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On the basins of attraction for a class of delay differential equations with non-monotone bistable nonlinearities ☆

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Abstract

We consider the delay differential equation (DDE) $\dot{x}(t) = -g(x(t)) + f(x(t-\tau))$ which shares the same equilibria with the corresponding ordinary differential equation (ODE) $\dot{x}(t) = -g(x(t)) + f(x(t))$. For the bistable case, both the DDE and ODE share three equilibria $x_0 = 0 < x_1 < x_2$ with x_0 and x_2 being stable and x_1 being unstable for the ODE. We are concerned with stability of these equilibria for the DDE and the basins of attraction of x_0 and x_2 when they are asymptotically stable for the DDE. Combining the idea of relating the dynamics of a map to the dynamics of a DDE and invariance arguments for the solution semiflow, we are able to characterize some subsets of basins of attraction of these equilibria for the DDE. In addition, existence of heteroclinic orbits is also explored. The general results are applied to a particular model equation describing the matured population of some species demonstrating the Allee effect.

Keywords: Bistability; Basin of attraction; Delay differential equation; Heteroclinic orbit

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1. Introduction

Many mathematical models in population biology and physiology fall into the form of the delay differential equation (DDE)

$$\dot{x}(t) = -g\big(x(t)\big) + f\big(x(t-\tau)\big), \tag{1.1}$$

where $\tau \geqslant 0$ and the functions $g, f : \mathbb{R}_+ \to \mathbb{R}_+$ are continuously differentiable where $\mathbb{R}_+ := [0, \infty)$, satisfying $g(0) = f(0) = 0$. The term $-g(x(t))$ in (1.1) assumes a decay in the absence of new recruitment/activation/production, and the nonlinear term $f(x(t-\tau))$ in (1.1) accounts for a delayed response in the corresponding practical problems. Among such models are several well-known equations resulted from appropriate choices for the functions g and f in (1.1). For example, the Nicholson equation for blowfly population growth corresponds to $g(x) = \mu x$ and $f(x) = pxe^{-qx}$, and the Mackey–Glass equation for regulation of hematopoiesis is a result of taking $g(x) = \mu x$ and $f(x) = \frac{ax}{b+x^m}$. For detailed derivation/explanation of such models, see, e.g., Murray [19] and Cooke et al. [3] and the references therein.

The usual assumption of $g(0) = f(0) = 0$ implies that (1.1) has the trivial equilibrium $x = 0$. For the case when $f(x)$ allows (1.1) to have *a unique positive equilibrium* x^+ (the so called *monostable* nonlinearity, i.e., $f(x) = g(x)$ has exactly one positive solution x^+), $x = 0$ is typically unstable and x^+ is either globally asymptotically stable in $\mathbb{R}_+$ (meaning that it attracts all positive solutions) under certain range of the model parameters, or it will lose its stability to periodic solutions around x^+ arising from Hopf bifurcation. Overall, Eq. (1.1) with various *monostable* nonlinearities has been extensively and intensively explored and the dynamics is well understood now. See, e.g., [1,3–5,9–13,15–17,20,23,24,27] and the references therein.

In this paper, we are interested in the dynamics of (1.1) for the case when $g(x) = f(x)$ has two positive roots $x_1 < x_2$, referred to as the *bistable case*, in addition to the trivial equilibrium $x_0 = 0$. To proceed conveniently, let us give some standard assumptions representing *bistable* case for (1.1):

(H1) $g'(x) > 0$ for all $x \in [0, \infty)$.
(H2) $f(\xi) \geqslant 0$ for all $\xi \geqslant 0$, and there exists a unique $\xi_0 > 0$ such that $f'(\xi) > 0$ if $0 < \xi < \xi_0$, $f'(\xi_0) = 0 = f'(0)$ and $f'(\xi) < 0$ if $\xi > \xi_0$; furthermore, there exists a unique $0 < \xi_1 < \xi_0$, such that $f''(\xi) > 0$ if $0 < \xi < \xi_1$, $f''(\xi_1) = 0$ and $f''(\xi) < 0$ if $\xi_0 > \xi > \xi_1$, and $\lim_{\xi\to\infty} f(\xi) = 0$.
(H3) In addition to the trivial root $x = 0$, $f(x) = g(x)$ has two positive roots $x_1 < x_2$ satisfying $f(x) < g(x)$ for $x \in (0, x_1) \cup (x_2, \infty)$ and $f(x) > g(x)$ for $x \in (x_1, x_2)$.

When $g(x)$ and $f(x)$ become tangential, x_1 and x_2 merge into a single one $x_1 = x_2$. A prototype of such f is $f(x) = px^2e^{-qx}$ representing the Allee effect in population biology, and a typical $g(x)$ is $g(x) = \mu x$.

To explain our motivations, let us first look at the corresponding ordinary differential equation (ODE) obtained by taking $\tau = 0$ in (1.1), that is,

$$\dot{x}(t) = -g\big(x(t)\big) + f\big(x(t)\big). \tag{1.2}$$

Note that under (H1)–(H3), (1.1) and (1.2) share the (same) three equilibria $0 < x_1 < x_2$. For (1.2), it is easy to show that 0 and x_2 are stable, and x_1 not only is unstable but also plays a role of defining the basins of attraction for 0 and x_2 in the sense that when the initial value $x_0 \in [0, x_1)$, the solution tends to the trivial equilibrium 0, while when $x_0 > x_1$, the solution converges to the largest equilibrium x_2, yielding a complete description of the dynamics of (1.2).

Some questions naturally arise: (i) for (1.1), do 0 and x_2 remain stable and is x_1 still unstable? (ii) if the answer to (i) is affirmative, what are the basins of the attraction of 0 and x_2, and what role does x_1 play in describing these basins? (iii) what is the impact of τ on questions (i) and (ii)? Addressing these questions constitutes the goal of this paper.

These questions are mathematically interesting and significant to the theory of delay differential equations, yet they are very challenging in the sense that finding complete answers seems to be very difficult, if not impossible. As an initial attempt, we realistically only seek partial answers in this paper. Motivated by Röst and Wu [20] where a monostable case is considered, we first use the domain decomposition method to obtain a series of invariant intervals. We point out that f can be non-monotone in these intervals, and hence the method in [20] cannot be applied in this case and this forces us to seek new approaches. More precisely, we will make use of some techniques for one-dimensional maps to give sufficient conditions that guarantee that all solutions converge to an equilibrium on these invariant intervals. These results allow us to describe the global dynamics of Allee-type model within certain range of parameters. Furthermore, we also obtain some results on Hopf bifurcation and the existence of heteroclinic orbits, including two types of heteroclinic orbits: orbit from one equilibrium to another one, and orbit from one equilibrium to a periodic orbit oscillating around the largest positive equilibrium.

We point out that the idea of relating the dynamics of a map to the dynamics of a delay differential equation has been used by some other researchers, among which are Mallet-Paret and Nussbaum [18], Ivanov and Sharkovsky [7], Hale and Verduyn Lunel [6, Section 12.7] and Liz [14]. Recently this idea has also been successfully employed to study some delay differential equations with spatial diffusion in Yi and Zou [29–32].

The rest of this paper is organized as follows. Some preliminaries are given in Section 2 where we present some basic definitions and notations. In Section 3, we identify some invariant sets and obtain some properties of the equilibria of model (1.1). Section 4 focuses on (1.1) with $g(x) = \mu x$ and $f(x) = px^2e^{-qx}$. By applying the results established in previous sections, we are able to obtain some more concrete results in terms of the model parameters. The paper is concluded by a discussion on some related topics, raising some interesting open problems.

2. Preliminaries

Let $C = C([-\tau, 0], \mathbb{R})$ be the Banach space of continuous functions defined in $[-\tau, 0]$ equipped with the usual supremum norm. The Banach space C contains the positive cone

$$C_+ = \{\phi \in C\colon\ \phi(s) \geqslant 0,\ -\tau \leqslant s \leqslant 0\},$$

which has non-empty interior $Int(C_+)$. Hence, it naturally induces the following order relations: For any given $\phi, \psi \in C$, we write $\phi \geqslant \psi$ if $\phi - \psi \in C_+$; $\phi > \psi$ if $\phi - \psi \in C_+ \setminus \{0\}$; $\phi \gg \psi$ if $\phi - \psi \in Int(C_+)$. Similarly, we can also define order relations $<$, $\leqslant$ and $\ll$.

For a given $\phi \in C_+$, by the method of steps together with (H1) and (H2), one can solve Eq. (1.1) inductively on $[0, \tau], [\tau, 2\tau], \ldots,$ giving a unique solution $x_t \in C_+$ of Eq. (1.1) defined for all $t \geqslant 0$. When we wish to emphasize the dependence of a solution on the initial data ϕ, we write $x_t(\phi)$ or $x(t,\phi)$. Under the assumptions (H1) and (H2), Eq. (1.1) generates a semiflow Φ on C_+ by $\Phi(t,\phi)=\Phi_t(\phi)=x_t(\phi)$, $t\geqslant 0$, $\phi\in C_+$.

Let $I\subseteq\mathbb{R}$ be a (possibly infinite) interval and $C_I\triangleq C([-\tau,0],I)$. For the sake of simplicity, we denote C_I by I when no confusion arises.

The semiflow Φ defined on C is said to be monotone or order preserving if

$$\Phi_t(\phi)\leqslant\Phi_t(\psi)\quad\text{whenever}\quad\phi\leqslant\psi\quad\text{and}\quad t\geqslant 0.$$

Let $\kappa: C\to\mathbb{R}$ be the functional on the right hand side of (1.1), i.e.,

$$\kappa(\phi):=-g\big(\phi(0)\big)+f\big(\phi(-\tau)\big),\quad\forall\phi\in C.$$

For any $y\in\mathbb{R}$, we also denote by y the constant function in C. The set of equilibria for (1.1) is then given by $E=\{y\in C\colon g(y)=f(y)\text{ and }y\in\mathbb{R}_+\}$.

For $\phi\in C$, let $O^+(\phi)=\{\Phi_t(\phi)\colon t\geqslant 0\}$ be the positive orbit of $\phi\in C$. The ω-limit set of $\phi\in C$ is defined by $\omega(\phi)=\bigcap_{t\geqslant 0}\overline{\bigcup_{s\geqslant t}\Phi_s(\phi)}$, i.e.,

$$\omega(\phi)=\left\{\psi\in C\colon\begin{array}{l}\text{there is a sequence }\{t_n\}_{n\geqslant 0}\text{ in }[0,\infty)\\ \text{satisfying }t_n\to\infty\text{ and }\Phi_{t_n}(\phi)\to\psi\text{ as }n\to\infty\end{array}\right\}.$$

By a negative orbit we mean a function $v:(-\infty,0]\to C$ such that $\Phi_t(v(s))=\Phi(v(t+s))$ for all $t\geqslant 0\geqslant s$ with $t+s\leqslant 0$. If $v(0)=\phi$, we say that v is a negative orbit of ϕ. For a negative orbit v, the α-limit set $\alpha(v)$ of v is the set of all limit points of v as $t\to-\infty$, i.e.,

$$\alpha(v)=\left\{\psi\in C\colon\begin{array}{l}\text{there is a sequence }\{t_n\}_{n\geqslant 0}\text{ in }(-\infty,0]\\ \text{with }t_n\to-\infty\text{ and }\Phi_{t_n}(v_0)\to\psi\text{ as }n\to\infty\end{array}\right\}.$$

See [26,10,6,28] for more details.

3. Invariance and stability analysis

When $g(x)=\mu x$ for $\mu>0$, Yi and Zou [30] established a fundamental lemma which played a crucial role in [30] (see [30, Lemma 3.7]). We can prove a similar version of that lemma for (1.1), as is done below.

Lemma 3.1. *Let $I\subseteq\mathbb{R}_+$ be a closed interval. Assume $g^{-1}\circ f(I)\subseteq I$, then the following statements are true*:

(i) $x_t(\varphi)\in C_I$ *for all* $(t,\varphi)\in\mathbb{R}_+\times C_I$;
(ii) $\omega(C_I)\subseteq\bigcap_{n\geqslant 0}(g^{-1}\circ f)^n(I)$, *where* $\omega(C_I)\triangleq\bigcap_{s\geqslant 0}Cl(\bigcup_{t\geqslant s}\Phi_t(C_I))$. *That is,* $\omega(\varphi)\subseteq\bigcap_{n\geqslant 0}(g^{-1}\circ f)^n(I)$ *for all* $\varphi\in C_I$.

Proof. (i) Without loss of generality, we assume that $I=[a,b]$ and $g^{-1}\circ f(I)=[a,b]$. For any $\varphi\in C_+$, if $\varphi(0)=a$, we have

$$-g\big(\varphi(0)\big)+f\big(\varphi(-\tau)\big)\geqslant 0;$$

and if $\varphi(0)=b$, we obtain

$$-g\bigl(\varphi(0)\bigr)+f\bigl(\varphi(-\tau)\bigr)\leqslant 0.$$

By Remark 5.2.1 in [21], we conclude that $x_t(\varphi)\in C_I$ for all $(t,\varphi)\in\mathbb{R}_+\times C_I$.

(ii) From (i), we have $\omega(C_I)\subseteq C_I$. We shall show that $\omega(C_I)\subseteq g^{-1}\circ f(I)$. Suppose it is not true, then $\omega(C_I)\setminus g^{-1}\circ f(I)$ is non-empty. Thus, there exists $\psi\in\omega(C_I)$ such that

$$\psi(0)=\inf\bigl\{\varphi(0)\colon\ \varphi\in\omega(C_I)\bigr\}\notin g^{-1}\circ f(I), \tag{3.1}$$

or

$$\psi(0)=\sup\bigl\{\varphi(0)\colon\ \varphi\in\omega(C_I)\bigr\}\notin g^{-1}\circ f(I). \tag{3.2}$$

Without loss of generality, we assume (3.1) holds. By the invariance of $\omega(C_I)$, there is a solution $x:\mathbb{R}\to\mathbb{R}$, such that $x_0=\psi$ and $x_t\in\omega(C_I)$ for all $t\in\mathbb{R}$. It is obvious that $\dot{x}(0)=0$. By $\dot{x}(0)=-g(x(0))+f(x(-\tau))$, we then have $g(x(0))=f(x(-\tau))$ implying that $\psi(0)=g^{-1}\circ f(\psi(-\tau))\in g^{-1}\circ f(I)$, which is a contradiction to (3.1).

Let $n^*=\sup\{n\geqslant 0\colon\ \omega(C_I)\subseteq\bigcap_{k=0}^{n}(g^{-1}\circ f)^k(I)\}$. Then we have $1\leqslant n^*\leqslant\infty$. To finish the proof, we only prove that $n^*=\infty$. Otherwise, $1\leqslant n^*<\infty$. Noting that $g^{-1}\circ f(\bigcap_{k=0}^{n^*}(g^{-1}\circ f)^k(I))\subseteq\bigcap_{k=0}^{n^*}(g^{-1}\circ f)^k(I)$, replacing $\bigcap_{k=0}^{n^*}(g^{-1}\circ f)^k(I)$ with I and applying the above discussion, we have $\omega(\bigcap_{k=0}^{n^*}(g^{-1}\circ f)^k(I))\subseteq\bigcap_{k=0}^{n^*+1}(g^{-1}\circ f)^k(I)$, and hence

$$\omega(C_I)\subseteq\omega\bigl(\omega(C_I)\bigr)\subseteq\omega\left(\bigcap_{k=0}^{n^*}\bigl(g^{-1}\circ f\bigr)^k(I)\right)\subseteq\bigcap_{k=0}^{n^*+1}\bigl(g^{-1}\circ f\bigr)^k(I),$$

a contradiction to the definition of n^*. This completes the proof. $\square$

Lemma 3.2. *Let $I\subseteq\mathbb{R}_+$ be a bounded closed interval. Assume that*

(H4) $\{x\in I\colon\ (g^{-1}\circ f)^2(x)=x\}=\{x^*\}$ *for some* $x^*\in I$.

Then, x^* is a globally asymptotically stable equilibrium point in C_I for Eq. (1.1).

Proof. By (H3) and Proposition 2.1 in [30], we have $\bigcap_{n\geqslant 0}(g^{-1}\circ f)^n(I)=\{x^*\}$. It follows from Lemma 3.1-(ii) that $\omega(C_I)=\{x^*\}$. Therefore, $x^*\in C_I$ is a globally asymptotically stable equilibrium point in C_I for Eq. (1.1). $\square$

Lemma 3.3. *Assume that $I\subseteq\mathbb{R}_+$ is an interval and $g^{-1}\circ f(I)\subseteq I$. Then, $x_t(\varphi)\in C_I$ for all $(t,\varphi)\in[2\tau,\infty)\times C_{\bar{I}}\setminus\{\{\inf I,\sup I\}\setminus I\}$.*

Proof. It is obvious that $g^{-1}\circ f(\bar{I})\subseteq\bar{I}$. Without lose of generality, we assume $I=[a,b)$; when I takes other form, the proof of this lemma is similar and we omit it.

It follows from Lemma 3.1-(i) that

$$x_t(\varphi)\in C_{\bar{I}}\quad\text{for all }(t,\varphi)\in\mathbb{R}_+\times C_{\bar{I}}.$$

Therefore,

$$x_t(\varphi) \in C_{[a,b]} \quad \text{for all } (t,\varphi) \in \mathbb{R}_+ \times C_{[a,b]} \setminus \{b\}.$$

Assume that $\varphi \in C_{[a,b]} \setminus \{b\}$, we first show there exists $\theta \in [0,\tau]$ such that $x(\theta,\varphi) < b$. If $\varphi(0) < b$, then $\varphi(\theta) < b$ with $\theta = 0$. If $\varphi(0) = b$, then there exists $\theta_1 \in [-\tau, 0]$ such that $\varphi(\theta_1) < b$. We conclude that $x(\theta_1 + \tau, \varphi) < b$. Otherwise, we have

$$x(\theta_1 + \tau, \varphi) = b \quad \text{and} \quad 0 = \dot{x}(\theta_1 + \tau, \varphi) = -g(b) + f\bigl(\varphi(\theta_1)\bigr).$$

Therefore, $g(b) = f(\varphi(\theta_1))$ and $b = g^{-1} \circ f(\varphi(\theta_1)) \in g^{-1} \circ f(I) \subseteq I$, which is a contradiction to $b \notin I$. Thus $x(\theta,\varphi) < b$ with $\theta = \theta_1 + \tau$.

Let $t^* = \inf\{t \geqslant \theta\colon x(t,\varphi) = b\}$. To complete the proof we must show $t^* = \infty$. If $t^* < \infty$, then $x(t^*,\varphi) = b$. From (H1), there exist $\varepsilon \in (0, \frac{t^*-\theta}{2})$ and $\mu^* > 0$ such that

$$g(b) - g\bigl(x(t,\varphi)\bigr) \leqslant \mu^*\bigl(b - x(t,\varphi)\bigr) \quad \text{for any } t \in \bigl(t^* - \varepsilon, t^* + \varepsilon\bigr).$$

Thus, for all $t \in (t^* - \varepsilon, t^* + \varepsilon)$, we have

$$\begin{aligned}
\dot{x}(t,\varphi) &= -g\bigl(x(t,\varphi)\bigr) + f\bigl(x(t-\tau,\varphi)\bigr)\\
&= g(b) - g\bigl(x(t,\varphi)\bigr) + f\bigl(x(t-\tau,\varphi)\bigr) - g(b)\\
&\leqslant g(b) - g\bigl(x(t,\varphi)\bigr)\\
&\leqslant \mu^*\bigl(b - x(t,\varphi)\bigr).
\end{aligned}$$

Therefore,

$$x(t,\varphi) \leqslant b - e^{-\mu^*(t^*-t-\frac{\varepsilon}{2})}\left[b - x\left(t^* - \frac{\varepsilon}{2}, \varphi\right)\right] \quad \text{for any } t \in \left[t^* - \frac{\varepsilon}{2}, t^* + \frac{\varepsilon}{2}\right].$$

As $\varepsilon \in (0, \frac{t^*-\theta}{2})$, we have $x(t^* - \frac{\varepsilon}{2}, \varphi) < b$, therefore, $x(t,\varphi) < b$ for all $t \in [t^* - \frac{\varepsilon}{2}, t^* + \frac{\varepsilon}{2}]$. In particular, we have $x(t^*,\varphi) < b$, which yields a contradiction with $x(t^*,\varphi) = b$. This completes the proof. □

Theorem 3.1. *Assume that $I \subseteq \mathbb{R}_+$ is an interval with $g^{-1} \circ f(I) \subseteq I$, and there is $x^* \in I$ satisfying* (H4). *Additionally, assume that one of the following conditions holds*:

(i) *$I = (a,b)$, and there exist sequences $\{a_k\}$ and $\{b_k\}$ in I such that $a_k \to a$, $b_k \to b$ as $k \to \infty$ and $g^{-1} \circ f([a_k, b_k]) \subseteq [a_k, b_k]$*;
(ii) *$I = [a,b)$, and there exists a sequence $\{b_k\}$ in I such that $b_k \to b$ as $k \to \infty$ and $g^{-1} \circ f([a, b_k]) \subseteq [a, b_k]$*;
(iii) *$I = (a,b]$, and there exists sequence $\{a_k\}$ in I such that $a_k \to a$ as $k \to \infty$ and $g^{-1} \circ f([a_k, b]) \subseteq [a_k, b]$.*

Then, x^* is an asymptotically stable equilibrium point and it attracts all $\phi \in \{x^*\} \cup C_{\bar{I}} \setminus \{a,b\}$ for system (1.1).

Proof. We only give the proof under (i), since the conclusions under (ii) and (iii) can be proved by similar arguments.

From Lemma 3.3, we have $x_t(\varphi) \in C_I$ for all $\varphi \in C_{\bar{I}} \setminus \{a, b\}$ and $t \geqslant 2\tau$. Clearly, we only need to prove that x^* attracts every $\phi \in C_I$. In fact, from (H4) and the assumptions in (i), we have $x^* \in (a_{k_0}, b_{k_0})$ for some $k_0 \geqslant 1$. From Lemma 3.2, x^* attracts every $\phi \in C_{[a_{k_0}, b_{k_0}]}$. As $x^* \in (a_{k_0}, b_{k_0})$, it is obvious that x^* is stable in C_I. For any $\varphi \in C_I$, by Lemma 3.3, we have $x(2\tau, \varphi) \in [a_{k_1}, b_{k_1}]$ for some $k_1 \geqslant 1$. Again, Lemma 3.2 implies that $\omega(\varphi) = \{x^*\}$, hence, x^* is globally asymptotically stable in C_I. This completes the proof. □

Define $\hat{F}(x) = \hat{f}^{-1} \circ f(x)$, where $\hat{f}(\cdot)$ denotes the restriction of f to the interval $[\xi_0, \infty)$. Then, $\hat{F}(x) = x$ for $x \in [\xi_0, \infty)$ and $\hat{F}(x) > \xi_0 > x$ for $x \in [0, \xi_0)$. Under the assumptions (H1) and (H2), we can obtain the positively invariant sets as stated in the following proposition.

Proposition 3.1. *Assume that the assumptions* (H1) *and* (H2) *hold. Suppose* (1.1) *has a positive equilibrium* $x^+ \in [0, \xi_0]$. *Then,*

(i) *the order interval* $C_{[0, g^{-1} \circ f(\xi_0)]}$ *is positively invariant to the semiflow*;
(ii) *if* $g(\hat{F}(x^+)) \geqslant f(\xi_0)$, *then the order interval* $C_{[x^+, \hat{F}(x^+)]}$ *is also positively invariant to the semiflow.*

Proof. (i) Let $I = [0, g^{-1} \circ f(\xi_0)]$. Then we have $g(I) = [g(0), f(\xi_0)] \supseteq f([0, \xi_0]) \supseteq f(I)$, which together with Lemma 3.1-(i) implies that $C_{[0, g^{-1} \circ f(\xi_0)]}$ is positively invariant.

(ii) Let $I = [x^+, \hat{F}(x^+)]$. Then we have $g(I) = [g(x^+), g(\hat{F}(x^+))] \supseteq [f(x^+), f(\xi_0)] \supseteq f([x^+, \hat{F}(x^+)]) = f(I)$, hence, $g^{-1} \circ f(I) \subseteq I$. This, combined with Lemma 3.1-(i) implies that C_I is positively invariant. □

We are now in the position to state and prove our main theorems.

Theorem 3.2. *Assume that the assumptions* (H1) *and* (H2) *hold. If* 0 *is the unique equilibrium for* (1.1), *then it is globally asymptotically stable in* C_+.

Proof. The uniqueness of equilibrium for (1.1) shows that $g(x) > f(x)$ for all $x > 0$. Therefore, $g^{-1} \circ f([0, k]) \subseteq [0, k]$ for any integer $k \geqslant 1$. It is easy to see that $\{x \geqslant 0\colon (g^{-1} \circ f)^2(x) = x\} = \{0\}$. Thus, by Theorem 3.1-(ii), we know that 0 is globally asymptotically stable in C_+. □

Theorem 3.3. *Assume that the assumptions* (H1)–(H3) *hold. If* $x_2 < \xi_0$, *then,*

(i) 0 *attracts every* $\varphi \in C_{[0, x_1]} \setminus \{x_1\}$ *and thus is asymptotically stable in* C_+;
(ii) x_2 *is asymptotically stable attracting every* $\varphi \in C_{[x_1, \hat{F}(x_1)]} \setminus \{x_1, \hat{F}(x_1)\}$;
(iii) x_1 *is unstable.*

Proof. (i) Let $I = [0, x_1)$. If $x \in I \setminus \{0\}$, then $g^{-1} \circ f(x) < x$, in particular, $g^{-1} \circ f(I) \subseteq I$ and $g^{-1} \circ f([0, x]) \subseteq [0, x]$. Moreover, $(g^{-1} \circ f)^2(x) < x$ for $x \in I \setminus \{0\}$, that is $\{x \in I\colon (g^{-1} \circ f)^2(x) = x\} = \{0\}$. Therefore, Theorem 3.1-(ii) leads to (i).

(ii) Let $I = (x_1, \hat{F}(x_1))$. Then we have $x < g^{-1} \circ f(x) < x_2$ for any $x \in (x_1, x_2)$, $x_2 < g^{-1} \circ f(x) < x$ for any $x \in (x_2, \hat{F}(x_2))$, $x_1 < g^{-1} \circ f(x) < x_2$ for any $x \in (\hat{F}(x_2), \hat{F}(x_1))$ and $g^{-1} \circ f(\hat{F}(x_2)) = x_2$. If $(g^{-1} \circ f)^2(x) = x$ for some $x \in I$, then $x = x_2$ follows from the above discussions. Thus (H4) holds. Let $a \in (x_1, x_2)$ and $b = \hat{F}(a)$. Claim that $g^{-1} \circ f([a, b]) \subseteq [a, b]$. In fact, if $x \in [a, x_2)$, then $x < g^{-1} \circ f(x) < x_2$, and thus $g^{-1} \circ f(x) \subseteq [a, b]$; if $x \in [x_2, \hat{F}(x_2)]$, then $x_2 \leqslant g^{-1} \circ f(x) < x$, and thus $g^{-1} \circ f(x) \subseteq [a, b]$; if $x \in [\hat{F}(x_2), b]$, then $a < g^{-1} \circ f(x) < x_2$, and thus $g^{-1} \circ f(x) \subseteq [a, b]$. Therefore, it easy to see that (ii) follows from Theorem 3.1-(i).

(iii) follows from (i) and (ii) and the proof is completed. □

Theorem 3.4. *Assume that the assumptions* (H1)–(H3) *hold with* $x_2 \geqslant \xi_0$. *Let* $B = g^{-1} \circ f(\xi_0)$, $A = g^{-1} \circ f(B)$. *Then the following statements are true.*

(i) 0 *is asymptotically stable in* C_+ *and it attracts every* $\varphi \in C_{[0,x_1]} \setminus \{x_1\}$;
(ii) $A \leqslant B$ *and* $x_2 \in [A, B]$;
(iii) $\omega(\varphi) \leqslant B$ *for all* $\varphi \in C_+$;
(iv) *If* $A \geqslant x_1$, *then* $C_{[A,B]}$ *is positively invariant*;
(v) *If* $A > x_1$, *then* $\omega(\varphi) \geqslant A$ *for any* $\varphi \in C_{[x_1, \hat{F}(x_1)]} \setminus \{x_1, \hat{F}(x_1)\}$;
(vi) *If* $x_2 = \xi_0$, *then* x_2 *is asymptotically stable in* C_+ *and it attracts every* $\varphi \in C_{[x_1, \hat{F}(x_1)]} \setminus \{x_1, \hat{F}(x_1)\}$.

Proof. The proof of (i) is the same as the proof for Theorem 3.3-(i).

If $x_2 > \xi_0$, then $A = g^{-1} \circ f(B) < g^{-1} \circ f(x_2) = g^{-1} \circ g(x_2) = x_2 = g^{-1} \circ f(x_2) < g^{-1} \circ f(\xi_0) = B$. If $x_2 = \xi_0$, it is obvious that $A = x_2 = B$ holds. Hence, when $x_2 \geqslant \xi_0$, we have $A \leqslant x_2 \leqslant B$, proving (ii).

If $x \in \mathbb{R}_+$, then $0 \leqslant g^{-1} \circ f(x) \leqslant g^{-1} \circ f(\xi_0) = B$, thus $g^{-1} \circ f(\mathbb{R}_+) \subseteq [0, B]$. This together with Lemma 3.1-(ii) implies that $\omega(\varphi) \leqslant B$ for all $\varphi \in C_+$, proving (iii).

We claim that $g^{-1} \circ f([A, B]) \subseteq [A, B]$. Otherwise, there is $x \in [A, B]$ such that $g^{-1} \circ f(x) < A$ or $g^{-1} \circ f(x) > B$. If $g^{-1} \circ f(x) < A$, then $f(x) < g(A) = f(B)$. But $f(B) = \min_{A \leqslant x \leqslant B}\{f(x)\}$, a contradiction. If $g^{-1} \circ f(x) > B$, then $f(x) > g(B) = f(\xi_0)$. But $f(\xi_0) = \max_{A \leqslant x \leqslant B}\{f(x)\}$, a contradiction. Therefore, by Lemma 3.1-(i), $C_{[A,B]}$ is positively invariant, proving (iv).

In this case, we have $g^{-1} \circ f([A, B]) \subseteq [A, B]$, $x < g^{-1} \circ f(x) \leqslant B$ for all $x \in (x_1, A)$ and $g^{-1} \circ f((B, \hat{F}(x_1))) \subseteq (x_1, A)$. Let $a \in (x_1, x_3)$, $b = \hat{F}(a)$, where $x_3 = \check{f}^{-1} \circ g(A)$, and $\check{f}(\cdot)$ denotes the restriction of f to the interval $[0, \xi_0]$. Then $b \in (B, \hat{F}(x_1))$, $g^{-1} \circ f([a, b]) \subseteq [a, b]$ and $g^{-1} \circ f([a, b]) \subseteq [a, B]$. Let $I = \bigcap_{n \geqslant 1}(g^{-1} \circ f)^n([a, B])$. Then $I \subseteq [A, B]$. Otherwise, $I \setminus [A, B] \neq \emptyset$, that is, there is $x \in I \setminus [A, B]$. Let $x_4 = \inf I$. Then $x_4 < A$. The invariance of I under $g^{-1} \circ f$ implies that there is $x_5 \in I$ such that $g^{-1} \circ f(x_5) = x_4$. This together with $g^{-1} \circ f([A, B]) \subseteq [A, B]$, implies that $x_5 \in [x_4, A)$ and thus $g^{-1} \circ f(x_5) > x_5$. Thus $x_4 > x_5$, a contradiction. So, $I \subseteq [A, B]$. This, combined with Lemma 3.1-(ii) and Lemma 3.3, shows that $\omega(\varphi) \geqslant A$ for any $\varphi \in C_{[x_1, \hat{F}(x_1)]} \setminus \{x_1, \hat{F}(x_1)\}$, proving (v).

Finally, if $x_2 = \xi_0$, then $x_2 = \xi_0 = B = A > x_1$. By (ii)–(v), we conclude that ξ_0 attracts every $\varphi \in C_{[x_1, \hat{F}(x_1)]} \setminus \{x_1, \hat{F}(x_1)\}$ and thus is asymptotically stable, proving (vi). The proof is completed. □

For the tangential case where the two positive equilibria x_1 and x_2 merge into a single one, we have the following theorem.

Theorem 3.5. *Assume that the assumptions* (H1) *and* (H2) *hold. Suppose that in addition to the trivial equilibrium,* (1.1) *has a unique positive equilibrium* x_1. *Then,*

(i) $\omega(\varphi) \subseteq [0, \xi_0]$ *for every* $\varphi \in C_+$;
(ii) 0 *is asymptotically stable and it attracts every* $\varphi \in C_{[0,x_1]} \setminus \{x_1\}$;
(iii) x_1 *is unstable in* C_+ *but it attracts every* $\varphi \in C_{[x_1, \hat{F}(x_1)]}$.

Proof. By the uniqueness of the positive equilibrium, we know that $x_1 \leqslant \xi_0$ and $g(x) > f(x)$ for $x \in (0, x_1) \cup (x_1, \infty)$. Since $g^{-1} \circ f(\mathbb{R}_+) \subseteq g^{-1} \circ f([0, \xi_0]) \subseteq [0, \xi_0]$, by Lemma 3.1-(ii), we conclude that $\omega(\varphi) \subseteq [0, \xi_0]$ for any $\varphi \in C_+$, proving (i).

Let $I = [0, x_1]$ and choose $b_k = x_1 - \frac{1}{k}$ where k is a positive integer. Then we have $g^{-1} \circ f(I) \subseteq I$ and $g^{-1} \circ f([0, b_k]) \subseteq [0, b_k]$ for sufficiently large k. If $x \in I \setminus \{0\}$, then $g^{-1} \circ f(x) < x$, and hence $(g^{-1} \circ f)^2(x) < x$, which implies $\{x \in I\colon (g^{-1} \circ f)^2(x) = x\} = \{0\}$. It follows from Theorem 3.1-(ii) that $\lim_{t\to\infty} x(t, \varphi) = 0$ for $\varphi \in C_{[0,x_1]} \setminus \{x_1\}$, hence 0 attracts every $\varphi \in C_{[0,x_1]} \setminus \{x_1\}$ and thus is asymptotically stable in C_+, proving (ii).

By (i), it suffices to prove that x_1 attracts every $\varphi \in C_{[x_1, \hat{F}(x_1)]}$. Indeed, let $I = [x_1, \hat{F}(x_1)]$. Then we have $f(I) \subseteq [f(x_1), f(\xi_0)] \subseteq [g(x_1), g(\xi_0)] = g([x_1, \xi_0]) \subseteq g(I)$. If $x \in I \setminus \{x_1\}$, then $g^{-1} \circ f(x) < x$, and hence $(g^{-1} \circ f)^2(x) < x$, implying $\{x \in I\colon (g^{-1} \circ f)^2(x) = x\} = \{x_1\}$. This, combined with Lemma 3.2 yields (iii). $\square$

Remark 3.1. From Theorem 3.5-(ii), if x_1 is the unique positive equilibrium for Eq. (1.1), it is easy to see that every solution eventually enters the order interval $C_{[0,\xi_0]}$, which is positively invariant, and confined to which the semiflow is monotone. Since $\lim_{t\to\infty} x(t, \xi_0) = x_1$, it is obvious that $\limsup_{t\to\infty} x(t, \psi) \leqslant \lim_{t\to\infty} x(t, \xi_0) = x_1$ for all $\psi \in C_+$. In other words, $x(t, \psi)$ cannot 'oscillate' between the intervals $[0, x_1]$ and $(x_1, \xi_0]$ infinitely (i.e., oscillate about x_1) since it is attracted to $[0, x_1]$.

4. Application to a model with Allee effect

In this section, we apply the general results obtained in Section 3 to the following delay differential equation

$$\frac{\mathrm{d}N(t)}{\mathrm{d}t} = -\mu N(t) + a_1 N^2(t-\tau) e^{-a_2 N(t-\tau)}, \quad \text{for } t \geqslant 0, \tag{4.1}$$

where μ, a_1, a_2 are positive constants, the variable $N(t)$ stands for the matured population at time t and $\tau > 0$ is the maturation time of the species under consideration. In this model, the death function $g(N) = \mu N$ and the birth function $f(N) = a_1 N^2 e^{-a_2 N}$ reflects the so called Allee effect (see, e.g., [2,25]). Obviously, the functions $g(N) = \mu N$ and $f(N) = a_1 N^2 e^{-a_2 N}$ satisfy the assumptions (H1) and (H2) and f reaches the maximum value $4a_1/a_2^2 e^{-2}$ at the point $\xi_0 = 2/a_2$.

The equilibria of (4.1) are determined by the following scalar equation

$$\mu N = a_1 N^2 e^{-a_2 N}. \tag{4.2}$$

Analyzing (4.2), we can easily obtain the structure of the equilibria of (4.1), which is summarized in the following proposition.

Proposition 4.1. *Eq.* (4.1) *has a trivial equilibrium* $N_0 = 0$. *In addition,*

(i) *if* $\mu > \frac{a_1}{a_2 e}$, *Eq.* (4.1) *has no positive equilibrium*;

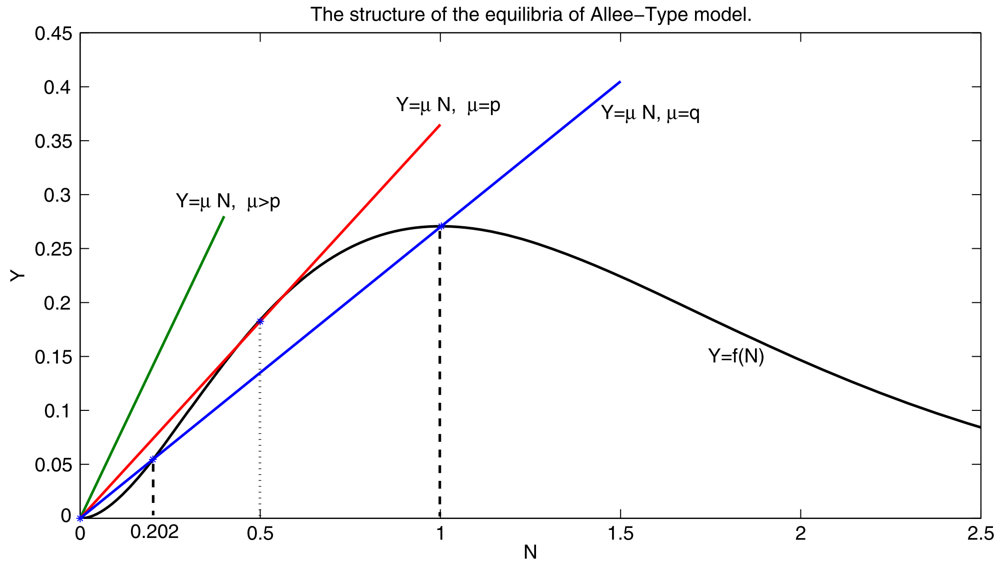

Fig. 1. Illustration of equilibria of Eq. (4.1), where $a_1 = a_2 = 2$, $p = \frac{a_1}{a_2 e} = \frac{1}{e}$, $q = \frac{2}{a_2} = 1$.

(ii) *if* $\mu = \frac{a_1}{a_2 e}$, *Eq.* (4.1) *has exactly one positive equilibrium* $N_1 = \frac{1}{a_2}$;

(iii) *if* $0 < \mu < \frac{a_1}{a_2 e}$ *Eq.* (4.1) *has exactly two positive equilibria* $N_1 < N_2$. *Moreover,*

   (iii)-1 *if* $\frac{2a_1}{a_2 e^2} < \mu < \frac{a_1}{a_2 e}$, *then* $0 < N_1 < \frac{1}{a_2} < N_2 < \frac{2}{a_2}$;

   (iii)-2 *if* $\mu = \frac{2a_1}{a_2 e^2}$, *then* $0 < N_1 < \frac{1}{a_2} < N_2 = \frac{2}{a_2}$;

   (iii)-3 *if* $0 < \mu < \frac{2a_1}{a_2 e^2}$, *then* $0 < N_1 < \frac{1}{a_2} < \frac{2}{a_2} < N_2$.

The above results are visually demonstrated in Fig. 1.

Applying Theorem 3.2 and Proposition 4.1, we immediately have the following theorem.

Theorem 4.1. *If* $\mu > \frac{a_1}{a_2 e}$, *then* $N_0 = 0$ *is a globally asymptotically stable equilibrium of Eq.* (4.1).

For the case (iii)-1 in Proposition 4.1, we have the following result.

Theorem 4.2. *Assume that* $\frac{2a_1}{a_2 e^2} \leqslant \mu < \frac{a_1}{a_2 e}$. *Then,*

(i) $N_0 = 0$ *is asymptotically stable in* C_+ *and it attracts every* $\varphi \in C_{[0,N_1]} \setminus \{N_1\}$;

(ii) N_1 *is unstable, and* N_2 *is asymptotically stable in* C_+ *attracting every* $\varphi \in C_{[N_1,\hat{N}_1]} \setminus \{N_1, \hat{N}_1\}$ *where* $\hat{N}_1 \in [\frac{2}{a_2}, \infty)$ *satisfies* $f(\hat{N}_1) = f(N_1)$;

(iii) *there exist two heteroclinic orbits* $N^{(1)}(t)$ *and* $N^{(2)}(t)$, *connecting* N_0 *and* N_1, *and* N_1 *and* N_2 *respectively.*

Proof. (i) and (ii) directly follow from Proposition 4.1-(iii)-1 and Theorem 3.3.

(iii) Let $K = \{N_1\}$. Clearly, K is an isolated and unstable compact invariant set in $C_{[0,N_1]}$ and $C_{[N_1,\hat{N}_1]}$, respectively. By applying Corollary 2.9 in [28] to $\Phi|_{\mathbb{R}_+ \times C_{[0,N_1]}}$ and $\Phi|_{\mathbb{R}_+ \times C_{[N_1,\hat{N}_1]}}$, respectively, there exist two pre-compact full orbits $N^{(1)} : \mathbb{R} \to C_{[0,N_1]} \setminus \{0\}$ and $N^{(2)} : \mathbb{R} \to C_{[N_1,\hat{N}_1]} \setminus \{0\}$ such that $\alpha(N^{(1)}) = \alpha(N^{(2)}) = K$. These together with statements (i) and (ii), imply $\omega(N^{(1)}) = \{0\}$ and $\omega(N^{(2)}) = \{N_2\}$. In other words, there exist two heteroclinic orbits

$N^{(1)}(t)$ and $N^{(2)}(t)$ with the first connecting N_0 and N_1 and the second connecting N_1 and N_2. This completes the proof. $\square$

To consider the cases 2 and 3 in Proposition 4.1-(iii), we first calculate

$$B = g^{-1} \circ f(\xi_0) = \frac{4a_1}{\mu a_2^2 e^2}, \qquad A = g^{-1} \circ f(B) = \frac{16a_1^3}{\mu^3 a_2^4 e^4} e^{-\frac{4a_1}{\mu a_2 e^2}}. \tag{4.3}$$

Theorem 4.3. *Assume that $0 < \mu \leqslant \frac{2a_1}{a_2 e^2}$ and let A and B be given by (4.3). Then,*

(i) *$N_0 = 0$ is asymptotically stable in C_+ and it attracts every $\varphi \in C_{[0,N_1]} \setminus \{N_1\}$, and thus N_1 is unstable in C_+;*
(ii) *$A \leqslant N_2 \leqslant B$;*
(iii) *$\omega(\varphi) \leqslant B$ for every $\varphi \in C_+$;*
(iv) *if $A \geqslant N_1$, then $C_{[A,B]}$ is positively invariant for (4.1);*
(v) *if $A > N_1$, then $\omega(\varphi) \geqslant A$ for every $\varphi \in C_{[N_1,\hat{N}_1]} \setminus \{N_1, \hat{N}_1\}$ where $\hat{N}_1 \in [\frac{2}{a_2}, \infty)$ satisfies $f(\hat{N}_1) = f(N_1)$;*
(vi) *if $\mu = \frac{2a_1}{a_2 e^2}$, then N_2 is asymptotically stable in C_+ and it attracts every $\varphi \in C_{[N_1,\hat{N}_1]} \setminus \{N_1, \hat{N}_1\}$ where $\hat{N}_1 \in [\frac{2}{a_2}, \infty)$ satisfies $f(\hat{N}_1) = f(N_1)$;*
(vii) *there exists a heteroclinic orbit N, which connects N_0 and N_1.*

Proof. Clearly, by cases 2, 3 in Proposition 4.1-(iii), we easily see that statements (i)–(vi) follow from corresponding conclusions of Theorem 3.4.

(vii) Let $K = \{N_1\}$. Clearly, by the second result in statement (i), we know that K is an isolated and unstable compact invariant set in $C_{[0,N_1]}$. By applying Corollary 2.9 in [28] to $\Phi|_{\mathbb{R}_+ \times C_{[0,N_1]}}$, there exists a pre-compact full orbits $N : \mathbb{R} \to C_{[0,N_1]} \setminus \{0\}$ such that $\alpha(N) = K$. This together with the first result in statement (i) gives $\omega(N) = \{N_0\}$. In other words, there exists a heteroclinic orbit N which connects N_0 and N_1. This completes the proof. $\square$

Under the conditions of Theorem 4.3, if $A > \frac{2}{a_2}$ is further satisfied, then we can conclude a bit more than that in Theorem 4.3.

Theorem 4.4. *If $0 < \mu < \frac{2a_1}{a_2 e^2}$ and $A > \frac{2}{a_2}$, then for every $\phi \in C_{[N_1,\hat{F}(N_1)]} \setminus \{N_1, \hat{F}(N_1)\}$, $\omega(\phi)$ is either $\{N_2\}$ or a periodic orbit oscillating about N_2.*

Proof. Clearly, Proposition 3.1-(i) and Theorem 4.3-(iv) imply that $C_{[N_1,\hat{N}_1]}$ and $C_{[A,B]}$ are positively invariant. Again by Theorem 4.3-(iii) and (v), we know that $\omega(\varphi) \subseteq C_{[A,B]}$ for all $\varphi \in C_{[N_1,\hat{N}_1]} \setminus \{N_1, \hat{N}_1\}$ with $B = 4a_1/(\mu a_2^2 e^2) < \hat{N}_1$.

Choose a constant η such that $2/a_2 < \eta < A$ and consider the following auxiliary functional differential equation

$$\frac{\mathrm{d}N(t)}{\mathrm{d}t} = -\mu N(t) + h\big(N(t-\tau)\big), \quad \text{for } t \geqslant 0, \tag{4.4}$$

where

$$h(N) := \begin{cases} f(N) & \text{if } N \geqslant \eta, \\ f(\eta) - f'(\eta) + f'(\eta)e^{N-\eta} & \text{if } N < \eta. \end{cases} \tag{4.5}$$

Eq. (4.4) generates a semiflow $\hat{\Phi}$ on the whole space C. It is easy to check that $h(N) \to f(\eta) - f'(\eta)$ as $N \to -\infty$, $h(N) \to 0$ as $N \to \infty$, and h is differentiable and monotonically decreasing on $\mathbb{R}$. Hence the corresponding semiflow $\hat{\Phi}$ is generated by a scalar delay differential equation with delayed negative feedback. By the Poincaré–Bendixson type theorem in [26, Theorem 10.1], we know that for every $\psi \in C$, $\omega(\psi; \hat{\Phi})$ (ω-limit set with respect to $\hat{\Phi}$) is either $\{N_2\}$ or a periodic orbit oscillating about N_2. On the other hand, for any $\varphi \in C_{[N_1, \hat{N}_1]} \setminus \{N_1, \hat{N}_1\}$, there exists a $T_0 > 0$ such that $\Phi(t, \varphi) > \eta$ for all $t \geqslant T_0$. Since f and h coincide on $[\eta, \infty)$, we have $\hat{\Phi}(t, \Phi(T_0, \varphi)) = \Phi(t, \Phi(T_0, \varphi)) = \Phi(t + T_0, \varphi)$ for all $t \geqslant 0$, and thus $\omega(\varphi) = \omega(\varphi; \hat{\Phi})$. Therefore, the ω-limit set of φ with respect to Φ is also either $\{N_2\}$ or a periodic orbit oscillating about N_2. This completes the proof. □

For the tangential case for (4.1), we have the following theorem.

Theorem 4.5. *If $\mu = \frac{a_1}{a_2 e}$, then*

(i) $\omega(\varphi) \subseteq [0, \frac{2}{a_2}]$ *for every* $\varphi \in C_+$;
(ii) $N_0 = 0$ *is asymptotically stable in* C_+ *and it attracts every* $\varphi \in C_{[0, \frac{1}{a_2}]} \setminus \{\frac{1}{a_2}\}$;
(iii) *the unique positive equilibrium* $N_1 = \frac{1}{a_2}$ *is unstable in* C_+ *but it attracts every* $\varphi \in C_{[\frac{1}{a_2}, \hat{F}(\frac{1}{a_2})]}$;
(iv) *there exists a heteroclinic orbit* $N : \mathbb{R} \to \mathbb{R}_+$ *such that* $N(\infty) = N_1$ *and* $N(-\infty) = N_0$.

Proof. (i)–(iii) directly follow from Proposition 4.1 and Theorem 3.5.

(iv) Let $K = \{N_1\}$. Clearly, by statement (iii), we know that K is an isolated and unstable compact invariant set in $C_{[0, N_1]}$. By applying Corollary 2.9 in [28] to $\Phi|_{\mathbb{R}_+ \times C_{[0, N_1]}}$, there exists a pre-compact full orbit $N : \mathbb{R} \to C_{\mathbb{R}_+ \times [0, N_1]} \setminus \{0\}$ such that $\alpha(N) = K$. This together with statement (ii) gives $\omega(N) = \{N_0\}$. In other words, there exists a heteroclinic orbit $N(t)$ which connects N_0 and N_1. This completes the proof. □

From the above theorems, we see that when $\frac{2a_1}{a_2 e^2} \leqslant \mu \leqslant \frac{a_1}{a_2 e}$, the stability/instability of the three equilibrium $N_0 = 0$, N_1 and N_2 remain the same as in the corresponding equation obtained by dropping the delay τ. This is because all three equilibria are in the invariant set $C_{[0, \xi_0]}$ and on this set, the solution semiflow is monotone. Thus, by the theory on monotone delay differential equations (see [21]), delay has no effect on the stability/instability of the three equilibrium. However, when $\mu < \frac{2a_1}{a_2 e^2}$, we have $N_2 > \xi_0$ and hence N_2 is no longer in $C_{[0, \xi_0]}$. In such a case, it is possible that the delay may destroy the stability of N_2 through Hopf bifurcation causing periodic solutions. A standard Hopf bifurcation analysis on N_2 can confirm such periodic solutions (see, e.g., [20]) around N_2 caused by large delay.

5. Discussion

We have obtained some preliminary results on comparing the stability of the common equilibria of the DDE (1.1) and the corresponding ODE (1.2) with bistable nonlinearities, which give some partial answers to the questions raised in the introduction. Roughly speaking, by using dynamical system approach (mainly invariance arguments), we have shown that within certain range of parameters, the stability/instability of the equilibria for the DDE (1.1) remain the same as the corresponding ODE (1.2), and the middle equilibrium x_1 plays a sort of separating role in the sense that for an initial function φ having an order relation with x_1, the corresponding solution converges either to the trivial equilibrium $x_0 = 0$ (if $\varphi < N_1$), or to the largest equilibrium x_2 (if $\varphi > N_1$). There are also ranges of parameters for which x_2 is asymptotically stable for (1.2) but is unstable for the DDE (1.1) due to Hopf bifurcation.

We point out that as far as the basin of attraction is concerned, our results can only characterize some subsets of the basins for both stable equilibria x_0 and x_2. When an initial function φ does not have an order relation with the middle equilibrium x_1 (i.e., $\varphi(\theta)$ crosses x_1 on $[-\tau, 0]$), it seems to be very difficult to determine the tendency of the corresponding solution. Even in the tangential case when x_1 and x_2 merge into a single positive equilibrium, there is also a similar problem: determining the tendency of solutions with initial function crossing x_1 on $[-\tau, 0]$. It seems that some averaging technique needs to be developed which should combine the pattern of the initial function and the functions $g(x)$ and $f(x)$. We have to leave this challenging yet interesting problem as future research project(s).

In this paper, we only consider *scalar* equations for which the equilibria are relative easier to determine than for *systems*. Similar situation may also occur in systems, particularly in competitive systems, even without delay. Indeed, Smith and Thieme [22] showed that for a competitive system, assuming that there is a unique co-existence equilibrium (which is a saddle, destroying competition exclusion), the two single-population steady states for the system would both be locally asymptotically stable; moreover, there exists a separatrix that separate the basins of the attraction of these two stable single-population steady states. Noticing that competitive systems are monotone, Jiang et al. [8] generalized Smith and Thieme's results to the general monotone semiflows and some reaction–diffusion systems. Completely determining such a separatrix also remains a challenge for most, if not all, model systems.

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