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Asymptotic profiles of steady states for a diffusive SIS epidemic model with mass action infection mechanism ☆

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Abstract

Mass action and standard incidence are two major infection mechanisms in modelling spread of infectious diseases. Spatial heterogeneity plays an important role in spread of infectious diseases, and hence, motivates and advocates diffusive models for disease dynamics. By analyzing a diffusive SIS model with the standard incidence infection mechanism, some recent works [2,12] have investigated the asymptotical profiles of the endemic steady state for large and small diffusion rates, and the results show that controlling the diffusion rate of the susceptible individuals can help eradicate the infection, while controlling the diffusion rate of the infectious individuals cannot. This paper aims to reveal the difference between the two infection mechanisms in a spatially heterogeneous environment. To this end, we consider a diffusive SIS model of the same structure but with the mass action infection adopted, and explore the asymptotic profiles of the endemic steady state for small and large diffusion rates. It turns out that the new model poses some new challenges due to the nonlocal term in the equilibrium problem and the unboundedness of the nonlinear term. Our results on this new model reveal some fundamental differences between the two transmission mechanisms in such spatial models, which may provide some implications on disease modelling and controls.

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1. Introduction

In modelling disease dynamics, an infection mechanism needs to be assumed and adopted. Among the various infection mechanisms are the most frequently used mass action and standard incidence mechanisms. Indeed, in the classic and pioneering Kermack–McKendrick model [9], the mass action term βSI was used to describe the interactions between susceptible and infected individuals. Much later, in de Jong et al. [8], the standard incidence transmission term $\beta SI/N$ was suggested as an alternative to mass action. In [11], McCallum et al. compared the mass action and the standard incidence theoretically by looking at some host-pathogen models. According to McCallum et al., the basic reproduction number is dependent on the population size N in mass action models, which can be used to explain why population culling is a common strategy for the control of the outbreak of many diseases. But if the transmission obeys the standard incidence mechanism, the basic reproduction number is independent on the total population. However, despite of the above mentioned advantage of mass action, they also observed that many models that used simple mass action did not fit the empirical data as accurately as the models that used standard incidence mechanism.

The aforementioned studies are for a spatially homogeneous environment, meaning that only ODE models are involved. On the other hand, it is known that spatial heterogeneity and diffusion are ubiquitous in the real world and they play important roles in the spread of many diseases. A very natural question arises: Would incorporation of spatial heterogeneity and diffusion lead to any new phenomenon in disease spread under different infection mechanisms? Answering such a question may not only give insights into disease spread and control in reality, but also suggest new aspects and considerations for modelling spatial-temporal dynamics of infectious diseases. Indeed, these or similar questions have attracted many researchers in recent years, and there have been quite a few publications along this line. See, e.g., [1,2,5,6,12–17] and the references therein.

Among the above works, Allen et al. [2] proposed a simple diffusive SIS model with space-dependent disease transmission rate $\beta(x)$ and recovery rate $\gamma(x)$, given by

$$\begin{cases} S_t = d_S \Delta S - \dfrac{\beta(x)SI}{S+I} + \gamma(x)I, \\ I_t = d_I \Delta I + \dfrac{\beta(x)SI}{S+I} - \gamma(x)I, \end{cases} \qquad x \in \Omega, \quad t > 0, \tag{1.1}$$

where $\Omega \subset R^m$ is a bounded domain with smooth boundary $\partial\Omega$ and $N = \int_\Omega (S+I)dx$. They showed that if the disease is of low risk in some part of the habitat, the disease can be potentially controllable by limiting the diffusion rate of the susceptible individuals. More precisely, they proved that if there are some low risk spots in the domain (i.e., $\beta(x) < \gamma(x)$ for some $x \in \Omega$), then the disease component of the endemic equilibrium vanishes as the diffusion rate of the susceptible individuals approaches zero. In a subsequent work [12], Peng showed that for this model, limiting the diffusion rate of the infected individuals cannot help annihilate the disease.

Thus the results in [2,12] suggest that the diffusion rate of susceptible individuals should be target of control in order to eradicate the disease.

Note that above conclusions are for the model (1.1) where the standard incidence infection mechanism is adopted. One naturally wonder what happens if standard incidence is replaced by the mass action. Considering the fact that the mass action is still dominantly adopted by biologists and mathematicians in the host-pathogen and host-parasite models, answering this question is of both theoretical and practical importance. As an initial attempt, Deng and Wu [5] proposed and analyzed the following model

$$\begin{cases} S_t = d_S \Delta S - \beta(x) SI + \gamma(x) I, \\ I_t = d_I \Delta I + \beta(x) SI - \gamma(x) I, \end{cases} \qquad x \in \Omega, \ \ t > 0, \tag{1.2}$$

which is parallel to (1.1). Here, as in (1.1), $\Omega \subset R^m$ is a bounded domain with smooth boundary $\partial\Omega$ and $N = \int_\Omega (S + I)dx$ denoting the total population. The main results in [5] are the threshold dynamics in terms of the model's basic reproduction number $\mathcal{R}_0$. This paper is a continuation of [5] and [2,12], aiming to explore the asymptotic profiles of the endemic steady state (under $\mathcal{R}_0 > 1$) for large and small diffusion rates. To this end, we will firstly summarize, in Section 2, some preliminary and relevant results on the dynamics of (1.2) from [5], and then present our main results on the asymptotic profiles of the endemic equilibrium for large and small diffusion rate. We give the proofs of the main results in Section 3. Finally we conclude the paper by Section 4 in which we briefly discuss the biological interpretations of our results and also compare them with the results for (1.1), revealing some differences between the two transmission mechanisms with spatial effects presented.

We point out that unlike in ODE models, adoption of mass action in (1.2) makes the analysis more difficult and challenging than the analysis of (1.1) on the above mentioned topics. For example, the equilibrium problem for (1.1) can be reduced to a local elliptic problem while the corresponding problem for (1.2) is a *nonlocal* elliptic problem. Moreover the standard incidence term $\beta SI/(S + I)$ assumes bounded infection force while the mass action term βSI uses a unbounded infection force.

2. The model and main results

Suppose that the host individuals live in an open and bounded domain $\Omega \subseteq \mathbb{R}^m$ with smooth boundary $\partial\Omega$. Let $\bar{S}(x,t)$ and $\bar{I}(x,t)$ be the populations of susceptible and infectious individuals at position x and time t, respectively. The individuals are assumed to randomly move around in the domain with diffusion rates d_S and d_I for susceptible and infectious individuals, respectively. Let $\beta(x)$ and $\gamma(x)$ be the disease transmission and recovery rates, respectively, which are assumed to be dependent on position x. For biological reason, we assume that they are non-negative and for mathematical tractability, we suppose that they are Hölder continuous in Ω. As in [2], we consider a fast disease by ignoring the demography of the host; but unlike in [2] we will adopt the mass action infection mechanism. These leads to the SIS model system with diffusion (1.2) (see [5]):

$$\begin{cases} \bar{S}_t = d_S \Delta \bar{S} - \beta(x) \bar{S}\bar{I} + \gamma(x) \bar{I}, \\ \bar{I}_t = d_I \Delta \bar{I} + \beta(x) \bar{S}\bar{I} - \gamma(x) \bar{I}, \end{cases} \qquad x \in \Omega, \ \ t > 0. \tag{2.1}$$

We consider a scenario that the domain Ω is isolated from outside for the host, implying the homogeneous Neumann boundary condition:

$$\frac{\partial \bar{S}}{\partial n} = \frac{\partial \bar{I}}{\partial n} = 0, \qquad x \in \partial\Omega, \quad t > 0. \tag{2.2}$$

For initial conditions, we pose the following

(H1) $\bar{S}(x,0)$ and $\bar{I}(x,0)$ are nonnegative continuous functions in $\overline{\Omega}$, and $\int_\Omega \bar{I}(x,0)\,dx > 0$.

The positivity of the integral means that there are infectious individuals initially at $t = 0$ in the region.

Adding the two equations in (2.1) and integrating over Ω, we find that

$$\frac{\partial}{\partial t}\int_\Omega (\bar{S} + \bar{I})dx = 0.$$

Thus, if we assume

(H2) $\int_\Omega (\bar{S}(x,0) + \bar{I}(x,0))\,dx \equiv N > 0$,

then the total population remains the constant N, i.e.,

$$\int_\Omega (\bar{S} + \bar{I})dx = N \quad \text{for all } t \geq 0. \tag{2.3}$$

Steady state solutions of (2.1) are governed by the following elliptic system:

$$\begin{cases} d_S \Delta S - \beta(x)SI + \gamma(x)I = 0, & x \in \Omega, \\ d_I \Delta I + \beta(x)SI - \gamma(x)I = 0, & x \in \Omega, \end{cases} \tag{2.4}$$

with the same zero flux boundary condition as (2.2):

$$\frac{\partial S}{\partial n} = \frac{\partial I}{\partial n} = 0, \quad x \in \partial\Omega, \tag{2.5}$$

and subject to the same constraint on the total population as in (2.3):

$$\int_\Omega (S + I)dx = N. \tag{2.6}$$

System (2.4)–(2.6) always has a solution $E_0 = (N/|\Omega|, 0)$, which is the unique disease free equilibrium (DFE). A nonnegative solution $E_1 = (S, I)$ of (2.4)–(2.6) is an endemic equilibrium (EE) of (2.1)–(2.2) if $I(x) > 0$ for some $x \in \overline{\Omega}$. In [5], it is shown that (S, I) is an EE if and only if I is a positive solution to the nonlocal elliptic problem

$$
\begin{aligned}
& d_I \Delta I + I\left(\frac{N}{|\Omega|}\beta - \gamma - \left(1-\frac{d_I}{d_S}\right)\frac{\beta}{|\Omega|}\int_\Omega I dx - \frac{d_I \beta}{d_S} I\right) = 0, \quad x \in \Omega, \\
& \frac{\partial I}{\partial n} = 0, \quad x \in \partial\Omega,
\end{aligned}
\tag{2.7}
$$

and S is given by

$$
S = \frac{N}{|\Omega|} - \left(1-\frac{d_I}{d_S}\right)\frac{\int_\Omega I dx}{|\Omega|} - \frac{d_I}{d_S} I. \tag{2.8}
$$

As for (1.1) in [2], for (2.1) the basic reproduction number $\mathcal{R}_0$ is given by the variational formula

$$
\mathcal{R}_0 = \sup\left\{\frac{\frac{N}{|\Omega|}\int_\Omega \beta\varphi^2 \, dx}{\int_\Omega (d_I|\nabla\varphi|^2 + \gamma\varphi^2)\, dx} : \ \varphi \in H^1(\Omega) \text{ and } \varphi \neq 0\right\}.
$$

Let λ^* denote the principal eigenvalue of the following eigenvalue problem:

$$
\begin{aligned}
& d_I \Delta\phi + (\frac{N}{|\Omega|}\beta - \gamma)\phi + \lambda\phi = 0, && \text{in } \Omega, \\
& \frac{\partial\phi}{\partial n} = 0, && \text{on } \partial\Omega.
\end{aligned}
$$

It has been shown in [5] that $\mathcal{R}_0$ and λ^* are related and they both serve as threshold parameters in the sense stated in the following lemma.

Lemma 2.1. *The following statements about λ^* and $\mathcal{R}_0$ hold.*

(i) $1 - \mathcal{R}_0$ *and* λ^* *have the same sign;*
(ii) *If* $\int_\Omega \frac{N}{|\Omega|}\beta \, dx \geq \int_\Omega \gamma \, dx$, *then* $\lambda^* \leq 0$ *for all* $d_I > 0$;
(iii) *If* $\frac{N}{|\Omega|}\beta - \gamma$ *changes sign on* Ω *but* $\int_\Omega \frac{N}{|\Omega|}\beta \, dx < \int_\Omega \gamma \, dx$, *then there exists* $d_I^* > 0$ *such that* $\lambda^* = 0$ *when* $d_I = d_I^*$, $\lambda^* < 0$ *when* $d_I < d_I^*$, *and* $\lambda^* > 0$ *when* $d_I > d_I^*$.

Motivated by [2], we define the high-risk region and low-risk region respectively by

$$
\Omega^+ = \{x \in \Omega : \frac{N}{|\Omega|}\beta(x) - \gamma(x) > 0\}
$$

and

$$
\Omega^- = \{x \in \Omega : \frac{N}{|\Omega|}\beta(x) - \gamma(x) < 0\}.
$$

We say the domain Ω is a high-risk domain if $\int_\Omega (N\beta/|\Omega| - \gamma) dx > 0$ and it is a low-risk domain if $\int_\Omega (N\beta/|\Omega| - \gamma) dx < 0$.

In [5], the following result on the existence and non-existence of an endemic equilibrium (EE) has been proved.

Theorem 2.2. *When $d_S \geq d_I$, then the EE does not exist if $\mathcal{R}_0 \leq 1$, and there exists a unique EE if $\mathcal{R}_0 > 1$; when $d_S < d_I$, then the EE does not exist if $\mathcal{R}_0 \leq d_S/d_I$, and there exists an EE if $\mathcal{R}_0 > 1$.*

By Lemma 2.1 and Theorem 2.2, we can see that if Ω is a high-risk domain, then the EE always exists. If Ω is a lower risk domain but there are also high risk locations in Ω (i.e., Ω^+ is non-empty), then there exists $d_I^* > 0$ such that the EE exists for $d_I < d_I^*$.

In the rest of this paper, we always assume $\mathcal{R}_0 > 1$ so that (2.1)–(2.2) has a unique EE. For convenience, when no confusion arises, we simply use (S, I) to denote the EE of (2.1)–(2.2) (i.e., the positive solution of (2.4)–(2.6)). We now investigate the asymptotic profile of the EE as the diffusion rates approaching zero. We first consider the case $d_S \to 0$ and have the following main theorem for this case.

Theorem 2.3. *Suppose that $\mathcal{R}_0 > 1$. Then, for any fixed $d_I > 0$, there exists a sequence $\{d_{S_n}\}$ with $d_{S_n} \to 0$ such that the corresponding EE (S_n, I_n) of (2.1)–(2.2) satisfies $(S_n, I_n) \to (S^*, I^*)$ in $C(\overline{\Omega})$, where S^* is a positive function and I^* is a nonnegative constant. Moreover, either (a)*

$$(S^*, I^*) = \left(\frac{\gamma(x)}{\beta(x)}, \frac{N}{|\Omega|} - \frac{1}{|\Omega|} \int_\Omega \frac{\gamma(x)}{\beta(x)} dx \right),$$

or (b) $I^* = 0$ and S^* is the positive solution of the following problem

$$\begin{aligned} -\Delta S &= \alpha(-\beta S + \gamma), && x \in \Omega, \\ \frac{\partial S}{\partial n} &= 0, && x \in \partial\Omega, \\ \int_\Omega S dx &= N, \end{aligned}$$

where α is some positive continuous function on $\overline{\Omega}$ satisfying

$$-d_I \Delta \alpha = (\beta S^* - \gamma)\alpha, \quad x \in \Omega, \tag{2.9}$$

$$\frac{\partial \alpha}{\partial n} = 0, \quad x \in \partial\Omega. \tag{2.10}$$

If $N - \int_\Omega [\gamma(x)/\beta(x)] dx < 0$, then (i) in Theorem 2.3 is excluded, leading to the following corollary which indicates that the disease can be eradicated by limiting the mobility of the susceptible individuals.

Corollary 2.4. *Suppose that $\mathcal{R}_0 > 1$ and $N - \int_\Omega [\gamma(x)/\beta(x)] dx < 0$. If d_I is fixed, then there exists a sequence $\{d_{S_n}\}$ with $d_{S_n} \to 0$ as $n \to \infty$ such that the corresponding EE of (2.4)–(2.6) satisfies that $(S, I) \to (S^*, 0)$ in $C(\overline{\Omega})$, where S^* is the positive solution of the following problem*

$$\begin{cases} -\Delta S = \alpha(-\beta S + \gamma), & x \in \Omega, \\ \frac{\partial S}{\partial n} = 0, & x \in \partial\Omega, \\ \int_\Omega S dx = N, \end{cases}$$

where α is some positive continuous function on $\overline{\Omega}$.

If $N - \int_\Omega [\gamma(x)/\beta(x)]dx > 0$, we conjecture that (ii) of Theorem 2.3 is impossible. At the present, we are only able to prove this for a special case, that is, when β is a positive constant, as stated in the following corollary.

Corollary 2.5. *Suppose that β is a positive constant with $N - \int_\Omega [\gamma(x)/\beta]dx > 0$. If d_I is fixed, then as $d_S \to 0$ the corresponding EE satisfies that $(S, I) \to (S^*, I^*)$ in $C(\overline{\Omega})$, where*

$$(S^*, I^*) = \left(\frac{\gamma(x)}{\beta}, \frac{N}{|\Omega|} - \frac{1}{|\Omega|} \int_\Omega \frac{\gamma(x)}{\beta} dx \right).$$

Remark 2.6. When β is a constant, the condition $N - \int_\Omega [\gamma(x)/\beta]dx > 0$ actually implies that Ω is a high-risk domain. The above two corollaries tell that in order for the strategy of limiting the mobility of susceptible individuals to succeed in eradicating the disease, if and only if the whole region Ω is a lower-risk domain. This result is considerably different from the corresponding conclusion for the model system (1.1) obtained in [2], where it is shown that the same strategy will succeed as long as the lower-risk region Ω^- is non-empty, regardless it is the whole region Ω or not.

Next, we explore the asymptotical profile of the EE as $d_I \to 0$ and $d_I/d_S \to d > 0$. As is customary, for any function h defined on $\overline{\Omega}$, we use Ω^+ to denote the function $\Omega^+(x) = \max\{h(x), 0\}$.

Theorem 2.7. *Assume that Ω^+ is non-empty. Then the following statements hold:*

(a) If $d_I \to 0$ and $d_I/d_S \to d \in (0, \infty)$, then $(S, I) \to (S^*, I^*)$ in $C(\overline{\Omega})$, where I^* is the unique positive solution of

$$\left\{ \frac{N}{|\Omega|}\beta - \gamma - \frac{(1-d)\beta}{|\Omega|} \int_\Omega I^* dx \right\}^+ - d\beta I^* = 0, \tag{2.11}$$

and S^* is given by

$$S^* = \frac{N}{|\Omega|} - (1-d)\frac{\int_\Omega I^* dx}{|\Omega|} - dI^*.$$

(b) If $d_I \to 0$ and $d_I/d_S \to 1$, then $(S, I) \to (S^*, I^*)$ in $C(\overline{\Omega})$, where

$$S^* = \frac{N}{|\Omega|} - \left(\frac{N}{|\Omega|} - \frac{\gamma}{\beta}\right)^+ \quad \text{and} \quad I^* = \left(\frac{N}{|\Omega|} - \frac{\gamma}{\beta}\right)^+.$$

(c) Let d and (S^*, I^*) be as in (a). If $d \in (0, 1)$, then $\{x \in \Omega : I^*(x) > 0\} \subsetneqq \Omega^+$; if $d \in (1, \infty)$, then $\{x \in \Omega : I^*(x) > 0\} \supseteqq \Omega^+$ and this inclusion is strict if Ω^- is non-empty.

We now consider the profile when the diffusion rates are large.

Theorem 2.8. *The following statements hold.*

(a) Suppose that Ω is a high-risk domain. If $d_S \to \infty$ and $d_I \to \infty$, then $(S, I) \to (S^*, I^*)$ in $C^2(\overline{\Omega})$, where

$$(S^*, I^*) = \left(\frac{\int_\Omega \gamma(x)dx}{\int_\Omega \beta(x)dx}, \frac{N}{|\Omega|} - \frac{\int_\Omega \gamma(x)dx}{\int_\Omega \beta(x)dx}\right).$$

(b) Suppose that $\mathcal{R}_0 > 1$. If d_I is fixed and $d_S \to \infty$, then $(S, I) \to (S^*, I^*)$ in $C^2(\overline{\Omega})$, where I^* is the unique positive solution of the following problem

$$\begin{cases} -d_I \Delta I = I\left(\frac{N}{|\Omega|}\beta - \gamma - \frac{\beta}{|\Omega|}\int_\Omega I dx\right), & x \in \Omega, \\ \frac{\partial I}{\partial n} = 0, & x \in \partial\Omega, \end{cases} \tag{2.12}$$

and

$$S^* = \frac{N - \int_\Omega I^* dx}{|\Omega|}.$$

(c) Suppose that Ω is a high-risk domain. If d_S is fixed, then there exists a sequence $\{d_{I_n}\}$ with $d_{I_n} \to \infty$ as $n \to \infty$ such that the corresponding EE $(S_n, I_n) \to (S^*, I^*)$ in $C^2(\overline{\Omega})$, where I^* is a positive constant and S^* is the positive solution of the following problem

$$\begin{cases} -d_S \Delta S = (-\beta S + \gamma)I^*, & x \in \Omega, \\ \frac{\partial S}{\partial n} = 0, & x \in \partial\Omega, \\ \int_\Omega S dx = N - I^*|\Omega|. \end{cases} \tag{2.13}$$

Furthermore, there exists a sequence $\{d_{S_n}\}$ with $d_{S_n} \to 0$ as $n \to \infty$ such that the corresponding solution (S_n^*, I_n^*) of (2.13) satisfies $(S_n^*, I_n^*) \to (\tilde{S}^*, \tilde{I}^*)$ in $C(\overline{\Omega})$, where $\tilde{S}^*$ is a positive function and $\tilde{I}^*$ is a nonnegative constant satisfying the two alternatives in Theorem 2.3.

Part (c) of the above Theorem 2.8 shows that if we let $d_I \to \infty$ first and then take $d_S \to 0$, the asymptotic profile is consistent with what obtained in Theorem 2.3 by directly letting $d_S \to 0$ (see [12] for similar observations). Naturally, one would expect a similar situation with $d_I \to 0$ in (2.12). Of course, to fully understand the asymptotic profile of EE when d_I is small, one needs to study (2.7) by letting $d_I \to 0$. We leave this as an open problem.

Theorem 2.9. *Assume that Ω^+ is nonempty. Let $k_0 > 0$ be such that*

$$\max_{x\in\bar{\Omega}} \left\{ \frac{(N-k_0)}{|\Omega|}\beta(x) - \gamma(x) \right\} = 0,$$

and define

$$\mathcal{M} = \left\{ x \in \bar{\Omega} : \frac{(N-k_0)}{|\Omega|}\beta(x) - \gamma(x) = 0 \right\}.$$

Then there exists $\hat{d}_I > 0$ such that for each $d_I < \hat{d}_I$, the problem (2.12) has a unique positive solution I^* with $\int_\Omega I^* dx \le N$. If further letting $d_I \to 0$, the solution satisfies $\int_\Omega I^* dx \to k_0$. Moreover, there exists a sequence $\{d_{I_n}\}$ with $d_{I_n} \to 0$ as $n \to \infty$ such that the corresponding solution I_n^* satisfies $I_n^* \to k_0\mu$ weakly, where μ is a probability measure with support contained in $\mathcal{M}$.

3. Proof of the main results

3.1. Preliminary results

In this section, we present the proofs of our main results. Before that, we collect several useful lemmas. If $a \in L^\infty(\Omega)$ and $d > 0$, we denote by $\lambda_1(d, a)$ the principal eigenvalue of

$$\begin{cases} d\Delta\phi + a\phi + \lambda\phi = 0, & \text{in } \Omega, \\ \dfrac{\partial\phi}{\partial n} = 0, & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

It is well-known that $\lambda_1(d, a)$ is given by the following variational formula

$$\lambda_1(d,a) = \min\left\{ \int_\Omega (d|\nabla\varphi|^2 - a\varphi^2)dx : \varphi \in H^1(\Omega) \text{ and } \int_\Omega \varphi^2 dx = 1 \right\}.$$

The following result about the principal eigenvalue can be found in [2,3].

Lemma 3.1. *If $a_1(x) \le a_2(x)$ in Ω with $a_i \in L^\infty(\Omega)$ for $i = 1, 2$, then $\lambda_1(d, a_1) \ge \lambda_1(d, a_2)$ with equality holds if and only if $a_1 = a_2$ a.e. in Ω. If $a \in L^\infty(\Omega)$ is non-constant, then $\lambda_1(d_1, a) < \lambda_1(d_2, a)$ if $d_1 < d_2$. Moreover, $\lambda_1(d, a)$ depends continuously on a and d, and it satisfies*

$$\lim_{d\to 0} \lambda_1(d,a) \to \min\{-a(x) : x \in \overline{\Omega}\} \quad \text{and} \quad \lim_{d\to\infty} \lambda_1(d,a) \to -\bar{a}, \tag{3.2}$$

where $\bar{a}$ is the spatial average of a, i.e. $\bar{a} = (\int_\Omega a(x)dx)/|\Omega|$.

Remark 3.2. Regarding the principal eigenvalue $\lambda_1(d,a)$, Allen et al [2] obtained the following more information. Assume that $a \in C(\overline{\Omega})$ is non-constant and $a(x_0) > 0$ for some $x_0 \in \overline{\Omega}$ in the previous lemma. If $\int_\Omega a(x)dx \geq 0$, then Lemma 3.1 implies that $\lambda_1(d,a) < 0$ for all $d > 0$; if $\int_\Omega a(x)dx < 0$, then there exists $d^* > 0$ such that $\lambda_1(d,a) < 0$ for all $d < d^*$ and $\lambda_1(d,a) > 0$ for all $d > d^*$.

The main reason that we are interested in the sign of $\lambda_1(d,a)$ is that it can be used to determine the existence of unique positive solution of the related elliptic problem. The following lemma is such an example, which can be found in [3], or can be directly proved by an upper/lower solution argument.

Lemma 3.3. *Suppose that $a, b \in C^\alpha(\overline{\Omega})$ with $b(x) > 0$ for $x \in \overline{\Omega}$. Then the following statements hold about the problem:*

$$\begin{aligned} 0 &= d\Delta u + [a(x) - b(x)u]u && x \in \Omega, \\ 0 &= \frac{\partial u}{\partial n} && x \in \partial\Omega. \end{aligned} \tag{3.3}$$

(a) If $\lambda_1(d,a) \geq 0$, then the problem (3.3) has no positive solution;
(b) If $\lambda_1(d,a) < 0$, then the problem (3.3) has a unique positive solution in $C^{2+\alpha}(\overline{\Omega})$.

Remark 3.4. The asymptotic profile (as $d \to 0$ or $d \to \infty$) of the positive solution of (3.3) is also well-known. Let $u(d,a,b)$ be the unique positive solution of (3.3) if it exists. If $a(x_0) > 0$ for some $x_0 \in \overline{\Omega}$ in the previous lemma, then the positive solution $u(d,a,b)$ exists for small d and it satisfies that $u(d,a,b) \to (a/b)^+$ as $d \to 0$; if $a(x)$ is non-constant and $\int_\Omega a(x)dx \geq 0$, then the positive solution $u(d,a,b)$ exists for all $d > 0$ and it satisfies $u(d,a,b) \to \bar{a}/\bar{b}$ as $d \to \infty$.

The following maximum/minimum principle is from Lou and Ni [10].

Lemma 3.5. *Suppose that $g \in C(\overline{\Omega} \times \mathbb{R})$.*

- *Assume that $w \in C^2(\Omega) \cap C^1(\overline{\Omega})$ and satisfies*

$$\begin{aligned} &\Delta w(x) + g(x, w(x)) \geq 0 && x \in \Omega, \\ &\frac{\partial w}{\partial n} \leq 0 && x \in \partial\Omega, \end{aligned}$$

  and $w(x_0) = \max_{x \in \overline{\Omega}} w(x)$, then $g(x_0, w(x_0)) \geq 0$.
- *Assume that $w \in C^2(\Omega) \cap C^1(\overline{\Omega})$ and satisfies*

$$\begin{aligned} &\Delta w(x) + g(x, w(x)) \leq 0 && x \in \Omega, \\ &\frac{\partial w}{\partial n} \geq 0 && x \in \partial\Omega, \end{aligned}$$

  and $w(x_0) = \min_{x \in \overline{\Omega}} w(x)$, then $g(x_0, w(x_0)) \leq 0$.

We also need the following Harnack's inequality from [4].

Lemma 3.6. *Let $w \in C^2(\Omega) \cap C^1(\overline{\Omega})$ be a positive solution of*

$$\begin{aligned} &\Delta w(x) + c(x)w = 0 && x \in \Omega \subset \mathbb{R}^m, \\ &\frac{\partial w}{\partial n} = 0 && x \in \partial\Omega, \end{aligned}$$

where $c \in C(\overline{\Omega})$. Then there exists a positive constant $C = C(m, \Omega, \|c\|_\infty)$ such that

$$\sup_\Omega w \le C \inf_\Omega w.$$

The above two lemmas are also collected in [12].

3.2. Proof of Theorem 2.3

Note that $\mathcal{R}_0$ is independent of d_S. Thus, under $\mathcal{R}_0 > 1$, EE (S, I) exists for any $d_S > 0$ by Theorem 2.2. By (2.6), we have $\int_\Omega I dx \le N$. Hence there exists a sequence $\{d_{S_n}\}$ with $d_{S_n} \to 0$ as $n \to \infty$ such that the corresponding EE (S_n, I_n) satisfies

$$\int_\Omega I_n dx \to k, \quad \text{for some } \ k \ge 0.$$

It then follows that

$$F_n \equiv \left(\frac{N}{|\Omega|}\beta - \gamma\right) d_{S_n} + (d_I - d_{S_n}) \frac{\beta}{|\Omega|} \int_\Omega I_n dx \to d_I \beta \frac{k}{|\Omega|} \ \text{ as } \ n \to \infty.$$

We claim that

$$I_n \to \frac{k}{|\Omega|} \text{ uniformly on } \overline{\Omega} \text{ as } n \to \infty. \tag{3.4}$$

To show this, we first note that by (2.7), I_n satisfies

$$\begin{aligned} d_{S_n} d_I \Delta I_n + I_n (F_n - d_I \beta I_n) = 0, \quad & x \in \Omega, \\ \frac{\partial I_n}{\partial n} = 0, \quad & x \in \partial\Omega. \end{aligned} \tag{3.5}$$

Now for any $\epsilon > 0$, there exists $n_1 > 0$ such that

$$\frac{d_I \beta}{|\Omega|}(k - \epsilon) \le F_n \le \frac{d_I \beta}{|\Omega|}(k + \epsilon) \ \text{ for all } \ n > n_1.$$

This implies that I_n is a lower solution of the problem

$$
\begin{aligned}
d_{S_n} d_I \Delta I + I\left(\frac{d_I\beta}{|\Omega|}(k+\epsilon) - d_I\beta I\right) &= 0, \quad x\in\Omega,\\
\frac{\partial I}{\partial n} &= 0, \quad x\in\partial\Omega,
\end{aligned}
\tag{3.6}
$$

and an upper solution of the problem

$$
\begin{aligned}
d_{S_n} d_I \Delta I + I\left(\frac{d_I\beta}{|\Omega|}(k-\epsilon) - d_I\beta I\right) &= 0, \quad x\in\Omega,\\
\frac{\partial I}{\partial n} &= 0, \quad x\in\partial\Omega.
\end{aligned}
\tag{3.7}
$$

Observing that $I = (k+\varepsilon)/|\Omega|$ solves (3.6) and $I = (k-\varepsilon)/|\Omega|$ solves (3.7), we then have

$$
(k-\varepsilon)/|\Omega| \le I_n \le (k+\varepsilon)/|\Omega| \;\text{ for all }\; n > n_1 \tag{3.8}
$$

Since $\epsilon > 0$ is arbitrary, (3.8) indeed implies $I_n \to k/|\Omega|$ uniformly on $\overline{\Omega}$ as $n\to\infty$, confirming the claim.

If $k > 0$, we consider the equations involving the S-component of the EE:

$$
\begin{aligned}
-d_{S_n}\Delta S_n &= (-\beta S_n + \gamma) I_n, \quad x\in\Omega,\\
\frac{\partial S_n}{\partial n} &= 0, \quad x\in\partial\Omega.
\end{aligned}
\tag{3.9}
$$

Noticing $I_n \to k/|\Omega|$ and using a standard singular perturbation theory technique (for example, see [7]), we can prove that

$$
S_n \to \frac{\gamma}{\beta} \;\text{ uniformly on }\; \overline{\Omega} \;\text{ as }\; n\to\infty.
$$

It then follows from (2.6) that we must have

$$
I_n \to \frac{k}{|\Omega|} = \frac{N}{|\Omega|} - \frac{1}{|\Omega|}\int_\Omega \frac{\gamma(x)}{\beta(x)}\,dx.
$$

We now consider the possibility that $k = 0$, i.e. $I_n \to 0$ uniformly on $\overline{\Omega}$ as $n\to\infty$. Passing to a subsequence if needed, we then have either (i) $\|I_n\|_\infty/d_{S_n} \to 0$, or (ii) $\|I_n\|_\infty/d_{S_n} \to \infty$, or (iii) $\|I_n\|_\infty/d_{S_n} \to C_0$ with C_0 being a positive constant. We explore these three cases individually below.

If (i) occurs, i.e., $\|I_n\|_\infty/d_{S_n} \to 0$, then $(\int_\Omega I_n dx)/d_{S_n} \to 0$. Let $\hat{I}_n = I_n/d_{S_n}$. Then $\hat{I}_n$ satisfies

$$
\begin{aligned}
&d_I\Delta\hat{I}_n + \hat{I}_n\left(\frac{N}{|\Omega|}\beta - \gamma + (d_I - d_{S_n})\frac{\beta}{|\Omega|}\frac{1}{d_{S_n}}\int_\Omega I_n dx - d_I\beta\hat{I}_n\right) = 0, && x\in\Omega,\\
&\frac{\partial \hat{I}_n}{\partial n} = 0, && x\in\partial\Omega.
\end{aligned}
$$

We claim that $\hat{I}_n \to \hat{I}$ in $C(\overline{\Omega})$ as $n \to \infty$, where $\hat{I}$ is the unique positive solution of the following problem

$$\begin{cases} d_I \Delta \hat{I} + \hat{I}\left(\frac{N}{|\Omega|}\beta - \gamma - d_I \beta \hat{I}\right) = 0, & x \in \Omega, \\ \frac{\partial \bar{I}}{\partial n} = 0, & x \in \partial\Omega. \end{cases} \tag{3.10}$$

To see this, we first note that $\mathcal{R}_0 > 1$ implies $\lambda_1(d_I, N\beta/|\Omega| - \gamma) < 0$ by Lemma 2.1. It then follows from Lemma 3.3 that Problem (3.10) has a unique positive solution. Since $[(d_I - d_{S_n})\beta/(|\Omega| d_{S_n})] \int_\Omega I_n dx > 0$ for large n, $\hat{I}_n$ is an upper solution of (3.10). Let ϕ be a positive eigenvector of the following problem

$$\begin{cases} d_I \Delta\phi + (\frac{N}{|\Omega|}\beta - \gamma)\phi + \lambda\phi = 0, & \text{in } \Omega, \\ \frac{\partial \phi}{\partial n} = 0, & \text{on } \partial\Omega \end{cases}$$

corresponding to the principal eigenvalue $\lambda_1(d_I, N\beta/|\Omega| - \gamma)$. Since $\lambda_1(d_I, N\beta/|\Omega| - \gamma) < 0$, one can easily check that $\varepsilon\phi$ is a lower solution of (3.10) if $\varepsilon > 0$ is small. Then by the method of upper/lower solutions and the uniqueness of the positive solution of (3.10), we find that $\epsilon\phi \le \hat{I} \le \hat{I}_n$. It then follows from $[(d_I - d_{S_n})\beta/(|\Omega| d_{S_n})] \int_\Omega I_n dx \to 0$ that $\hat{I}_n \to \hat{I}$ in $C(\overline{\Omega})$ as $n \to \infty$. But this is a contradiction because the positivity of $\hat{I}$ would imply

$$(d_I - d_{S_n})\frac{\beta}{|\Omega|}\int_\Omega I_n dx / d_{S_n} \to d_I \frac{\beta}{|\Omega|}\int_\Omega \hat{I} dx > 0.$$

Hence $\|I_n\|_\infty / d_{S_n} \to 0$ is impossible.

If (ii) holds, i.e., $\|I_n\|_\infty / d_{S_n} \to \infty$, we first prove the uniform boundedness of S_n in $C(\overline{\Omega})$. Noticing that S_n satisfies

$$\begin{cases} -d_{S_n}\Delta S_n = (-\beta S_n + \gamma) I_n, & x \in \Omega, \\ \frac{\partial S_n}{\partial n} = 0, & x \in \partial\Omega, \end{cases}$$

it then follows from Lemma 3.5 that

$$\min\left\{\frac{\gamma(x)}{\beta(x)} : x \in \overline{\Omega}\right\} \le S_n(x) \le \max\left\{\frac{\gamma(x)}{\beta(x)} : x \in \overline{\Omega}\right\}.$$

Let $\tilde{I}_n = I_n / \|I_n\|_\infty$. Then, by (2.4), $\tilde{I}_n$ satisfies that

$$-d_I \Delta \tilde{I}_n = (\beta S_n - \gamma)\tilde{I}_n, \quad x \in \Omega, \tag{3.11}$$

$$\frac{\partial \tilde{I}_n}{\partial n} = 0, \quad x \in \partial\Omega. \tag{3.12}$$

Since $\|\tilde{I}_n\|_\infty = 1$ and S_n is uniformly bounded, by the standard elliptic estimate, $\tilde{I}_n$ is uniformly bounded in $C^1(\overline{\Omega})$. So passing to a subsequence if necessary, we have $\tilde{I}_n \to \tilde{I}$ in $C(\overline{\Omega})$ with $\|\tilde{I}\|_\infty = 1$. Moreover, $\tilde{I}$ is strictly positive on $\overline{\Omega}$. To see this, by the uniform boundedness of S_n and the Harnack inequality, there is a positive constant K independent of n such that

$$1 = \sup_{x\in\overline{\Omega}} \tilde{I}_n(x) \le K \inf_{x\in\overline{\Omega}} \tilde{I}_n(x).$$

Hence $\inf_{x\in\overline{\Omega}} \tilde{I} \ge 1/K > 0$ and so $\tilde{I}$ is strictly positive. We now turn to the equation for S_n:

$$\begin{cases} -d_{S_n}/\|I_n\|_\infty \Delta S_n = (-\beta S_n + \gamma) I_n/\|I_n\|_\infty, & x \in \Omega, \\ \dfrac{\partial S_n}{\partial n} = 0, & x \in \partial\Omega. \end{cases}$$

It then follows from $d_{S_n}/\|I_n\|_\infty \to 0$, $I_n/\|I_n\|_\infty \to \tilde{I}$ and the standard singular perturbation method that

$$S_n \to \frac{\gamma}{\beta} \quad \text{uniformly on } \overline{\Omega} \text{ as } n \to \infty.$$

Moreover by (2.6), we compute

$$I_n \to \frac{N}{|\Omega|} - \frac{1}{|\Omega|} \int_\Omega \frac{\gamma(x)}{\beta(x)} dx.$$

If $N - \int_\Omega (\gamma/\beta)\, dx \neq 0$, this is a contradiction since $I_n \to 0$; If $N - \int_\Omega (\gamma/\beta)\, dx = 0$, we arrive at the alternative (a) of Theorem 2.3.

Lastly we consider case (iii), i.e., $\|I_n\|_\infty/d_{S_n} \to C_0$ for some positive constant C_0. From (2.4), one knows that

$$\begin{cases} -\Delta S_n = (-\beta S_n + \gamma) \dfrac{I_n}{\|I_n\|_\infty} \dfrac{\|I_n\|_\infty}{d_{S_n}}, & x \in \Omega, \\ \dfrac{\partial S_n}{\partial n} = 0, & x \in \partial\Omega. \end{cases}$$

By a similar argument to that in case (ii), we obtain

$$\frac{I_n}{\|I_n\|_\infty} \frac{\|I_n\|_\infty}{d_{S_n}} \to \frac{\tilde{I}}{C_0}, \quad \text{in } C(\overline{\Omega}) \text{ as } n \to \infty$$

for some positive $\tilde{I}$. Hence the conclusion (b) of the theorem holds with $\alpha = \tilde{I}/C_0$. Moreover by (3.11)–(3.12), it is not easy to see that α satisfies (2.9)–(2.10). The proof of Theorem 2.3 is completed.

Corollary 2.4 is a direct consequence of Theorem 2.3, and we give the proof of Corollary 2.5 below.

Proof of Corollary 2.5. By Lemma 2.1 and the assumption that β is a positive constant with $N - \int_\Omega [\gamma(x)/\beta] dx > 0$, we know that $\mathcal{R}_0 > 1$ and that the EE (S, I) exists for all $d_I > 0$. Dividing both sides of the first equation of (2.7) by I and integrating it over Ω, we obtain

$$d_I \int_\Omega \frac{|\nabla I|^2}{I^2} dx + \int_\Omega \left(\frac{N}{|\Omega|}\beta - \gamma - \left(1 - \frac{d_I}{d_S}\right) \frac{\beta}{|\Omega|} \int_\Omega I\, dx - \frac{d_I \beta}{d_S} I \right) = 0,$$

which implies that

$$\int_\Omega \left(\frac{N}{|\Omega|}\beta - \gamma - \left(1 - \frac{d_I}{d_S}\right) \frac{\beta}{|\Omega|} \int_\Omega I\, dx - \frac{d_I \beta}{d_S} I \right) \le 0.$$

When β is a constant, the above inequality implies

$$\int_\Omega I\, dx \ge N - \int_\Omega \frac{\gamma(x)}{\beta} dx. \tag{3.13}$$

On the other hand, by Theorem 2.3, there exists a sequence $\{d_{S_n}\}$ with $d_{S_n} \to 0$ as $n \to \infty$ such that the corresponding EE of (2.4)–(2.6) satisfies that

$$(S_n, I_n) \to (S^*, I^*) \quad \text{in } C(\overline{\Omega}),$$

where S^* is a positive function and I^* is a nonnegative positive *constant*. The estimate (3.13) implies that

$$I^* \ge \frac{N}{|\Omega|} - \frac{1}{|\Omega|} \int_\Omega \frac{\gamma(x)}{\beta}\, dx > 0.$$

Hence the second alternative of Theorem 2.3 is impossible and the proof is complete. □

3.3. Proof of Theorem 2.7

Before proving Theorem 2.7, we need the following result.

Lemma 3.7. *Assume that Ω^+ is non-empty and d is a positive constant. Then the following equation*

$$\left(\frac{N}{|\Omega|}\beta - \gamma - (1-d)\frac{\beta}{|\Omega|} \int_\Omega I\, dx \right)^+ = d\beta I, \quad x \in \overline{\Omega} \tag{3.14}$$

has a unique nonnegative solution.

Proof. The case $d = 1$ is trivial. Let

$$G_\tau = \left(\frac{N}{|\Omega|}\beta - \gamma - (1-d)\frac{\beta}{|\Omega|}\tau\right)^+ / d\beta.$$

If $d \in (0, 1)$, then $\int_\Omega G_\tau dx$ is non-increasing in τ for $\tau \geq 0$ with $\int_\Omega G_\tau dx = 0$ for sufficiently large τ. Since Ω^+ is non-empty, we have $\int_\Omega G_0 dx > 0$. So there exists a unique $\tau^* > 0$ such that $\int_\Omega G_{\tau^*} dx = \tau^*$. Then G_{τ^*} is the unique non-negative solution of (3.14).

If $d > 1$, then $\int_\Omega G_\tau dx$ is non-decreasing in τ for $\tau \geq 0$ with $\int_\Omega G_\tau dx \to \infty$ as $\tau \to \infty$. We notice that

$$\int_\Omega G_\tau dx \leq \frac{1}{d}\int_\Omega \left(\frac{N}{|\Omega|} - \frac{\gamma}{\beta}\right)^+ dx + (1 - \frac{1}{d})\tau. \tag{3.15}$$

Since the right hand side of (3.15) is linear in τ with slope less than 1, there exists $\tau^* > 0$ such that $\int_\Omega G_{\tau^*} dx = \tau^*$, and so G_{τ^*} is a solution of (3.14). Moreover, it is clear that $\int_\Omega G_\tau dx$ is concave up in τ. Hence τ^* is the unique solution of $\int_\Omega G_\tau dx = \tau$ and therefore, (3.14) have a unique non-negative solution. The proof is completed. □

Proof of Theorem 2.7. Since Ω^+ has positive measure, the EE (S, I) (i.e., positive solution of (2.4)–(2.6)) exists if d_I is small by Lemma 2.1. We first prove (a) for the case $d < 1$. We claim that $\int_\Omega I dx \to \int_\Omega I^* dx$ as $d_I \to 0$ and $d_I/d_S \to d$, where I^* is the unique solution of (3.14). Since $\int_\Omega I dx \leq N$, there exist two sequences $\{d_{I_n}\}$ and $\{d_{S_n}/d_{I_n}\}$ with $d_{I_n} \to 0$ and $d_{S_n}/d_{I_n} \to d$ as $n \to \infty$ such that the corresponding EE (S_n, I_n) satisfies $\int_\Omega I_n dx \to k \in [0, N]$. Let $\varepsilon > 0$ be given. Then there exists $n^* > 0$ such that $k - \varepsilon < \int I_n dx < k + \varepsilon$ and $d - \varepsilon < d_{S_n}/d_{I_n} < d + \varepsilon$ for all $n > n^*$. So I_n is an upper solution of the following problem

$$\begin{cases} d_{I_n}\Delta I + I\left(\dfrac{N}{|\Omega|}\beta - \gamma - (1-d+\varepsilon)\dfrac{\beta}{|\Omega|}(k+\varepsilon) - (d+\varepsilon)\beta I\right) = 0, & x \in \Omega, \\ \dfrac{\partial I}{\partial n} = 0, & x \in \partial\Omega \end{cases} \tag{3.16}$$

and it is also a lower solution of

$$\begin{cases} d_{I_n}\Delta I + I\left(\dfrac{N}{|\Omega|}\beta - \gamma - (1-d-\varepsilon)\dfrac{\beta}{|\Omega|}(k-\varepsilon) - (d-\varepsilon)\beta I\right) = 0, & x \in \Omega, \\ \dfrac{\partial I}{\partial n} = 0, & x \in \partial\Omega \end{cases} \tag{3.17}$$

for all $n > n^*$. Denote by $I_{n,\varepsilon}$ ($I_{n,-\varepsilon}$) the unique positive solution of (3.16) ((3.17)) if it exists; otherwise, let $I_{n,\varepsilon} = 0$ ($I_{n,-\varepsilon} = 0$). Then by an upper–lower solution argument, we have $I_{n,\varepsilon} \leq I_n \leq I_{n,-\varepsilon}$ for all $n > n^*$. By Remark 3.4, we know that $\lim_{n\to\infty} I_{n,\pm\varepsilon} = I_{\pm\varepsilon}$ in $C(\overline{\Omega})$, where

$$I_{\pm\varepsilon}=\left(\frac{\frac{N}{|\Omega|}\beta-\gamma-(1-d\pm\varepsilon)\frac{\beta}{|\Omega|}(k\pm\varepsilon)}{(d\pm\varepsilon)\beta}\right)^+.$$

Since $\varepsilon>0$ is arbitrary, we obtain

$$k=\lim_{n\to\infty}\int_\Omega I_n dx=\int_\Omega \frac{1}{d}\left(\frac{N}{|\Omega|}-\frac{\gamma}{\beta}-(1-d)\frac{k}{|\Omega|}\right)^+.$$

By Lemma 3.7, we then have $k=\int_\Omega I^* dx$.

We now prove $I\to I^*$ as $d_I\to 0$ and $d_I/d_S\to d$. By the claim, we have $\int_\Omega I dx\to\int_\Omega I^* dx$, so we can set $k=\int_\Omega I^* dx$ in the previous arguments to get:

$$\lim_{d_I\to 0, d_I/d_S\to d} I=\left(\frac{\frac{N}{|\Omega|}\beta-\gamma-(1-d)\frac{\beta}{|\Omega|}\int_\Omega I^* dx}{d\beta}\right)^+=I^*.$$

The proof of the case $d\ge 1$ is similar, so we omit it here. (b) is obtained by taking $d=1$ in (a). The conclusion in (c) is easily observed from equation (3.14). The proof is completed. $\square$

3.4. Proof of Theorem 2.8

We first prove (a). Since Ω is a high-risk domain, the EE (S,I) always exists. Similar to the proof of Theorem 2.3, we can show that S is uniformly bounded in $C(\overline{\Omega})$ for all $d_S, d_I>0$. Applying Lemma 3.6 to

$$\begin{aligned} -d_I\Delta I&=(\beta S-\gamma)I, && x\in\Omega,\\ \frac{\partial I}{\partial n}&=0, && x\in\partial\Omega, \end{aligned}$$

we know that there exists a positive number C such that

$$\sup_{x\in\overline{\Omega}} I(x)\le C\inf_{x\in\overline{\Omega}} I(x),$$

for all $d_S>0$ and $d_I\ge 1$. By (2.6), we then have

$$N\ge\int_\Omega I dx\ge|\Omega|\inf_\Omega I\ge|\Omega|(\sup_\Omega I)/C.$$

This implies that $\|I\|_\infty\le CN/|\Omega|$, and hence, I is uniformly bounded in $C(\overline{\Omega})$ for all $d_S>0$ and $d_I\ge 1$. Now, by (2.4), the elliptic estimate, and the Sobolev embedding theorem, S and I are uniformly bounded in $C^{2+\alpha}(\overline{\Omega})$ for all $d_S, d_I\ge 1$. It then follows from the compactness of the embedding $C^{2+\alpha}(\overline{\Omega})\subset C^2(\overline{\Omega})$ that there exist sequences $\{d_{S_n}\}$ and $\{d_{I_n}\}$ with $d_{S_n}\to\infty$ and $d_{I_n}\to\infty$ as $n\to\infty$ such that the corresponding EE $(S_n, I_n)\to(S^*, I^*)$ in $C^2(\overline{\Omega})$, where (S^*, I^*) satisfies

$$\Delta S^* = \Delta I^* = 0, \quad \text{in } \Omega,$$
$$\frac{\partial S^*}{\partial n} = \frac{\partial I^*}{\partial n} = 0, \quad \text{on } \partial\Omega.$$

By the maximum principle, S^* and I^* are constants. Let $\tilde{I}_n = I_n/\|I_n\|_\infty$. By (2.4), we have

$$-d_{I_n}\Delta\tilde{I}_n = (\beta S_n - \gamma)\tilde{I}_n, \quad \text{in } \Omega, \tag{3.18}$$
$$\frac{\partial \tilde{I}_n}{\partial n} = 0, \quad \text{on } \partial\Omega. \tag{3.19}$$

Since $\|\tilde{I}_n\|_\infty = 1$ and S_n is uniformly bounded, $\tilde{I}_n$ is uniformly bounded in $C^{2+\alpha}(\overline{\Omega})$ for $n \geq 1$ by the elliptic estimate and the Sobolev embedding theorem. Passing to a subsequence if necessary, we have $\tilde{I}_n \to \tilde{I}^*$ in $C^2(\overline{\Omega})$, where $\tilde{I}^*$ satisfies

$$\Delta\tilde{I}^* = 0, \quad \text{in } \Omega,$$
$$\frac{\partial \tilde{I}^*}{\partial n} = 0, \quad \text{on } \partial\Omega.$$

Again by the maximum principle, $\tilde{I}^*$ is a constant. Noticing $\|\tilde{I}_n\|_\infty = 1$, we have $\tilde{I}^* = 1$. Integrating both sides of (3.18), we find

$$\int_\Omega (\beta S_n - \gamma)\tilde{I}_n dx = 0.$$

Letting $n \to \infty$ in the above leads to

$$S^* = \frac{\int_\Omega \gamma(x)dx}{\int_\Omega \beta(x)dx}.$$

Lastly, by (2.6), we obtain

$$I^* = \frac{N}{|\Omega|} - \frac{\int_\Omega \gamma(x)dx}{\int_\Omega \beta(x)dx},$$

proving (a).

To prove (b), we first show that (2.12) has a unique positive solution. Note that $\mathcal{R}_0 > 1$ is equivalent to $\lambda_1(d_I, N\beta/|\Omega| - \gamma) < 0$ by Lemma 2.1. By Lemma 3.1 and the variational formula, $\lambda_1(d_I, N\beta/|\Omega| - \gamma - k\beta/|\Omega|)$ is increasing in k with $\lambda_1(d_I, N\beta/|\Omega| - \gamma - k\beta/|\Omega|) \to \infty$ as $k \to \infty$. Hence there exists $\hat{k} > 0$ such that $\lambda_1(d_I, N\beta/|\Omega| - \gamma - \hat{k}\beta/|\Omega|) = 0$. Let φ be an eigenvector corresponding to the principal eigenvalue (which is zero by $\lambda_1(d_I, N\beta/|\Omega| - \gamma - \hat{k}\beta/|\Omega|) = 0$) of the problem

$$\begin{aligned} -d_I\Delta\varphi &= \varphi\left(\frac{N}{|\Omega|}\beta - \gamma - \frac{\hat{k}}{|\Omega|}\beta\right) + \lambda\varphi, && x \in \Omega, \\ \frac{\partial I}{\partial n} &= 0, && x \in \partial\Omega. \end{aligned} \tag{3.20}$$

Then $\hat{k}\varphi/\int_\Omega \varphi dx$ is a positive solution of (2.12). To prove the uniqueness, we suppose that I_1 and I_2 are two positive solutions of (2.12). Then the positivity of I_1 and I_2 implies that

$$\lambda_1\left(d_I, \frac{N}{|\Omega|}\beta - \gamma - \frac{\beta}{|\Omega|}\int_\Omega I_1 dx\right) = \lambda_1\left(d_I, \frac{N}{|\Omega|}\beta - \gamma - \frac{\beta}{|\Omega|}\int_\Omega I_2 dx\right) = 0.$$

Then by Lemma 3.1, we have $\int_\Omega I_1 dx = \int_\Omega I_2 dx$. Let φ be an eigenvector corresponding to the principal eigenvalue of (3.20) with $\hat{k}$ replaced by $\int_\Omega I_1 dx$, then we have

$$I_1 = \frac{\int_\Omega I_1 dx}{\int_\Omega \varphi dx}\varphi = I_2,$$

confirming the uniqueness of the positive solution of (2.12).

Now, since $\mathcal{R}_0 > 1$, the EE (S, I) exists for all $d_S > 0$. As before, we have the uniform boundedness of S in $C(\overline{\Omega})$ for all $d_S > 0$ by Lemma 3.5. And then, by the Harnack inequality, I is also uniformly bounded in $C(\overline{\Omega})$. It then follows from (2.4), the elliptic estimate and the Sobolev embedding theorem that there exists a sequence $\{d_{S_n}\}$ with $d_{S_n} \to \infty$ as $n \to \infty$ such that the corresponding EE $(S_n, I_n) \to (S^*, I^*)$ in $C^2(\overline{\Omega})$. By (2.7), I^* satisfies (2.12). But we have just proved that (2.12) has a unique positive solution, thus, either I^* is the positive solution of (2.12) or $I^* = 0$. By (2.7) and the positivity of I_n, we have

$$\lambda_1\left(d_I, \frac{N}{|\Omega|}\beta - \gamma - \left(1 - \frac{d_I}{d_{S_n}}\right)\frac{\beta}{|\Omega|}\int_\Omega I_n dx - \frac{d_I}{d_{S_n}}\beta I_n\right) = 0.$$

If $I^* = 0$, taking $n \to \infty$ in the above, we have $\lambda_1(d_I, N\beta/|\Omega| - \gamma) = 0$, which contradicts $\mathcal{R}_0 > 1$ by Lemma 2.1. Hence I^* is the positive solution of (2.12). By (2.4), S^* satisfies

$$\begin{aligned} \Delta S^* &= 0, \quad \text{in } \Omega, \\ \frac{\partial S^*}{\partial n} &= 0, \quad \text{on } \partial\Omega. \end{aligned}$$

It follows from the maximum principle that S^* is a constant. Noting (2.6), we conclude that

$$S^* = \frac{N - \int_\Omega I^* dx}{|\Omega|}.$$

The proof of (c) is similar to (a), so we only sketch it here. Since Ω is a high-risk domain, the EE (S, I) always exists. We can show that S is uniformly bounded in $C(\overline{\Omega})$ by Lemma 3.5, and I is also uniformly bounded in $C(\overline{\Omega})$ for all $d_I > 1$ by the Harnack inequality and $\int_\Omega I \le N$. Then by the elliptic estimate, the Sobolev embedding theorem and the maximum principle, there exists a sequence $\{d_{I_n}\}$ with $d_{I_n} \to \infty$ as $n \to \infty$ such that the corresponding EE $(S_n, I_n) \to (S^*, I^*)$

in $C^2(\overline{\Omega})$, where I^* is a constant. To see $I^* \neq 0$, we also introduce $\tilde{I}_n = I_n/\|I_n\|_\infty$. Then we can prove $\tilde{I}_n \to 1$ in $C^2(\overline{\Omega})$ as $n \to \infty$. As in the proof of (a), this leads to

$$\int_\Omega (\beta S^* - \gamma)dx = 0.$$

If $I^* = 0$, one can see from (2.6) that $S^* = N/|\Omega|$. Hence we have $\int_\Omega (N\beta/|\Omega| - \gamma)dx = 0$, which contradicts the assumption that Ω is a high-risk domain. Therefore I^* is positive. Note that we do not have the uniqueness of I^*, so different from (a) and (b), our result here in (c) is for a sequence $\{d_{I_n}\}$ only. Finally, there exists a sequence $\{d_{S_n}\}$ with $d_{S_n} \to 0$ as $n \to \infty$ such that the corresponding solution (S_n^*, I_n^*) of (2.13) satisfies either $d_{S_n}/I_n^* \to 0$, or $d_{S_n}/I_n^* \to \infty$, or $d_{S_n}/I_n^* \to C$ for some positive constant C. If $d_{S_n}/I_n^* \to 0$, we have $(S_n^*, I_n^*) \to (\tilde{S}^*, \tilde{I}^*)$ in $C(\overline{\Omega})$ with $(\tilde{S}^*, \tilde{I}^*)$ satisfying alternative (a) of Theorem 2.3. If $d_{S_n}/I_n^* \to C$, we arrive at alternative (b) with $\alpha = 1/C$. If $d_{S_n}/I_n^* \to \infty$, up to a sequence, we have $(S_n^*, I_n^*) \to (\tilde{S}^*, 0)$ with $\tilde{S}^*$ constant. By the last equation of (2.13), we actually have $\tilde{S}^* = N/|\Omega|$. Integrating both sides of the first equation in (2.13), we get $\int_\Omega (-\beta S_n + \gamma)dx = 0$. Taking $n \to \infty$, we find $\int_\Omega (-N\beta/|\Omega| + \gamma)dx = 0$, which contradicts the assumption that Ω is a high-risk domain. Hence $d_{S_n}/I_n^* \to \infty$ is not possible. The proof of Theorem 2.8 is completed.

3.5. Proof of Theorem 2.9

In the proof of the existence and uniqueness of a positive solution to (2.12) in Theorem 2.8-(b), we used the condition $\lambda_1(d_I, N\beta/|\Omega| - \gamma) < 0$. Now, the assumption that Ω^+ is non-empty implies $\lambda_1(d_I, N\beta/|\Omega| - \gamma) < 0$ when $d_I < \hat{d}_I$ for some $\hat{d}_I > 0$, and therefore, ensures the existence of a unique positive solution I^* to (2.12) when $d_I < \hat{d}_I$.

To prove $\int_\Omega I^* dx \leq N$, we suppose to the opposite: $\int_\Omega I^* dx > N$. Then we have $N\beta/|\Omega| - \gamma - (\beta/|\Omega|)\int_\Omega I^* dx < 0$ and thus, by Lemma 3.1, we have $\lambda_1(d_I, N/|\Omega|\beta - \gamma - (\beta/|\Omega|)\int_\Omega I^* dx) > \lambda_1(d_I, 0) = 0$, which is a contradiction to

$$\lambda_1(d_I, N/|\Omega|\beta - \gamma - (\beta/|\Omega|)\int_\Omega I^* dx) = 0. \tag{3.21}$$

Also from (3.21) and Lemma 3.1, we know that $\int_\Omega I^* dx$ is non-decreasing in d_I. This together with the fact that $0 \leq \int_\Omega I^* dx \leq N$ implies that $\lim_{d_I \to 0} \int_\Omega I^* dx$ exists. Denote this limit by k_1. We now show that $k_1 = k_0$. Let $\varepsilon > 0$ be given. Then there exists $d_1 > 0$ such that $k_1 - \varepsilon < \int_\Omega I^* dx < k_1 + \varepsilon$ for all $d < d_1$. By Lemma 3.1 and (3.21), we have

$$\lambda_1(d_I, N/|\Omega|\beta - \gamma - (\beta/|\Omega|)(k_1 - \varepsilon)) < 0 < \lambda_1(d_I, N/|\Omega|\beta - \gamma - (\beta/|\Omega|)(k_1 + \varepsilon)).$$

Taking $d_I \to 0$ and by (3.2), we get

$$-\max\{N/|\Omega|\beta - \gamma - (\beta/|\Omega|)(k_1 - \varepsilon)\} \le 0 \le -\max\{N/|\Omega|\beta - \gamma - (\beta/|\Omega|(k_1 + \varepsilon))\}.$$

Since $\varepsilon > 0$ is arbitrary, we have

$$\max\{N/|\Omega|\beta - \gamma - (\beta/|\Omega|)k_1\} = 0,$$

and it then follows from the definition of k_0 that $k_1 = k_0$, i.e., $\int_\Omega I^* dx \to k_0$ as $d_I \to 0$, which implies that there exists a sequence $\{d_{I_n}\}$ with $d_{I_n} \to 0$ as $n \to \infty$ such that $I_n^* \to k_0\mu$ weakly, where μ is a probability measure, in the sense that

$$\lim_{n\to\infty} \int_\Omega I_n^*(x)\psi(x)dx = \int_\Omega \psi(x)d\mu(x), \qquad \text{for all } \psi \in C(\overline{\Omega}).$$

It then remains to show that the support of μ is contained in $\mathcal{M}$. If $\mathcal{M} = \overline{\Omega}$, then the statement holds trivially. So suppose that $\mathcal{M} \subsetneqq \overline{\Omega}$. Pick a point $x_0 \in \overline{\Omega}/\mathcal{M}$. For simplicity, assume that x_0 is an interior point of Ω. By the definition of $\mathcal{M}$, there exist small positive numbers ε, δ such that

$$\frac{N - k_0}{|\Omega|}\beta(x) - \gamma(x) < -\delta \qquad \text{for all} \qquad x \in B(x_0, \varepsilon),$$

where $B(x_0, \varepsilon)$ is the open ball centred at x_0 with radius ε such that $B(x_0, \varepsilon) \subseteq \Omega/\mathcal{M}$. Choose a smooth cutoff function ψ with $0 \le \psi \le 1$ such that

$$\psi(x) = \begin{cases} 1, & \text{on } B(x_0, \varepsilon/3) \\ 0, & \text{on } \Omega/B(x_0, 2\varepsilon/3). \end{cases} \tag{3.22}$$

Multiplying both sides of (2.12) by ψ and integrating it over $B(x_0, 2\varepsilon/3)$, we find that

$$0 = d_{I_n} \int_{B(x_0,2\varepsilon/3)} I_n^* \Delta\psi dx + \int_{B(x_0,2\varepsilon/3)} \psi I_n^* \left(\frac{N}{|\Omega|}\beta - \gamma - \frac{\beta}{|\Omega|}\int_\Omega I_n^* dx\right) dx.$$

Taking $n \to \infty$, we have that

$$0 = \int_{B(x_0,2\varepsilon/3)} \psi \left(\frac{N - k_0}{|\Omega|}\beta - \gamma\right) d\mu \le -\delta\mu(B(x_0, \varepsilon/3)),$$

which implies $\mu(B(x_0, \varepsilon/3)) = 0$. This completes the proof.

4. Discussion

In this section, we discuss some biological implications of our results, and most importantly, we will compare our results with the parallel results for the model (1.1) obtained in [2,12]. Our discussion can partially answer the questions in the introduction and provide some insights into the strategy of disease control.

As we pointed out in the introduction, McCallum et al. [11] found that without considering the spatial effect, the basic reproduction number of the SIS model is dependent of the total population which explains, to some extent, why population culling can be a disease control strategy. Now, with spatial effect considered, Allen et al. [2] found that for the model (1.1) with the standard incidence transmission, the basic reproduction number is defined as

$$\tilde{\mathcal{R}}_0 = \sup\left\{ \frac{\int_\Omega \beta\varphi^2\, dx}{\int_\Omega (d_I|\nabla\varphi|^2 + \gamma\varphi^2)\, dx} : \ \varphi \in H^1(\Omega) \text{ and } \varphi \neq 0 \right\}$$

and the high-risk region and low-risk region are defined by

$$\tilde{\Omega}^+ = \{x \in \Omega : \beta(x) - \gamma(x) > 0\} \quad \text{and} \quad \tilde{\Omega}^- = \{x \in \Omega : \beta(x) - \gamma(x) < 0\},$$

respectively. These definitions are all independent of the total population N. However, the corresponding definitions $\mathcal{R}_0$, Ω^+ and Ω^- for the model (1.2) are dependent not only on the total population N but also the size of the domain $|\Omega|$, in terms of $N/|\Omega|$, the population density with respect to the space (population per unit space). This explains why it is *easier* for a disease to become endemic in a more crowded population (larger $N/|\Omega|$) than in a sparse population (smaller $N/|\Omega|$). An implication is even considering the spatial effect, as long as the mass action infection term is adopted, that population culling may till be an effective control strategy, it should be combined with the size of the region though.

In [2,12], the authors explored the impact of the movements of the individuals and the spatial heterogeneity on the profiles of the EE for (1.1), and their results offer many interesting implications on disease control. We now compare our results for (1.2) in this paper with those parallel ones in [2,12], hoping to reveal more differences between the two infection mechanisms in the "spatial" setting.

One of the main conclusions in [2] is that if there are lower risk sites in the domain Ω (i.e. $\tilde{\Omega}^-$ is non-empty), then the I-component of the EE of the model (1.1) approaches zero if the diffusion rate of the susceptible individuals tends to zero. This implies that the disease may be controlled by limiting the movement of the susceptible individuals reflected by d_S. This is still the case for the model (1.2) but with an extra requirement on the total population, i.e., $N < \int_\Omega (\gamma/\beta)dx$, as indicated by Corollary 2.4. However, the disease may not be controllable by limiting d_S when the total population is large. For example, when the disease recovery rate γ is heterogeneous while the disease transmission rate β is homogeneous (constant), Corollaries 2.4 and 2.5 imply that the disease is not controllable by limiting d_S when the domain is a high-risk one (i.e. $\int_\Omega (N\beta/|\Omega| - \gamma)dx > 0$, equivalent to $N > \int_\Omega (\gamma/\beta)dx > 0$), regardless of whether Ω^- is non-empty or not. It remains an open problem to examine the case when both β and γ are heterogeneous. We conjecture that the sign of $N - \int_\Omega (\gamma/\beta)dx$ will play a threshold role here.

In [12], Peng considered the asymptotic profiles of the EE of (1.1) when the movement rates d_S and d_I are small or large in various cases. In particular, it was shown that if $d_I \to 0$ and $d_I/d_S \to d \in [0, \infty)$, then the infected individuals will reside exactly in the high-risk region (see Corollary 1.1 of [12]). Combining this result with the one in [2], Peng claims that limiting the movement of susceptible individuals is a better control strategy than limiting the movement of infected individuals. However, for the model (1.2), the ratio d plays a very interesting role. As is shown in Theorem 2.7, if $d = 1$ then the infected individuals will live exactly in the high-risk region; if $d < 1$ then the residence area of infected individuals is strictly contained in the high-risk region; if $d > 1$ then there are infected individuals who will even persist in the lower-risk site. Here the ratio d actually determines the size of the residence area of infected individuals. If the total population is below a certain level (i.e. $N < \int_\Omega (\gamma/\beta)dx$), we can conclude that limiting d_S is a better control strategy than limiting d_I. However if the total population is large, then limiting d_I *may be a better one* because it will at least eliminate the disease in certain area. From the above discussion, we see that for the model (1.2) with mass action transmission mechanism, we can no longer conclude that limiting d_S is always a better control strategy than limiting d_I, in strong contrast to the conclusion for (1.1).

Another interesting thing for (1.2) is the occurrence of the concentration phenomenon when $d_I \to 0$, which has not been observed from (1.1). Here we consider $d_I \to 0$ by looking at (2.12) instead of (2.7) and this phenomenon can also be observed from equation (2.11) by letting $d \to 0$. Theorem 2.9 suggests that the infected individuals may concentrate on certain sites, characterized by $\mathcal{M}$, when the movement of infected individuals is limited, and those sites can be considered as the highest risk sites in the domain. In particular if $\mathcal{M}$ is a singleton, then the infected individuals will concentrate at a single point when $d_I \to 0$.

Finally, Theorem 2.8 implies that large diffusion rate of either susceptible or infected individuals tends to homogenize the spatial distribution of the corresponding component of the endemic equilibrium of (1.2), and this coincides with the situation for (1.1).

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