B_m = T(t)[\phi]$. So, statement (ii) holds.

(iii) Fix ϕ in $\mathcal{X}_m$ and let $\psi = P_m[\phi]$. Suppose that $(t, x, y) \in \mathbb{R}_+ \times \Omega_m$. Clearly, $P_m \circ \mathcal{T}_m(t)[\phi] = T(t) \circ P_m[\phi]$ when $t = 0$. If $t > 0$, then it follows from the definition of ψ and the linear transformations of variables that

$$\begin{aligned} T(t)[\psi](x, y) &= \frac{1}{4\pi t}\int_{\mathbb{R}^2} \psi(\tilde{x}, \tilde{y}) \exp(-\frac{||(x, y) - (\tilde{x}, \tilde{y})||^2}{4t})\mathrm{d}\tilde{x}\mathrm{d}\tilde{y} \\ &= \frac{1}{4\pi t}\sum_{l=0}^{m-1}[\int_{B_m^l[\Omega_m]} \psi(\tilde{x}, \tilde{y}) \exp(-\frac{||(x, y) - (\tilde{x}, \tilde{y})||^2}{4t})\mathrm{d}\tilde{x}\mathrm{d}\tilde{y} \\ &\quad + \int_{B_m^l[A[\Omega_m]]} \psi(\tilde{x}, \tilde{y}) \exp(-\frac{||(x, y) - (\tilde{x}, \tilde{y})||^2}{4t})\mathrm{d}\tilde{x}\mathrm{d}\tilde{y}] \\ &= \frac{1}{4\pi t}\sum_{l=0}^{m-1}[\int_{\Omega_m} \psi((\tilde{x}, \tilde{y})B_m^l) \exp(-\frac{||(x, y) - (\tilde{x}, \tilde{y})B_m^l||^2}{4t})\mathrm{d}\tilde{x}\mathrm{d}\tilde{y} \end{aligned}$$

$$+\int_{\Omega_m}\psi((\tilde{x},\tilde{y})B_m^l A)\exp(-\frac{||(x,y)-(\tilde{x},\tilde{y})B_m^l A||^2}{4t})\mathrm{d}\tilde{x}\mathrm{d}\tilde{y}]$$

$$=\frac{1}{4\pi t}\sum_{l=0}^{m-1}[\int_{\Omega_m}\psi(\tilde{x},\tilde{y})\exp(-\frac{||(x,y)B_m^{-l}-(\tilde{x},\tilde{y})||^2}{4t})\mathrm{d}\tilde{x}\mathrm{d}\tilde{y}$$

$$-\int_{\Omega_m}\psi(\tilde{x},\tilde{y})\exp(-\frac{||(x,y)A^{-1}B_m^{-l}-(\tilde{x},\tilde{y})||^2}{4t})\mathrm{d}\tilde{x}\mathrm{d}\tilde{y}]$$

$$=\int_{\Omega_m}\psi(\tilde{x},\tilde{y})k_{t,m}(x,y,\tilde{x},\tilde{y})\mathrm{d}\tilde{x}\mathrm{d}\tilde{y}$$

$$=\mathcal{T}_m(t)[\phi](x,y).$$

This together with statement (ii) and the definitions of ψ and P_m, implies (iii).

(iv) follows from Lemma A.1-(i)-(ii), statements (ii) and (iii).

(v) By statements (i) and (iii), we know that for given $\phi\in\mathcal{X}_m$, $\mathcal{T}_m(t)[\phi](x,y)$ solves the following initial-boundary problem,

$$\begin{cases}\frac{\partial u}{\partial t}(t,x,y)=\Delta u(t,x,y), & (t,x,y)\in(0,\infty)\times\Omega_m,\\ u(t,x,y)=0, & (x,y)\in\mathbb{R}_+\times\partial\Omega_m,\\ u(0,x,y)=\phi(x,y), & (x,y)\in\Omega_m.\end{cases} \tag{A.1}$$

This together with the Phragmén-Lindelöf type maximum principle in [33], implies $\mathcal{T}_m(t)[\phi](x,y)\geq 0$ for all $(t,x,y)\in\mathbb{R}_+\times\Omega_m$. That is, $\mathcal{T}_m(t)[\mathcal{X}_m^+]\subseteq\mathcal{X}_m^+$ for all $t\in\mathbb{R}_+$ and thus by continuity of $k_{t,m}$ and the definition of $\mathcal{T}_m(t)$, $k_{t,m}(x,y,\tilde{x},\tilde{y})\geq 0$ for all $(t,x,y,\tilde{x},\tilde{y})\in\mathbb{R}_+\times\Omega_m\times\Omega_m$.

By the strong maximum principle, we easily see that $\mathcal{T}_m(t)[\phi](x,y)>0$ for all $(t,x,y)\in(0,\infty)\times Int(\Omega_m)$ and $\phi\in\mathcal{X}_m^+\setminus\{0\}$.

Finally, the Hopf boundary lemma implies that $\frac{\partial\mathcal{T}_m(t)[\phi]}{\partial\nu}(x,y)<0$ for all $(x,y)\in\partial\Omega_m\setminus\{(0,0)\}$ and $(t,\phi)\in(0,\infty)\times(\mathcal{X}_m^+\setminus\{0\})$.

(vi) follows from the definition of $P_m[\mathcal{X}_m^+]$ and statements (ii), (iii) and (v).

This completes the proof. □

For any $L>0$, let $\mathbb{X}_L=\{\phi\in C([0,L]^2,\mathbb{R}):\phi(x,y)=0$ for all $(x,y)\in\partial([0,L]^2)\}$, $\mathbb{X}_L^+=\{\phi\in\mathbb{X}_L:\phi(x,y)\geq 0$ for all $(x,y)\in[0,L]^2\}$ and $g_L(t,x,y,\tilde{x},\tilde{y})=\frac{1}{L^2}\sum_{n=1}^{\infty}\{[cos(\frac{n\pi(x-\tilde{x})}{L})-cos(\frac{n\pi(x+\tilde{x})}{L})]\times[cos(\frac{n\pi(y-\tilde{y})}{L})-cos(\frac{n\pi(y+\tilde{y})}{L})]\times e^{-2(\frac{n\pi}{L})^2t}\}$ for all $L>0$ and $(t,x,y,\tilde{x},\tilde{y})\in(0,\infty)\times[0,L]^4$. Let us define $\mathbb{S}_L(0)[\phi](x,y)=\phi(x,y)$ and $\mathbb{S}_L(t)[\phi](x,y)=\int_{[0,L]^2}g_L(t,x,y,\tilde{x},\tilde{y})\phi(\tilde{x},\tilde{y})\mathrm{d}\tilde{x}\mathrm{d}\tilde{y}$ for all $t\in(0,\infty)$ and $(x,y,\phi)\in[0,L]^2\times\mathbb{X}_L$ Note that $\{\mathbb{S}_L(t)\}_{t\geq 0}$ is an analytic semigroup on $\mathbb{X}_L$ generated by the $\mathbb{X}_L$-realization $\Delta_{\mathbb{X}_L}$ of Δ. Moreover, for given $\phi\in X$, $\mathbb{S}_L(t)[\phi](x,y)$ solves the following initial-boundary problem for $t>0$,

$$\begin{cases}\frac{\partial u}{\partial t}=\Delta u, & t>0,\\ u(t,x,y)=0, & (x,y)\in\partial([0,L]^2),\\ u(0,x,y)=\phi(x,y), & (x,y)\in[0,L]^2.\end{cases} \tag{A.2}$$

According to the definitions of $\mathcal{T}_m(t)$ and $\mathbb{S}_L(t)$, the operators $\mathcal{T}_m(t)$, $\mathbb{S}_L(t)$ can be extended to linear operators from $L^\infty(\Omega_m)$, $L^\infty([0, L]^2)$ into itself, respectively, which are also order preserving in the sense of the pointwise order.

In addition to the basic properties for $\mathcal{T}_m(t)$ and $\mathbb{S}_L(t)$, these two operators enjoy some further properties which are very useful for estimating the $\omega-$limit sets near $\partial\Omega_m$ in the vicinity of the origin and at infinity locations.

Lemma A.3. *For a given positive integer m, we have the following results:*

(i) $\mathcal{T}_m(t)[\phi] \le \mathcal{T}_{m^*}(t)[\tilde{\phi}]|_{\Omega_m}$, *where* $0 \le m^* \le m$, $(t, \phi) \in \mathbb{R}_+ \times \mathcal{X}_m^+$ *and* $\tilde{\phi} \in \mathcal{X}_{m^*}^+$ *is defined by* $\tilde{\phi}(x, y) = \phi(x, y)$ *for all* $(x, y) \in \Omega_m$ *and* $\tilde{\phi}(x, y) = 0$ *for all* $(x, y) \in \Omega_{m^*} \setminus \Omega_m$*;*

(ii) *Let* $\phi, \psi \in \mathcal{X}_m^+$ *and let* $(a, b) \in \mathbb{R}^2$ *with* $(a, b) + \Omega_m \subseteq \Omega_m$. *If* $\phi(a + x, b + y) \ge \psi(x, y)$ *for all* $(x, y) \in \Omega_m$, *then* $\mathcal{T}_m(t)[\phi](a + x, b + y) \ge \mathcal{T}_m(t)[\psi](x, y)$ *for all* $(t, x, y) \in \mathbb{R}_+ \times \Omega_m$*;*

(iii) *If* $x_0 > 0$ *and* $L > 0$ *such that* $L \le x_0 \tan(\frac{\pi}{m})$ *for all* $m \ge 3$ *and* $L \le x_0$ *for all* $m \le 2$, *then* $\mathcal{T}_m(t)[\phi]|_{(x_0,0)+[0,L]^2} \ge \mathbb{S}_L(t)[\phi(\cdot + x_0, \cdot)|_{[0,L]^2}](\cdot - x_0, \cdot)|_{(x_0,0)+[0,L]^2}$ *for all* $(t, \phi) \in \mathbb{R}_+ \times \mathcal{X}_m^+$*;*

(iv) *Let* $a(t, T, x, y) = \sup\{a \ge 0 : \mathcal{T}_m(t)[\zeta](x, y) \ge a\mathcal{T}_m(t)[\eta](x, y) + a\mu \int_0^t \exp(\mu s)\mathcal{T}_m(t - s)[\eta](x, y)\mathrm{d}s$ *for all* $\zeta, \eta \in \mathcal{X}_m^1$ *with* $\zeta \ge h_m^{T,\infty}\}$ *and* $b(r, t, T) = \inf\{a(t, T, x, y) : (x, y) \in \Omega_m \setminus \Omega_m^{T,\infty}$ *with* $\sqrt{x^2 + y^2} \ge r\}$ *for all* $(r, t, T) \in (0, \infty)^3$ *and* $(x, y) \in \Omega_m$. *Then* $b(r, t, T) > 0$ *for all* $(r, t, T) \in (0, \infty)^3$.

Proof. Clearly, (i), (ii) and (iii) follow from the Phragmén-Lindelöf type maximum principle in [33].

(iv) Fix $(r, t, T) \in (0, \infty)^3$. Note that $a(t, T, \cdot, \cdot)$ is a continuous and positive function in $\Omega \setminus \{(0, 0)\}$ due to Lemma A.2-(v) in the Appendix.

By letting $L = \min\{2T, 2T \tan(\frac{\pi}{2m})\}$, we have $(x_0 - \frac{L}{2}, 0) + [0, L] \times [\frac{L}{2}, L] \subseteq \Omega_m^{T,\infty}$ for all $x_0 \ge 5T$. It follows from Lemma A.3-(iii) that for any $\zeta \in \mathcal{X}_m^1$ with $\zeta \ge h_m^{T,\infty}$ and $(x, y) \in (x_0 - \frac{L}{2}, 0) + [0, L]^2$ with $x_0 \ge 5T$, we have

$$\begin{aligned}\mathcal{T}_m(t)[\zeta](x, y) &\ge \mathbb{S}_L(t)[\zeta(\cdot + x_0 - \frac{L}{2}, \cdot)|_{[0,L]^2}](x - x_0 + \frac{L}{2}, y)\\ &= \int\limits_{[0,L]^2} \zeta(\tilde{x} + x_0 - \frac{L}{2}, \tilde{y})g_L(t, x - x_0 + \frac{L}{2}, y, \tilde{x}, \tilde{y})\mathrm{d}\tilde{x}\mathrm{d}\tilde{y}\\ &\ge \int\limits_{[0,L]\times[\frac{L}{2},L]} g_L(t, x - x_0 + \frac{L}{2}, y, \tilde{x}, \tilde{y})\mathrm{d}\tilde{x}\mathrm{d}\tilde{y}.\end{aligned}$$

Thus, for any $\zeta \in \mathcal{X}_m^1$ with $\zeta \ge h_m^{T,\infty}$ and $(x, y) \in [5T, \infty) \times (0, \frac{L}{2}]$, by taking $x_0 = x - \frac{L}{2}$ and applying the previous inequality, we have

$$\mathcal{T}_m(t)[\zeta](x, y) \ge \int\limits_{[0,L]\times[\frac{L}{2},L]} g_L(t, \frac{L}{2}, y, \tilde{x}, \tilde{y})\mathrm{d}\tilde{x}\mathrm{d}\tilde{y}$$

$$=\sum_{n\geq 1}\{\frac{4(1+(-1)^{n+1})sin(\frac{n\pi}{2})\times sin(\frac{n\pi y}{L})\times e^{-2(\frac{n\pi}{L})^2 t}}{(n\pi)^2}\}>0.$$

On the other hand, by applying Lemma 2.1-(iv) in [35], Lemma A.3-(i) and Fubini's theorem, it follows that for any $\eta\in\mathcal{X}_m^1$ and $(x,y)\in[5T,\infty)\times(0,\frac{L}{2}]$, we have

$$\mathcal{T}_m(t)[\eta](x,y)+\mu\int_0^t \exp(\mu s)\mathcal{T}_m(t-s)[\eta](x,y)\mathrm{d}s$$

$$\leq \mathcal{T}_1(t)[1](x,y)+\mu\int_0^t \exp(\mu s)\mathcal{T}_1(t-s)[1](x,y)\mathrm{d}s$$

$$=\frac{1}{4\pi t}\int_{\mathbb{R}\times\mathbb{R}_+}\exp(-\frac{(x-\tilde{x})^2}{4t})[\exp(-\frac{(y-\tilde{y})^2}{4t})-\exp(-\frac{(y+\tilde{y})^2}{4t})]\mathrm{d}\tilde{x}\mathrm{d}\tilde{y}$$

$$+\mu\exp(\mu t)\int_0^t\{\frac{\exp(-\mu s)}{4\pi s}\int_{\mathbb{R}\times\mathbb{R}_+}\exp(-\frac{(x-\tilde{x})^2}{4s})[\exp(-\frac{(y-\tilde{y})^2}{4s})-\exp(-\frac{(y+\tilde{y})^2}{4s})]\mathrm{d}\tilde{x}\mathrm{d}\tilde{y}\}\mathrm{d}s$$

$$\leq\frac{1}{\sqrt{4\pi t}}\int_{-y}^{y}\exp(-\frac{\tilde{y}^2}{4t})\mathrm{d}\tilde{y}+\mu\exp(\mu t)\int_{\mathbb{R}_+}\frac{\exp(-\mu s)}{\sqrt{4\pi s}}\int_{-y}^{y}\exp(-\frac{\tilde{y}^2}{4s})\mathrm{d}\tilde{y}\mathrm{d}s$$

$$\leq\frac{1}{\sqrt{4\pi t}}\int_{-y}^{y}\exp(-\frac{\tilde{y}^2}{4t})\mathrm{d}\tilde{y}+\frac{\sqrt{\mu}\exp(\mu t)}{2}\int_{-y}^{y}\exp(-\sqrt{\mu\tilde{y}^2})\mathrm{d}\tilde{y}.$$

Define $\rho:[0,\frac{L}{2}]\to\mathbb{R}$ by

$$\rho(0)=\frac{\sum_{n\geq 1}\frac{4(1+(-1)^{n+1})sin(\frac{n\pi}{2})\times e^{-2(\frac{n\pi}{L})^2 t}}{nL\pi}}{\frac{1}{\sqrt{\pi t}}+\sqrt{\mu}\exp(\mu t)}$$

and for any $y\in(0,\frac{L}{2}]$,

$$\rho(y)=\frac{\sum_{n\geq 1}\frac{4(1+(-1)^{n+1})sin(\frac{n\pi}{2})\times sin(\frac{n\pi y}{L})\times e^{-2(\frac{n\pi}{L})^2 t}}{(n\pi)^2}}{\frac{1}{\sqrt{4\pi t}}\int_{-y}^{y}\exp(-\frac{\tilde{y}^2}{4t})\mathrm{d}\tilde{y}+\frac{\sqrt{\mu}\exp(\mu t)}{2}\int_{-y}^{y}\exp(-\sqrt{\mu\tilde{y}^2})\mathrm{d}\tilde{y}}.$$

Clearly, ρ is a continuous and positive function, and thus there exist $\rho^*>0$ such that $\rho(y)\geq\rho^*$ for all $y\in[0,\frac{L}{2}]$. By the definitions of $a(t,T,x,y)$ and $\rho(y)$, we have $\rho^*\leq a(t,T,x,y)$ for all $(x,y)\in[5T,\infty)\times[0,\frac{L}{2}]\}$. Similarly, we have $\rho^*\leq a(t,T,(x,y)AB_{2m})$ for all $(x,y)\in$

$[5T, \infty) \times [0, \frac{L}{2}]\}$. Thus, $a(t, T, x, y) \geq \rho^*$ for all $(x, y) \in \Omega_m \setminus \Omega_m^{T,\infty}$ with $\sqrt{x^2 + y^2} \geq 6T$. This together with the continuity and positivity of $a(t, T, \cdot, \cdot)$, implies that $b(r, t, T) > 0$ for all $(r, t, T) \in (0, \infty)^3$. This completes the proof. □

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