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Global attractivity of positive periodic solution to periodic Lotka–Volterra competition systems with pure delay ☆

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Abstract

We consider a *periodic* Lotka–Volterra competition system without instantaneous negative feedbacks (i.e., pure-delay systems)

$$\dot{x}_i(t) = x_i(t)\left[r_i(t) - \sum_{j=1}^{n} a_{ij}(t)x_j\big(t-\tau_{ij}(t)\big)\right], \quad i = 1, 2, \ldots, n. \tag{$*$}$$

We establish some 3/2-type criteria for global attractivity of a positive periodic solution of the system, which generalize the well-known Wright's 3/2 criteria for the autonomous delay logistic equation, and thereby, address the open problem proposed by both Kuang [Y. Kuang, Global stability in delayed nonautonomous Lotka–Volterra type systems without saturated equilibria, Differential Integral Equations 9 (1996) 557–567] and Teng [Z. Teng, Nonautonomous Lotka–Volterra systems with delays, J. Differential Equations 179 (2002) 538–561].

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1. Introduction

One of the most celebrated models for population dynamics is the Lotka–Volterra system. Due to its theoretical and practical significance, the Lotka–Volterra system has been extensively and intensively studied (see, e.g., [1–55]). In order to reflect the seasonal fluctuations, it is reasonable to study the Lotka–Volterra system with *periodic coefficients*. A very basic and important ecological problem associated with the study of multi-species population interaction in a periodic environment is the existence and global attractivity of a positive periodic solution. Such a problem also arises in many other contexts.

In this paper, we investigate the following *periodic* n-species Lotka–Volterra competition system with delays

$$\dot{x}_i(t) = x_i(t)\left[r_i(t) - \sum_{j=1}^{n} a_{ij}(t)x_j\big(t - \tau_{ij}(t)\big)\right], \quad i = 1, 2, \dots, n, \tag{1.1}$$

where $r_i, a_{ij}, \tau_{ij} \in C(\mathbb{R},\ \mathbb{R}^+ = [0, \infty))$ are ω-periodic functions ($\omega > 0$) with

$$\bar{r}_i = \frac{1}{\omega}\int_0^{\omega} r_i(s)\,ds > 0, \quad \bar{a}_{ij} = \frac{1}{\omega}\int_0^{\omega} a_{ij}(s)\,ds \geqslant 0, \quad i, j = 1, 2, \dots, n. \tag{1.2}$$

The existence and attractivity of the positive periodic solutions of some special cases of Eq. (1.1) have been studied extensively. Many important results can be found in [1–19,22–32,34–55] and references cited therein. In those works the method of Liapunov functions [5,21], the theory of monotone semiflows generated by functional differential equations [39,40], the fixed point theory [9], and so on are extensively applied. Recently, the un-delayed version of (1.1), i.e., system (1.1) with $\tau_{ij}(t) \equiv 0$, $i, j = 1, 2, \dots, n$, was studied by Redheffer [34,35] and Tineo [51]. Under remarkably weak conditions (see [34, conditions (a)–(e)] and [51, condition (0.2)]), the boundedness, permanence, global attractivity, and existence of positive periodic solutions are obtained (see [34, Theorem 1], [51, Sections 3 and 4]). Recently, Teng [47] extended the main results in Redheffer [34] to the following delayed system

$$\dot{x}_i(t) = x_i(t)\left[r_i(t) - a_{ii}(t)x_i(t) - \sum_{j\neq i}^{n} a_{ij}(t)x_j\big(t - \tau_{ij}(t)\big)\right], \quad i = 1, 2, \dots, n, \tag{1.3}$$

which is a special form of system (1.1) with $\tau_{ii}(t) \equiv 0$, $i = 1, 2, \dots, n$. In [47], the author fully took advantage of the fact that there is no delay in the negative feedback terms $a_{ii}(t)x_i(t)$, $i = 1, 2, \dots, n$ (i.e., the system has instantaneous negative feedbacks). Therefore, the methods used in [47] would fail when applying to the pure-delay system (1.1), due to the lack of the instantaneous negative feedbacks.

On the *existence* of positive periodic solutions of system (1.1), Tang and Zou [46] recently obtained the following rather general results.

Theorem 1.1. [46] *Assume that*

(C1) *the linear system*

$$\sum_{j=1}^{n} \bar{a}_{ij} x_j = \bar{r}_i, \quad i = 1, 2, \ldots, n, \tag{1.4}$$

has a positive solution.

Then system (1.1) *has at least one positive ω-periodic solution.*

Theorem 1.2. [46] *Assume that*

$$a_{ij}(t) \equiv a_{ij}, \quad \tau_{ij}(t) \equiv \tau_{ij}, \quad i, j = 1, 2, \ldots, n. \tag{1.5}$$

Then Eq. (1.1) *has at least one positive ω-periodic solution if and only if* (C1) *holds* (*in this case* $\bar{a}_{ij} = a_{ij}$, $i, j = 1, 2, \ldots, n$).

Nevertheless, to the best of our knowledge, there is no results on the *global attractivity* of the positive periodic solutions of system (1.1) when $\tau_{ii}(t) \not\equiv 0$, $i = 1, 2, \ldots, n$ (i.e., pure-delay system). Based on this fact, both Kuang [27] and Teng [47] proposed the following important open problem.

Open problem. To study the global attractivity of the positive periodic solution of system (1.1) when $\tau_{ii}(t) \not\equiv 0$, $i = 1, 2, \ldots, n$.

When $n = 1$, (1.1) reduces to the following delayed *periodic* logistic equation

$$\dot{x}(t) = x(t)\big[r(t) - a(t)x\big(t - \tau(t)\big)\big]. \tag{1.6}$$

It was shown in Li [31] that Eq. (1.6) always has positive ω-periodic solution if $r, a, \tau \in C(\mathbb{R}, \mathbb{R}^+)$ are ω-periodic functions with $\int_0^\omega r(s)\,ds > 0$ and $\int_0^\omega a(s)\,ds > 0$ (this can also be obtained from Theorem 1.1 as a special case $n = 1$). In particular, if Eq. (1.6) has a trivial positive periodic solution x^* (i.e., positive equilibrium x^* which exists if $r(t)$ and $a(t)$ are positively proportional), then in view of the result in So and Yu [41], the positive equilibrium x^* is a global attractor for Eq. (1.6) provided that

$$\int_{g(t)}^{t} r(s)\,ds \leqslant \frac{3}{2}, \quad \text{for all large } t, \tag{1.7}$$

where $g(t) = \min\{s - \tau(s)\colon t \leqslant s < \infty\}$.

The above so-called 3/2-type condition (1.7) for the global attractivity of a positive equilibrium of Eq. (1.6) is the extension of the well-known Wright's 3/2 criteria [54] for the corresponding autonomous delay logistic equation. Recently, Tang and Zou [44,45] have extended such 3/2-type conditions to, respectively, *autonomous* and *non-autonomous pure delay* Lotka–Volterra systems.

The main purpose of this paper is to further extend the above so-called 3/2-type conditions to the *periodic pure-delay* system (1.1) for the global attractivity of a positive periodic solution, and thereby, address the above open problem to some extent.

For convenience, we give some conditions related with (C1) which will be used in the rest of the paper:

(C2) the linear system (1.4) has a *unique* positive solution $x^* = (x_1^*, x_2^*, \dots, x_n^*)$.
(C3) there exist positive constants x_i^*, $i = 1, 2, \dots, n$, such that

$$\sum_{j=1}^{n} a_{ij}(t)x_j^* \equiv r_i(t), \quad i = 1, 2, \dots, n. \tag{1.8}$$

The remainder of the paper is organized as follows. In Section 2, we give the main results. In Section 3, we establish some preliminary lemmas, which address the persistence and dissipativity of system (1.1), and therefore, which themselves are of some interest and importance. In Section 4, by combining these lemmas with the "sandwiching" technique and some subtle integration and inequality skills, we give the proofs of the main theorems.

Throughout this paper, we say a vector $x = (x_1, x_2, \dots, x_n)$ is positive if $x_i > 0$, $i = 1, 2, \dots, n$. Set

$$g_i(t) = \min\{s - \tau_{ii}(s):\ t \leqslant s < \infty\}, \quad i = 1, 2, \dots, n. \tag{1.9}$$

Clearly, $g_i(t)$ is nondecreasing and $t - \tau_{ii}^u \leqslant g_i(t) \leqslant t - \tau_{ii}(t)$ for $i = 1, 2, \dots, n$, where $\tau_{ii}^u = \max_{t\in[0,\omega]} \tau_{ii}(t)$, $i = 1, 2, \dots, n$.

2. Main results

If system (1.1) has a positive ω-periodic solution, we always denote it by $x^*(t) = (x_1^*(t), x_2^*(t), \dots, x_n^*(t))$. Since $x^*(t)$ is a positive ω-periodic solution of Eq. (1.1), we can choose $\xi_i^*, \eta_i^* \in [0, \omega]$ such that

$$a_i := x_i^*(\xi_i^*) = \min_{s\in[0,\omega]} x_i^*(s), \quad x_i^*(\eta_i^*) = \max_{s\in[0,\omega]} x_i^*(s), \quad i = 1, 2, \dots, n. \tag{2.1}$$

If $\xi_i^* \leqslant \eta_i^*$, then from (1.1),

$$\ln\left(\frac{x_i^*(\eta_i^*)}{x_i^*(\xi_i^*)}\right) = \int_{\xi_i^*}^{\eta_i^*} \frac{\dot{x}_i^*(s)}{x_i^*(s)}\,ds \leqslant \int_{\xi_i^*}^{\eta_i^*} r_i(s)\,ds \leqslant \int_{\xi_i^*}^{\xi_i^*+\omega} r_i(s)\,ds = \bar{r}_i\omega.$$

If $\xi_i^* > \eta_i^*$, then from (1.1),

$$\ln\left(\frac{x_i^*(\eta_i^*)}{x_i^*(\xi_i^*)}\right) = \ln\left(\frac{x_i^*(\eta_i^* + \omega)}{x_i^*(\xi_i^*)}\right) = \int_{\xi_i^*}^{\eta_i^*+\omega} \frac{\dot{x}_i^*(s)}{x_i^*(s)}\,ds \leqslant \int_{\xi_i^*}^{\eta_i^*+\omega} r_i(s)\,ds \leqslant \bar{r}_i\omega.$$

Combining the above two inequalities, we have

$$\frac{x_i^*(\eta_i^*)}{x_i^*(\xi_i^*)} \leqslant e^{\bar{r}_i\omega}, \quad i = 1, 2, \ldots, n.$$

It follows that

$$e^{-\bar{r}_i\omega} \leqslant \frac{x_i^*(t_1)}{x_i^*(t_2)} \leqslant e^{\bar{r}_i\omega}, \quad i = 1, 2, \ldots, n, \ \forall t_1, t_2 \in \mathbb{R}. \tag{2.2}$$

On the other hand, integrating (1.1) from 0 to ω, we get

$$\int_0^\omega r_i(t)\,dt = \int_0^\omega \sum_{j=1}^n a_{ij}(t) x_j^*\big(t - \tau_{ij}(t)\big)\,dt, \quad i = 1, 2, \ldots, n, \tag{2.3}$$

which, together with (2.1) and (2.2), yields

$$\sum_{j=1}^n \bar{a}_{ij} a_j \leqslant \bar{r}_i \leqslant \sum_{j=1}^n \bar{a}_{ij} a_j e^{\bar{r}_j\omega}, \quad i = 1, 2, \ldots, n. \tag{2.4}$$

Set

$$b_i(t) = r_i(t) - \sum_{j=1}^n a_{ij}(t) x_j^*\big(t - \tau_{ij}(t)\big) = \frac{\dot{x}_i^*(t)}{x_i^*(t)}, \quad i = 1, 2, \ldots, n, \tag{2.5}$$

and

$$y_i(t) = \big[x_i(t) - x_i^*(t)\big] \exp\left(-\int_{\xi_i^*}^t b_i(s)\,ds\right), \quad i = 1, 2, \ldots, n. \tag{2.6}$$

Then it follows from (2.2), (2.5) and (2.6) that

$$\big|y_i(t)\big| \leqslant \big|x_i(t) - x_i^*(t)\big| e^{\bar{r}_i\omega}, \quad i = 1, 2, \ldots, n, \tag{2.7}$$

$$\begin{aligned} a_i + y_i(t) &= x_i^*\big(\xi_i^*\big) + y_i(t) = x_i^*(t) \exp\left(-\int_{\xi_i^*}^t b_i(s)\,ds\right) + y_i(t) \\ &= x_i(t) \exp\left(-\int_{\xi_i^*}^t b_i(s)\,ds\right) > 0, \quad i = 1, 2, \ldots, n, \end{aligned} \tag{2.8}$$

and (1.1) is transformed into

$$\dot{y}_i(t) = -\big[a_i + y_i(t)\big] \sum_{j=1}^n \tilde{a}_{ij}(t) y_j\big(t - \tau_{ij}(t)\big), \quad i = 1, 2, \ldots, n, \tag{2.9}$$

where

$$\tilde{a}_{ij}(t) = a_{ij}(t)\exp\left(\int_{\xi_i^*}^{t-\tau_{ij}(t)} b_i(s)\,ds\right), \quad i, j = 1, 2, \ldots, n. \tag{2.10}$$

Note that

$$\tilde{a}_{ij}(t+\omega) = a_{ij}(t)\exp\left(\int_{\xi_i^*}^{t+\omega-\tau_{ij}(t)} b_i(s)\,ds\right) = \tilde{a}_{ij}(t), \quad i, j = 1, 2, \ldots, n.$$

Thus, $\tilde{a}_{ij}(t)$, $i, j = 1, 2, \ldots, n$, are still ω-periodic functions.

When (1.5) holds, it follows from (2.3) that

$$\bar{r}_i = \sum_{j=1}^{n} a_{ij}\left(\frac{1}{\omega}\int_0^{\omega} x_j^*(s)\,ds\right) = \sum_{j=1}^{n} a_{ij}\bar{x}_j^*, \quad i = 1, 2, \ldots, n. \tag{2.11}$$

Further, if (C2) also holds, then $\bar{x}_i^* = x_i^*$, $i = 1, 2, \ldots, n$, and

$$0 < a_i \leqslant x_i^*, \quad i = 1, 2, \ldots, n. \tag{2.12}$$

When condition (C3) holds, i.e., system (1.1) has a trivial positive periodic solution $x^* = (x_1^*, x_2^*, \ldots, x_n^*)$, which is also the positive equilibrium of system (1.1). In this case,

$$a_i = x_i^*, \quad b_i(t) \equiv 0, \quad i = 1, 2, \ldots, n. \tag{2.13}$$

For general case, it follows from (2.4) that

$$0 < a_i \leqslant \bar{r}_i/\bar{a}_{ii}, \quad i = 1, 2, \ldots, n. \tag{2.14}$$

For the sake of convenience, we give some "diagonal dominant" conditions:

(DD1) there exist positive constants ν_i, $i = 1, 2, \ldots, n$, such that

$$\nu_i\tilde{a}_{ii}(t) > \sum_{j\neq i}^{n} \nu_j\tilde{a}_{ij}(t), \quad t \in [0, \omega],\ i = 1, 2, \ldots, n;$$

(DD2) there exist positive constants ν_i, $i = 1, 2, \ldots, n$, such that

$$\nu_i a_{ii}(t) > \sum_{j\neq i}^{n} \nu_j a_{ij}(t), \quad t \in [0, \omega],\ i = 1, 2, \ldots, n;$$

(DD3) there exist positive constants ν_i, $i = 1, 2, \ldots, n$, such that

$$\nu_i a_{ii}(t) > e^{\bar{r}_i \omega} \sum_{j \neq i}^{n} \nu_j a_{ij}(t), \quad t \in [0, \omega],\ i = 1, 2, \ldots, n.$$

We are now in the position to state our main results.

Theorem 2.1. *In addition to* (C2)*, assume that* $\tau_{ii}(t) \equiv 0$, $\dot{\tau}_{ij}(t) < 1$, $i, j = 1, 2, \ldots, n$, *and that there exist* $\nu_i > 0$, $i = 1, 2, \ldots, n$, *such that*

$$\nu_i a_{ii}(t) > \sum_{j \neq i}^{n} \frac{\nu_j a_{ji}(\xi_{ji}^{-1}(t))}{1 - \dot{\tau}_{ji}(\xi_{ji}^{-1}(t))}, \quad i = 1, 2, \ldots, n, \tag{2.15}$$

where $\xi_{ij}^{-1}(t)$ *is the inverse function of* $\xi_{ij}(t) = t - \tau_{ij}(t)$, $i, j = 1, 2, \ldots, n$. *Then system* (1.1) *has a unique positive* ω*-periodic solution* $x^*(t) = (x_1^*(t), x_2^*(t), \ldots, x_n^*(t))$ *which attracts all positive solutions* $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))$ *of system* (1.1)*, that is,*

$$\lim_{t \to \infty} \left| x_i(t) - x_i^*(t) \right| = 0, \quad i = 1, 2, \ldots, n. \tag{2.16}$$

Remark 2.1. This theorem improves the corresponding results in Teng [46], since [46] requires some other extra conditions in addition to (C2) and (2.15).

Theorem 2.2. *In addition to* (C2)*, assume that* $\dot{\tau}_{ij}(t) < 1$, $i, j = 1, 2, \ldots, n$, *and that there exist* $\nu_i > 0$, $i = 1, 2, \ldots, n$, *such that*

$$\frac{\nu_i a_{ii}(\xi_{ii}^{-1}(t))}{1 - \dot{\tau}_{ii}(\xi_{ii}^{-1}(t))} \left(1 - \int_{t - \tau_{ii}(t)}^{t} \frac{h_i(\xi_{ii}^{-1}(s))}{1 - \dot{\tau}_{ii}(\xi_{ii}^{-1}(s))} \, ds \right) > \sum_{j \neq i}^{n} \frac{\nu_j a_{ji}(\xi_{ji}^{-1}(t))}{1 - \dot{\tau}_{ji}(\xi_{ji}^{-1}(t))},$$
$$i = 1, 2, \ldots, n, \tag{2.17}$$

where $\xi_{ij}^{-1}(t)$, $i = 1, 2, \ldots, n$, *are the same as in Theorem* 2.1 *and*

$$h_i(t) = \frac{2 a_{ii}(\xi_{ii}^{-1}(t)) \Delta_i}{1 - \dot{\tau}_{ii}(\xi_{ii}^{-1}(t))} \exp\left(\Delta_i \int_{t - \tau_{ii}(t)}^{t} \frac{a_{ii}(\xi_{ii}^{-1}(s))}{1 - \dot{\tau}_{ii}(\xi_{ii}^{-1}(s))} \, ds \right), \tag{2.18}$$

$$\Delta_i := \max_{t \in [0, \omega]} \left[\frac{r_i(t)}{a_{ii}(t)} \exp\left(\int_{t - \tau_{ii}(t)}^{t} r_i(s) \, ds \right) \right], \quad i = 1, 2, \ldots, n. \tag{2.19}$$

Then system (1.1) *has a unique positive* ω*-periodic solution* $x^*(t)$*, which attracts all positive solutions of system* (1.1).

Remark 2.2. When $\tau_{ii}(t) \equiv 0$, $i = 1, 2, \ldots, n$, Theorem 2.2 reproduces Theorem 2.1.

Theorem 2.3. *Assume that* (C2) *and* (DD1) *hold, and that*

$$\int_{g_i(t)}^{t} \nu_i \tilde{a}_{ii}(s)\,ds \leqslant \frac{3(1-\mu)}{2a(1+\mu_i)} + \frac{(1-\mu)(\mu+\mu_i)}{2a(1+\mu_i)^2}, \quad i = 1, 2, \ldots, n, \tag{2.20}$$

where

$$a := \max\{\nu_i^{-1} a_i \colon i = 1, 2, \ldots, n\}, \tag{2.21}$$

$$\mu_i := \max_{t \in [0,\omega]} \left\{ \frac{1}{\nu_i \tilde{a}_{ii}(t)} \sum_{j \neq i}^{n} \nu_j \tilde{a}_{ij}(t) \right\}, \quad i = 1, 2, \ldots, n, \quad \text{and} \tag{2.22}$$

$$\mu := \max\{\mu_i \colon i = 1, 2, \ldots, n\}. \tag{2.23}$$

Then system (1.1) *has a unique positive ω-periodic solution $x^*(t)$, which attracts all positive solutions of system* (1.1).

Theorem 2.4. *Assume that* (C2) *and* (DD1) *hold, and that*

$$\int_{g_i(t)}^{t} \tilde{a}_{ii}(s)\,ds \leqslant d_i, \quad i = 1, 2, \ldots, n, \tag{2.24}$$

and for $i = 1, 2, \ldots, n$,

$$(a_i + a\nu_i\mu_i)d_i \exp\left[(a_i + a\nu_i\mu_i)d_i + e^{-(a_i + a\nu_i\mu_i)d_i} - 1\right] < \begin{cases} \frac{3-\mu_i}{2(1+\mu_i)}, & \mu_i \leqslant \frac{1}{3}, \\ \sqrt{\frac{2(1-\mu_i)}{1+\mu_i}}, & \mu_i > \frac{1}{3}. \end{cases} \tag{2.25}$$

Then system (1.1) *has a unique positive ω-periodic solution $x^*(t)$, which attracts all positive solutions of system* (1.1).

When (C3) holds, we have $a_i = x_i^*$ and $b_i(t) \equiv 0$ for $i = 1, 2, \ldots, n$. Therefore, from Theorems 2.3 and 2.4, we have the following two corollaries.

Corollary 2.1. *Assume that* (C3) *and* (DD2) *hold, and that*

$$\int_{g_i(t)}^{t} \nu_i a_{ii}(s)\,ds \leqslant \frac{3(1-\mu^*)}{2a^*(1+\mu_i^*)} + \frac{(1-\mu^*)(\mu^*+\mu_i^*)}{2a^*(1+\mu_i^*)^2}, \quad i = 1, 2, \ldots, n, \tag{2.26}$$

where

$$a^* := \max\left\{\nu_i^{-1}x_i^*:\ i=1,2,\ldots,n\right\}, \tag{2.27}$$

$$\mu_i^* := \max_{t\in[0,\omega]}\left\{\frac{1}{\nu_i a_{ii}(t)}\sum_{j\neq i}^{n}\nu_j a_{ij}(t)\right\},\quad i=1,2,\ldots,n,\quad \text{and} \tag{2.28}$$

$$\mu^* := \max\left\{\mu_i^*:\ i=1,2,\ldots,n\right\}. \tag{2.29}$$

Then the positive equilibrium x^* of system (1.1) *is a global attractor, i.e., for every positive solutions $x(t)=(x_1(t),x_2(t),\ldots,x_n(t))$ of system* (1.1),

$$\lim_{t\to\infty} x_i(t) = x_i^*,\quad i=1,2,\ldots,n. \tag{2.30}$$

Corollary 2.2. *Assume that* (C3) *and* (DD2) *hold, and that*

$$\int_{g_i(t)}^{t} a_{ii}(s)\,ds \leqslant d_i^*,\quad i=1,2,\ldots,n, \tag{2.31}$$

and for $i=1,2,\ldots,n$,

$$\left(x_i^*+a^*\nu_i\mu_i^*\right)d_i^*\exp\left[\left(x_i^*+a^*\nu_i\mu_i^*\right)d_i^*+e^{-(x_i^*+a^*\nu_i\mu_i^*)d_i^*}-1\right] < \begin{cases}\frac{3-\mu_i^*}{2(1+\mu_i^*)}, & \mu_i^*\leqslant\frac{1}{3},\\ \sqrt{\frac{2(1-\mu_i^*)}{1+\mu_i^*}}, & \mu_i^*>\frac{1}{3},\end{cases} \tag{2.32}$$

where a^*, μ_i^*, $i=1,2,\ldots,n$, are the same as in Corollary 2.1. *Then the positive equilibrium x^* of system* (1.1) *is a global attractor.*

Note that

$$\int_t^{t+\omega} b_i(s)\,ds = \int_0^{\omega} b_i(s)\,ds = 0,\quad i=1,2,\ldots,n,\quad \text{and}$$

$$\tilde{a}_{ij}(t)\leqslant e^{\bar{r}_i\omega}a_{ij}(t),\quad i,j=1,2,\ldots,n.$$

Therefore from (2.10), Theorems 2.3 and 2.4, we have the following two corollaries.

Corollary 2.3. *Assume that there exist integers k_{ij}, $i,j=1,2,\ldots,n$, such that*

$$\tau_{ij}(t)=\tau_{ii}(t)+k_{ij}\omega,\quad i,j=1,2,\ldots,n. \tag{2.33}$$

Further suppose that (C2) *and* (DD2) *hold, and that*

$$e^{\bar{r}_i\omega}\int_{g_i(t)}^{t}\nu_i a_{ii}(s)\,ds\leqslant\frac{3(1-\mu^*)}{2\hat{a}(1+\mu_i^*)}+\frac{(1-\mu^*)(\mu^*+\mu_i^*)}{2\hat{a}(1+\mu_i^*)^2},\quad i=1,2,\ldots,n,\tag{2.34}$$

where μ^*, μ_i^*, $i=1,2,\ldots,n$, *are the same as in Corollary* 2.1 *and*

$$\hat{a}=\max\left\{\frac{\bar{r}_i}{\nu_i\bar{a}_{ii}}\colon\, i=1,2,\ldots,n\right\}.\tag{2.35}$$

Then system (1.1) *has a unique positive ω-periodic solution* $x^*(t)$, *which attracts all positive solutions of system* (1.1).

Corollary 2.4. *Assume that* (2.33), (C2) *and* (DD2) *hold, and that*

$$e^{\bar{r}_i\omega}\int_{g_i(t)}^{t}a_{ii}(s)\,ds\leqslant\hat{d}_i,\quad i=1,2,\ldots,n,\tag{2.36}$$

and for $i=1,2,\ldots,n$,

$$\begin{aligned}&\big(\bar{r}_i/\bar{a}_{ii}+\hat{a}\nu_i\mu_i^*\big)\hat{d}_i\exp\big[\big(\bar{r}_i/\bar{a}_{ii}+\hat{a}\nu_i\mu_i^*\big)\hat{d}_i+e^{-(\bar{r}_i/\bar{a}_{ii}+\hat{a}\nu_i\mu_i^*)\hat{d}_i}-1\big]\\&\quad<\begin{cases}\frac{3-\mu_i^*}{2(1+\mu_i^*)}, & \mu_i^*\leqslant\frac{1}{3},\\ \sqrt{\frac{2(1-\mu_i^*)}{1+\mu_i^*}}, & \mu_i^*>\frac{1}{3},\end{cases}\end{aligned}\tag{2.37}$$

where $\hat{a}$, μ_i^*, $i=1,2,\ldots,n$, *are the same as in Corollary* 2.3. *Then system* (1.1) *has a unique positive ω-periodic solution* $x^*(t)$, *which attracts all positive solutions of system* (1.1).

For general case, we have the following corollaries.

Corollary 2.5. *Assume that* (C2) *and* (DD3) *hold, and that*

$$e^{\bar{r}_i\omega}\int_{g_i(t)}^{t}\nu_i a_{ii}(s)\,ds\leqslant\frac{3(1-\hat{\mu})}{2\hat{a}(1+\hat{\mu}_i)}+\frac{(1-\hat{\mu})(\hat{\mu}+\hat{\mu}_i)}{2\hat{a}(1+\hat{\mu}_i)^2},\quad i=1,2,\ldots,n,\tag{2.38}$$

where $\hat{a}$ *defined as in* (2.35),

$$\hat{\mu}_i:=e^{\bar{r}_i\omega}\max_{t\in[0,\omega]}\left\{\frac{1}{\nu_i a_{ii}(t)}\sum_{j\neq i}^{n}\nu_j a_{ij}(t)\right\},\quad i=1,2,\ldots,n,\tag{2.39}$$

and

$$\hat{\mu} := \max\{\hat{\mu}_i\colon\ i = 1, 2, \dots, n\}. \tag{2.40}$$

Then system (1.1) *has a unique positive ω-periodic solution $x^*(t)$, which attracts all positive solutions of system* (1.1).

Corollary 2.6. *Assume that* (2.36), (C2) *and* (DD3) *hold, and that for $i = 1, 2, \dots, n$,*

$$\begin{aligned}&(\bar{r}_i/\bar{a}_{ii} + \hat{a}\nu_i\hat{\mu}_i)\hat{d}_i \exp\bigl[(\bar{r}_i/\bar{a}_{ii} + \hat{a}\nu_i\hat{\mu}_i)\hat{d}_i + e^{-(\bar{r}_i/\bar{a}_{ii}+\hat{a}\nu_i\hat{\mu}_i)\hat{d}_i} - 1\bigr]\\ &\quad < \begin{cases}\frac{3-\hat{\mu}_i}{2(1+\hat{\mu}_i)}, & \hat{\mu}_i \leqslant \frac{1}{3},\\ \sqrt{\frac{2(1-\hat{\mu}_i)}{1+\hat{\mu}_i}}, & \hat{\mu}_i > \frac{1}{3},\end{cases}\end{aligned} \tag{2.41}$$

where $\hat{a}$, $\hat{\mu}_i$, $\hat{d}_i$, $i = 1, 2, \dots, n$, are the same as in Corollary 2.5. *Then system* (1.1) *has a unique positive ω-periodic solution $x^*(t)$, which attracts all positive solutions of system* (1.1).

Remark 2.3. It is easily seen that Corollary 2.1 will reproduce the 3/2-type condition (1.7) for the nonautonomous delayed logistic equation (1.6).

Furthermore, applying Corollary 2.5 to Eq. (1.6) directly, we have

Corollary 2.7. *Assume that $\bar{r} > 0$, $a(t) > 0$, and that*

$$\int_{g(t)}^{t} r(s)\,ds \leqslant \frac{3\bar{a}}{2\bar{r}} e^{-\bar{r}\omega}, \tag{2.42}$$

where $g(t) = \min\{s - \tau(s)\colon\ t \leqslant s < \infty\}$. Then Eq. (1.6) *has a unique positive ω-periodic solution $x^*(t)$, which attracts all positive solutions of Eq.* (1.6).

3. Preliminary lemmas

In this section, we give some preliminary lemmas, which are useful in the proofs of the main theorems. They themselves are of some interests and importance. The first lemma is directly from [45]

Lemma 3.1. [45] *Let $a > 0$, $0 < \mu < 1$. Then system of inequalities*

$$\begin{cases} y \leqslant (a + \mu x)\exp\bigl[\frac{1-\mu}{a}x - \frac{(1-\mu)^2(1+2\mu)}{6a^2(1+\mu)}x^2\bigr] - a,\\ x \leqslant a - (a - \mu y)\exp\bigl[-\frac{1-\mu}{a}y - \frac{(1-\mu)^2(1+2\mu)}{6a^2(1+\mu)}y^2\bigr]\end{cases} \tag{3.1}$$

has a unique solution $(x, y) = (0, 0)$ in the region $D = \{(x, y)\colon\ 0 \leqslant x < a,\ 0 \leqslant y < a/\mu\}$.

Remark 3.1. Comparing [45, Lemma 3.1] and Lemma 3.1, the restriction $0 < a < 1$ is replaced by $a > 0$. In fact, the restriction $a < 1$ is not necessary.

The next three lemmas establish the persistence and dissipativity of system (1.1) or (2.9).

Lemma 3.2. *Assume that* $a_{ii}(t) > 0$, $i = 1, 2, \ldots, n$. *Let* $(x_1(t), x_2(t), \ldots, x_n(t))$ *be the positive solution of* (1.1). *Then we have eventually*

$$0 < x_i(t) \leqslant \Delta_i, \quad i = 1, 2, \ldots, n, \tag{3.2}$$

where Δ_i *is defined by* (2.19).

Proof. Set

$$\delta_i = \max_{t\in[0,\omega]} \left(\frac{r_i(t)}{a_{ii}(t)}\right), \quad i = 1, 2, \ldots, n.$$

From (1.1), we have

$$\dot{x}_i(t) \leqslant x_i(t)\big[r_i(t) - a_{ii}(t)x_i\big(t - \tau_{ii}(t)\big)\big], \quad i = 1, 2, \ldots, n. \tag{3.3}$$

If $x_i(t) \leqslant \delta_i$ eventually, then (3.2) holds naturally for large t. If $x_i(t) > \delta_i$ eventually, then (3.3) implies that $x_i(t)$ is decreasing eventually. Let $\lim_{t\to\infty} x_i(t) = c_i$. Then $x_i(t) > c_i \geqslant \delta_i$ for large t and from (3.3), we have

$$\dot{x}_i(t) \leqslant x_i(t)\big[r_i(t) - c_i a_{ii}(t)\big] \leqslant x_i(t)a_{ii}(t)(\delta_i - c_i), \quad i = 1, 2, \ldots, n,$$

for large t. Integrating the above from t to ∞, we obtain

$$\ln c_i - \ln x_i(t) \leqslant (\delta_i - c_i)\int_t^\infty a_{ii}(s)\,ds, \quad i = 1, 2, \ldots, n,$$

which, together with the fact that $\int_t^\infty a_{ii}(s)\,ds = \infty$, implies that $c_i = \delta_i$, $i = 1, 2, \ldots, n$. Hence, (3.2) holds also. In the sequel, we only consider the case when $x_i(t) - \delta_i$ is oscillatory. For this case, let t_i^* be an arbitrary local left maximum point of $x_i(t)$ such that $x_i(t_i^*) > \delta_i$. Then $\dot{x}_i(t_i^*) \geqslant 0$, it follows from (3.3) that there exists $x_i(t_i^* - \tau_{ii}(t_i^*)) \leqslant r_i(t_i^*)/a_{ii}(t_i^*)$. Integrating (3.3) from $t_i^* - \tau_{ii}(t_i^*)$ to t_i^* and using (2.4), we have

$$\begin{aligned} x_i\big(t_i^*\big) &\leqslant x_i\big(t_i^* - \tau_{ii}\big(t_i^*\big)\big)\exp\left(\int_{t_i^*-\tau_{ii}(t_i^*)}^{t_i^*} r_i(s)\,ds\right) \leqslant \frac{r_i(t_i^*)}{a_{ii}(t_i^*)}\exp\left(\int_{t_i^*-\tau_{ii}(t_i^*)}^{t_i^*} r_i(s)\,ds\right) \\ &\leqslant \Delta_i, \quad i = 1, 2, \ldots, n. \end{aligned}$$

It follows that (3.2) holds eventually. The proof is complete. □

Lemma 3.3. *Assume that* (2.24) *and* (DD1) *hold. Let* $(x_1(t), x_2(t), \dots, x_n(t))$ *be the positive solution of Eq.* (1.1) *and let* $y_i(t)$, $i = 1, 2, \dots, n$, *be defined by* (2.6). *Then we have eventually*

$$a_i + y_i(t) \leqslant (a_i + a\nu_i\mu_i)\exp\left[(a_i + a\nu_i\mu_i)d_i + e^{-(a_i+a\nu_i\mu_i)d_i} - 1\right], \quad i = 1, 2, \dots, n. \tag{3.4}$$

Proof. It follows from (2.8) that $y_i(t) > -a_i$, $t \geqslant 0$, $i = 1, 2, \dots, n$. Hence, by (2.9),

$$\begin{aligned}
\dot{y}_i(t) &\leqslant \left[a_i + y_i(t)\right]\left[-\tilde{a}_{ii}(t)y_i\left(t - \tau_{ii}(t)\right) + \sum_{j=1}^{n} \tilde{a}_{ij}(t)a_j\right] \\
&\leqslant \tilde{a}_{ii}(t)\left[a_i + y_i(t)\right]\left[-y_i\left(t - \tau_{ii}(t)\right) + \frac{a}{\tilde{a}_{ii}(t)}\sum_{j=1}^{n} \nu_j\tilde{a}_{ij}(t)\right] \\
&\leqslant \tilde{a}_{ii}(t)\left[a_i + y_i(t)\right]\left[-y_i\left(t - \tau_{ii}(t)\right) + a\nu_i\mu_i\right] \\
&\leqslant (a_i + a\nu_i\mu_i)\tilde{a}_{ii}(t)\left[a_i + y_i(t)\right], \quad i = 1, 2, \dots, n.
\end{aligned} \tag{3.5}$$

If $y_i(t) \leqslant a\nu_i\mu_i$ eventually, then (3.4) holds naturally for large t. Otherwise, let t_i^* be an arbitrary local left maximum point of $y_i(t)$ such that $y_i(t_i^*) > a\nu_i\mu_i$. Then $\dot{y}_i(t_i^*) \geqslant 0$, it follows from the third inequality in (3.5) that there exists $\xi_i \in [t_i^* - \tau_{ii}(t_i^*), t_i^*] \subseteq [g_i(t_i^*), t_i^*]$ such that $y_i(\xi_i) = a\nu_i\mu_i$ and $a\nu_i\mu_i < y_i(t) \leqslant y_i(t_i^*)$ for $\xi_i < t \leqslant t_i^*$. For $t \in [\xi_i, t_i^*]$ and $t - \tau_{ii}(t) \leqslant \xi_i$, integrating (3.5) from $t - \tau_{ii}(t)$ to ξ_i, we get

$$\begin{aligned}
-\ln\left(\frac{a_i + y_i(t - \tau_{ii}(t))}{a_i + a\nu_i\mu_i}\right) &\leqslant (a_i + a\nu_i\mu_i)\int_{t-\tau_{ii}(t)}^{\xi_i} \tilde{a}_{ii}(u)\,du \\
&\leqslant (a_i + a\nu_i\mu_i)\int_{g_i(t)}^{\xi_i} \tilde{a}_{ii}(u)\,du, \quad \xi_i \leqslant t \leqslant t_i^*.
\end{aligned}$$

It follows that

$$a_i + y_i\left(t - \tau_{ii}(t)\right) \geqslant (a_i + a\nu_i\mu_i)\exp\left(-(a_i + a\nu_i\mu_i)\int_{g_i(t)}^{\xi_i} \tilde{a}_{ii}(s)\,ds\right), \quad \xi_i \leqslant t \leqslant t_i^*. \tag{3.6}$$

For $t \in [\xi_i, t_i^*]$ and $t - \tau_{ii}(t) > \xi_i$, we have $y_i(t) > a\nu_i\mu_i$, in this case, (3.6) obviously holds. Substituting (3.6) into (3.5), we obtain

$$\frac{\dot{y}_i(t)}{a_i + y_i(t)} \leqslant (a_i + a\nu_i\mu_i)\tilde{a}_{ii}(t)\left[1 - \exp\left(-(a_i + a\nu_i\mu_i)\int_{g_i(t)}^{\xi_i} \tilde{a}_{ii}(s)\,ds\right)\right], \quad \xi_i \leqslant t \leqslant t_i^*. \tag{3.7}$$

Integrating (3.7) from ξ_i to t_i^* and using (2.1), we have

$$\begin{aligned}
&\ln\left(\frac{a_i + y_i(t_i^*)}{a_i + a\nu_i\mu_i}\right) \\
&\quad\leqslant (a_i + a\nu_i\mu_i)\int_{\xi_i}^{t_i^*} \tilde{a}_{ii}(t)\left[1 - \exp\left(-(a_i + a\nu_i\mu_i)\int_{g_i(t)}^{\xi_i} \tilde{a}_{ii}(s)\,ds\right)\right]dt \\
&\quad\leqslant (a_i + a\nu_i\mu_i)\left\{\int_{\xi_i}^{t_i^*} \tilde{a}_{ii}(s)\,ds - \int_{\xi_i}^{t_i^*} \tilde{a}_{ii}(t)\exp\left[(a_i + a\nu_i\mu_i)\left(-d_i + \int_{\xi_i}^{t} \tilde{a}_{ii}(s)\,ds\right)\right]dt\right\} \\
&\quad= (a_i + a\nu_i\mu_i)\int_{\xi_i}^{t_i^*} \tilde{a}_{ii}(s)\,ds + e^{-(a_i+a\nu_i\mu_i)d_i} - \exp\left[(a_i + a\nu_i\mu_i)\left(-d_i + \int_{\xi_i}^{t_i^*} \tilde{a}_{ii}(s)\,ds\right)\right] \\
&\quad\leqslant (a_i + a\nu_i\mu_i)d_i + e^{-(a_i+a\nu_i\mu_i)d_i} - 1,
\end{aligned}$$

which implies that

$$a_i + y_i\big(t_i^*\big) \leqslant (a_i + a\nu_i\mu_i)\exp\big[(a_i + a\nu_i\mu_i)d_i + e^{-(a_i+a\nu_i\mu_i)d_i} - 1\big], \quad i = 1, 2, \ldots, n.$$

It follows that for large t

$$a_i + y_i(t) \leqslant (a_i + a\nu_i\mu_i)\exp\big[(a_i + a\nu_i\mu_i)d_i + e^{-(a_i+a\nu_i\mu_i)d_i} - 1\big), \quad i = 1, 2, \ldots, n.$$

The proof is complete. □

Lemma 3.4. *Assume that* (2.20) *and* (DD1) *hold. Let* $(x_1(t), x_2(t), \ldots, x_n(t))$ *be the positive solution of Eq.* (1.1) *and let* $y_i(t)$, $i = 1, 2, \ldots, n$, *be defined by* (2.6)*. Then*

$$-\nu_i a < \liminf_{t\to\infty} y_i(t) \leqslant \limsup_{t\to\infty} y_i(t) < \infty, \quad i = 1, 2, \ldots, n. \tag{3.8}$$

Proof. By (2.20) and Lemma 3.3 with $\nu_i d_i = \frac{(1-\mu)(3+\mu+4\mu_i)}{2a(1+\mu_i)^2}$, $i = 1, 2, \ldots, n$, there exists $T > 0$ such that

$$\begin{aligned}
y_i(t) &\leqslant -a_i + (a_i + a\nu_i\mu_i)\exp\big[(a_i + a\nu_i\mu_i)d_i + e^{-(a_i+a\nu_i\mu_i)d_i} - 1\big] \\
&\leqslant -a_i + (a_i + a\nu_i\mu_i)\exp\left[\frac{(a_i + a\nu_i\mu_i)^2 d_i^2}{2}\right] \\
&\leqslant -\nu_i a + \nu_i a(1 + \mu_i)\exp\left[\frac{a^2(1 + \mu_i)^2(\nu_i d_i)^2}{2}\right] \\
&= -\nu_i a + \nu_i a(1 + \mu_i)\exp\left[\frac{(1 - \mu)^2(3 + \mu + 4\mu_i)^2}{8(1 + \mu_i)^2}\right] \\
&\leqslant -\nu_i a + \nu_i a(1 + \mu)\exp\left[\frac{(1 - \mu)^2(3 + 5\mu)^2}{8(1 + \mu)^2}\right], \quad i = 1, 2, \ldots, n,\ t \geqslant T - \tau_M.
\end{aligned} \tag{3.9}$$

It follows that

$$\begin{aligned}\sum_{j\neq i}^{n}\tilde{a}_{ij}(t)y_j\big(t-\tau_{ij}(t)\big) &\leqslant a\left\{(1+\mu)\exp\left[\frac{(1-\mu)^2(3+5\mu)^2}{8(1+\mu)^2}\right]-1\right\}\sum_{j\neq i}^{n}\nu_j\tilde{a}_{ij}(t)\\ &\leqslant a\left\{(1+\mu)\exp\left[\frac{(1-\mu)^2(3+5\mu)^2}{8(1+\mu)^2}\right]-1\right\}\mu_i\nu_i\tilde{a}_{ii}(t)\\ &\leqslant a\mu\left\{(1+\mu)\exp\left[\frac{(1-\mu)^2(3+5\mu)^2}{8(1+\mu)^2}\right]-1\right\}\nu_i\tilde{a}_{ii}(t)\\ &\leqslant a\mu\big[(1+\mu)e^{2(1-\mu)^2}-1\big]\nu_i\tilde{a}_{ii}(t)\\ &=\theta a\nu_i\tilde{a}_{ii}(t),\quad i=1,2,\ldots,n,\ t\geqslant T,\end{aligned}\tag{3.10}$$

where

$$\theta:=\mu\big[(1+\mu)e^{2(1-\mu)^2}-1\big]<1.$$

Substituting (3.10) into (2.9), we have

$$\dot{y}_i(t)\geqslant\big[a_i+y_i(t)\big]\tilde{a}_{ii}(t)\big[-y_i\big(t-\tau_i(t)\big)-\theta a\nu_i\big],\quad i=1,2,\ldots,n,\ t\geqslant T.\tag{3.11}$$

If $\dot{y}_i(t)>0$ eventually, then $a_i+y_i(t)$ is increasing eventually. It follows from the fact that $a_i+y_i(t)>0$ and $a=\max\{\nu_i^{-1}a_i\colon i=1,2,\ldots,n\}$ that (3.8) holds. If $\dot{y}_i(t)<0$ eventually or $\dot{y}_i(t)$ is oscillatory, then there exists a sequence $\{t_k\}$ such that

$$T<t_1<t_2<\cdots,\quad t_k\to\infty,\qquad \dot{y}_i(t_k)\leqslant 0,\qquad \liminf_{t\to\infty}y_i(t)=\liminf_{k\to\infty}y_i(t_k).$$

It follows from (3.11) that $y_i(t_k-\tau_i(t_k))\geqslant-\theta a\nu_i$. Set

$$M_i=-\nu_i a+\nu_i a(1+\mu)\exp\left[\frac{(1-\mu)^2(3+5\mu)^2}{8(1+\mu)^2}\right],\quad i=1,2,\ldots,n.$$

Then from (3.11), we have

$$\begin{aligned}\dot{y}_i(t)&\geqslant-\big[a_i+y_i(t)\big]\tilde{a}_{ii}(t)(M_i+\theta a\nu_i)\\ &\geqslant-\big[\nu_i a+y_i(t)\big]\tilde{a}_{ii}(t)(M_i+\theta a\nu_i),\quad i=1,2,\ldots,n,\ t\geqslant T.\end{aligned}$$

Integrating the above from $t_k-\tau_i(t_k)$ to t_k, we obtain

$$\begin{aligned}\ln\left(\frac{\nu_i a+y_i(t_k)}{\nu_i a+y_i(t_k-\tau_{ii}(t_k))}\right)&\geqslant-\int_{t_k-\tau_{ii}(t_k)}^{t_k}\tilde{a}_{ii}(t)(M_i+\theta a\nu_i)\,dt\\ &\geqslant-(M_i+\theta a\nu_i)d_i,\quad i=1,2,\ldots,n,\end{aligned}$$

which yields

$$\begin{aligned}\nu_i a + y_i(t_k) &\geqslant \left[\nu_i a + y_i\big(t_k - \tau_{ii}(t_k)\big)\right]e^{-(M_i+\theta a \nu_i)d_i}\\ &\geqslant (1-\theta)\nu_i a e^{-(M_i+\theta a\nu_i)d_i}, \quad i = 1, 2, \ldots, n.\end{aligned}$$

This shows that (3.8) holds. The proof is complete. □

4. The proofs of main results

Proof of Theorem 2.1. Set $\xi_{ij}(s) = t$. Then

$$\xi_{ij}(s+\omega) = s + \omega - \tau_{ij}(s+\omega) = \xi_{ij}(s) + \omega = t + \omega,$$

and so

$$\xi_{ij}^{-1}(t+\omega) = s + \omega = \xi_{ij}^{-1}(t) + \omega.$$

Thus, $a_{ij}(\xi_{ij}^{-1}(t))$ and $\dot{\tau}_{ij}(\xi_{ji}^{-1}(t))$ are still ω-periodic functions for $i, j = 1, 2, \ldots, n$. Set

$$\theta_i = \max_{t\in[0,\omega]}\left\{\frac{1}{\nu_i a_{ii}(t)}\sum_{j\neq i}^{n}\frac{\nu_j a_{ji}(\xi_{ji}^{-1}(t))}{1-\dot{\tau}_{ji}(\xi_{ji}^{-1}(t))}\right\}, \quad i = 1, 2, \ldots, n.$$

Then it follows from (2.15) that $0 \leqslant \theta_i < 1$, $i = 1, 2, \ldots, n$, and

$$\theta_i \nu_i a_{ii}(t) \geqslant \sum_{j\neq i}^{n}\frac{\nu_j a_{ji}(\xi_{ji}^{-1}(t))}{1-\dot{\tau}_{ji}(\xi_{ji}^{-1}(t))}, \quad i = 1, 2, \ldots, n. \tag{4.1}$$

Let $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))$ be any positive solution of Eq. (1.1). Set

$$V(t) = \sum_{i=1}^{n}\nu_i\left[\left|\ln\left(\frac{x_i(t)}{x_i^*(t)}\right)\right| + \sum_{j\neq i}^{n}\int\limits_{t-\tau_{ij}(t)}^{t}\frac{a_{ij}(\xi_{ij}^{-1}(s))}{1-\dot{\tau}_{ij}(\xi_{ij}^{-1}(s))}\left|x_j(s) - x_j^*(s)\right| ds\right], \quad t \geqslant 0. \tag{4.2}$$

A direct calculation of the upper right-hand derivative $D^+V(t)$ of $V(t)$ leads to

$$\begin{aligned}D^+V(t) &\leqslant \sum_{i=1}^{n}\nu_i\left[-a_{ii}(t)\left|x_i(t) - x_i^*(t)\right| + \sum_{j\neq i}^{n}\frac{a_{ij}(\xi_{ij}^{-1}(t))}{1-\dot{\tau}_{ij}(\xi_{ij}^{-1}(t))}\left|x_j(t) - x_j^*(t)\right|\right]\\ &= \sum_{i=1}^{n}\left[-\nu_i a_{ii}(t) + \sum_{j\neq i}^{n}\frac{\nu_j a_{ji}(\xi_{ji}^{-1}(t))}{1-\dot{\tau}_{ji}(\xi_{ji}^{-1}(t))}\right]\left|x_i(t) - x_i^*(t)\right|\\ &\leqslant -\sum_{i=1}^{n}\nu_i(1-\theta_i)a_{ii}(t)\left|x_i(t) - x_i^*(t)\right|, \quad t \geqslant 0.\end{aligned}$$

The above shows that $V(t)$ is decreasing in $[0,\infty)$ and so the limit $v=\lim_{t\to\infty}V(t)$ exists. Furthermore, from the above, we have

$$\sum_{i=1}^{n}\nu_i(1-\theta_i)\int_0^\infty a_{ii}(s)\left|x_i(s)-x_i^*(s)\right|ds\leqslant V(0)<\infty,$$

and so

$$\sum_{i=1}^{n}\nu_i\int_0^\infty a_{ii}(s)\left|x_i(s)-x_i^*(s)\right|ds<\infty. \tag{4.3}$$

Note that

$$\int_0^\infty a_{ii}(s)\,ds=\infty,\quad i=1,2,\dots,n.$$

It follows from (4.3) that

$$\liminf_{t\to\infty}\sum_{i=1}^{n}\nu_i\left|x_i(t)-x_i^*(t)\right|=0. \tag{4.4}$$

Again from (4.1) and (4.3), we obtain

$$\begin{aligned}
&\sum_{j\neq i}^{n}\int_{t-\tau_{ij}(t)}^{t}\frac{a_{ij}(\xi_{ij}^{-1}(s))}{1-\dot\tau_{ij}(\xi_{ij}^{-1}(s))}\left|x_j(s)-x_j^*(s)\right|ds\\
&\quad\leqslant\int_{t-\tau_M}^{t}\sum_{j\neq i}^{n}\frac{a_{ij}(\xi_{ij}^{-1}(s))}{1-\dot\tau_{ij}(\xi_{ij}^{-1}(s))}\left|x_j(s)-x_j^*(s)\right|ds\\
&\quad\leqslant\frac{1}{\nu}\int_{t-\tau_M}^{t}\sum_{j\neq i}^{n}\nu_j a_{jj}(s)\left|x_j(s)-x_j^*(s)\right|ds\to0,\quad t\to\infty,\ i=1,2,\dots,n,
\end{aligned}\tag{4.5}$$

where $\tau_M=\max\{\tau_{ij}(t)\colon\ t\in[0,\omega],\ i,j=1,2,\dots,n\}$ and $\nu=\min\{\nu_i\colon\ i=1,2,\dots,n\}$. Combining (4.2) and (4.5), we have

$$\lim_{t\to\infty}\sum_{i=1}^{n}\nu_i\left|\ln\left(\frac{x_i(t)}{x_i^*(t)}\right)\right|=v, \tag{4.6}$$

which yields

$$x_i(t)\geqslant e^{-(v+1)/\nu}x_i^*(t),\quad \text{for large } t,\ i=1,2,\dots,n. \tag{4.7}$$

By (4.4), (4.6) and (4.7),

$$\begin{aligned} v &= \liminf_{t\to\infty}\sum_{i=1}^n \nu_i\left|\ln\left(\frac{x_i(t)}{x_i^*(t)}\right)\right| \leqslant \liminf_{t\to\infty}\sum_{i=1}^n \frac{\nu_i}{\min\{x_i(t), x_i^*(t)\}}\left|x_i(t)-x_i^*(t)\right| \\ &\leqslant e^{(v+1)/\nu}\liminf_{t\to\infty}\sum_{i=1}^n \frac{\nu_i}{x_i^*(t)}\left|x_i(t)-x_i^*(t)\right| \leqslant \frac{e^{(v+1)/\nu}}{m}\liminf_{t\to\infty}\sum_{i=1}^n \nu_i\left|x_i(t)-x_i^*(t)\right| \\ &= 0, \end{aligned} \tag{4.8}$$

where $m=\min\{x_i^*(t)\colon\ t\in[0,\omega],\ i=1,2,\dots,n\}$. Hence, it follows from (4.6) that

$$\lim_{t\to\infty}\left|x_i(t)-x_i^*(t)\right| = 0, \quad i=1,2,\dots,n. \tag{4.9}$$

The proof is complete. □

Proof of Theorem 2.2. Set

$$\begin{aligned} R_i(t) &= \frac{\nu_i a_{ii}(\xi_{ii}^{-1}(t))}{1-\dot\tau_{ii}(\xi_{ii}^{-1}(t))}\left(1-\int_{t-\tau_{ii}(t)}^{t}\frac{h_i(\xi_{ii}^{-1}(s))}{1-\dot\tau_{ii}(\xi_{ii}^{-1}(s))}\,ds\right) - \sum_{j\neq i}^{n}\frac{\nu_j a_{ji}(\xi_{ji}^{-1}(t))}{1-\dot\tau_{ji}(\xi_{ji}^{-1}(t))}, \\ & i=1,2,\dots,n. \end{aligned} \tag{4.10}$$

Then, in view of the proof of Theorem 2.1, the functions $R_i(t)$, $i=1,2,\dots,n$, are ω-periodic functions. Furthermore, it follows from (2.17) that there exists a constant $\eta>0$ such that

$$R_i(t)\geqslant \eta, \quad i=1,2,\dots,n. \tag{4.11}$$

Let $x(t)=(x_1(t),x_2(t),\dots,x_n(t))$ be any positive solution of Eq. (1.1). Define

$$\begin{aligned} V(t) = \sum_{i=1}^n \nu_i \Bigg[&\left|\ln\left(\frac{x_i(t)}{x_i^*(t)}\right) - \int_{t-\tau_{ii}(t)}^{t}\frac{a_{ii}(\xi_{ii}^{-1}(s))}{1-\dot\tau_{ii}(\xi_{ii}^{-1}(s))}\left[x_i(s)-x_i^*(s)\right]ds\right| \\ &+ \int_{t-\tau_{ii}(t)}^{t}\frac{h_i(\xi_{ii}^{-1}(s))}{1-\dot\tau_{ii}(\xi_{ii}^{-1}(s))}\int_s^t \frac{a_{ii}(\xi_{ii}^{-1}(u))}{1-\dot\tau_{ii}(\xi_{ii}^{-1}(u))}\left|x_i(u)-x_i^*(u)\right|du\,ds \\ &+ \sum_{j\neq i}^{n}\int_{t-\tau_{ij}(t)}^{t}\frac{a_{ij}(\xi_{ij}^{-1}(s))}{1-\dot\tau_{ij}(\xi_{ij}^{-1}(s))}\left|x_j(s)-x_j^*(s)\right|ds\Bigg], \quad t\geqslant 0. \end{aligned} \tag{4.12}$$

Note that $(\ln\alpha-\ln\beta)(\alpha-\beta)>0$ for any $\alpha,\beta>0$. A direct calculation of the upper right-hand derivative $D^+V(t)$ of $V(t)$ leads to

$$
\begin{aligned}
D^+V(t) \leqslant \sum_{i=1}^{n} \nu_i \Bigg[& -\frac{a_{ii}(\xi_{ii}^{-1}(t))}{1-\dot{\tau}_{ii}(\xi_{ii}^{-1}(t))} \\
& \times \left| x_i(t) - x_i^*(t) \exp\left(\int_{t-\tau_{ii}(t)}^{t} \frac{a_{ii}(\xi_{ii}^{-1}(s))}{1-\dot{\tau}_{ii}(\xi_{ii}^{-1}(s))} \big[x_i(s) - x_i^*(s)\big]\, ds \right) \right| \\
& + \frac{a_{ii}(\xi_{ii}^{-1}(t)) x_i^*(t)}{1-\dot{\tau}_{ii}(\xi_{ii}^{-1}(t))} \left| \exp\left(\int_{t-\tau_{ii}(t)}^{t} \frac{a_{ii}(\xi_{ii}^{-1}(s))}{1-\dot{\tau}_{ii}(\xi_{ii}^{-1}(s))} \big[x_i(s) - x_i^*(s)\big]\, ds \right) - 1 \right| \\
& - h_i(t) \int_{t-\tau_{ii}(t)}^{t} \frac{a_{ii}(\xi_{ii}^{-1}(s))}{1-\dot{\tau}_{ii}(\xi_{ii}^{-1}(s))} \big|x_i(s) - x_i^*(s)\big|\, ds \\
& + \frac{a_{ii}(\xi_{ii}^{-1}(t))}{1-\dot{\tau}_{ii}(\xi_{ii}^{-1}(t))} \big|x_i(t) - x_i^*(t)\big| \int_{t-\tau_{ii}(t)}^{t} \frac{h_i(\xi_{ii}^{-1}(s))}{1-\dot{\tau}_{ii}(\xi_{ii}^{-1}(s))}\, ds \\
& + \sum_{j\neq i}^{n} \frac{a_{ij}(\xi_{ij}^{-1}(t))}{1-\dot{\tau}_{ij}(\xi_{ij}^{-1}(t))} \big|x_j(t) - x_j^*(t)\big| \Bigg] \\
\leqslant \sum_{i=1}^{n} \nu_i \Bigg[& -\frac{a_{ii}(\xi_{ii}^{-1}(t))}{1-\dot{\tau}_{ii}(\xi_{ii}^{-1}(t))} \big|x_i(t) - x_i^*(t)\big| \\
& + \frac{2a_{ii}(\xi_{ii}^{-1}(t)) x_i^*(t)}{1-\dot{\tau}_{ii}(\xi_{ii}^{-1}(t))} \left| \exp\left(\int_{t-\tau_{ii}(t)}^{t} \frac{a_{ii}(\xi_{ii}^{-1}(s))}{1-\dot{\tau}_{ii}(\xi_{ii}^{-1}(s))} \big[x_i(s) - x_i^*(s)\big]\, ds \right) - 1 \right| \\
& - h_i(t) \int_{t-\tau_{ii}(t)}^{t} \frac{a_{ii}(\xi_{ii}^{-1}(s))}{1-\dot{\tau}_{ii}(\xi_{ii}^{-1}(s))} \big|x_i(s) - x_i^*(s)\big|\, ds \\
& + \frac{a_{ii}(\xi_{ii}^{-1}(t))}{1-\dot{\tau}_{ii}(\xi_{ii}^{-1}(t))} \big|x_i(t) - x_i^*(t)\big| \int_{t-\tau_{ii}(t)}^{t} \frac{h_i(\xi_{ii}^{-1}(s))}{1-\dot{\tau}_{ii}(\xi_{ii}^{-1}(s))}\, ds \\
& + \sum_{j\neq i}^{n} \frac{a_{ij}(\xi_{ij}^{-1}(t))}{1-\dot{\tau}_{ij}(\xi_{ij}^{-1}(t))} \big|x_j(t) - x_j^*(t)\big| \Bigg] \\
\leqslant \sum_{i=1}^{n} \nu_i \Bigg[& -\frac{a_{ii}(\xi_{ii}^{-1}(t))}{1-\dot{\tau}_{ii}(\xi_{ii}^{-1}(t))} \big|x_i(t) - x_i^*(t)\big| \\
& + \frac{2a_{ii}(\xi_{ii}^{-1}(t)) x_i^*(t)}{1-\dot{\tau}_{ii}(\xi_{ii}^{-1}(t))} \exp\left(\int_{t-\tau_{ii}(t)}^{t} \frac{a_{ii}(\xi_{ii}^{-1}(s))}{1-\dot{\tau}_{ii}(\xi_{ii}^{-1}(s))} x_i(s)\, ds \right)
\end{aligned}
$$

$$
\begin{aligned}
&\times \int_{t-\tau_{ii}(t)}^{t} \frac{a_{ii}(\xi_{ii}^{-1}(s))}{1-\dot{\tau}_{ii}(\xi_{ii}^{-1}(s))}\left|x_i(s)-x_i^*(s)\right| ds \\
&- h_i(t) \int_{t-\tau_{ii}(t)}^{t} \frac{a_{ii}(\xi_{ii}^{-1}(s))}{1-\dot{\tau}_{ii}(\xi_{ii}^{-1}(s))}\left|x_i(s)-x_i^*(s)\right| ds \\
&+ \frac{a_{ii}(\xi_{ii}^{-1}(t))}{1-\dot{\tau}_{ii}(\xi_{ii}^{-1}(t))}\left|x_i(t)-x_i^*(t)\right| \int_{t-\tau_{ii}(t)}^{t} \frac{h_i(\xi_{ii}^{-1}(s))}{1-\dot{\tau}_{ii}(\xi_{ii}^{-1}(s))} ds \\
&+ \sum_{j\neq i}^{n} \frac{a_{ij}(\xi_{ij}^{-1}(t))}{1-\dot{\tau}_{ij}(\xi_{ij}^{-1}(t))}\left|x_j(t)-x_j^*(t)\right|\Bigg] \\
\leqslant &\sum_{i=1}^{n} \nu_i \Bigg[-\frac{a_{ii}(\xi_{ii}^{-1}(t))}{1-\dot{\tau}_{ii}(\xi_{ii}^{-1}(t))}\Bigg(1-\int_{t-\tau_{ii}(t)}^{t} \frac{h_i(\xi_{ii}^{-1}(s))}{1-\dot{\tau}_{ii}(\xi_{ii}^{-1}(s))} ds\Bigg)\left|x_i(t)-x_i^*(t)\right| \\
&+ \sum_{j\neq i}^{n} \frac{a_{ij}(\xi_{ij}^{-1}(t))}{1-\dot{\tau}_{ij}(\xi_{ij}^{-1}(t))}\left|x_j(t)-x_j^*(t)\right|\Bigg] \\
= &\sum_{i=1}^{n} \Bigg[-\frac{\nu_i a_{ii}(\xi_{ii}^{-1}(t))}{1-\dot{\tau}_{ii}(\xi_{ii}^{-1}(t))}\Bigg(1-\int_{t-\tau_{ii}(t)}^{t} \frac{h_i(\xi_{ii}^{-1}(s))}{1-\dot{\tau}_{ii}(\xi_{ii}^{-1}(s))} ds\Bigg) \\
&+ \sum_{j\neq i}^{n} \frac{\nu_j a_{ji}(\xi_{ji}^{-1}(t))}{1-\dot{\tau}_{ji}(\xi_{ji}^{-1}(t))}\Bigg]\left|x_i(t)-x_i^*(t)\right| \\
= &-\sum_{i=1}^{n} R_i(t)\left|x_i(t)-x_i^*(t)\right| \leqslant -\eta \sum_{i=1}^{n}\left|x_i(t)-x_i^*(t)\right|, \quad t \geqslant 0.
\end{aligned}
$$

The above shows that $V(t)$ is decreasing in $[0,\infty)$ and so the limit $v = \lim_{t\to\infty} V(t)$ exists. Furthermore, from the above, we have

$$
\int_0^\infty \left|x_i(s)-x_i^*(s)\right| ds \leqslant \frac{V(0)}{\eta} < \infty, \quad i = 1,2,\ldots,n. \tag{4.13}
$$

It follows that

$$
\liminf_{t\to\infty}\left|x_i(t)-x_i^*(t)\right| = 0, \quad i = 1,2,\ldots,n, \tag{4.14}
$$

$$
\lim_{t\to\infty} \int_{t-\tau_{ii}(t)}^{t} \frac{a_{ii}(\xi_{ii}^{-1}(s))}{1-\dot{\tau}_{ii}(\xi_{ii}^{-1}(s))}\left[x_i(s)-x_i^*(s)\right] ds = 0, \quad i = 1,2,\ldots,n, \tag{4.15}
$$

$$\lim_{t\to\infty}\int\limits_{t-\tau_{ii}(t)}^{t}\frac{h_i(\xi_{ii}^{-1}(s))}{1-\dot\tau_{ii}(\xi_{ii}^{-1}(s))}\int\limits_{s}^{t}\frac{a_{ii}(\xi_{ii}^{-1}(u))}{1-\dot\tau_{ii}(\xi_{ii}^{-1}(u))}\bigl|x_i(u)-x_i^*(u)\bigr|\,du\,ds=0,$$
$$i=1,2,\ldots,n, \tag{4.16}$$

and

$$\lim_{t\to\infty}\sum_{j\neq i}^{n}\int\limits_{t-\tau_{ij}(t)}^{t}\frac{a_{ij}(\xi_{ij}^{-1}(s))}{1-\dot\tau_{ij}(\xi_{ij}^{-1}(s))}\bigl|x_j(s)-x_j^*(s)\bigr|\,ds=0,\quad i=1,2,\ldots,n. \tag{4.17}$$

Combining (4.12), (4.15)–(4.17), we have

$$\lim_{t\to\infty}\sum_{i=1}^{n}\nu_i\left|\ln\left(\frac{x_i(t)}{x_i^*(t)}\right)\right|=v, \tag{4.18}$$

which yields

$$x_i(t)\geqslant e^{-(v+1)/v}x_i^*(t),\quad \text{for large } t,\ i=1,2,\ldots,n. \tag{4.19}$$

By (4.14), (4.18) and (4.19),

$$\begin{aligned}
v=\liminf_{t\to\infty}\sum_{i=1}^{n}\nu_i\left|\ln\left(\frac{x_i(t)}{x_i^*(t)}\right)\right| &\leqslant \liminf_{t\to\infty}\sum_{i=1}^{n}\frac{\nu_i}{\min\{x_i(t),x_i^*(t)\}}\bigl|x_i(t)-x_i^*(t)\bigr| \\
&\leqslant e^{(v+1)/v}\liminf_{t\to\infty}\sum_{i=1}^{n}\frac{\nu_i}{x_i^*(t)}\bigl|x_i(t)-x_i^*(t)\bigr| \\
&\leqslant \frac{e^{(v+1)/v}}{m}\liminf_{t\to\infty}\sum_{i=1}^{n}\nu_i\bigl|x_i(t)-x_i^*(t)\bigr|=0,
\end{aligned}$$

where $m=\min\{x_i^*(t)\colon t\in[0,\omega],\ i=1,2,\ldots,n\}$. Hence, it follows from (4.18) that

$$\lim_{t\to\infty}\bigl|x_i(t)-x_i^*(t)\bigr|=0,\quad i=1,2,\ldots,n. \tag{4.20}$$

The proof is complete. □

Proof of Theorem 2.3. Let $x(t)=(x_1(t),x_2(t),\ldots,x_n(t))$ be any positive solution of Eq. (1.1). Set

$$z_i(t)=\nu_i^{-1}y_i(t),\quad i=1,2,\ldots,n. \tag{4.21}$$

Then we can rewrite (2.9) as

$$\dot z_i(t)=-\bigl[\nu_i^{-1}a_i+z_i(t)\bigr]\sum_{j=1}^{n}\nu_j\tilde a_{ij}(t)z_j\bigl(t-\tau_{ij}(t)\bigr),\quad i=1,2,\ldots,n. \tag{4.22}$$

Clearly, we only need to prove that

$$\lim_{t\to\infty} z_i(t) = 0, \quad i = 1, 2, \ldots, n. \tag{4.23}$$

In what follows, we divide into two cases to prove (4.23).

Case 1. $\dot{z}_i(t)$, $i = 1, 2, \ldots, n$, are all nonoscillatory. In this case, $z_i(t)$, $i = 1, 2, \ldots, n$, are monotone eventually. By Lemma 3.3, we have $z_i(t) \to c_i$ as $t \to \infty$ and $\nu_i^{-1} a_i + c_i \geqslant 0$ for $i = 1, 2, \ldots, n$. For the sake of simplicity, it is harmless to assume that $|c_1| = \max\{|c_i|\colon\ i = 1, 2, \ldots, n\}$. Choose $T > 0$ such that $\dot{z}_1(t) > 0$, $t \geqslant T$ or $\dot{z}_1(t) < 0$, $t \geqslant T$. If $c_1 = -\nu_1^{-1} a_1$, then $\dot{z}_1(t) < 0$ for $t \geqslant T$. Choose $\epsilon > 0$ such that $(\nu_1^{-1} a_1 + \epsilon)\mu_1 < \nu_1^{-1} a_1 - 2\epsilon$. For the given ϵ, we can choose $T_1 > T$ such that

$$z_1\bigl(t - \tau_{11}(t)\bigr) < -\nu_1^{-1} a_1 + \epsilon \quad \text{and} \quad z_j\bigl(t - \tau_{1j}(t)\bigr) < \nu_1^{-1} a_1 + \epsilon, \quad t \geqslant T_1. \tag{4.24}$$

Hence, from (2.22), (4.22) and (4.24), we have

$$\begin{aligned}
-\dot{z}_1(t) &= \bigl[\nu_1^{-1} a_1 + z_1(t)\bigr] \sum_{j=1}^{n} \nu_j \tilde{a}_{1j}(t) z_j\bigl(t - \tau_{1j}(t)\bigr) \\
&< \bigl[\nu_1^{-1} a_1 + z_1(t)\bigr] \Biggl[\bigl(-\nu_1^{-1} a_1 + \epsilon\bigr)\nu_1 \tilde{a}_{11}(t) + \bigl(\nu_1^{-1} a_1 + \epsilon\bigr) \sum_{j\neq 1}^{n} \nu_j \tilde{a}_{1j}(t)\Biggr] \\
&\leqslant \bigl[\nu_1^{-1} a_1 + z_1(t)\bigr]\nu_1 \tilde{a}_{11}(t)\bigl[-\nu_1^{-1} a_1 + \epsilon + \bigl(\nu_1^{-1} a_1 + \epsilon\bigr)\mu_1\bigr] \\
&< -\epsilon\bigl[\nu_1^{-1} a_1 + z_1(t)\bigr]\nu_1 \tilde{a}_{11}(t) < 0, \quad t \geqslant T_1.
\end{aligned}$$

This contradicts the fact that $-\dot{z}_i(t) > 0$ for $t \geqslant T$. If $c_1 > -\nu_1^{-1} a_1$, then, integrating (4.22) from T to ∞, we obtain

$$\begin{aligned}
\infty &> \left|\ln\left(\frac{\nu_1^{-1} a_1 + c_1}{\nu_1^{-1} a_1 + z_1(T)}\right)\right| \\
&= \int_T^{\infty} \nu_1 \tilde{a}_{11}(t) \left| z_1\bigl(t - \tau_{11}(t)\bigr) + \frac{1}{\nu_1 \tilde{a}_{11}(t)} \sum_{j\neq 1}^{n} \nu_j \tilde{a}_{1j}(t) z_j\bigl(t - \tau_{1j}(t)\bigr)\right| dt.
\end{aligned} \tag{4.25}$$

Note that $\int_T^\infty \tilde{a}_{11}(t)\,dt = \infty$ and

$$\begin{aligned}
&\liminf_{t\to\infty} \left| z_1\bigl(t - \tau_{11}(t)\bigr) + \frac{1}{\nu_1 \tilde{a}_{11}(t)} \sum_{j\neq 1}^{n} \nu_j \tilde{a}_{1j}(t) z_j\bigl(t - \tau_{1j}(t)\bigr)\right| \\
&\quad \geqslant \liminf_{t\to\infty} \Biggl[\bigl|z_1\bigl(t - \tau_{11}(t)\bigr)\bigr| - \frac{1}{\nu_1 \tilde{a}_{11}(t)} \sum_{j\neq 1}^{n} \nu_j \tilde{a}_{1j}(t) \bigl|z_j\bigl(t - \tau_{1j}(t)\bigr)\bigr| \Biggr]
\end{aligned}$$

$$\geqslant |c_1|\left[1-\limsup_{t\to\infty}\left(\frac{1}{\nu_1\tilde{a}_{11}(t)}\sum_{j\neq 1}^{n}\nu_j\tilde{a}_{1j}(t)\right)\right]\geqslant (1-\mu_1)|c_1|.$$

It follows from (4.25) that $c_1=0$, and so $c_1=c_2=\cdots=c_n=0$. Hence, (4.23) is true.

Case 2. $\dot{z}_l(t)$ is oscillatory for some $l\in\{1,2,\dots,n\}$. Then there exists an infinity sequence $\{t_k\}$ with $t_k\uparrow\infty$ such that

$$\sum_{j=1}^{n}\nu_j\tilde{a}_{lj}(t_k)z_j\big(t_k-\tau_{lj}(t_k)\big)=0,\quad k=1,2,\dots. \tag{4.26}$$

Set

$$V_i=\liminf_{t\to\infty}z_i(t)\quad\text{and}\quad U_i=\limsup_{t\to\infty}z_i(t),\quad i=1,2,\dots,n. \tag{4.27}$$

In view of Lemma 3.4,

$$-a<V_i\leqslant U_i<\infty,\quad i=1,2,\dots,n. \tag{4.28}$$

Let

$$-V=\min\{V_1,V_2,\dots,V_n\}\quad\text{and}\quad U=\max\{U_1,U_2,\dots,U_n\}.$$

Then from (4.26)–(4.28), we have

$$0\leqslant V<a,\qquad 0\leqslant U<\infty. \tag{4.29}$$

In what follows, we show that V and U satisfy the inequalities

$$a+U\leqslant (a+\mu V)\exp\left[\frac{1-\mu}{a}V-\frac{(1-\mu)^2(1+2\mu)}{6a^2(1+\mu)}V^2\right],\quad\text{and} \tag{4.30}$$

$$a-V\geqslant (a-\mu U)\exp\left[-\frac{1-\mu}{a}U-\frac{(1-\mu)^2(1+2\mu)}{6a^2(1+\mu)}U^2\right]. \tag{4.31}$$

For the sake of simplicity, we set

$$A_i=\frac{3(1-\mu)}{2a(1+\mu_i)}+\frac{(1-\mu)(\mu+\mu_i)}{2a(1+\mu_i)^2},\quad i=1,2,\dots,n.$$

Without loss of generality, we may assume that $U=U_i$ and $V=-V_j$. Let $\epsilon>0$ be sufficiently small such that $v_1\equiv V+\epsilon<a$. Choose $T>0$ such that

$$-v_1<z_k(t)<U+\epsilon\equiv u_1,\quad t\geqslant T-\tau_M,\ k=1,2,\dots,n, \tag{4.32}$$

where $\tau_M=\max\{\tau_{ij}(t)\colon\ t\in[0,\omega],\ i,j=1,2,\dots,n\}$. First, we prove that (4.30) holds. If $U\leqslant\mu_i V$, then (4.30) obviously holds. Therefore, we will prove (4.30) only in the case when

$U > \mu_i V$. For the sake of simplicity, it is harmless to assume $U > \mu_i v_1$. Set $v_2 = (1+\mu_i)v_1$ and $u_2 = (1+\mu_j)u_1$. Then from (2.22) and (4.22), we have

$$\frac{\dot{z}_i(t)}{\nu_i^{-1}a_i + z_i(t)} \leqslant \nu_i \tilde{a}_{ii}(t)\left[-z_i\big(t-\tau_{ii}(t)\big) + \frac{v_1}{\nu_i \tilde{a}_{ii}(t)} \sum_{k\neq i}^{n} \nu_k \tilde{a}_{ik}(t)\right]$$
$$\leqslant \nu_i \tilde{a}_{ii}(t)\big[-z_i\big(t-\tau_{ii}(t)\big) + \mu_i v_1\big] \leqslant \nu_i \tilde{a}_{ii}(t) v_2, \quad t \geqslant T, \tag{4.33}$$

and

$$-\frac{\dot{z}_j(t)}{\nu_j^{-1}a_j + z_j(t)} \leqslant \nu_j \tilde{a}_{jj}(t)\left[z_j\big(t-\tau_{jj}(t)\big) + \frac{u_1}{\nu_j \tilde{a}_{jj}(t)} \sum_{k\neq j}^{n} \nu_i \tilde{a}_{jk}(t)\right]$$
$$\leqslant \nu_j \tilde{a}_{jj}(t)\big[z_j\big(t-\tau_{jj}(t)\big) + \mu_j u_1\big] \leqslant \nu_j \tilde{a}_{jj}(t) u_2, \quad t \geqslant T. \tag{4.34}$$

Since $U > \mu_i v_1$, we cannot have $z_i(t) \leqslant \mu_i v_1$ eventually. On the other hand, if $z_i(t) \geqslant \mu_i v_1$ eventually, then it follows from the fact that $\int_0^\infty \tilde{a}_{ii}(t)\,dt = \infty$ and the second inequality in (4.33) that $z_i(t)$ is nonincreasing and $U = \lim_{t\to\infty} z_i(t) = \mu_i v_1$. This is also impossible. Therefore, it follows that $z_i(t)$ oscillates about $\mu_i v_1$.

Let $\{p_k\}$ be an increasing sequence such that $p_k \geqslant T + \tau_M$, $\dot{z}_i(p_k) = 0$, $z_i(p_k) \geqslant \mu_i v_1$, $\lim_{k\to\infty} p_k = \infty$ and $\lim_{k\to\infty} z_i(p_k) = U$. By (4.33), there exists $\xi_k \in [p_k - \tau_{ii}(p_k), p_k] \subseteq [g_i(p_k), p_k]$ such that $z_i(\xi_k) = \mu_i v_1$ and $z_i(t) > \mu_i v_1$ for $\xi_k < t \leqslant p_k$. We claim that

$$z_i\big(t-\tau_{ii}(t)\big) \geqslant -\nu_i^{-1}a_i + \big(\nu_i^{-1}a_i + \mu_i v_1\big)\exp\left(-v_2 \int_{g_i(t)}^{\xi_k} \nu_i \tilde{a}_{ii}(s)\,ds\right), \quad \xi_k \leqslant t \leqslant p_k. \tag{4.35}$$

In fact, if $t \in [\xi_k, p_k]$ and $t - \tau_{ii}(t) \leqslant \xi_k$, integrating (4.33) from $t - \tau_{ii}(t)$ to ξ_k we get

$$-\ln \frac{\nu_i^{-1}a_i + z_i(t-\tau_{ii}(t))}{\nu_i^{-1}a_i + z_i(\xi_k)} \leqslant v_2 \int_{t-\tau_{ii}(t)}^{\xi_k} \nu_i \tilde{a}_{ii}(s)\,ds \leqslant v_2 \int_{g_i(t)}^{\xi_k} \nu_i \tilde{a}_{ii}(s)\,ds.$$

It follows that (4.35) holds. If $t \in [\xi_k, p_k]$ and $t - \tau_{ii}(t) > \xi_k$, then $z_i(t) > \mu_i v_1$, which implies that (4.35) also holds. Substituting (4.35) into the second inequality in (4.33), we obtain

$$\frac{\dot{z}_i(t)}{\nu_i^{-1}a_i + z_i(t)} \leqslant \big(\nu_i^{-1}a_i + \mu_i v_1\big)\nu_i \tilde{a}_{ii}(t)\left[1 - \exp\left(-v_2 \int_{g_i(t)}^{\xi_k} \nu_i \tilde{a}_{ii}(s)\,ds\right)\right], \quad \xi_k \leqslant t \leqslant p_k.$$

Combining this with (4.33), we have

$$\frac{\dot{z}_i(t)}{a + z_i(t)} \leqslant \nu_i \tilde{a}_{ii}(t) \min\left\{v_2, (a + \mu_i v_1)\left[1 - \exp\left(-v_2 \int_{g_i(t)}^{\xi_k} \nu_i \tilde{a}_{ii}(s)\,ds\right)\right]\right\},$$
$$\xi_k \leqslant t \leqslant p_k. \tag{4.36}$$

From (2.20) and (4.36), using the same type of reasoning as in the proof of Theorem 2.1 in [45], we can prove (4.30). Next, we will prove that (4.31) holds as well. If $V = 0$, then (4.31) holds naturally. In what follows, we assume that $V > 0$. Then from (4.30), we have

$$U < a(1+\mu)e^{1-\mu} - a < 2a, \qquad \mu U < a\mu\big[(1+\mu)e^{1-\mu} - 1\big] < a. \tag{4.37}$$

Thus we may assume, without loss of generality, that $V > \mu_j u_1$. In view of this and (4.23), we can show that neither $z_j(t) \geqslant -\mu_j u_1$ eventually nor $z_j(t) \leqslant -\mu_j u_1$ eventually. Therefore, $z_j(t)$ oscillates about $-\mu_j u_1$.

Let $\{q_k\}$ be an increasing sequence such that $q_k \geqslant T + \tau_M$, $\dot z_j(q_k) = 0$, $z_j(q_k) \leqslant -\mu_j u_1$, $\lim_{k\to\infty} q_k = \infty$ and $\lim_{k\to\infty} z_j(q_k) = -V$. By (4.23), there exists $\eta_k \in [q_k - \tau_{jj}(q_k), q_k] \subseteq [g_i(q_k), q_k]$ such that $z_j(\eta_k) = -\mu_j u_1$ and $z_j(t) < -\mu_j u_1$ for $\eta_k < t \leqslant q_k$. Similar to the above proof, from (4.23), we have

$$z_j\big(t - \tau_{jj}(t)\big) \leqslant \big[\nu_j^{-1} a_j - \mu_j u_1\big] \exp\left(u_2 \int_{g_j(t)}^{\eta_k} \nu_j \tilde a_{jj}(s)\, ds\right) - \nu_j^{-1} a_j, \quad \eta_k \leqslant t \leqslant q_k.$$

Substituting this into the second inequality in (4.23), we obtain

$$-\frac{\dot z_j(t)}{\nu_j^{-1} a_j + z_j(t)} \leqslant \big[\nu_j^{-1} a_j - \mu_j u_1\big] \nu_j \tilde a_{jj}(t) \left[\exp\left(u_2 \int_{g_j(t)}^{\eta_k} \nu_j \tilde a_{jj}(s)\, ds\right) - 1\right],$$
$$\eta_k \leqslant t \leqslant q_k.$$

Combining this with (4.23), we have

$$-\frac{\dot z_j(t)}{a + z_j(t)} \leqslant \nu_j \tilde a_{jj}(t) \min\left\{u_2, \big[a - \mu_j u_1\big]\left[\exp\left(u_2 \int_{g_j(t)}^{\eta_k} \nu_j \tilde a_{jj}(s)\, ds\right) - 1\right]\right\},$$
$$\eta_k \leqslant t \leqslant q_k. \tag{4.38}$$

The rest is the same as in the proof of Theorem 2.1 in [45]. In view of Lemma 3.1, it follows from (4.30) and (4.31) that $U = V = 0$. Thus, (4.23) holds. The proof is complete. □

Proof of Theorem 2.4. Let $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))$ be any positive solution of Eq. (1.1). Set $y_i(t)$ as in (2.6). We only need to prove that

$$\lim_{t\to\infty} y_i(t) = 0, \quad i = 1, 2, \ldots, n. \tag{4.39}$$

By Lemma 3.3, there exists $T > 0$ such that

$$a_i + y_i(t) \leqslant (a_i + a\nu_i\mu_i)\exp\big[(a_i + a\nu_i\mu_i)d_i + e^{-(a_i + a\nu_i\mu_i)d_i} - 1\big]$$
$$:= D_i, \quad t \geqslant T - \tau_M,\ i = 1, 2, \ldots, n. \tag{4.40}$$

Set

$$v_i = \frac{1}{\nu_i} \limsup_{t\to\infty} \left| y_i(t) \right|, \quad i = 1, 2, \ldots, n. \tag{4.41}$$

Then by Lemma 3.3, $0 \leqslant v_i < \infty$, $i = 1, 2, \ldots, n$. To complete the proof, we only show that $v_1 = v_2 = \cdots = v_n = 0$. Without loss of generality, assume that $v_1 = \max\{v_j\colon\ j = 1, 2, \ldots, n\} > 0$. Then there are two possible cases.

Case 1. $\dot{y}_1(t)$ is positive eventually or negative eventually. In this case, the limit $c_1 := \lim_{t\to\infty} y_1(t)$ exists and $c_1 + a_1 \geqslant 0$. Choose $T_1 > T$ such that $\dot{y}_1(t) > 0$, $t \geqslant T_1$ or $\dot{y}_1(t) < 0$, $t \geqslant T_1$. If $c_1 > -a_1$, then, integrating (2.9) from T_1 to ∞, we obtain

$$\begin{aligned} \infty &> \left| \ln\left(\frac{a_1 + c_1}{a_1 + y_1(T_1)} \right) \right| \\ &= \int_T^\infty \tilde{a}_{11}(t) \left| y_1\bigl(t - \tau_{11}(t)\bigr) + \frac{1}{\tilde{a}_{11}(t)} \sum_{j\neq 1}^n \tilde{a}_{1j}(t) y_j\bigl(t - \tau_{1j}(t)\bigr) \right| dt. \end{aligned} \tag{4.42}$$

Note that

$$\begin{aligned} &\liminf_{t\to\infty} \left| y_1\bigl(t - \tau_{11}(t)\bigr) + \frac{1}{\tilde{a}_{11}(t)} \sum_{j\neq 1}^n \tilde{a}_{1j}(t) y_j\bigl(t - \tau_{1j}(t)\bigr) \right| \\ &\quad \geqslant \liminf_{t\to\infty} \left[\left| y_1\bigl(t - \tau_{11}(t)\bigr) \right| - \frac{1}{\tilde{a}_{11}(t)} \sum_{j\neq 1}^n \tilde{a}_{1j}(t) \left| y_j\bigl(t - \tau_{1j}(t)\bigr) \right| \right] \\ &\quad \geqslant \nu_1 v_1 \left[1 - \limsup_{t\to\infty} \left(\frac{1}{\nu_1 \tilde{a}_{11}(t)} \sum_{j\neq 1}^n \nu_j \tilde{a}_{1j}(t) \right) \right] \geqslant (1 - \mu_1) \nu_1 v_1. \end{aligned}$$

It follows from (4.42) and the fact that $\int_0^\infty \tilde{a}_{11}(t)\,dt = \infty$ that $v_1 = 0$, which is a contradiction. If $c_1 = -a_1$, then $\dot{y}_1(t) < 0$ for $t \geqslant T$ and $v_1 = -\nu_1^{-1} c_1$. Choose $\epsilon > 0$ such that $(v_1 + \epsilon)\mu_1 < v_1 - 2\epsilon$. For the given ϵ, by (4.41) and $v_1 = \max\{v_j\colon\ j = 1, 2, \ldots, n\} > 0$, we can choose $T_1 > T$ such that

$$y_1\bigl(t - \tau_{11}(t)\bigr) < -\nu_1(v_1 - \epsilon) \quad \text{and} \quad y_j\bigl(t - \tau_{1j}(t)\bigr) < \nu_j(v_1 + \epsilon), \quad t \geqslant T_1. \tag{4.43}$$

Hence, from (2.22) and (4.43), we have

$$\begin{aligned} -\dot{y}_1(t) &= \bigl[a_1 + y_1(t)\bigr] \sum_{j=1}^n \tilde{a}_{1j}(t) y_j\bigl(t - \tau_{1j}(t)\bigr) \\ &< \bigl[a_1 + y_1(t)\bigr] \left[-\nu_1(v_1 - \epsilon)\tilde{a}_{11}(t) + (v_1 + \epsilon) \sum_{j\neq 1}^n \nu_j \tilde{a}_{1j}(t) \right] \end{aligned}$$

$$
\begin{aligned}
&= \big[a_1 + y_1(t)\big]\nu_1\tilde{a}_{11}(t)\left[-v_1 + \epsilon + \frac{v_1+\epsilon}{\nu_1\tilde{a}_{11}(t)}\sum_{j\neq 1}^{n}\nu_j\tilde{a}_{1j}(t)\right] \\
&\leqslant \big[a_1 + y_1(t)\big]\nu_1\tilde{a}_{11}(t)\big[-v_1 + \epsilon + \mu_1(v_1+\epsilon)\big] \\
&\leqslant -\epsilon\big[a_1 + y_1(t)\big]\nu_1\tilde{a}_{11}(t) < 0, \quad t \geqslant T_1.
\end{aligned}
$$

This contradicts the fact that $-\dot{y}_i(t) > 0$ for $t \geqslant T_1$.

Case 2. $\dot{y}_1(t)$ is oscillatory. For any $\epsilon \in (0, (1-\mu_1)v_1/(1+\mu_1))$, there exist $T_2 > T_1 + \tau_M$ and a sequence $\{t_k\}$ with $t_k \uparrow \infty$ and $t_k > T_2$ such that

$$
\big|y_1(t_k)\big| \to \nu_1 v_1 \quad \text{as } k \to \infty, \quad \dot{y}_1(t_k) = 0, \qquad \big|y_1(t_k)\big| > \nu_1(v_1 - \epsilon), \quad k = 1, 2, \ldots, \tag{4.44}
$$

$$
\text{and} \quad \big|y_j(t)\big| < \nu_j(v_1 + \epsilon) \quad \text{for } t \geqslant T_2 - \tau_M, \ j = 1, 2, \ldots, n, \tag{4.45}
$$

where $\tau_M = \max\{\tau_{ij}(t)\colon t \in [0,\omega], i, j = 1, 2, \ldots, n\}$. We only consider the case when $|y_1(t_k)| = y_1(t_k)$ (the case when $|y_1(t_k)| = -y_1(t_k)$ is similar by using $-y_1(t)$ instead of $y_1(t)$). Then from (2.9), (2.22), (4.44) and (4.45), we have

$$
\begin{aligned}
0 &= -\sum_{j=1}^{n}\tilde{a}_{1j}(t_k)y_j\big(t_k - \tau_{1j}(t_k)\big) \\
&\leqslant -\tilde{a}_{11}(t_k)y_1\big(t_k - \tau_{11}(t_k)\big) + (v_1+\epsilon)\sum_{j\neq 1}^{n}\nu_j\tilde{a}_{1j}(t_k) \\
&= \tilde{a}_{11}(t_k)\left[-y_1\big(t_k - \tau_{11}(t_k)\big) + \frac{v_1+\epsilon}{\tilde{a}_{11}(t_k)}\sum_{j\neq 1}^{n}\nu_j\tilde{a}_{1j}(t_k)\right] \\
&\leqslant \tilde{a}_{11}(t_k)\big[-y_1\big(t_k - \tau_{11}(t_k)\big) + \nu_1\mu_1(v_1+\epsilon)\big], \quad \text{or}
\end{aligned}
$$

$$
y_1\big(t_k - \tau_{11}(t_k)\big) \leqslant \mu_1\nu_1(v_1+\epsilon),
$$

which, together with the fact $y_1(t_k) > \nu_1(v_1-\epsilon) > \mu_1\nu_1(v_1+\epsilon)$, implies that there exists $\xi_k \in [t_k - \tau_{11}(t_k), t_k) \subseteq [g_1(t_k), t_k]$ such that $y_1(\xi_k) = \mu_1\nu_1(v_1+\epsilon)$ and $y_1(t) > \mu_1\nu_1(v_1+\epsilon)$ for $\xi_k < t \leqslant t_k$. Hence from (2.9), (2.22), (4.40) and (4.45), we have

$$
\begin{aligned}
\dot{y}_1(t) &= -\big[a_1 + y_1(t)\big]\sum_{j=1}^{n}\tilde{a}_{1j}(t)y_j\big(t - \tau_{1j}(t)\big) \\
&\leqslant \tilde{a}_{11}(t)\big[a_1 + y_1(t)\big]\big[-y_1\big(t - \tau_{11}(t)\big) + \mu_1\nu_1(v_1+\epsilon)\big] \\
&\leqslant \nu_1\tilde{a}_{11}(t)(1+\mu_1)(v_1+\epsilon)D_1, \quad t \geqslant T_2.
\end{aligned} \tag{4.46}
$$

By (4.46) and the fact that $y_1(t) > \mu_1\nu_1(v_1+\epsilon)$ for $\xi_k < t \leqslant t_k$, we have

$$
\mu_1\nu_1(v_1+\epsilon) - y_1\big(t - \tau_{11}(t)\big) \leqslant \nu_1(1+\mu_1)(v_1+\epsilon)D_1\int_{g_1(t)}^{\xi_k}\tilde{a}_{11}(u)\,du, \quad \xi_k \leqslant t \leqslant t_k.
$$

Substituting this into the first inequality in (4.46) and using (4.40), we obtain

$$\dot{y}_1(t) \leqslant \nu_1(1+\mu_1)(v_1+\epsilon)D_1^2\tilde{a}_{11}(t)\int_{g_1(t)}^{\xi_k}\tilde{a}_{11}(s)\,ds, \quad \xi_k \leqslant t \leqslant t_k.$$

Combining this and (4.46), we have

$$\dot{y}_1(t) \leqslant \nu_1(1+\mu_1)(v_1+\epsilon)D_1\tilde{a}_{11}(t)\min\left\{1, D_1\int_{g_1(t)}^{\xi_k}\tilde{a}_{11}(s)\,ds\right\}, \quad \xi_k \leqslant t \leqslant t_k. \tag{4.47}$$

Set

$$\theta = \begin{cases} \max\left\{d_1D_1 - \frac{1}{2}, \frac{1}{2}\right\}(1+\mu_1), & \mu_1 < \frac{1}{3}, \\ \frac{1}{2}(1+\mu_1)(d_1D_1)^2, & \mu_1 \geqslant \frac{1}{3}. \end{cases}$$

Then by (2.25)

$$\theta < 1 - \mu_1. \tag{4.48}$$

We will show that

$$y_1(t_k) - y_1(\xi_k) \leqslant \nu_1\theta(v_1+\epsilon). \tag{4.49}$$

To this end, we consider the following three subcases:

Case 2.1. $\mu_1 < 1/3$ and $D_1\int_{\xi_k}^{t_k}\tilde{a}_{11}(s)\,ds \leqslant 1$. In this case, by (2.24) and (4.47), we have

$$\begin{aligned} &y_1(t_k) - y_1(\xi_k) \\ &\quad\leqslant \nu_1(1+\mu_1)(v_1+\epsilon)D_1^2\int_{\xi_k}^{t_k}\tilde{a}_{11}(t)\int_{g_1(t)}^{\xi_k}\tilde{a}_{11}(s)\,ds\,dt \\ &\quad\leqslant \nu_1(1+\mu_1)(v_1+\epsilon)D_1^2\int_{\xi_k}^{t_k}\tilde{a}_{11}(t)\left(d_1 - \int_{\xi_k}^{t}\tilde{a}_{11}(s)\,ds\right)dt \\ &\quad= \nu_1(1+\mu_1)(v_1+\epsilon)\left[d_1D_1^2\int_{\xi_k}^{t_k}\tilde{a}_{11}(s)\,ds - \frac{1}{2}\left(D_1\int_{\xi_k}^{t_k}\tilde{a}_{11}(s)\,ds\right)^2\right] \\ &\quad\leqslant \nu_1(1+\mu_1)(v_1+\epsilon)\left(\max\{d_1D_1, 1\} - \frac{1}{2}\right) \\ &\quad= \nu_1(1+\mu_1)(v_1+\epsilon)\max\left\{d_1D_1 - \frac{1}{2}, \frac{1}{2}\right\} = \nu_1\theta(v_1+\epsilon). \end{aligned}$$

Case 2.2. $\mu_1 < 1/3$ and $D_1 \int_{\xi_k}^{t_k} \tilde{a}_{11}(s)\,ds > 1$. In this case, there exists $\eta_k \in (\xi_k, t_k)$ such that $D_1 \int_{\eta_k}^{t_k} \tilde{a}_{11}(s)\,ds = 1$. Then by (2.24) and (4.47), we have

$$
\begin{aligned}
&y_1(t_k) - y_1(\xi_k) \\
&\quad \leqslant \nu_1(1+\mu_1)(v_1+\epsilon)D_1\left[\int_{\xi_k}^{\eta_k} \tilde{a}_{11}(s)\,ds + D_1 \int_{\eta_k}^{t_k} \tilde{a}_{11}(t) \int_{g_1(t)}^{\xi_k} \tilde{a}_{11}(s)\,ds\,dt\right] \\
&\quad = \nu_1(1+\mu_1)(v_1+\epsilon)D_1^2\left[\int_{\eta_k}^{t_k} \tilde{a}_{11}(t)\,dt \int_{\xi_k}^{\eta_k} \tilde{a}_{11}(s)\,ds + \int_{\eta_k}^{t_k} \tilde{a}_{11}(t) \int_{g_1(t)}^{\xi_k} \tilde{a}_{11}(s)\,ds\,dt\right] \\
&\quad = \nu_1(1+\mu_1)(v_1+\epsilon)D_1^2 \int_{\eta_k}^{t_k} \tilde{a}_{11}(t) \int_{g_1(t)}^{\eta_k} \tilde{a}_{11}(s)\,ds\,dt \\
&\quad \leqslant \nu_1(1+\mu_1)(v_1+\epsilon)D_1^2 \int_{\eta_k}^{t_k} \tilde{a}_{11}(t)\left(d_1 - \int_{\xi_k}^{t} \tilde{a}_{11}(s)\,ds\right) dt \\
&\quad = \nu_1(1+\mu_1)(v_1+\epsilon)\left[d_1 D_1^2 \int_{\eta_k}^{t_k} \tilde{a}_{11}(s)\,ds - \frac{1}{2}\left(D_1 \int_{\eta_k}^{t_k} \tilde{a}_{11}(s)\,ds\right)^2\right] \\
&\quad = \nu_1(1+\mu_1)(v_1+\epsilon)\left(d_1 D_1 - \frac{1}{2}\right) \leqslant \nu_1\theta(v_1+\epsilon).
\end{aligned}
$$

Case 2.3. $\mu_1 \geqslant 1/3$. In this case, $\int_{\xi_k}^{t_k} \tilde{a}_{11}(s)\,ds \leqslant d_1$, hence, by (2.24) and (4.47), we have

$$
\begin{aligned}
&y_1(t_k) - y_1(\xi_k) \\
&\quad \leqslant \nu_1(1+\mu_1)(v_1+\epsilon)D_1^2 \int_{\xi_k}^{t_k} \tilde{a}_{11}(t) \int_{g_1(t)}^{\xi_k} \tilde{a}_{11}(s)\,ds\,dt \\
&\quad \leqslant \nu_1(1+\mu_1)(v_1+\epsilon)D_1^2 \int_{\xi_k}^{t_k} \tilde{a}_{11}(t)\left(d_1 - \int_{\xi_k}^{t} \tilde{a}_{11}(s)\,ds\right) dt \\
&\quad = \nu_1(1+\mu_1)(v_1+\epsilon)\left[d_1 D_1^2 \int_{\xi_k}^{t_k} \tilde{a}_{11}(s)\,ds - \frac{1}{2}\left(D_1 \int_{\xi_k}^{t_k} \tilde{a}_{11}(s)\,ds\right)^2\right] \\
&\quad \leqslant \frac{1}{2}\nu_1(1+\mu_1)(v_1+\epsilon)(d_1 D_1)^2 = \nu_1\theta(v_1+\epsilon).
\end{aligned}
$$

Cases 2.1–2.3 show (4.49) holds. Let $\epsilon \to 0$ in (4.49). Then we conclude that $v_1 < v_1$. This is also a contradiction. The proof is complete. □

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