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# Global dynamics of a delay differential equation with spatial non-locality in an unbounded domain ☆

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ABSTRACT

In this paper, we study the global dynamics of a class of differential equations with temporal delay and spatial non-locality in an unbounded domain. Adopting the compact open topology, we describe the delicate asymptotic properties of the nonlocal delayed effect and establish some *a priori* estimate for nontrivial solutions which enables us to show the permanence of the equation. Combining these results with a dynamical systems approach, we determine the global dynamics of the equation under appropriate conditions. Applying the main results to the model with Ricker's birth function and Mackey–Glass's hematopoiesis function, we obtain threshold results for the global dynamics of these two models. We explain why our results on the global attractivity of the positive equilibrium in $C_+ \setminus \{0\}$ under the *compact open topology* becomes invalid in $C_+ \setminus \{0\}$ with respect to the *usual supremum norm*, and we identify a subset of $C_+ \setminus \{0\}$ in which the positive equilibrium remains attractive with respect to the *supremum norm*.


## 1. Introduction

There has been much interest in population dynamics described by models with age and spatial structures [19]. Gourley and Wu [6] provide a nice survey on models with temporal delay and spatial non-locality, as well as some existing results on such models. Among these models is the following delayed reaction–diffusion equation with a spatial non-locality, which was derived and studied by So

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et al. [24]:

$$\frac{\partial u(t,x)}{\partial t} = D\frac{\partial^2 u(t,x)}{\partial x^2} - \delta u(t,x) + \varepsilon \int_{\mathbb{R}} \Gamma_\alpha(x-y) b\big(u(t-\tau, y)\big)\,\mathrm{d}y, \quad t \geqslant 0,\ x \in \mathbb{R}. \tag{1.1}$$

Here $u(t,x)$ is the mature population of a species at time $t$ and location $x$, $D$ and $\delta$ are the diffusion rate and the death rate of the mature population; the other two indirect parameters $\varepsilon$ and $\alpha$ are defined by $\varepsilon = e^{-\delta_i \tau}$ and $\alpha = D_i \tau$ where $D_i$ and $\delta_i$ are the diffusion rate and the death rate of the immature population of the species; $b(u)$ is a birth function and the kernel function $f_\alpha(u)$ parameterized by $\alpha$ is given by

$$\Gamma_\alpha(u) = \frac{1}{\sqrt{4\pi\alpha}} e^{-u^2/4\alpha}.$$

Obviously, $\alpha$ measures the mobility of the immature individuals, $\varepsilon$ measures the probability that a new born can survive the immature period, and $f_\alpha(x-y)$ accounts for the probability that an individual born at location $y$ at time $t-\tau$ will be at location $x$ at time $t$.

We point out that when considering a species that habitats in a bounded domain $\Omega$, similar models have also been derived/proposed in Liang et al. [16] and Xu and Zhao [30], in which the model equations are also in the form of (1.1) except that the integrals are in a bounded domain. Moreover, depending on the forms of boundary condition associated to the differential equation, the kernel function $\Gamma_\alpha(u)$ will take different forms.

In nature, there are species, such as birds, whose immature individuals do not move around but the mature ones do. For such species, letting $\alpha \to 0$, the model (1.1) reduces to the following spatially local model

$$\frac{\partial u(t,x)}{\partial t} = D\frac{\partial^2 u(t,x)}{\partial x^2} - \delta u(t,x) + \varepsilon b\big(u(t-\tau, x)\big), \quad t \geqslant 0,\ x \in \mathbb{R}. \tag{1.2}$$

For this equation, traveling wavefront is an important class of solutions, which may explain spatial invasion of the species [3–5,26]. When the birth function is taken to be Ricker's function $b(u) = pue^{-qu}$, the existence and stability of traveling wavefronts solutions have been investigated in [26,18].

There are also species whose immature individuals diffuse but their mature individuals do not. Barnacles as well as mussels are among such species. Taking barnacles as an example, they are exclusively marine and tend to live in shallow and tidal waters, typically in erosive settings [29]. Barnacles have two distinct larval stages, the nauplius and the cyprid, before developing into a mature adult. A fertilized egg hatches into a nauplius: a one eyed larva comprising a head and a telson, without a thorax or abdomen. This undergoes *six months* of growth before transforming into the bivalved cyprid stage. Nauplii are typically initially brooded by the parent, and released as free-swimming larvae after the first moult. The cyprid stage lasts from *days to weeks*. During this part of the life cycle, the barnacle searches for a place to settle. It explores potential surfaces with modified antennules; once it has found a potentially suitable spot, it attaches head-first using its antennules and a secreted glycoproteinous substance, and then cements down permanently with another proteinacous compound. Once this accomplished, the barnacle undergoes metamorphosis into a juvenile barnacle. In the spot, the barnacle further grow mature by developing six hard calcareous plates to surround and protect their bodies. For the rest of their lives they are cemented to the ground, using their feathery legs (cirri) to capture plankton, and producing offspring by laying eggs in the same spot. See [29] for more details. From the life cycle of barnacles, it is clear that their population can be described by the equation resulted from letting $D = 0$ in (1.1), that is,

$$\frac{\partial u(t,x)}{\partial t} = -\delta u(t,x) + \varepsilon \int_{\mathbb{R}} \Gamma_\alpha(x-y) b\big(u(t-\tau, y)\big)\,\mathrm{d}y, \quad t \geqslant 0,\ x \in \mathbb{R}. \tag{1.3}$$

The above information also tells that the maturation delay $\tau$ for a barnacle species can be very large (six months).

Note that further letting $\alpha \to 0$ in (1.3) leads to the following point-wise delayed ODE

$$\frac{\partial u(t)}{\partial t} = -\delta u(t) + \varepsilon b\big(u(t-\tau)\big), \quad t \geqslant 0, \tag{1.4}$$

whose dynamics have been extensively and intensively studied, see e.g., [1,2,7,8,10–12,14,15,22,28] and the references therein. One naturally wonders how the global dynamics of the delayed ODE model (1.4) is related to that of the model equation (1.1) or (1.2) or (1.3) where a spatial variable $x$ is involved. Note that the spatial domain in (1.1), (1.2), (1.3) is unbounded. The lack of compactness of this domain $\mathbb{R}$ gives rise to a challenge in applying methods and/or tools from dynamical system theory to explore the global dynamics of these equations. This may explain (at least partially) why results on global dynamics of such equations are so scare in literature and most research works on such equations are only on traveling wave solutions (a special class of solutions) and spread speeds.

In this paper, we make an attempt in this direction—the direction of studying the global dynamics, but we only confine ourselves to (1.3). For convenience, we consider the following re-scaled version of (1.3):

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = -\mu u(t,x) + \mu \displaystyle\int_{\mathbb{R}} k(x-y) f\big(u(t-1,y)\big)\,\mathrm{d}y, & \text{for all } (t,x) \in (0,\infty)\times\mathbb{R}, \\ u(\theta,x) = \phi(\theta,x) \quad \text{for all } (\theta,x) \in [-\tau,0]\times\mathbb{R}, \end{cases} \tag{1.5}$$

where $\mu > 0$, $f:\mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function with $f(0) = 0$, and $k:\mathbb{R} \to \mathbb{R}_+$ is a continuous function with $\int_{\mathbb{R}} k(y)\,\mathrm{d}y = 1$. Here, an initial condition is incorporated with $\phi(\theta,x) \geqslant 0$ being continuous in $[-\tau,0]\times\mathbb{R}$.

The non-compactness of the spatial domain prohibit us from employing many nice results in dynamical system theory. To overcome this difficulty of non-compactness, we first introduce the compact open topology. We then prove that each solution of (1.5) will be attracted to a uniformly bounded and positively invariant set. Then we show that the solution map of (1.5) induces a continuous semiflow on each uniformly bounded and positively invariant set in which each orbit is pre-compact, but this semiflow is not a compact semiflow (see Remark 2.1). Thirdly, by describing the delicate asymptotic properties of the nonlocal delayed effect, we establish some *a priori* estimate for nontrivial solutions which enables us to show the permanence of the equation. Combine these results with a dynamical systems approach, we are able to determine the global dynamics of the equation under appropriate conditions. Applying the main results to two particular models with Ricker's birth function and Mackey–Glass's hematopoiesis function, we obtain threshold results on the global dynamics of these models. We point out that the global attractivity of the trivial equilibrium is within the positive cone $C_+$ with respect to the usual *supremum norm*, while the global attractivity of the positive equilibrium $u^*$ is within $C_+ \setminus \{0\}$ with respect to the *compact open topology*. Finally in Section 5, we will show that with respect to the *supremum norm*, $u^*$ cannot be globally attractive in $C_+ \setminus \{0\}$ itself, and will illustrate this by considering the local version (1.2) with Ricker's birth function. Indeed, existence of traveling wave fronts for this equation will prevent $u^*$ from attracting all solutions in the sense of the usual *supremum norm* in $C_+ \setminus \{0\}$. Motivated by the role of traveling wave fronts, we identify a subset, denoted by $C_+^{>} \setminus \{0\}$, of $C_+ \setminus \{0\}$ in which, the positive equilibrium $u^*$ is also globally attractive with respect to the *supremum norm*.

## 2. Preliminaries and basic hypothesis

We first introduce some notations. Let $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{R}_+$ be the sets of all positive integers, reals, and nonnegative reals, respectively. Let $X = BC(\mathbb{R},\mathbb{R})$ be the normed vector space of all bounded and continuous functions from $\mathbb{R}$ to $\mathbb{R}$ with the usual compact open topology induced by norm $\|\phi\|_X \triangleq \sum_{n\geqslant 1} 2^{-n}\sup\{|\phi(x)|:\ x\in[-n,n]\}$ for all $\phi \in X$. Let $X_+ = \{\phi \in X:\ \phi(x) \geqslant 0 \text{ for all } x \in \mathbb{R}\}$ and $X_+^o =$

$\{\phi \in X\colon\ \phi(x) > 0 \text{ for all } x \in \mathbb{R}\}$. It follows that $X_+$ is a closed cone in the normed vector space $X$, but $X_+^o \neq Int(X_+)$ due to the non-compactness of the spatial domain $\mathbb{R}$. Let $C = C([-1,0], X)$ be the normed vector space of continuous functions from $[-1,0]$ into $X$ with the usual compact open topology induced by norm $\|\varphi\|_C = \sup\{\|\varphi(\theta)\|_X\colon\ \theta \in [-1,0]\}$, let $C_+ = C([-1,0], X_+)$ and let $C_+^o = C([-1,0], X_+^o)$. Then $C_+$ is a closed cone of $C$, but $C_+^o \neq Int(C_+)$.

For convenience, we shall identify an element $\varphi \in C$ as a function from $[-1,0] \times \mathbb{R}$ into $\mathbb{R}$. For $a \in R$, $\hat{a} \in X$ is defined as $\hat{a}(x) = a$ for all $x \in \mathbb{R}$. Similarly, $\hat{\hat{a}} \in C$ is defined as $\hat{\hat{a}}(\theta) = \hat{a}$ for all $\theta \in [-1,0]$. For any $\phi, \psi \in X$, we write $\phi \geqslant_X \psi$ if $\phi - \psi \in X_+$, $\phi >_X \psi$ if $\phi \geqslant_X \psi$ and $\phi \neq \psi$. For any $\xi, \eta \in C$, we write $\xi \geqslant_C \eta$ if $\xi - \eta \in X_+$, $\xi >_C \eta$ if $\xi \geqslant_C \eta$ and $\xi \neq \eta$. For simplicity of notations, we shall write $a \triangleq \hat{a}$ and $a \triangleq \hat{\hat{a}}$, and write $\geqslant$, $>$ and $\|\cdot\|$ for $\geqslant_*$, $>_*$ and $\|\cdot\|_*$ respectively, where $*$ stands for $X$ or $C$.

For a real interval $I$, let $I + [-1,0] = \{t + \theta\colon\ t \in I \text{ and } \theta \in [-1,0]\}$. For $u : (I + [-1,0]) \times \mathbb{R} \to \mathbb{R}$ and $t \in I$, we write $u_t(\cdot,\cdot)$ for the element of $C$ defined by $u_t(\theta, x) = u(t+\theta, x)$, for all $\theta \in [-1,0]$ and $x \in \mathbb{R}$.

Let $\mu > 0$ and $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous function with $f(0) = 0$. Consider the following scalar equation with temporal delay and spatial non-locality

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = -\mu u(t,x) + \mu \displaystyle\int_{\mathbb{R}} f\big(u(t-1,y)\big)k(x-y)\,\mathrm{d}y, & \text{for all } (t,x) \in (0,\infty) \times \mathbb{R}, \\ u(\theta,x) = \varphi(\theta,x), \quad \text{for all } (\theta,x) \in [-\tau,0] \times \mathbb{R}, \end{cases} \tag{2.1}$$

where $\varphi \in C_+$. The kernel function is always assumed to satisfy the following:

(A) $k \in C(\mathbb{R}, \mathbb{R}_+)$, $k(0) > 0$, $\int_{\mathbb{R}} k(y)\,\mathrm{d}y = 1$ and $k(x) = k(-x)$ for all $x \in \mathbb{R}$.

By an argument of steps, we know that for any given $\varphi \in C_+$, (2.1) has a unique solution in $C_+$ for all $t \geqslant 0$. Denote this solution by $u^\varphi(t,x)$.

Define $F : C_+ \to X_+$ by $F(\varphi)(x) = \int_{\mathbb{R}} f(\varphi(-1,y))k(x-y)\,\mathrm{d}y$ for all $x \in \mathbb{R}$ and $\varphi \in C_+$. Then, as usual, associated to (2.1) is the following integral equation with the given initial function:

$$\begin{cases} u(t,\cdot) = e^{-\mu t}\varphi(0,\cdot) + \displaystyle\int_0^t \mu e^{-\mu(t-s)} F(u_s)\,\mathrm{d}s, \quad t \geqslant 0, \\ u_0 = \varphi \in C_+. \end{cases} \tag{2.2}$$

In the sequel, we will mainly use (2.2) to investigate the asymptotic behavior of solution (2.1).

To proceed, we always assume, in the rest of this paper, that $f$ satisfies the following conditions:

(H1) There exists $M > 0$ such that $f(x) \in (0, M]$ for all $x \in (0,\infty)$.
(H2) $f$ is a continuously differentiable function on some right-neighborhood of 0.

**Lemma 2.1.** *Let $B_r = \{\phi \in X\colon\ |\phi(x)| \leqslant r \text{ for all } x \in \mathbb{R}\}$ and $d_r(\phi,\psi) = \|\phi - \psi\|$, where $r > 0$. Then the following statements are true*:

(i) *If $\phi_n, \phi \in B_r$ with $n \in \mathbb{N}$, then $\lim_{n\to\infty} d_r(\phi_n, \phi) = 0$ if and only if $\lim_{t\to\infty} \sup\{|\phi_n(x) - \phi(x)|\colon\ x \in I\} = 0$ for any bounded and closed interval $I \subseteq \mathbb{R}$.*
(ii) *Let $A \subseteq B_r$. Then $A$ is pre-compact if and only if $A_I = \{\phi|_I\colon\ \phi \in A\}$ is a family of equicontinuous functions, for any bounded and closed interval $I \subseteq \mathbb{R}$.*

**Lemma 2.2.** *Define $K : X \to X$ by $K(\phi)(x) = \int_{\mathbb{R}} \phi(y)k(x-y)\,\mathrm{d}y$ for all $x \in \mathbb{R}$ and $\phi \in X$. Then the linear operator $K$ satisfies the following*:

(i) $K(X_+) \subseteq X_+$.
(ii) *There is* $a^* = a^*(K) > 0$ *such that for any interval* $I \equiv [c,d] \subseteq \mathbb{R}$ *and* $n \in \mathbb{N}$, $K^n(X_+^o(I)) \subseteq X_+^o([c - na^*, d + na^*])$, *where* $X_+^o(I) = \{\phi \in X_+ \colon \phi(x) > 0$ *for all* $x \in I\}$.
(iii) $K(B_1) \subseteq B_1$.
(iv) $K|_{B_1} : B_1 \to B_1$ *is a continuous and compact map, where* $B_1$ *with topology induced by* $d_1$.

**Proof.** (i) and (iii) immediately follow from the definition of $K$.

(ii): Obviously, by $k(0) > 0$, there exists $a^* > 0$ such that $k(x) > 0$ for all $x \in [-a^*, a^*]$. Suppose that $I \equiv [c,d] \subseteq \mathbb{R}$ and $\phi \in X_+^o(I)$. We claim that for any $n \in \mathbb{N}$, $K^n(\phi)(x) > 0$ for all $x \in [c - na^*, d + na^*]$. We shall finish the proof of this claim by induction. If $n = 1$, then $K(\phi)(x) = \int_{\mathbb{R}} \phi(x+y)k(-y)\,\mathrm{d}y$. For any $x \in [c - a^*, d + a^*]$, there exists $y^* \in [-a^*, a^*]$ such that $x + y^* \in [c,d]$, and hence $\phi(x + y^*)k(-y^*) > 0$ and $\int_{\mathbb{R}} \phi(x+y)k(-y)\,\mathrm{d}y > 0$. Thus, $K(\phi)(x) > 0$ for all $x \in [c - a^*, d + a^*]$. Now we assume that the claim holds when $n = i$. If $n = i + 1$, then by the assumption, $K^i(\phi)(x) > 0$ for all $x \in [c - ia^*, d + ia^*]$. By applying the claim with $n = 1$ to $K^i(\phi)$, we have $K^{i+1}(\phi)(x) = K(K^i(\phi))(x)$ for all $x \in [c - (i+1)a^*, d + (i+1)a^*]$. Thus, the claim holds and implies the statement (ii).

(iv): Firstly, the continuity of $K$ can be easily shown by employing some standard arguments together with Lemma 2.1(i) and the fact that $\int_{\mathbb{R}} k(y)\,\mathrm{d}y = 1$.

Next, we show that $K|_{B_1}$ is compact. It suffices to prove that $K(B_1)$ is a pre-compact subset in $(B_1, d_1)$. By Lemma 2.1(ii), for any bounded and closed interval $I \equiv [a,b] \subseteq \mathbb{R}$, it suffices to prove that $A_I \equiv \{\phi|_I \colon \phi \in K(B_r)\}$ is a family of equicontinuous functions in $C(I, \mathbb{R})$. Indeed, for any $\varepsilon > 0$, there exists $T = T(\varepsilon) > 0$ such that $\int_{|y| \geqslant T} k(y)\,\mathrm{d}y < \frac{\varepsilon}{6}$. Let $T^* = T + \max\{|a|, |b|, 0\}$ and $I^* = [-T^*, T^*]$. Then there exists $\delta = \delta(\varepsilon, I) \in (0,1)$ such that $|k(z) - k(\tilde{z})| < \frac{\varepsilon}{1+3T^*}$ when $z, \tilde{z} \in [-1 - T^*, 1 + T^*]$ and $|z - \tilde{z}| < \delta$. It follows from the definition of $K$ that for any $\phi \in K(B_1)$, $x, \tilde{x} \in I$ and $|x - \tilde{x}| < \delta$, we have

$$
\begin{aligned}
&\bigl|K(\phi)(x) - K(\phi)(\tilde{x})\bigr| \\
&\quad = \left| \int_{\mathbb{R}} \phi(y)\bigl(k(x-y) - k(\tilde{x}-y)\bigr)\,\mathrm{d}y \right| \\
&\quad \leqslant \int_{y \in [-T^*, T^*]} \bigl|\phi(y)\bigr| \cdot \bigl|k(x-y) - k(\tilde{x}-y)\bigr|\,\mathrm{d}y + \int_{y \notin [-T^*, T^*]} \bigl|\phi(y)\bigr| \cdot \bigl|k(x-y) - k(\tilde{x}-y)\bigr|\,\mathrm{d}y \\
&\quad \leqslant \int_{y \in [-T^*, T^*]} \bigl|\phi(y)\bigr| \cdot \bigl|k(x-y) - k(\tilde{x}-y)\bigr|\,\mathrm{d}y + 2\int_{y \notin [-T, T]} \bigl|\phi(x+y)\bigr|k(-y)\,\mathrm{d}y \\
&\quad \leqslant 2T^* \frac{\varepsilon}{1+3T^*} + \frac{2\varepsilon}{6} \\
&\quad < \varepsilon.
\end{aligned}
$$

Thus, $A_I \equiv \{\phi|_I \colon \phi \in K(B_1)\}$ is a family of equicontinuous functions, which shows that $K|_{B_1} : B_1 \to B_1$ is compact. This completes the proof. □

**Proposition 2.1.** *The following statements are true*:

(i) *For any* $\varphi \in C_+$, $(u^\varphi)_t \in C_+$ *for all* $t \in \mathbb{R}_+$. *Moreover, if* $\varphi \in C_+ \setminus \{0\}$, *then for any* $T > 0$, *there exists* $t^* = t^*(T, \varphi) > 0$ *such that* $u^\varphi(t,x) > 0$ *for all* $(t,x) \in [t^*, \infty) \times [-T, T]$.
(ii) *If* $\varphi \in C_+$ *and* $\varphi(\theta, x) \leqslant M + 1$ *for all* $(\theta, x) \in [-1, 0] \times \mathbb{R}$, *then* $u^\varphi(t,x) \in [0, M+1]$ *for all* $(t,x) \in [-1, \infty) \times \mathbb{R}$.
(iii) *For any* $\varphi \in C_+$, *there exists* $t^* = t^*(\varphi) > 0$ *such that* $u^\varphi(t,x) \leqslant M + 1$ *for all* $(t,x) \in [t^*, \infty) \times \mathbb{R}$.

**Proof.** (i): Clearly, from (2.2), (H1) and Lemma 2.2(i), we have $(u^\varphi)_t \in X_+$ for all $t \in \mathbb{R}_+$ and $\varphi \in C_+$. Now, suppose that $\varphi \in C_+ \setminus \{0\}$. Then there exist $\theta^* \in [-1, 0]$ and an interval $[c,d]$ such that $\varphi(\theta^*, x) > 0$ and thus $f(\varphi(\theta^*, x)) > 0$ for all $x \in [c, d]$. It follows from Lemma 2.2(ii) that $K(f(\varphi(\theta^*, \cdot)))(x) > 0$ for all $x \in [c-a^*, d+a^*]$, where $a^*$ is defined as in Lemma 2.2(ii). This combined with (2.2), yields

$$u^\varphi(t, x) \geqslant \int_0^t \mu e^{-\mu(t-s)} \int_{\mathbb{R}} f\big(u^\varphi(s-1, y)\big)k(x-y)\,\mathrm{d}y\,\mathrm{d}s > 0,$$

for all $(t, x) \in [1+\theta^*, \infty) \times [c-a^*, d+a^*]$. By applying Lemma 2.2(ii), the induction and the semigroup property of the solution map of (2.2), we obtain that $u^\varphi(t, x) > 0$ for all $(t, x) \in [n+\theta^*, \infty) \times [c - na^*, d + na^*]$. So, for any $T > 0$, by taking $t^* = t^*(T, \varphi) = 1 + \frac{\max\{1, T+c, T-d\}}{a^*} > 0$, we obtain that $u^\varphi(t, x) > 0$ for all $(t, x) \in [t^*, \infty) \times [-T, T]$.

(ii): Suppose that $\varphi \in C_+$ and $\varphi(\theta, x) \leqslant M+1$ for all $(\theta, x) \in [-1, 0] \times \mathbb{R}$. It follows from (2.2) and assumption (H1) that for any $t \in \mathbb{R}_+$, we have

$$\begin{aligned} u^\varphi(t, x) &= e^{-\mu t}\varphi(0, x) + \int_0^t \mu e^{-\mu(t-s)} \int_{\mathbb{R}} f\big(u^\varphi(s-1, y)\big)k(x-y)\,\mathrm{d}y\,\mathrm{d}s \\ &\leqslant (1+M)e^{-\mu t} + \int_0^t M\mu e^{-\mu(t-s)}\,\mathrm{d}s \\ &= M + e^{-\mu t}, \end{aligned}$$

which combined with the induction, implies the statement (ii).

(iii): By an argument similar to the proof of the statement (ii), we easily see that the statement (iii) holds.

The proof of the proposition is completed. □

In the following parts, let $Y = \{\varphi \in C_+ \colon \varphi(\theta, x) \leqslant 1+M$ for all $(\theta, x) \in [-1, 0] \times \mathbb{R}\}$. Then by Propositions 2.1(ii)–(iii), we know that $Y$ is a positively invariant set of the solution map and every point in $C_+$ is attracted by $Y$ in the sense of Hale [9].

Define $d \colon Y \times Y \to \mathbb{R}_+$, and $\Phi \colon \mathbb{R}_+ \times Y \to Y$ by $d(\zeta, \eta) = \|\zeta - \eta\|$, and $\Phi(t, \varphi) = (u^\varphi)_t$ for all $\zeta, \eta, \varphi \in Y$ and $t \in \mathbb{R}_+$, respectively.

**Lemma 2.3.** *Assume that $Y$ is defined as above. Then the following results are true.*

(i) *If $\varphi_n, \varphi \in Y$ with $n \in \mathbb{N}$, then $\lim_{n\to\infty} d(\varphi_n, \varphi) = 0$ if and only if $\lim_{t\to\infty} \sup\{|\varphi_n(\theta, x) - \varphi(\theta, x)| \colon (\theta, x) \in [-1, 0] \times I\} = 0$ for any bounded and closed interval $I \subseteq \mathbb{R}$.*
(ii) *Let $A \subseteq Y$. Then $A$ is pre-compact in $Y$ if and only if $A_I = \{\varphi|_{[-1,0]\times I} \colon \varphi \in A\}$ is a family of equicontinuous functions in $C([-1, 0] \times I, \mathbb{R})$, for any bounded and closed interval $I \subseteq \mathbb{R}$.*

**Proposition 2.2.** *Assume that $Y$ and $\Phi$ are defined as above. Then the following statements are true*:

(i) *$\Phi$ is a continuous semiflow on $Y$.*
(ii) *If $\varphi \in Y$, then $\{\Phi(t, \varphi) \colon t \in \mathbb{R}_+\}$ is a pre-compact subset of $Y$.*

**Proof.** (i): Obviously, the semigroup of $\Phi$ immediately follows from the definition of $\Phi$. By employing (2.2), Lemma 2.2(iv), the semigroup property of $\Phi$ in connection with some standard arguments, we may prove the continuity of $\Phi$.

(ii): Suppose that $\varphi \in Y$ and $I \subseteq \mathbb{R}$ is a bounded and closed interval. By (2.1), we know that $D \equiv 1 + \sup\{|\frac{\partial u}{\partial t}(t,x)|: (t,x) \in \mathbb{R}_+ \times \mathbb{R}\} \leqslant \mu(1+M) + \mu M < \infty$, and thus

$$\left|u^\varphi(t,x) - u^\varphi(\tilde{t},x)\right| \leqslant D|t-\tilde{t}| \quad \text{for all } (t,\tilde{t},x) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}.$$

On the other hand, for any $\varepsilon > 0$, by the continuity of $\varphi$, Lemmas 2.1(ii) and 2.2(iv), there exists $\delta^* = \delta^*(\varepsilon, I, \varphi) > 0$ such that $|\varphi(0,x) - \varphi(0,\tilde{x})| < \frac{\varepsilon}{3}$ and $|K(f(u^\varphi(s,\cdot)))(x) - K(f(u^\varphi(s,\cdot)))(\tilde{x})| < \frac{\varepsilon}{3}$ when $s \in [-1,\infty)$, $x,\tilde{x} \in I$ with $|x-\tilde{x}| < \delta^*$. It follows from (2.2) and the choice of $\delta^*$ that for any $(t,x), (t,\tilde{x}) \in \mathbb{R}_+ \times I$ with $|x-\tilde{x}| < \delta^*$, we have

$$\begin{aligned}
&\left|u^\varphi(t,x) - u^\varphi(t,\tilde{x})\right| \\
&\quad = \left| e^{-\mu t}\bigl(\varphi(0,x) - \varphi(0,\tilde{x})\bigr) + \int_0^t \mu e^{-\mu(t-s)} \int_{\mathbb{R}} f\bigl(u^\varphi(s-1,y)\bigr) k(x-y) - k(\tilde{x}-y)\,\mathrm{d}y\,\mathrm{d}s \right| \\
&\quad \leqslant e^{-\mu t}\left|\varphi(0,x) - \varphi(0,\tilde{x})\right| + \int_0^t \mu e^{-\mu(t-s)} \left|K\bigl(f\bigl(u^\varphi(s,\cdot)\bigr)\bigr)(x) - K\bigl(f\bigl(u^\varphi(s,\cdot)\bigr)\bigr)(\tilde{x})\right|\mathrm{d}s \\
&\quad \leqslant \frac{\varepsilon e^{-\mu t}}{3} + \frac{\varepsilon(1-e^{-\mu t})}{3} \\
&\quad < \frac{2\varepsilon}{3}.
\end{aligned}$$

Thus, the above discussions imply that for any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon, \varphi) > 0$ such that $|u^\varphi(t,x) - u^\varphi(\tilde{t},\tilde{x})| \leqslant |u^\varphi(t,x) - u^\varphi(\tilde{t},x)| + |u^\varphi(\tilde{t},x) - u^\varphi(\tilde{t},\tilde{x})| < D\delta + \frac{2\varepsilon}{3} \leqslant \varepsilon$ for all $(t,x), (\tilde{t},\tilde{x}) \in \mathbb{R}_+ \times I$ with $|t-\tilde{t}| < \delta$ and $|x-\tilde{x}| < \delta$. Therefore, Lemma 2.3(ii) yields the statement (ii). The proof of the theorem is completed. $\square$

**Remark 2.1.** We emphasize that $\Phi$ is not a compact semiflow for $t > 1$. Indeed, by Lemma 2.2(iv) and the proof of Proposition 2.2, we easily see that for any $t > 1$, the operator $Y \ni \varphi \mapsto G(t,\varphi) \in C_+$ is compact, where $G(t,\varphi)(\theta,x) = \int_0^{t+\theta} \mu e^{-\mu(t+\theta-s)} F((u^\varphi)_s)\,\mathrm{d}s$. But for any $t > 1$, the operator $Y \ni \varphi \mapsto e^{-\mu t}\varphi \in C_+$ is NOT compact. Thus, by (2.2), we know that $\Phi$ is not a compact semiflow for $t > 1$.

Given $\varphi \in Y$. We write $O(\varphi) = \{\Phi(t,\varphi): t \in \mathbb{R}_+\}$ for the positive semi-orbit through the point $\varphi \in Y$. The $\omega$-limit set of $O(\varphi)$ is defined by $\omega(\varphi) = \bigcap_{t \in \mathbb{R}_+} \overline{O(\Phi(t,\varphi))}$. Thus, for $\varphi \in C_+$, by Proposition 2.2, we know $\overline{O(\varphi)}$ is compact, and hence $\omega(\varphi) \subseteq Y$ is a nonempty, compact and connected subset of $C_+$ and also an invariant set of $\Phi$.

If $f(u)$ has a positive fixed point $u^*$, then $u^*$ is a positive constant equilibrium of (2.1). In such a case, we say that $u^* > 0$ is globally attractive in $C_+ \setminus \{0\}$ if $\omega(\varphi) = \{u^*\}$ for all $\varphi \in C_+ \setminus \{0\}$.

It is obvious that 0 is a constant equilibrium of (2.1) (since $f(0) = 0$). When $f'(0) \leqslant 1$, and $f(x) < x$ for all $x \in (0,\infty)$, by (2.2), we can easily see that 0 is a globally attractive equilibrium in $C_+$ with respect to the usual supremum norm. That is, for any $\varphi \in C_+$, $(u^\varphi)_t$ tends to 0 with respect to the supremum norm. However, when $f'(0) > 1$, 0 becomes unstable and there is a positive constant equilibrium for (2.1) under (H1)–(H2). In the next section, we explore the global attractivity of the positive equilibrium in $C_+ \setminus \{0\}$ with respect to the compact open topology, and in Section 5, we will further discuss the global attractivity of the positive equilibrium with respect to the supremum norm.

## 3. Global dynamics

In this section, we always assume that $f'(0) > 1$. In this case, to overcome the difficulty in describing the global dynamics due to the lack of non-compactness of the spatial domain, we first establish some *a priori* estimates for nontrivial solutions after describing a delicate asymptotic property of the nonlocal delayed effect. This estimate enables us to show the permanence of the equation. Then we obtain the global attractivity of the nontrivial equilibrium by employing standard dynamical system theoretical arguments.

Let $X(\varepsilon, T) = \{\phi \in X_+ \colon \phi(x) \geqslant \varepsilon$ for all $x \in [-T, T]\}$ and let $X^o(\varepsilon, T) = \{\phi \in X_+ \colon \phi(x) > \varepsilon$ for all $x \in [-T, T]\}$, where $\varepsilon > 0$ and $T \geqslant 0$.

For any bounded function $\phi : \mathbb{R} \to \mathbb{R}$, and $x \in \mathbb{R}$, we still denote $\int_{\mathbb{R}} \phi(y)k(x-y)\,\mathrm{d}y$ by $K(\phi)(x)$. Then $K$ is order-preserving with the point-wise order.

**Lemma 3.1.** *For any $n \in \mathbb{N}$ and $\delta \in (0, \frac{1}{2})$, there exists $T_{n,\delta} > 0$ such that $K^n(X(1,T)) \subseteq X(\delta, T)$ for all $T \geqslant T_{n,\delta}$, where $K^n$ represents the $n$-composition of $K$.*

**Proof.** Suppose that $n \in \mathbb{N}$ and $\delta \in (0, \frac{1}{2})$. Clearly, $K^n(1) \equiv 1$. By Fubini's Theorem, we obtain that for any $\phi \in X$ and $x \in \mathbb{R}$,

$$K^n(\phi) = \int_{\mathbb{R}^n} \phi(y_1)k(x-y_n) \prod_{i=1}^{n-1} \big(k(y_{i+1} - y_i)\big) \prod_{i=1}^{n} \mathrm{d}y_i.$$

It follows from the linear transformations of variables that

$$K^n(\phi) = \int_{\mathbb{R}^n} \phi\left(x + \sum_{i=1}^{n} y_i\right) \prod_{i=1}^{n} \big(k(-y_i)\big) \prod_{i=1}^{n} \mathrm{d}y_i.$$

Define $g : \mathbb{R}^n \to \mathbb{R}$ by $g(\mathbf{y}) = \prod_{i=1}^{n} k(y_i)$ for all $\mathbf{y} = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$. Then

$$K^n(\phi) = \int_{\mathbb{R}^n} \phi\left(x + \sum_{i=1}^{n} y_i\right) g(\mathbf{y})\,\mathrm{d}\mathbf{y} \quad \text{for all } \mathbf{y} = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n.$$

For any $T \in [0, \infty)$, let $I_+(T) = [0, T]$, $I_-(T) = [-T, 0]$ and

$$D_\pm(T) = \left\{\mathbf{y} = (y_1, y_2, \dots, z_n) \in \mathbb{R}^n \colon \sum_{i=1}^{n} y_i \in I_\pm(T)\right\}.$$

Define $a_\pm : \mathbb{R}_+ \to \mathbb{R}$ by

$$a_\pm(T) = \int_{D_\pm(T)} g(\mathbf{y})\,\mathrm{d}\mathbf{y} \quad \text{for all } T \in \mathbb{R}_+.$$

By the above definitions and the properties of $k$, we easily see that $a_-(T) \equiv a_+(T)$, $a_\pm(\pm\infty) = \frac{1}{2}$, and $a_+$, $a_-$ are both continuous and increasing functions on $\mathbb{R}_+$. Hence there exists $T_{n,\delta} > 0$ such that $a_\pm(T) \geqslant \delta$ for all $T \geqslant T_{n,\delta}$. If $T \geqslant T_{n,\delta}$ and $\phi \in X(1, T)$, then $K^n(\phi)(x) \geqslant a_-(T) \geqslant \delta$ for all $x \in [0, T]$ and $K^n(\phi)(x) \geqslant a_+(T) \geqslant \delta$ for all $x \in [-T, 0]$, and thus, $K^n(\phi)(x) \geqslant \delta$ for all $x \in [-T, T]$, which, combined with Lemma 2.2(i), implies that $K^n(\phi) \in X(\delta, T)$. This completes the proof. $\square$

The following result gives an *a priori* estimate for solutions for (2.2), which plays a key role in the proof of the permanence and global attractivity of (2.2).

**Lemma 3.2.** *Suppose that $f'(0) > 1$. Then there exist $\varepsilon_0 > 0$, $T_0 > 0$ and $T^* > 0$ such that for all $\varepsilon \in [0, \varepsilon_0]$, $T \in [T_0, \infty)$, $u : [-1, \infty) \times \mathbb{R} \to [0, 1 + M]$ is a solution of (2.2) such that $u(t, \cdot) \in X(\varepsilon, T)$ for all $t \in [-1, T^*]$, we have $u(t, \cdot) \in X(\varepsilon, T)$ for all $t \in [-1, \infty)$, and $u(t, \cdot) \in X^o(\varepsilon, T)$ for all $t \in (T^*, \infty)$.*

**Proof.** By $f'(0) > 1$ and (H1), there exist $\alpha_1 > 1$ and $\varepsilon_1 \in (0, 1 + M]$ such that $f(u) \geqslant \alpha_1 u$ for all $u \in [0, \varepsilon_1]$ and $f(u) \geqslant \alpha_1 \varepsilon_1$ for all $u \in [\varepsilon_1, 1 + M]$. It follows from $\alpha_1 > 1$ that there exists $n \in \mathbb{N}$ such that $(\alpha_1)^n > 2 \geqslant (\alpha_1)^{n-1}$.

By taking $\delta \in (1/(\alpha_1)^n, 1/2)$ and applying Lemma 3.1, we know that there exists $T_{n,\delta} > 0$ such that $K^n(X(1, T)) \subseteq X(\delta, T)$ for all $T \geqslant T_{n,\delta}$.

Now, we choose $\varepsilon_0 = \varepsilon_1/(\alpha_1)^{n+1}$, $T_0 = T_{n,\delta}$, $T_1 = 1 + \frac{1}{\mu} \ln(\alpha_1 \delta^{\frac{1}{n}}/(\alpha_1 \delta^{\frac{1}{n}} - 1))$ and $T^* = nT_1 + n - 1$.

In the following, we assume that $\varepsilon \in [0, \varepsilon_0]$, $T \in [T_0, \infty)$, $u : [-1, \infty) \times \mathbb{R} \to [0, 1 + M]$ is a solution of (2.2) such that $u(t, \cdot) \in X(\varepsilon, T)$ for all $t \in [-1, T^*]$. Then there exists $\phi \in X(\varepsilon, T)$ such that $\phi(x) \leqslant \varepsilon$ for all $x \in \mathbb{R}$ and $u(t, \cdot) \geqslant \phi$ for all $t \in [-1, T^*]$. We easily obtain that

$$f\bigl(\alpha_1^j K^j(\phi)\bigr) \geqslant \alpha_1^{j+1} K^j(\phi)$$

for all $j = 0, 1, \ldots, n$, due to the choices of $\phi$, $\varepsilon$ and $\alpha_1$.

Let $\varphi = u_0$. Then $u(t, x) = u^\varphi(t, x) = \Phi(t + 1, \varphi)(-1, x)$ for all $(t, x) \in [-1, \infty) \times \mathbb{R}$. We claim that

$$u^\varphi(t, \cdot) \geqslant \alpha_1^j \bigl(1 - e^{-\mu T_1}\bigr)^j K^j(\phi) \quad \text{for all } t \in \bigl[jT_1 + j - 1, 1 + T^*\bigr] \text{ and } j \in \{1, 2, \ldots, n\}.$$

Indeed, when $j = 1$ and $t \in [T_1, 1 + T^*]$, it follows from (2.2) that we have

$$\begin{aligned} u^\varphi(t, \cdot) &= e^{-\mu t} \varphi(0, \cdot) + \int_0^t \mu e^{-\mu(t-s)} F\bigl((u^\varphi)_s\bigr)\, \mathrm{d}s \\ &\geqslant \int_0^t \mu e^{-\mu(t-s)} K\bigl(f\bigl(u^\varphi(s-1, \cdot)\bigr)\bigr)\, \mathrm{d}s \\ &\geqslant \alpha_1 \int_0^t \mu e^{-\mu(t-s)} K(\phi)\, \mathrm{d}s \\ &\geqslant \alpha_1 \bigl(1 - e^{-\mu T_1}\bigr) K(\phi). \end{aligned}$$

Now suppose that the claim holds for $j = j_0 < n$. Thus, for any $t \in [(1 + j_0)T_1 + j_0, 1 + T^*]$ and $s \in [0, T_1]$, we have $u^\varphi(s + t - T_1 - 1, \cdot) \geqslant \alpha_1^{j_0}(1 - e^{-\mu T_1})^{j_0} K^{j_0}(\phi)$, which together with (2.2) and the semigroup property of $\Phi$ implies

$$\begin{aligned} u^\varphi(t, \cdot) &= u^\varphi\bigl(T_1, (u^\varphi)_{t-T_1}\bigr) \\ &= e^{-\mu T_1} u^\varphi(t - T_1, \cdot) + \int_0^{T_1} \mu e^{-\mu(T_1-s)} K\bigl(f\bigl(u^\varphi(s + t - T_1 - 1, \cdot)\bigr)\bigr)\, \mathrm{d}s \end{aligned}$$

$$\geqslant \int_0^{T_1} \mu e^{-\mu(T_1-s)} K\big(f\big(u^{\varphi}(s+t-T_1-1,\cdot)\big)\big)\,\mathrm{d}s$$

$$\geqslant \alpha_1^{1+j_0}\big(1-e^{-\mu T_1}\big)^{1+j_0} K^{1+j_0}(\phi).$$

The claim follows by the induction.

Applying the above claim with $j=n$ in connection with the choices of $T_1$ and $T^*$, we conclude that for any $t \in (T^*, T^*+1]$, $u(t,\cdot) \in X(\varepsilon\delta(\alpha_1)^n(1-e^{-\mu T_1})^n, T) \subseteq X^o(\varepsilon, T)$. This, together with the semigroup property of $\Phi$, implies that $u(t,\cdot) \in X(\varepsilon, T)$ for all $t \in [-1,\infty)$, and $u(t,\cdot) \in X^o(\varepsilon, T)$ for all $t \in (T^*, \infty)$. This completes the proof. $\square$

We are now in the position to state and prove our first main theorem, which, together with Proposition 2.1(ii), implies that (2.2) is permanent with respect to the compact open topology in $C_+$. We remark that (2.2) is NOT permanent with respect to the usual supremum norm, see the discussions in Section 5.

**Theorem 3.1.** *If $\varphi \in C_+ \setminus \{0\}$, then there exists $a > 0$ such that $\xi \geqslant \hat{a}$ for all $\xi \in \omega(\varphi)$.*

**Proof.** By Proposition 2.1(iii), we may assume that $\varphi \in Y \setminus \{0\}$. Choose $T_0, T^*$ and $\varepsilon_0$ as in Lemma 3.2. By Proposition 2.1(i), we may assume without loss of generality that $u^{\varphi}(t,x) > 0$ for all $(t,x) \in [-1,T^*] \times [-T_0, T_0]$. Let $\varepsilon_1 = \inf\{u(t,x)\colon (t,x) \in [-1,T^*] \times [-T_0,T_0]\}$ and $\varepsilon = \min\{\varepsilon_0, \varepsilon_1\}$. Then $\varepsilon_1 > 0$ and $\varepsilon > 0$.

By Lemma 3.2 and the choice of $T_0, T^*, \alpha_0$ and $\varepsilon_0$, we get $u^{\varphi}(t,\cdot) \in X(\varepsilon, T_0)$ for all $t \in [-1,\infty)$. This combined with the definition of $\omega(\varphi)$, implies $\xi(\theta,\cdot) \in X(\varepsilon, T_0)$ for all $\xi \in \omega(\varphi)$ and $\theta \in [-1,0]$.

For any $\xi \in \omega(\varphi)$, let $a_\xi = \sup\{a \in \mathbb{R}_+\colon \xi(\theta,x) \geqslant \varepsilon$ for all $(t,x) \in [-1,0] \times [-a,0]\}$ and $b_\xi = \sup\{b \in \mathbb{R}_+\colon \xi(\theta,x) \geqslant \varepsilon$ for all $(t,x) \in [-1,0] \times [0,b]\}$. Then $a_\xi, b_\xi \geqslant T_0$.

Let $I_\xi = [-a_\xi, b_\xi]$ if $a_\xi, b_\xi \in \mathbb{R}_+$; $I_\xi = [-a_\xi, \infty)$ if $a_\xi \in \mathbb{R}_+$ and $b_\xi = \infty$; $I_\xi = (-\infty, b_\xi]$ if $a_\xi = \infty$ and $b_\xi \in \mathbb{R}_+$; $I_\xi = \mathbb{R}$ if $a_\xi = b_\xi = \infty$. Let $I = \bigcap_{\xi \in \omega(\varphi)} I_\xi$. Then $I$ is a closed interval and $I \supseteq [-T_0, T_0]$. Indeed, $I = [-c_1, c_2]$ where $c_1 = \inf\{a_\xi\colon \xi \in \omega(\varphi)\} \in [T_0,\infty)$ and $c_2 = \inf\{b_\xi\colon \xi \in \omega(\varphi)\} \in [T_0,\infty)$. Thus, there are four possibilities: (i) $I = [-c_1, c_2]$ with $c_1, c_2 \in \mathbb{R}_+$; (ii) $I = [-c_1, \infty)$ with $c_1 \in \mathbb{R}_+$; (iii) $I = (-\infty, c_2]$ with $c_2 \in \mathbb{R}_+$; or (iv) $I = \mathbb{R}$. It turns out that (i)–(iii) are all impossible. Below we only show that (i) will lead to a contraction, since the exclusion of (ii) and (iii) are similar.

Without loss of generality, we may assume that $c_1 \geqslant c_2$. Taking $\xi \in \omega(\varphi)$, we obtain by the invariance of $\omega(\varphi)$, that $u^{\xi}(t,\cdot) \in X(\varepsilon, c_2)$ for all $t \in [-1,\infty)$. Again, by Lemma 3.2 and the choices of $T_0, T^*$ and $\varepsilon_0$, we obtain that $u^{\xi}(t,\cdot) \in X^o(\varepsilon, c_2)$ for all $(t,x) \in (T^*,\infty) \times \mathbb{R}$, in particular, there exists $T > c_2$ such that $u^{\xi}(t,\cdot) \in X^o(\varepsilon, T)$ for all $t \in [1+T^*, 2+2T^*]$. On the other hand, by the definition of $\omega(\varphi)$, there exists a sequence $\{s_n\}_{n\in\mathbb{N}}$ such that

$$\lim_{s_n\to\infty} \big\| \big(u^{\varphi}\big)_{s_n} - \xi \big\| = 0,$$

and hence

$$\lim_{s_n\to\infty} \big(\sup\big\{\big|u^{\varphi}(s_n+t,x) - u^{\xi}(t,x)\big|\colon (t,x) \in \big[1+T^*, 1+2T^*\big] \times [-T,T]\big\}\big) = 0.$$

So, there exists $n^* > 1$ such that $u^{\varphi}(s_{n^*}+t,\cdot) \in X(\varepsilon, T)$ for all $t \in [1+T^*, 1+2T^*]$. It follows from Lemma 3.2 that $u^{\varphi}(s_{n^*}+t,\cdot) \in X(\varepsilon, T)$ for all $t \in [1+T^*, \infty)$, and thus by the definition of $\omega(\varphi)$, we have $\xi \in X(\varepsilon, T)$ for all $\xi \in \omega(\varphi)$. So, we have $b_\xi \geqslant T > c_2$ for all $\xi \in \omega(\varphi)$. But, $c_2 = \inf\{b_\xi\colon \xi \in \omega(\varphi)\} \geqslant T > c_2$, a contradiction.

The above shows that $I = \mathbb{R}$, implying that $a = \varepsilon$ serves the purpose of the theorem. This completes the proof. $\square$

Under the assumptions (H1)–(H2), if $f'(0) > 1$ then $f$ has a positive fixed point $u^*$ which is also a fixed point of $f^2$ and a positive equilibrium of (2.2). To proceed further to study the global attractivity of this positive equilibrium for (2.2), we formulate the following non-monotone assumption on the nonlinearity $f$.

(H3) $f^2$ has a unique positive fixed point $u^*$.

Assumption (H3), together with the assumptions (H1)–(H2) and $f'(0) > 1$ implies (see Proposition 2.1 in [33]) that

$$\lim_{n\to\infty} dist\big(f^n\big([\epsilon,\infty)\big), u^*\big) = 0 \quad \text{for any } \epsilon > 0.$$

In [33], the authors have shown that the assumption (H3) plays a decisive role on the delay independent global stability of a positive equilibrium for a class of delay reaction–diffusion equations (DRDE) in a bounded domain with the homogeneous Neumann boundary condition by establishing the relation of the globally stable dynamics of the map and the delay reaction–diffusion equations. The next theorem shows that such a decisive role of (H3) remains valid for (2.2) with respect to the compact open topology. We will employ a different (from [31,33]) method to prove this theorem.

**Theorem 3.2.** *In addition to* (H1)–(H2)*, further assume that* (H3) *holds and* $f'(0) > 1$. *Then* $u^*$ *is globally attractive equilibrium in* $C_+ \setminus \{0\}$.

**Proof.** By Proposition 2.1(iii), we may assume that $\varphi \in Y \setminus \{0\}$. By Theorem 3.1, there exists $a > 0$ such that $\xi \geqslant a$ for all $\xi \in \omega(\varphi)$. It follows from the compactness and connectivity of $\omega(\varphi)$ that there exist $u_+, u_- \in [a, 1+M]$ such that $[u_-, u_+] = \{\xi(\theta, x)\colon\ \xi \in \omega(\varphi),\ \theta \in [-1, 0] \text{ and } x \in \mathbb{R}\}$. We prove that $u_- = u_+$. For the sake of contradiction, assume $u_- < u_+$. By (H1) and $f'(0) > 1$, there is $\epsilon \in (0, u_-]$ such that $f([\epsilon, M+1]) \subseteq [\epsilon, M+1]$ and $\lim_{n\to\infty} dist(f^n([\epsilon, M+1]), u^*) = 0$. So, there are $a_-, a_+ \in [\epsilon, M+1]$ such that $[u_-, u_+] \neq f([a_-, a_+]) \subseteq [a_-, a_+]$ and $[u_-, u_+] \subseteq [a_-, a_+]$. Thus, $u_- < f([a_-, a_+])$ or $u_+ > f([a_-, a_+])$. Without loss of generality, we may assume that $u_- < f([a_-, a_+])$. It follows from (2.2) that for any $\xi \in \omega(\varphi)$, we have

$$\begin{aligned} u^\xi(t,x) &= e^{-\mu t}\xi(x,0) + \int_0^t \mu e^{-\mu(t-s)} \int_{\mathbb{R}} f\big(u^\xi(s-1,y)\big)k(x-y)\,\mathrm{d}y\,\mathrm{d}s \\ &\geqslant u_- e^{-\mu t} + \int_0^t \mu e^{-\mu(t-s)} \min\big(f\big([u_-, u_+]\big)\big) \\ &\geqslant e^{-\mu t}u_- + \big(1 - e^{-\mu t}\big)\min\big(f\big([a_-, b_+]\big)\big), \end{aligned}$$

which implies that $u^\xi(t,x) \geqslant \min(f([a_-, b_+])) + e^{-\mu}(u_- - \min(f([a_-, b_+]))) > u_-$ for all $\xi \in \omega(\varphi)$, $t \in [1, \infty)$ and $x \in \mathbb{R}$. This combined with the invariance of $\omega(\varphi)$, shows that $\xi \geqslant \min(f([a_-, b_+])) + e^{-\mu}(u_- - \min(f([a_-, b_+]))) > u_-$ for all $\xi \in \omega(\xi)$, a contradiction to the choice of $u_-$.

Now, the fact that $u_- = u_+$, together with (H3) further implies that $\omega(\varphi)) = \{u^*\}$, concluding the global attractivity of $u^*$ in $C_+ \setminus \{0\}$. This completes the proof. $\square$

## 4. Examples

In this section, we first apply the results obtained in Section 3 to the model equation (1.3) with the birth function being Ricker's function $b(u) = pue^{-qu}$ which is a widely used birth function, e.g., for fish population as well as for blowfly population (see, e.g., [2,7,8,14,15,20,22,23,25,32]).

Let $\beta = \varepsilon p/\delta$ and $\mu = \delta\tau$ and $qu(t,x) \to u(t,x)$. Then (1.3) is re-scaled to the following equation corresponding to (2.1):

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = -\mu u(t,x) + \mu \displaystyle\int_{\mathbb{R}} \Gamma_\alpha(x-y)\beta u(t-1,y)e^{-u(t-1,y)}\,\mathrm{d}y, & (t,x) \in (0,\infty)\times\mathbb{R}, \\ u(\theta,x) = \phi(\theta,x) \quad \text{for } (\theta,x) \in [-1,0]\times\mathbb{R}. \end{cases} \tag{4.1}$$

Applying the results in Sections 2–3 to this model and making use of the proof of Theorem 4.1 and Remark 4.3 in [33], we can obtain the following threshold dynamics.

**Theorem 4.1.** *If $\beta \in (0, e^2]$, then the following statements are true*:

(i) *If $\beta \leqslant 1$, then the trivial equilibrium 0 is globally attractive in $C_+$ with respect to the usual supremum norm for* (4.1).
(ii) *If $\beta > 1$, then positive equilibrium $\ln\beta$ is a globally attractive in $C_+ \setminus \{0\}$ for* (4.1).

It is obvious that if $\Gamma_\alpha(x)$ is replaced by a general $k(x)$ satisfying (A), the conclusions of Theorem 4.1 remain true.

Next, we consider (2.1) with the nonlinear function $f$ being Mackey–Glass's hematopoiesis function $f:\mathbb{R}\to\mathbb{R}$ by $f(u) = \frac{pu}{1+u^n}$ for all $u \in R_+$. This function was initially used by Mackey and Glass in [17] to model the model of the blood cell production in an ordinary differential equation model, and then model has since been studied modified by many researchers. Among other topics for these models is the stability of a positive equilibrium, accounting for a long term stable blood concentration level. See, for example, Kuang [13] and Tang and Zou [27], and the references therein. However, to the authors' knowledge, no spatially nonlocal version has been discussed yet. Applying the results in Sections 2–3 to this model and taking advantage of the proof of Theorem 4.2 and Remark 4.3 in [33], we obtain the following theorem.

**Theorem 4.2.** *If $p > 0$ and $n > 0$, then the following statements are true*:

(i) *if $p \leqslant 1$, then the trivial equilibrium 0 is globally attractive in $C_+$ with respect to the usual supremum norm*;
(ii) *if either ($p > 1$ and $0 < n \leqslant 2$) or ($1 < p \leqslant \frac{n}{n-2}$ and $n > 2$), then the positive equilibrium $(p-1)^{\frac{1}{n}}$ is globally attractive in $C_+\backslash\{0\}$.*

## 5. Discussion

This section is devoted to some discussions. We always assume that, in addition to (H1)–(H2), (H3) holds and $f'(0) > 1$.

Theorem 3.2 confirms that under the above assumptions, (2.1) has a positive equilibrium $u^*$ which is globally attractive in $C_+ \setminus \{0\}$ with respect to the compact open topology. One naturally wonders if it is also globally attractive in $C_+ \setminus \{0\}$ with respect to the usual supremum norm. The answer to this question is no, and we explain why below by making a connection to existence of traveling wave front solutions to (2.1). Without loss of generality, let us consider the local version of (4.1), that is, the following equation resulted from letting $\alpha \to 0$ in (4.1):

$$\frac{\partial u}{\partial t}(t,x) = -\mu u(t,x) + \mu\beta u(t-1,x)e^{-u(t-1,x)}. \tag{5.1}$$

Now, considering the following delay differential equation,

$$u'(t) = -\mu u(t) + \mu\beta u(t-1)e^{-u(t-1)}, \tag{5.2}$$

we know that if $\beta \in (1, e^2]$, then by Theorem 1 in [21] and the global stability of $\ln\beta$ in $C([-1,0],\mathbb{R})\setminus\{0\}$, there is a full solution $\eta:\mathbb{R}\to\mathbb{R}_+$ of (5.2) such that $\eta(-\infty)=0$, $\eta(\infty)=\ln\beta$ and $u(t,x)=\eta(x+t)$ satisfies (5.1) for $(t,x)\in\mathbb{R}\times\mathbb{R}$. This implies that the positive equilibrium $u^*=\ln\beta$ cannot attract all positive solutions in respect to the supremum norm, because the positive solution $u(t,x)=\eta(x+t)$ cannot approach $u^*=\ln\beta$ in the supremum norm as $t\to\infty$ due to the fact that $\eta(-\infty)=0$.

However, the positive equilibrium $u^*$ can be attractive with respect to the supremum norm in a *subset* of $C_+\setminus\{0\}$. To see this, define

$$C_+^> = \{\varphi\in C_+\colon \text{there exists } \varepsilon_\varphi>0 \text{ such that } \varphi(0,x)>\varepsilon_\varphi \text{ for all } x\in\mathbb{R}\},$$

and let $\|\varphi\|_{sup}=\sup\{|\varphi(\theta,x)|\colon (\theta,x)\in[-1,0]\times\mathbb{R}\}$, for any $\varphi\in C$. If $\varphi\in C_+^>$, then we can easily show that $\lim_{t\to\infty}\|(u^\varphi)_t-u^*\|_{sup}=0$, and thus $u^*$ is attractive in $C_+^>$ with respect to the usual supremum norm $\|\cdot\|_{sup}$ for (2.2). Indeed, by Proposition 2.1(iii), we may assume that $\varphi\in C_+^>\cap Y$. It follows from (2.2), that there exists $\varepsilon_1>0$ such that $u^\varphi(t,x)>\varepsilon_1$ for all $(t,x)\in[0,1]\times\mathbb{R}$. By the assumption (H1) and $f'(0)>1$, there is $\varepsilon_2\in(0,\varepsilon_1]$ such that $f(u)\geqslant\varepsilon_2$ for all $u\in[\varepsilon_2,M+1]$. This together with (2.2) and the semigroup property of $\Phi$ implies that $u^\varphi(t,x)\geqslant e^{-\mu t}\varepsilon_2+\varepsilon_2(1-e^{-\mu t})=\varepsilon_2$ for all $(t,x)\in[1,2]\times\mathbb{R}$, and thus by applying the induction, we know that $u^\varphi(t,x)\geqslant\varepsilon_2$ for all $(t,x)\in[1,\infty)\times\mathbb{R}$. By appealing to the same arguments as in the proof of Theorem 3.2 with $u_-$ and $u_+$ replace by $\liminf_{t\to\infty}\|(u^\varphi)_t\|_{sup}$ and $\limsup_{t\to\infty}\|(u^\varphi)_t\|_{sup}$, respectively, we conclude that $\lim_{t\to\infty}\|(u^\varphi)_t-u^*\|_{sup}=0$.

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