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Existence, uniqueness and stability of travelling waves in a discrete reaction–diffusion monostable equation with delay☆

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Abstract

In this paper, we study the existence, uniqueness and asymptotic stability of travelling wavefronts of the following equation:

$$u_t(x,t) = D[u(x+1,t) + u(x-1,t) - 2u(x,t)] - du(x,t) + b(u(x,t-r)),$$

where $x \in \mathbb{R}$, $t > 0$, $D, d > 0$, $r \geqslant 0$, $b \in C^1(\mathbb{R})$ and $b(0) = dK - b(K) = 0$ for some $K > 0$ under monostable assumption. We show that there exists a minimal wave speed $c^* > 0$, such that for each $c > c^*$ the equation has exactly one travelling wavefront $U(x+ct)$ (up to a translation) satisfying $U(-\infty) = 0, U(+\infty) = K$ and $\limsup_{\xi \to -\infty} U(\xi)e^{-\Lambda_1(c)\xi} < +\infty$, where $\lambda = \Lambda_1(c)$ is the smallest solution to the equation $c\lambda - D[e^{\lambda} + e^{-\lambda} - 2] + d - b'(0)e^{-\lambda cr} = 0$. Moreover, the travelling wavefront is strictly monotone and asymptotically stable with phase shift in the sense that if an initial data $\varphi \in C(\mathbb{R} \times [-r, 0], [0, K])$ satisfies $\liminf_{x \to +\infty} \varphi(x, 0) > 0$

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and $\lim_{x\to-\infty}\max_{s\in[-r,0]}|\varphi(x,s)e^{-\Lambda_1(c)x}-\rho_0 e^{\Lambda_1(c)cs}|=0$ for some $\rho_0\in(0,+\infty)$, then the solution $u(x,t)$ of the corresponding initial value problem satisfies $\lim_{t\to+\infty}\sup_{\mathbb{R}}|u(\cdot,t)/U(\cdot+ct+\xi_0)-1|=0$ for some $\xi_0=\xi_0(U,\varphi)\in\mathbb{R}$.

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1. Introduction

Travelling wavefront solutions play an important role in describing the long-term behaviour of solutions to initial value problems in reaction and diffusion (both continuous and discrete) equations. Such solutions also have their own practical background, such as, transition between different states of a physical system, propagation of patterns, and domain invasion of species in population biology. When the nonlinear reaction term is of *monostable type*, that is, considering the R-D equation

$$w_t(x,t)=Dw_{xx}(x,t)+f(w(x,t)),\quad x\in\mathbb{R},\ t\geqslant 0, \tag{1.1}$$

with $f(w)$ satisfying

(A) $f(0)=f(k)=0$ for some $k>0$; and $f(w)>0$ for $w\in(0,k)$,

it has been known from long time ago that $c_{\min}=2\sqrt{Df'(0)}>0$ is the minimal wave speed in the sense that (i) for every $c>c_{\min}$ there exists a travelling wavefront of the form $w(x,t)=u(x+ct)$ with $u(s)$ increasing and $u(-\infty)=0$, $u(\infty)=k$; (ii) the wavefront is unique up to translation; (iii) for $c<c_{\min}$, there is no such monotone wavefront with speed c. Moreover, the wavefront cannot be stable with respect to general initial functions, it can, however, be stable in respect to some smaller class of initial functions (e.g., initial functions with compact support).

For a spatially discrete analogue of (1.1), one may consider the following lattice differential equations

$$u_n'(t)=D[u_{n+1}(t)+u_{n-1}(t)-2u_n(t)]+f(u_n(t)),\quad n\in\mathbb{Z},\ t>0. \tag{1.2}$$

System (1.2) can either be considered as a discretization of (1.1), or be derived directly from population models over patchy environments (see, e.g., [3,12,18]). Indeed, as mentioned in Bell and Cosner [3] and Keener [12], in many situations, one usually derives a discrete version like (1.2) first, and then, by taking limit, arrives at a continuous version like (1.1). When the nonlinear term in (1.2) is of *bistable type*, the study on travelling wavefronts of such lattice differential equations have been extensive and intensive, and has resulted in many interesting and significant results, some of which, have revealed some essential difference between a discrete model and its continuous version. For details, see, for example, Bates et al. [1], Bates and Chmaj [2], Bell and Cosner [3], Cahn et al. [4], Chow et al. [9], Keener [12], Mallet-Paret [14], Shen [16,17], Zinner [25,26], and the references therein. However, for (1.2) with a *monostable nonlinearity*, the results are still very limited. Zinner et al. [27] addressed the existence and minimal speed of travelling wavefront for discrete Fisher equation. Recently, Chen and Guo [7,8] discussed a more general class of system

$$u_n'(t) = g(u_{n+1}(t)) + g(u_{n-1}(t)) - 2g(u_n(t)) + f(u_n(t)), \quad n \in \mathbb{Z}, \ \ t > 0, \tag{1.3}$$

where $g(u)$ is increasing and $f(u)$ is monostable. Established in Chen and Guo [7,8], are such results as existence, uniqueness and stability (in some sense) as well as minimal wave speed for (1.3). Also in a very recent paper, Carr and Chmaj [5] established the uniqueness of travelling wavefronts for the nonloncal *monostable* ODE system

$$u_n' = (J * u)_n - u_n + f(u_n), \quad n \in \mathbb{Z}, \tag{1.4}$$

which reduces to the discrete reaction–diffusion system (1.2) when taking $(J * u)_n = \frac{1}{2}[u_{n+1} + u_{n-1}]$.

On the other hand, in modelling population growth, temporal delay seems to be inevitable, accounting for the maturation time of the species under consideration. Based on such a consideration, in recent years, delayed reaction–diffusion equations of the form

$$w_t(x,t) = Dw_{xx}(x,t) - dw(x,t) + b(w(x,t-r)), \tag{1.5}$$

have been widely investigated in the literature (see, e.g., So and Yang [21] and Yang and So [24] and the references therein). As a model, this equation describes the evolution of a single species population with two age classes and a fixed maturation period living in a spatially unbounded environment $x \in R$, where $D > 0$ and $d > 0$ denote the diffusion rate and death rate, respectively, of the matured population, the constant $r \geqslant 0$ is the maturation time for the species. A more general model containing spatially nonlocal interactions, induced *jointly* by maturation delay and the diffusivity of the immature population, is also derived and studied in So et al. [20]. When the immature individuals do not diffuse, this general model reduces to (1.5).

Recent work of Faria et al. [10] shows that the multiplicity (in some sense) of the travelling wavefronts of (1.5) with *large wave speed* coincide with the dimension of the unstable manifold of the corresponding delay ordinary differential equation

$$w'(t) = -dw(t) + b(w(t-r)) \tag{1.6}$$

at the unstable connecting equilibrium 0. This indicates that the uniqueness of travelling wavefronts for monostable equations (continuous or discrete) is not automatic, and thus, needs to be established individually. Although no similar results for delayed discrete reaction diffusion equations that are parallel to those in [10] have been established, we expect that the multiplicity of travelling wavefonts for such equations are also related to the dimension of the unstable manifold of (1.6) at 0. Encouraged by the recent work of Chen and Guo [7,8], in this paper, we consider the discrete analog of (1.5), which can be written in the form

$$u_n'(t) = D[u_{n+1}(t) + u_{n-1}(t) - 2u_n(t)] - du_n(t) + b(u_n(t-r)), \quad n \in \mathbb{Z}, \quad t > 0. \tag{1.7}$$

We point out that (1.7) is a special case of a more general system

$$\begin{aligned} u_n'(t) = \; & D[u_{n+1}(t) + u_{n-1}(t) - 2u_n(t)] - du_n(t) \\ & + \sum_{j=-\infty}^{\infty} \Gamma(n,j) b(u_j(t-r)), \quad n \in \mathbb{Z}, \quad t > 0, \end{aligned} \tag{1.8}$$

modelling the growth of the matured population of a single species over a patchy environment. System (1.8), parallel to the continuous nonlocal model in So et al. [20], is derived recently in [22] and (1.7) precisely corresponds to the situation when the immatured do not disperse between patches (implying $\Gamma(n,j) = 1$ for $j = 0$, and $\Gamma(n,j) = 0$ for all other j). For details, see Weng et al. [22].

Throughout this paper, we always assume that the birth function $b \in C^1(\mathbb{R}_+)$ and there exists a constant $K > 0$ such that $b(0) = dK - b(K) = 0$. Therefore, (1.7) has at least two spatially homogeneous equilibria 0 and K. Furthermore, we need the following assumptions:

(H1) $b'(0) > d$, $b'(u) \geqslant 0$ and $b'(0)u \geqslant b(u) > du$ for all $u \in (0, K)$;
(H2) $b'(0)u - b(u) \leqslant Mu^{1+\nu}$ for all $u \in (0, K)$, some $M > 0$ and some $\nu \in (0, 1]$;
(H3) $b'(K) < d$;
(H4) $|b'(u_1) - b'(u_2)| \leqslant L|u_1 - u_2|^\nu$ for all $u_1, u_2 \in (0, K)$ and some $L > 0$.

It is easily seen that if $b \in C^2([0, K])$, then (H2) and (H4) hold spontaneously. A prototype of such functions which has been widely used in the mathematical biology literature is $b(u) = pue^{-\alpha u}$ for a wide range of parameters $p > 0$ and $\alpha > 0$. For convenience of discussion, we extend and improve the birth function $b(u)$ to $\hat{b}(u) \in\in C^1(\mathbb{R})$ in a natural way: $\hat{b}(u) = b(u)$ for $u \in [0, K]$, and $\hat{b}'(u) = b'(0)$ for $u \leqslant 0$ and $\hat{b}'(u) = b'(K)$ for $u \geqslant K$. This can be achieved by modifying (if necessary) the definition of b outside the closed interval $[0, K]$, giving a increasing and smooth $\hat{h}(u)$ on $\mathbb{R}$, which will still be denoted by $b(u)$ in the rest of the paper.

As Chen and Guo [7] did to (1.3), for convenience, we embed (1.7) into its continuum version

$$\begin{aligned} u_t(x,t) = \; & D[u(x+1,t) + u(x-1,t) - 2u(x,t)] - du(x,t) \\ & + b(u(x,t-r)), \quad x \in \mathbb{R}, \quad t > 0. \end{aligned} \tag{1.9}$$

We are interested in monotonic travelling waves $u(x,t) = U(x+ct)$ of (1.9), with U saturating at 0 and K, and our main concerns are the existence, uniqueness and asymptotic stability of such travelling wavefronts. In order to address these questions, we need to find an increasing function $U(\xi)$, where $\xi = x + ct$ which is a solution of the following associated wave equation:

$$-cU'(\xi) + D[U(\xi+1) + U(\xi-1) - 2U(\xi)] - dU(\xi) + b(U(\xi - cr)) = 0, \quad (1.10)$$

subject to the boundary conditions

$$U(-\infty) = 0, \quad U(+\infty) = K. \quad (1.11)$$

The main results of this paper can be formulated as follows.

Theorem 1.1. *Assume that* (H1)–(H4) *hold. Then there exists a minimal wave speed* $c^* > 0$, *such that for each* $c > c^*$ *Eq.* (1.9) *has exactly one travelling wavefront* $U(x+ct)$ (*up to a translation*) *satisfying* $U(-\infty) = 0, U(+\infty) = K$ *and* $\limsup_{\xi\to-\infty} U(\xi)e^{-\Lambda_1(c)\xi} < +\infty$, *where* $\lambda = \Lambda_1(c)$ *is the smallest solution to the equation* $c\lambda - D[e^{\lambda} + e^{-\lambda} - 2] + d - b'(0)e^{-\lambda cr} = 0$. *Moreover, the travelling wavefront is strictly increasing and asymptotically stable with phase shift in the sense that if an initial data* $\varphi \in C(\mathbb{R}\times[-r,0],[0,K])$ *satisfies* $\liminf_{x\to+\infty} \varphi(x,0) > 0$ *and* $\lim_{x\to-\infty} \max_{s\in[-r,0]} |\varphi(x,s)e^{-\Lambda_1(c)x} - \rho_0 e^{\Lambda_1(c)cs}| = 0$ *for some* $\rho_0 \in (0,+\infty)$, *then the solution* $u(x,t)$ *of the corresponding initial value problem satisfies*

$$\lim_{t\to+\infty} \sup_{\mathbb{R}} \left| \frac{u(\cdot,t)}{U(\cdot + ct + \xi_0)} - 1 \right| = 0$$

for some $\xi_0 = \xi_0(U,\varphi) \in \mathbb{R}$.

Remark 1.1. The minimal wave speed $c^* = c^*(r)$ is determined by $\Delta(c,\lambda) = 0$, the characteristic equation of (1.10) at 0 and $\partial_\lambda\Delta(c,\lambda) = 0$, where $\Delta(c,\lambda)$ is defined by (2.1). By implicit differentiation and some tedious calculation, one can see that $c*(r)$ is decreasing in r. In the case $r = 0$, the results in Theorem 1.1 reduce to the corresponding ones in [7,8] for (1.3) in the case of $g(u) = u$ (linear diffusion). From $c^*(r) < c^*(0)$, one concludes that delay can induce (*slower*) travelling wavefronts, a phenomenon also observed in Zou [29] for a *continuous* delay reaction–diffusion equation.

Remark 1.2. Under (H1)–(H4), similar conclusions for delayed reaction–diffusion (1.5) can be obtained by the results in Schaaf [15].

Remark 1.3. In [22], in addition to isotropic property of solutions and the asymptotic speed of travelling wavefronts, Weng et al. also addressed the existence of travelling wavefronts, and existence and uniqueness of the associated initial value problem to (1.8) under assumptions similar to (H1)–(H4). However, they did not consider the uniqueness and stability of the travelling wavefronts, which are the main concerns of this paper (only to local model (1.7) though).

Remark 1.4. The assumption (H1) is a crucial one by which, the delayed term $b(u)$ is increasing on the interval $[0, K]$ and thus, the whole interaction term is quasi-monotone. Applying the upper-lower solutions and monotone iteration technique established in Wu and Zou [23], the existence of monotone travelling waves are also obtained for various quasi-monotone and monostable lattice differential equations with delays in Zou [28], Hsu and Lin [11], Ma et al. [13]. When K is such that $b(u)$ is not increasing on $[0, K]$, the problem becomes much harder due to lack of quasi-monotonicity. For such delayed equations without quasi-monotonicity, some existence results for travelling waves have been obtained in Wu and Zou [23] by using the idea of the so-called exponential ordering for delayed differential equations, Application of these results to particular model equations is not trivial as it requires construction of very demanding upper–lower solutions. Uniqueness and stability of travelling waves of such systems seem to be very interesting and challenging problems.

The rest of this paper is organized as follows. In Section 2, we establish the existence of a travelling wavefront by using super-sub solutions and monotone iteration technique developed in [23]. We point out that [22] also applied the same technique, and thus, our existence result essentially can be obtained from the corresponding ones in [22]. However, we still provide this section because we need some more specific information about the sup-sub solutions and the asymptotic behaviour of the travelling waves, which will be used in later sessions for proving the uniqueness and stability of the travelling wavefronts. In Section 3, we prove that the travelling wavefront obtained in Section 2 is unique up to a translation. In Section 4, we address the existence and uniqueness of solution to the corresponding initial value problem associated to (1.9). We point out that although a similar result was established by fixed point theorem for a contracting map in [22], we decide to follow the direction of Section 2 to use the technique of super-sub solutions and comparison technique to achieve the goal. As can be naturally expected, some by-products (lemmas) in this section will then be reused in Section 5 to prove the asymptotic stability. The application of such a squeezing technique is motivated by the work of [6,7,19].

2. Existence of travelling waves

In this section, we first establish the existence of travelling wavefronts of (1.9) by using the sub-super solutions technique and an iteration scheme.

Firstly, we set

$$\Delta(c, \lambda) := c\lambda - D\left[e^{\lambda} + e^{-\lambda} - 2\right] + d - b'(0)e^{-\lambda c r}. \tag{2.1}$$

Lemma 2.1. *Assume that $b'(0) > d$. Then there exists a unique $c^* > 0$ such that*

(i) *if $c \geqslant c^*$, then there exist two positive numbers $\Lambda_1(c)$ and $\Lambda_2(c)$ with $\Lambda_1(c) \leqslant \Lambda_2(c)$ such that*

$$\Delta(c, \Lambda_1(c)) = \Delta(c, \Lambda_2(c)) = 0;$$

(ii) *if $c < c^*$, then $\Delta(c, \lambda) < 0$ for all $\lambda \geqslant 0$;*

(iii) *if $c = c^*$, then $\Lambda_1(c^*) = \Lambda_2(c^*) := \Lambda^*$, and if $c > c^*$, then $\Lambda_1(c) < \Lambda^* < \Lambda_2(c)$ and*

$$\Delta(c, \cdot) > 0 \quad \text{in } (\Lambda_1(c), \Lambda_2(c)), \qquad \Delta(c, \cdot) < 0 \quad \text{in } \mathbb{R} \setminus [\Lambda_1(c), \Lambda_2(c)],$$

(iv) *if $c > c^*$, then $\Lambda_1'(c) < 0$, $\Lambda_2'(c) > 0$. Moreover,*

$$\lim_{c \searrow c^*} \Lambda_1'(c) = -\infty, \qquad \lim_{c \searrow c^*} \Lambda_2'(c) = +\infty.$$

The proof of Lemma 2.1 is easy and is thus omitted.

For any absolutely continuous function $\phi : \mathbb{R} \to \mathbb{R}$, we set

$$\begin{aligned} N_c[\phi](\xi) := c \lim_{h \searrow 0} \frac{\phi(\xi) - \phi(\xi - h)}{h} - D[\phi(\xi+1) + \phi(\xi-1) - 2\phi(\xi)] + d\phi(\xi) \\ -b(\phi(\xi - cr)). \end{aligned} \tag{2.2}$$

Definition 2.1. An absolutely continuous function $\phi : \mathbb{R} \to [0, K]$ is called a supersolution (a subsolution, resp.) of (1.10) if for almost every $\xi \in \mathbb{R}$, $N_c[\phi](\xi) \geqslant 0$ ($\leqslant 0$, resp.).

Lemma 2.2. *Assume that* (H1) *and* (H2) *hold. Let $c > c^*$ and $\Lambda_1(c), \Lambda_2(c)$ be defined as in Lemma* 2.1. *Then for every $\beta \in (1, \min\{1 + \nu, \frac{\Lambda_2(c)}{\Lambda_1(c)}\})$, where $\nu \in (0, 1]$ is as in* (H2), *there exists $Q(c, \beta) \geqslant 1$, such that for any $q \geqslant Q(c, \beta)$ and any $\xi^\pm \in \mathbb{R}$, the functions $\phi^\pm$ defined by*

$$\phi^+(\xi) := \min\left\{K, e^{\Lambda_1(c)(\xi+\xi^+)} + qe^{\beta\Lambda_1(c)(\xi+\xi^+)}\right\}, \quad \xi \in \mathbb{R} \tag{2.3}$$

and

$$\phi^-(\xi) := \max\left\{0, e^{\Lambda_1(c)(\xi+\xi^-)} - qe^{\beta\Lambda_1(c)(\xi+\xi^-)}\right\}, \quad \xi \in \mathbb{R} \tag{2.4}$$

are a supersolution and a subsolution to (1.10), *respectively.*

Proof. It is easily seen that there exists $\xi^* \leqslant -\xi^+ - \frac{1}{\beta\Lambda_1(c)} \ln \frac{q}{K}$, such that $\phi^+(\xi) = K$ for $\xi > \xi^*$ and $\phi^+(\xi) = e^{\Lambda_1(c)(\xi+\xi^+)} + qe^{\beta\Lambda_1(c)(\xi+\xi^+)}$ for $\xi \leqslant \xi^*$.

For $\xi > \xi^*$, we have

$$N_c[\phi^+](\xi) = -D[\phi^+(\xi-1) - K] + dK - b(\phi^+(\xi - cr)) \geqslant dK - b(K) = 0.$$

For $\xi \leqslant \xi^*$, we have

$$\begin{aligned} N_c[\phi^+](\xi) \geqslant\; & e^{\Lambda_1(c)(\xi+\xi^+)} \left[c\Lambda_1(c) - D(e^{\Lambda_1(c)} + e^{-\Lambda_1(c)} - 2) + d\right] + qe^{\beta\Lambda_1(c)(\xi+\xi^+)} \\ & \times\left[c\beta\Lambda_1(c) - D\left(e^{\beta\Lambda_1(c)} + e^{-\beta\Lambda_1(c)} - 2\right) + d\right] - b(\phi^+(\xi - cr)) \\ \geqslant\; & qe^{\beta\Lambda_1(c)(\xi+\xi^+)}\Delta\left(c, \beta\Lambda_1(c)\right) + b'(0)\phi^+(\xi - cr) - b(\phi^+(\xi - cr)\big) > 0. \end{aligned}$$

Therefore, ϕ^+ is a supersolution of (1.10).

Let $\xi_* = -\xi^- - \frac{1}{(\beta-1)\Lambda_1(c)} \ln q$. If $q \geqslant 1$, then $\xi_* \leqslant -\xi^-$. Clearly, $\phi^-(\xi) = 0$ for $\xi > \xi_*$ and $\phi^-(\xi) = e^{\Lambda_1(c)(\xi+\xi^-)} - qe^{\beta\Lambda_1(c)(\xi+\xi^-)}$ for $\xi \leqslant \xi_*$.

For $\xi > \xi_*$, we have

$$N_c[\phi^-](\xi) = -D\phi^-(\xi - 1) - b(\phi^-(\xi - cr)) \leqslant 0.$$

For $\xi \leqslant \xi_*$, we have $\xi + \xi^- \leqslant -\frac{1}{(\beta-1)\Lambda_1(c)} \ln q$, and hence

$$\begin{aligned} N_c[\phi^-](\xi) \leqslant\; & e^{\Lambda_1(c)(\xi+\xi^-)} \left[c\Lambda_1(c) - D(e^{\Lambda_1(c)} + e^{-\Lambda_1(c)} - 2) + d\right] - qe^{\beta\Lambda_1(c)(\xi+\xi^-)} \\ & \times\left[c\beta\Lambda_1(c) - D\left(e^{\beta\Lambda_1(c)} + e^{-\beta\Lambda_1(c)} - 2\right) + d\right] - b(\phi^-(\xi - cr)) \\ \leqslant\; & -qe^{\beta\Lambda_1(c)(\xi+\xi^-)}\Delta(c, \beta\Lambda_1(c)) + b'(0)\phi^-(\xi - cr) - b(\phi^-(\xi - cr)) \\ \leqslant\; & -qe^{\beta\Lambda_1(c)(\xi+\xi^-)}\Delta(c, \beta\Lambda_1(c)) + M[\phi^-(\xi - cr)]^{1+\nu} \\ \leqslant\; & -qe^{\beta\Lambda_1(c)(\xi+\xi^-)}\Delta(c, \beta\Lambda_1(c)) + Me^{(1+\nu)\Lambda_1(c)(\xi+\xi^-)} \\ \leqslant\; & \left\{-q\Delta(c, \beta\Lambda_1(c)) + Me^{(1+\nu-\beta)\Lambda_1(c)(\xi+\xi^-)}\right\} e^{\beta\Lambda_1(c)(\xi+\xi^-)} \\ \leqslant\; & \left\{-q\delta(c, \beta\Lambda_1(c)) + Me^{-\frac{1+\nu-\beta}{\beta-1}\ln q}\right\} e^{\beta\Lambda_1(c)(\xi+\xi^-)} \\ =\; & \left\{-q^{\frac{\nu}{\beta-1}}\Delta(c, \beta\Lambda_1(c)) + M\right\} q^{-\frac{1+\nu-\beta}{\beta-1}} e^{\beta\Lambda_1(c)(\xi+\xi^-)} \leqslant 0, \end{aligned}$$

provided that $q \geqslant Q(c,\beta) := \max\left\{1, \left[\frac{M}{\Delta(c,\beta\Lambda_1(c))}\right]^{\frac{\beta-1}{\nu}}\right\}$. Therefore, ϕ^- is a subsolution of (1.10). The proof is completed. $\square$

Remark 2.1. In particular, we may choose $\beta = \min\{1+\nu/2, \Lambda^*/\Lambda_1(c)\} \in (1, \min\{1+\nu, \frac{\Lambda_2(c)}{\Lambda_1(c)}\})$ in Lemma 2.2. As $\lim_{c\searrow c^*} \Lambda_1(c) = \Lambda^*$, we see that $\beta = \Lambda^*/\Lambda_1(c)$ if $c - c^*$ is small enough. Therefore, we have

$$Q(c,\beta) = \max\left\{1, \left[\frac{M}{\Delta(c,\Lambda^*)}\right]^{\frac{\Lambda^*-\Lambda_1(c)}{\nu\Lambda_1(c)}}\right\}.$$

Let $c = c(\lambda)$, $\lambda > 0$ be defined by $\Delta(c(\lambda),\lambda) \equiv 0$. Then it is easily seen that $c'(\Lambda^*) = 0$ and $c''(\Lambda^*) > 0$. Hence, we have

$$\begin{aligned}
\lim_{c\searrow c^*} \ln[(c-c^*)\Lambda^*]^{\Lambda^*-\Lambda_1(c)} &= \lim_{c\searrow c^*} (\Lambda^* - \Lambda_1(c)) \ln[c - c^*] \\
&= \lim_{\lambda\nearrow\Lambda^*} (\Lambda^* - \lambda) \ln[c(\lambda) - c^*] \\
&= \lim_{\lambda\nearrow\Lambda^*} \frac{c'(\lambda)(\lambda-\Lambda^*)^2}{c(\lambda)-c^*} \\
&= \frac{2c'(\Lambda^*)}{c''(\Lambda^*)} = 0.
\end{aligned}$$

Since $\Delta(c,\Lambda^*) \geqslant (c-c^*)\Lambda^* > 0$, we find

$$\liminf_{c\searrow c^*} [\Delta(c,\Lambda^*)]^{\Lambda^*-\Lambda_1(c)} \geqslant \lim_{c\searrow c^*} [(c-c^*)\Lambda^*]^{\Lambda^*-\Lambda_1(c)} = 1.$$

Therefore, $\limsup_{c\searrow c^*} Q(c,\beta) < +\infty$. Thus we can assume, without loss of generality, that q is independent of c if $c - c^*$ is small enough.

The following is our main result for the existence of travelling waves.

Theorem 2.1. *Assume* (H1) *and* (H2) *hold. Let $c^* > 0$ be as in Lemma* 2.1. *Then for each $c \geqslant c^*$,* (1.9) *admits a travelling wave solution $u(x,t) = U(x+ct)$ satisfying $U' > 0$ on $\mathbb{R}$. Furthermore, for $c > c^*$, U also satisfies*

$$\lim_{\xi\to-\infty} U(\xi)e^{-\lambda\xi} = 1, \qquad \lim_{\xi\to-\infty} U'(\xi)e^{-\lambda\xi} = \lambda, \tag{2.5}$$

where $\lambda = \Lambda_1(c)$ is the smallest solution to the equation

$$\Delta(c,\lambda) = c\lambda - D\left[e^{\lambda} + e^{-\lambda} - 2\right] + d - b'(0)e^{-\lambda c r} = 0.$$

For every $c < c^*$, (1.9) *has no travelling wave solutions satisfying* (2.5) *with $\lambda > 0$.*

Proof. For $c > c^*$, by virtue of Lemma 2.2, ϕ^+ and ϕ^- with $\xi^\pm = 0$ are a supersolution and a subsolution to (1.10), respectively. Since $\phi^-(\xi) \leqslant \phi^+(\xi)$ for all $\xi \in \mathbb{R}$, the iteration scheme

$$\phi_{n+1}(\xi) := \frac{1}{c} e^{-\frac{2D+d}{c}\xi} \int_{-\infty}^{\xi} e^{\frac{2D+d}{c}s} \left\{ D[\phi_n(s+1) + \phi_n(s-1)] + b(\phi_n(s-cr)) \right\} ds,$$

with $\phi_0(\xi) = \phi^+(\xi)$, shows that there exists a nondecreasing solution $U_c(\xi)$ to (1.10) and (1.11), which will be denoted by (U_c, c) and satisfies

$$e^{\Lambda_1(c)\xi} - qe^{\beta\Lambda_1(c)\xi} \leqslant U_c(\xi) \leqslant e^{\Lambda_1(c)\xi} + qe^{\beta\Lambda_1(c)\xi}, \quad \xi \in \mathbb{R}. \tag{2.6}$$

Clearly, (U_c, c) is also a weak solution of (1.10), i.e., for any $\phi \in C_0^\infty(\mathbb{R})$, we have

$$\begin{aligned} c \int_{\mathbb{R}} U_c \phi' + \int_{\mathbb{R}} \{D[U_c(\cdot + 1) + U_c(\cdot - 1) - 2U_c] - dU_c\} \phi \\ + \int_{\mathbb{R}} b(U_c(\cdot))\phi(\cdot + cr) = 0. \end{aligned} \tag{2.7}$$

Take $u^* \in (0, K)$, then for each $c > c^*$, there exists $\xi_c \in \mathbb{R}$ such that $U_c(\xi_c) = u^*$. By Helly's Theorem, there exists a sequence $c_m > c^*$ with $c_m \searrow c^*$ as $m \to +\infty$, such that $\tilde{U}_{c_m}(\cdot) := U_{c_m}(\cdot + \xi_{c_m})$ converges pointwise to a nondecreasing function U_{c^*} as $m \to +\infty$.

Applying the Lebesgue's Dominated Convergence Theorem to (2.7) with c replaced by c_m and U_c replaced by $\tilde{U}_{c_m}$ then gives

$$\begin{aligned} c^* \int_{\mathbb{R}} U_{c^*} \phi' + \int_{\mathbb{R}} \{D[U_{c^*}(\cdot + 1) + U_{c^*}(\cdot - 1) - 2U_{c^*}] - dU_{c^*}\} \phi \\ + \int_{\mathbb{R}} b(U_{c^*}(\cdot))\phi(\cdot + c^* r) = 0 \end{aligned} \tag{2.8}$$

for all $\phi \in C_0^\infty(\mathbb{R})$. Since $c^* > 0$, (2.8) implies that $U_{c^*} \in W^{1,\infty}(\mathbb{R})$, and hence, a bootstrap argument shows that U_{c^*} is of class C^1 and thus a solution of (1.10). Since $U_{c^*}(0) = u^* \in (0, K)$ and $b(u) > du$ for $u \in (0, K)$, it follows that $U_{c^*}(-\infty) = 0$ and $U_{c^*}(+\infty) = K$.

Next, we show that for each $c \geqslant c^*$, $U_c' > 0$ on $\mathbb{R}$. Suppose for the contrary that $U_c'(x_0) = 0$ for some $x_0 \in \mathbb{R}$. Since $U_c' \geqslant 0$ on $\mathbb{R}$, we have $U_c''(x_0) = 0$, and hence

$$\begin{aligned} 0 = cU_c''(x_0) &= D\left[U_c'(x_0+1) + U_c'(x_0-1)\right] + b'(U_c(x_0 - cr))U_c'(x_0 - cr) \\ &\geqslant D\left[U_c'(x_0+1) + U_c'(x_0-1)\right] \geqslant 0, \end{aligned}$$

which together with the fact that $b'(0) > d > 0$ implies that $U_c'(x_0+1) = U_c'(x_0-1) = U_c'(x_0) = 0$ and $U_c'(x_0 - cr) = 0$ if $-x_0 > 0$ is sufficiently large. So by using an induction argument, we conclude that

$$U_c'(x_0 + n - mcr) = 0 \quad \text{for all } n, m \in \mathbb{Z} \text{ with } m \geqslant 0.$$

Let $w_{n,m}(t) := U_c'(x_0 + n - mcr + t)$, then $w_{n,m}$ satisfies the initial value problem

$$\begin{aligned} w_{n,m}' &= \frac{D}{c}\left[w_{n+1,m} + w_{n-1,m} - 2w_{n,m}\right] - \frac{d}{c}\, w_{n,m} \\ &\quad + \frac{1}{c}\, b'(U_c(x_0 + n - (m+1)cr + t))w_{n,m+1}, \\ &\qquad w_{n,m}(0) = 0, \end{aligned}$$

where $n, m \in \mathbb{Z}$ with $m \geqslant 0$. By the uniqueness of the initial value problem, we have $w_{n,m}(t) \equiv 0$, and hence $U \equiv$ const., which is a contradiction.

If $c > c^*$, it then follows from (2.6) that

$$\lim_{\xi \to -\infty} |U_c(\xi)e^{-\Lambda_1(c)\xi} - 1| \leqslant \lim_{\xi \to -\infty} q e^{(\beta-1)\Lambda_1(c)\xi} = 0.$$

Since $0 \leqslant b'(0)u - b(u) \leqslant Mu^{1+\nu}$ for $u \in (0, K)$, we have

$$\begin{aligned} &\lim_{\xi \to -\infty} |b(U_c(\xi - cr)) - b'(0)U_c(\xi - cr)|e^{-\Lambda_1(c)\xi} \\ &\quad \leqslant \lim_{\xi \to -\infty} M[U_c(\xi - cr)]^{1+\nu} e^{-\Lambda_1(c)\xi} = 0. \end{aligned}$$

Hence, for $c > c^*$, we also have

$$\begin{aligned} &\lim\nolimits_{\xi \to -\infty} U_c'(\xi)e^{-\Lambda_1(c)\xi} \\ &\quad = \frac{1}{c} \lim\nolimits_{\xi \to -\infty} \{D[U_c(\xi+1) + U_c(\xi-1) - 2U_c(\xi)] - dU_c(\xi) \\ &\qquad + b(U_c(\xi - cr))\}\, e^{-\Lambda_1(c)\xi} \\ &\quad = \frac{1}{c}\left\{D\left[e^{\Lambda_1(c)} + e^{-\Lambda_1(c)} - 2\right] - d + b'(0)e^{-\Lambda_1(c)cr}\right\} \\ &\quad = \Lambda_1(c). \end{aligned}$$

Finally, if $c < c^*$, and $U(x+ct)$ is a solution to (1.9) satisfying (2.5) with $\lambda > 0$. Then U satisfies (1.10). Multiplying (1.10) by $e^{-\lambda\xi}$ and sending $\xi \to -\infty$ then gives $\Delta(c, \lambda) = 0$, a contradiction. This completes the proof. $\square$

3. Uniqueness of travelling waves

In this section, we prove that the travelling wavefront obtained in Section 2 is unique up to a translation.

Theorem 3.1. *Assume* (H1)–(H3) *hold. For each* $c > c^*$, *let* (U, c) *be the solution to* (1.10) *and* (1.11) *as given in Theorem* 2.1. *Let* $(\hat{U}, c)$ *be another solution to* (1.10) *and* (1.11) *satisfying*

$$\limsup_{\xi \to -\infty} \hat{U}(\xi) e^{-\Lambda_1(c)\xi} < +\infty. \tag{3.1}$$

Then there exists $\bar{z} \in \mathbb{R}$ *such that* $\hat{U}(\cdot) = U(\cdot + \bar{z})$.

Proof. Firstly, we observe that if $(\hat{U}, c)$ is a solution to (1.10) and (1.11), then

$$\hat{U} \leqslant K. \tag{3.2}$$

Otherwise, suppose that there exists x_0 so that $\hat{U}(x_0) > K$ and $\hat{U}(x) \leqslant \hat{U}(x_0)$ for all $x \in \mathbb{R}$. Then, we have $\hat{U}'(x_0) = 0$ and so

$$\begin{aligned}
0 &\geqslant -c\hat{U}'(x_0) + D\left[\hat{U}(x_0+1) + \hat{U}(x_0-1) - 2\hat{U}(x_0)\right] \\
&= d\hat{U}(x_0) - b\left(\hat{U}(x_0 - cr)\right) \\
&\geqslant d\hat{U}\left(x_0) - b(\hat{U}(x_0)\right) > 0,
\end{aligned}$$

which is a contradiction.

In what follows, we denote by (U, c) the solution of (1.10) and (1.11) given in Theorem 2.1. Since $b'(K) < d$, we can choose $\alpha > 0$ and $\kappa > 0$ such that

$$d > \alpha e^{\Lambda_1(c)cr} + b'(\eta) \quad \text{for } \eta \in [K - \kappa, K + \kappa]. \tag{3.3}$$

Take $M_1 > cr$ sufficiently large so that

$$U(\xi) \geqslant K - \kappa/2 \quad \text{for } \xi \geqslant M_1 - cr. \tag{3.4}$$

Since $\lim_{x\to-\infty} U'(x)e^{-\Lambda_1(c)x} = \Lambda_1(c) > 0$, we can take $M_2 > 0$ sufficiently large such that

$$U'(x)e^{-\Lambda_1(c)x} \geqslant \tfrac{1}{2}\,\Lambda_1(c) \quad \text{for } x \leqslant -M_2. \tag{3.5}$$

Denote

$$\varrho := \min\{U'(\xi); -M_2 \leqslant \xi \leqslant M_1\} > 0.$$

Let $\mu \in (0, \kappa/2)$ and define

$$B = \max\left\{\frac{\mu}{\alpha\varrho}\, b'_{\max} e^{\Lambda_1(c)M_1}, \frac{3\mu}{\alpha\Lambda_1(c)} b'_{\max}\right\}. \tag{3.6}$$

We claim that for $\mu \in (0, \kappa/2)$ given above, there exists $z \geqslant M_1$, such that

$$U(x+z) + \mu \min\left\{1, e^{\Lambda_1(c)x}\right\} > \hat{U}(x) \quad \text{for all } x \in \mathbb{R}. \tag{3.7}$$

In fact, we can first choose $z_1 \geqslant M > 0$ such that $e^{\Lambda_1(c)z_1} > \rho := \limsup_{x\to-\infty} \hat{U}(x) e^{-\Lambda_1(c)x}$. Since

$$\lim_{x\to-\infty} U(x+z_1)e^{-\Lambda_1(c)x} = e^{\Lambda_1(c)z_1} > \rho,$$

there exists $M_3 > 0$ such that

$$U(x+z_1) > \hat{U}(x) \quad \text{for } x \leqslant -M_3.$$

Take $M_4 > 0$ sufficiently large so that

$$U(x) + \mu e^{-\Lambda_1(c)M_3} > K \quad \text{for } x \geqslant M_4.$$

Let $z = z_1 + M_3 + M_4$, then for $x \leqslant -M_3$, we have

$$U(x+z) + \mu \min\left\{1, e^{\Lambda_1(c)x}\right\} - \hat{U}(x) > U(x+z_1) - \hat{U}(x) > 0$$

and for $x \geqslant -M_3$, we have $x + z \geqslant M_4$, and hence, (3.2) implies that

$$\begin{aligned} &U(x+z) + \mu \min\left\{1, e^{\Lambda_1(c)x}\right\} - \hat{U}(x) \\ &\quad \geqslant U(x+z) + \mu e^{-\Lambda_1(c)M_3} - \hat{U}(x) > K - \hat{U}(x) \geqslant 0. \end{aligned}$$

Define

$$w(x,t) = U\left(x + z + B\left(1 - e^{-\alpha t}\right)\right) + \mu \min\left\{1, e^{\Lambda_1(c)x}\right\} e^{-\alpha t} - \hat{U}(x), \tag{3.8}$$

then we have

$$w(x,0) = U(x+z) + \mu \min\left\{1, e^{\Lambda_1(c)x}\right\} - \hat{U}(x) > 0.$$

We claim that $w(x,t) > 0$ for all $x \in \mathbb{R}$ and $t \geqslant 0$. To see this, suppose that there exist $x_0 \in \mathbb{R}$ and $t_0 > 0$ such that

$$w(x_0,t_0) = U(P_0) + \mu \min\left\{1, e^{\Lambda_1(c)x_0}\right\} e^{-\alpha t_0} - \hat{U}(x_0) = 0 \leqslant w(x,t) \tag{3.9}$$

for all $x \in \mathbb{R}$ and $t \in [0, t_0]$, where

$$P_0 = x_0 + z + B(1 - e^{-\alpha t_0}).$$

Clearly, if $x_0 = 0$, then

$$w_x(x_0-, t_0) = U'(P_0) - \hat{U}'(x_0) + \mu \Lambda_1(c) e^{\Lambda_1(c)x_0} e^{-\alpha t_0} \leqslant 0$$

and

$$w_x(x_0+, t_0) = U'(P_0) - \hat{U}'(x_0) \geqslant 0,$$

which is impossible. So we have $x_0 \neq 0$, and hence

$$w_x(x_0, t_0) = U'(P_0) - \hat{U}'(x_0) + \mu \Lambda_1(c) e^{\Lambda_1(c)x_0} e^{-\alpha t_0} = 0 \quad \text{if } x_0 < 0 \tag{3.10}$$

and

$$w_x(x_0, t_0) = U'(P_0) - \hat{U}'(x_0) = 0 \quad \text{if } x_0 > 0. \tag{3.11}$$

In the case where $x_0 > 0$, we have

$$\begin{aligned}
0 \;&\geqslant\; w_t(x_0,t_0) - D[w(x_0+1,t_0) + w(x_0-1,t_0) - 2w(x_0,t_0)] \\
&=\; -\alpha\mu e^{-\alpha t_0} + \alpha B U'(P_0) e^{-\alpha t_0} - \mu D\left[1 + \min\left\{1, e^{\Lambda_1(c)(x_0-1)}\right\} - 2\right] e^{-\alpha t_0} \\
&\quad -D[U(P_0+1) + U(P_0-1) - 2U(P_0)] + D\left[\hat{U}(x_0+1) + \hat{U}(x_0-1) - 2\hat{U}(x_0)\right]
\end{aligned}$$

$$\begin{aligned}
&\geqslant [-\alpha\mu + \alpha B U'(P_0)]e^{-\alpha t_0} - cU'(P_0) - dU(P_0) + b(U(P_0 - cr)) \\
&\quad + c\hat{U}'(x_0) + d\hat{U}(x_0) - b(\hat{U}(x_0 - cr)) \\
&= [d\mu - \alpha\mu + \alpha B U'(P_0)]e^{-\alpha t_0} + b(U(P_0 - cr)) - b(\hat{U}(x_0 - cr)) \\
&\geqslant [d\mu - \alpha\mu + \alpha B U'(P_0)]e^{-\alpha t_0} + b(U(P_0 - cr)) - b\left(U(P_0 - cr) + \mu e^{-\alpha t_0}\right) \\
&= \left[d - \alpha + \frac{\alpha B}{\mu} U'(P_0) - b'(\eta)\right] \mu e^{-\alpha t_0},
\end{aligned} \tag{3.12}$$

where $\eta \in (U(P_0 - cr), U(P_0 - cr) + \mu)$. Since $P_0 > z \geqslant M_1$, it follows from (3.4) that $\eta \geqslant U(P_0 - cr) \geqslant K - \kappa/2$, and hence, by (3.3), the right-hand side of (3.12) is positive, which is a contradiction.

In the case where $x_0 < 0$, we have

$$\begin{aligned}
0 &\geqslant w_t(x_0, t_0) - D[w(x_0 + 1, t_0) + w(x_0 - 1, t_0) - 2w(x_0, t_0)] \\
&= -\alpha\mu e^{\Lambda_1(c)x_0} e^{-\alpha t_0} + \alpha B U'(P_0) e^{-\alpha t_0} - \mu D \left[\min\left\{1, e^{\Lambda_1(c)(x_0+1)} t\right\} + e^{\Lambda_1(c)(x_0-1)} \right. \\
&\quad \left. -2e^{\Lambda_1(c)x_0}\right] e^{-\alpha t_0} - D[U(P_0 + 1) + U(P_0 - 1) - 2U(P_0)] \\
&\quad + D\left[\hat{U}(x_0 + 1) + \hat{U}(x_0 - 1) - 2\hat{U}(x_0)\right] \\
&\geqslant \left[-\alpha\mu e^{\Lambda_1(c)x_0} + \alpha B U'(P_0)\right] e^{-\alpha t_0} - \mu D e^{\Lambda_1(c)x_0} \left[e^{\Lambda_1(c)} + e^{-\Lambda_1(c)} - 2\right] e^{-\alpha t_0} \\
&\quad - cU'(P_0) - dU(P_0) + b(U(P_0 - cr)) + c\hat{U}'(x_0) + d\hat{U}(x_0) - b\left(\hat{U}(x_0 - cr)\right) \\
&\geqslant \left[-\alpha\mu e^{\Lambda_1(c)x_0} + \alpha B U'(P_0)\right] e^{-\alpha t_0} - \mu D e^{\Lambda_1(c)x_0} \left[e^{\Lambda_1(c)} + e^{-\Lambda_1(c)} - 2\right] e^{-\alpha t_0} \\
&\quad + \mu c \Lambda_1(c) e^{\Lambda_1(c)x_0} e^{-\alpha t_0} + d\mu e^{\Lambda_1(c)x_0} e^{-\alpha t_0} \\
&\quad + b(U(P_0 - cr)) - b\left(U(P_0 - cr) + \mu \min\left\{1, e^{\Lambda_1(c)(x_0 - cr)}\right\} e^{-\alpha t_0}\right) \\
&= \left[-\alpha\mu e^{\Lambda_1(c)x_0} + \alpha B U'(P_0)\right] e^{-\alpha t_0} - \mu b'(\eta) e^{\Lambda_1(c)(x_0 - cr)} e^{-\alpha t_0} \\
&\quad + \left[c\Lambda_1(c) + d - D\left(e^{\Lambda_1(c)} + e^{-\Lambda_1(c)} - 2\right)\right] \mu e^{\Lambda_1(c)x_0} e^{-\alpha t_0} \\
&\geqslant \left[-\alpha + \frac{\alpha B}{\mu} U'(P_0) e^{-\Lambda_1(c)P_0} + (b'(0) - b'(\eta)) e^{-\Lambda_1(c)cr}\right] \mu e^{\Lambda_1(c)x_0} e^{-\alpha t_0} \\
&\geqslant \left[d e^{-\Lambda_1(c)cr} - \alpha + \frac{\alpha B}{\mu} U'(P_0) e^{-\Lambda_1(c)P_0} - b'(\eta) e^{-\Lambda_1(c)cr}\right] \mu e^{\Lambda_1(c)x_0} e^{-\alpha t_0},
\end{aligned} \tag{3.13}$$

where $\eta \in (U(P_0 - cr), U(P_0 - cr) + \mu)$.

In this case, if $P_0 \leqslant -M_2$, then (3.5) and (3.6) imply that $\frac{\alpha B}{\mu} U'(P_0) e^{-\Lambda_1(c)P_0} - b'(\eta)e^{-\Lambda_1(c)cr} \geqslant \frac{\alpha B \Lambda_1(c)}{2\mu} - b'_{\max} \geqslant 0$, and hence, by (3.3), the right-hand side of (3.13) is positive, which is a contradiction.

If $P_0 \in [-M_2, M_1]$, then by (3.6), we have $\frac{\alpha B}{\mu} U'(P_0) e^{-\Lambda_1(c)P_0} - b'(\eta)e^{-\Lambda_1(c)cr} \geqslant \frac{\alpha B \varrho}{\mu} e^{-\Lambda_1(c)M_1} - b'_{\max} \geqslant 0$, and hence the right-hand side of (3.13) is positive, which is a contradiction.

If $P_0 \geqslant M_1$, then it follows from (3.4) that $\eta \geqslant U(P_0 - cr) \geqslant K - \kappa/2$, and hence, by (3.3), $de^{-\Lambda_1(c)cr} - \alpha - b'(\eta)e^{-\Lambda_1(c)cr} > 0$. So the right-hand side of (3.13) is positive, which is also a contradiction.

Taking the limit $t \to +\infty$ in (3.8), we get

$$U(x+z+B) \geqslant \hat{U}(x) \quad \text{for all } x \in \mathbb{R}.$$

Thus there exists a minimal $\bar{z}$ such that

$$U(x) \geqslant \hat{U}(x-z) \quad \text{for all } x \in \mathbb{R} \text{ and } z \geqslant \bar{z}. \tag{3.14}$$

We assert that if $U(x) \neq \hat{U}(x-\bar{z})$ for some x, then $U(x) > \hat{U}(x-\bar{z})$ for all $x \in \mathbb{R}$. Otherwise, suppose that for some x_0, $U(x_0) = \hat{U}(x_0 - \bar{z})$. Let $w(x) = U(x) - \hat{U}(x-\bar{z})$. Then we have $w'(x_0) = 0$ and $w(x) \geqslant w(x_0) = 0$ for all $x \in \mathbb{R}$, and hence

$$\begin{aligned}
0 &\leqslant D[w(x_0+1) + w(x_0-1) - 2w(x_0)] \\
&= -cw'(x_0) + D[w(x_0+1) + w(x_0-1) - 2w(x_0)] - dw(x_0) \\
&= -cU'(x_0) + D[U(x_0+1) + U(x_0-1) - 2U(x_0)] - dU(x_0) \\
&\quad + c\hat{U}'(x_0 - \bar{z}) - D\left[\hat{U}(x_0+1-\bar{z}) + \hat{U}(x_0-1-\bar{z}) - 2\hat{U}(x_0-\bar{z})\right] + d\hat{U}(x_0-\bar{z}) \\
&= -b\left(U(x_0 - cr)\right) + b\left(\hat{U}(x_0 - \bar{z} - cr)\right) \\
&= -b'(\eta) w(x_0 - cr) \leqslant 0,
\end{aligned}$$

where $\eta \in (\hat{U}(x_0 - \bar{z} - cr), U(x_0 - cr))$. Hence, notice that $b'(0) > d > 0$, we find $w(x_0+1) = w(x_0-1) = w(x_0) = 0$ and $w(x_0 - cr) = U(x_0 - cr) - \hat{U}(x_0 - \bar{z} - cr) = 0$ if $-x_0 > 0$ is sufficiently large. From which, by an induction argument, we can show that

$$w(x_0 - mcr + n) = 0 \quad \text{for all } n, m \in \mathbb{Z} \text{ with } m \geqslant 0. \tag{3.15}$$

Let $v_{n,m}(t) = w(x_0 - mcr + n + ct), n \in \mathbb{Z}, m \geqslant 0$, then by the Mean Value Theorem, it is easily seen that $v_{n,m}(t)$ satisfies the initial value problem

$$v'_{n,m} = D\left[v_{n+1,m} + v_{n-1,m} - 2v_{n,m}\right] - dv_{n,m} + P_{n,m+1}(t)v_{n,m+1},$$

$$v_{n,m}(0) = 0,$$

where $n \in \mathbb{Z}, m \geqslant 0$ and

$$\begin{aligned} P_{n,m}(t) \ = \ & \int_0^1 b'[U(x_0 - mcr + n + ct) + \alpha\left(\hat{U}(x_0 - mcr + n - \bar{z} + ct\right) \\ & -U(x_0 - mcr + n + ct))]\, d\alpha. \end{aligned}$$

By the uniqueness of solutions to the initial value problem, we conclude that $v_{n,m}(t) \equiv 0$, and hence $w(x) \equiv 0$, which leads to a contradiction and establish the assertion.

In what follows, we suppose that $U(x) > \hat{U}(x - \bar{z})$ for all $x \in \mathbb{R}$. It follows that

$$1 \geqslant \rho e^{-\Lambda_1(c)\bar{z}}, \tag{3.16}$$

where $\rho = \limsup_{x \to -\infty} \hat{U}(x)e^{-\Lambda_1(c)x}$.

Let $\varepsilon > 0$ and define

$$w(x,t) = U(x - \varepsilon(1 - e^{-\alpha t})) - \hat{U}(x - \bar{z}), \quad x \in \mathbb{R}, \quad t \in \mathbb{R}.$$

Then $w(x, 0) = U(x) - \hat{U}(x - \bar{z}) > 0$ for all $x \in \mathbb{R}$. Suppose that there exist $t_0 > 0$ and $x_0 \in \mathbb{R}$ such that

$$w(x_0, t_0) = U(x_0 - \varepsilon(1 - e^{-\alpha t_0})) - \hat{U}(x_0 - \bar{z}) = 0 < w(x,t) \quad \text{for } x \in \mathbb{R} \text{ and } \ t \in [0, t_0).$$

Then

$$w_x(x_0, t_0) = U'(x_0 - \varepsilon(1 - e^{-\alpha t_0})) - \hat{U}'(x_0 - \bar{z}) = 0.$$

Therefore, we have

$$\begin{aligned} 0 \ \leqslant \ & D[w(x_0 + 1, t_0) + w(x_0 - 1, t_0) - 2w(x_0, t_0)] \\ = \ & D[U(P_1 + 1) + U(P_1 - 1) - 2U(P_1)] \\ & -D\left[\hat{U}(x_0 + 1 - \bar{z}) + \hat{U}(x_0 - 1 - \bar{z}) - 2\hat{U}(x_0 - \bar{z})\right] \end{aligned}$$

$$
\begin{aligned}
&= c\left[U'(P_1)-\hat{U}'(x_0-\bar{z})\right]+d\left[U(P_1)-\hat{U}(x_0-\bar{z})\right]-b(U(P_1-cr))+b\left(\hat{U}(x_0-\bar{z}-cr)\right)\\
&= -b'(\eta)w(x_0-cr,t_0)\\
&\leqslant 0,
\end{aligned}
$$

where $P_1 = x_0 - \varepsilon(1 - e^{-\alpha t_0})$ and $\eta \in (\hat{U}(x_0 - cr), U(P_1 - cr))$. Since $b'(0) > d > 0$, it follows that $w(x_0 + 1, t_0) = w(x_0 - 1, t_0) = w(x_0, t_0) = 0$ and $w(x_0 - cr, t_0) = U(P_1 - cr) - \hat{U}(x_0 - cr) = 0$ if $-x_0 > 0$ is sufficiently large. By using a induction argument, it can be shown that

$$w(x_0 - mcr + n, t_0) = 0 \quad \text{for all } n, m \in \mathbb{Z} \text{ with } \ m \geqslant 0.$$

A similar argument as used above shows that

$$w(x, t_0) = U(x - \varepsilon(1 - e^{-\alpha t_0})) - \hat{U}(x - \bar{z}) \quad \text{for all } x \in \mathbb{R}.$$

Therefore, we have

$$
\begin{aligned}
e^{-\Lambda_1(c)\varepsilon(1-e^{-\alpha t_0})} &= \lim_{x\to-\infty} U(x - \varepsilon(1 - e^{-\alpha t_0})e^{-\Lambda_1(c)x}\\
&= \limsup_{x\to-\infty} \hat{U}(x - \bar{z})e^{-\Lambda_1(c)x}\\
&= \rho e^{-\Lambda_1(c)\bar{z}}.
\end{aligned} \tag{3.17}
$$

If $\rho e^{-\Lambda_1(c)\bar{z}} = 1$, then (3.17) leads to a contradiction. If $\rho e^{-\lambda_1(c)\bar{z}} < 1$, then we can choose $\varepsilon > 0$ in such a way that

$$e^{-\Lambda_1(c)\varepsilon} > \rho e^{-\Lambda_1(c)\bar{z}},$$

therefore, it follows from (3.17) that $e^{\Lambda_1(c)\varepsilon e^{-\alpha t_0}} < 1$, which is also a contradiction. So we have

$$w(x, t) = U(x - \varepsilon(1 - e^{-\alpha t})) - \hat{U}(x - \bar{z}) > 0 \quad \text{for all } x \in \mathbb{R} \text{ and } t \geqslant 0. \tag{3.18}$$

Passing to the limit as $t \to +\infty$ in (3.18) gives

$$U(x) \geqslant \hat{U}(x - (\bar{z} - \varepsilon)) \quad \text{for all } x \in \mathbb{R},$$

contradicting to the minimality of $\bar{z}$ and proving that $U(x) = \hat{U}(x - \bar{z})$ for all $x \in \mathbb{R}$. The proof is complete. □

As a direct consequence of Theorem 3.1, we have the following

Corollary 3.1. *For $c > c^*$, there are no solutions $(\hat{U}, c)$ of* (1.10) *and* (1.11) *satisfying*

$$\limsup_{\xi \to -\infty} \hat{U}(\xi) e^{-\Lambda_1(c)\xi} \leqslant 0.$$

4. The initial value problem

To study the asymptotic stability of the travelling waves, we first study the initial value problem

$$\begin{aligned} u_t(x,t) &= F[u](x,t), \quad x \in \mathbb{R}, \ t > 0, \\ u(x,s) &= \varphi(x,s), \quad x \in \mathbb{R}, \ s \in [-r, 0]. \end{aligned} \tag{4.1}$$

Here and in what follows, $F[u](x,t) = D[u(x+1,t)+u(x-1,t)-2u(x,t)]-du(x,t)+b(u(x,t-r))$.

For the existence of solutions to the initial value problem (4.1), we have the following result.

Lemma 4.1. *For every initial data $\varphi \in C(\mathbb{R} \times [-r,0],[0,K])$,* (4.1) *admits a unique solution $u \in C(\mathbb{R} \times [0,+\infty),[0,K])$ satisfying*

$$u(x \pm j, t) \geqslant D^{|j|}\varphi(x,0)t^{|j|}e^{-(2D+d)t}/|j|! \quad \textit{for all } x \in \mathbb{R}, \ j \in \mathbb{Z} \textit{ and } t > 0. \tag{4.2}$$

Proof. Clearly, (4.1) is equivalent to

$$\begin{aligned} u(x,t) &= \varphi(x,0)e^{-(2D+d)t} + \int_0^t e^{(2D+d)(\tau-t)}\{D[u(x+1,\tau)+u(x-1,\tau)] \\ &\quad + b(u(x,\tau-r))\}\,d\tau. \end{aligned} \tag{4.3}$$

The existence of solutions then follows by Picard's iteration and the monotonicity of the operator $T[u](x,t) := D[u(x+1,t)+u(x-1,t)]+b(u(x,t-r))$.

It follows from (4.3) that $u(x,t) \geqslant \varphi(x,0)e^{-(2D+d)t}$ and $u(x,t) \geqslant D\int_0^t e^{(2D+d)(\tau-t)} u(x \pm 1,\tau)\,d\tau$ for all $t > 0$. Therefore, (4.2) follows by an induction argument. This completes the proof. $\square$

Next, we establish some comparison results for solutions of the initial value problem (4.1).

Lemma 4.2. *Assume that u^1 and u^2 are continuous functions on $\mathbb{R} \times [-r,+\infty)$ such that $u^1 \geqslant 0$ and $u^2 \leqslant K$ on $\mathbb{R} \times [-r,+\infty)$, that $u^2 \leqslant u^1$ on $\mathbb{R} \times [-r,0]$ and that*

$$u_t^1(x,t) - F[u^1](x,t) \geqslant u_t^2(x,t) - F[u^2](x,t) \tag{4.4}$$

on $\mathcal{D} := \{(x,t) \in \mathbb{R} \times (0,+\infty) | u^2(x,t) > 0,\ u^1(x,t) < K\}$. *Then* $\min\{K, u^1\} \geqslant \max\{0, u^2\}$ *on* $\mathbb{R} \times (0,+\infty)$.

Proof. Clearly, we only need to show that $u^1 \geqslant u^2$ on $\mathcal{D}$. Since $w := u^2 - u^1$ is continuous and bounded from above by K, $\omega(t) := \sup_{\mathbb{R}} w(\cdot,t)$ is continuous on $[-r,+\infty)$. Suppose the assertion is not true. Let $M_0 > 0$ be such that $M_0 + d - b'_{\max} e^{-M_0 r} > 0$, then there exists $t_0 > 0$ such that $\omega(t_0) > 0$ and

$$\omega(t_0) e^{-M_0 t_0} = \sup_{t \geqslant -r} \left\{ \omega(t) e^{-M_0 t} \right\} > \omega(\tau) e^{-M_0 \tau} \quad \text{for all } \tau \in [-r, t_0). \tag{4.5}$$

Let $\{x_j\}_{j=1}^{\infty}$ be a sequence on $\mathbb{R}$ such that $w(x_j, t_0) > 0$ for all $j \geqslant 1$ and $\lim_{j \to +\infty} w(x_j, t_0) = \omega(t_0)$. Let $\{t_j\}_{j=1}^{\infty}$ be a sequence in $(0, t_0]$ such that

$$e^{-M_0 t_j} w(x_j, t_j) = \max_{t \in [0, t_0]} \left\{ e^{-M_0 t} w(x_j, t) \right\}. \tag{4.6}$$

As $w(x_j, t_0) > 0$, we have $w(x_j, t_j) = u^2(x_j, t_j) - u^1(x_j, t_j) > 0$, and hence $(x_j, t_j) \in \mathcal{D}$.

It follows from (4.5) that $\lim_{j \to +\infty} t_j = t_0$. Since

$$e^{-M_0 t_0} w(x_j, t_0) \leqslant e^{-M_0 t_j} w(x_j, t_j) \leqslant e^{-M_0 t_j} \omega(t_j) \leqslant e^{-M_0 t_0} \omega(t_0),$$

we have

$$e^{-M_0 (t_0 - t_j)} w(x_j, t_0) \leqslant w(x_j, t_j) \leqslant e^{-M_0 (t_0 - t_j)} \omega(t_0),$$

which yields $\lim_{j \to +\infty} w(x_j, t_j) = \omega(t_0)$.

In view of (4.6), for each $j \geqslant 1$, we obtain

$$\begin{aligned}
0 \ &\leqslant \ \underline{\mathrm{D}}_t \left\{ e^{-M_0 t} w(x_j, t) \right\} |_{t = t_j -} \\
&= \ \liminf_{h \searrow 0} \frac{e^{-M_0 t_j} w(x_j, t_j) - e^{-M_0 (t_j - h)} w(x_j, t_j - h)}{h} \\
&\leqslant \ \lim_{h \searrow 0} \frac{e^{-M_0 t_j} - e^{-M_0 (t_j - h)}}{h} w(x_j, t_j) + \liminf_{h \searrow 0} \\
&\quad \times e^{-M_0 (t_j - h)} \frac{w(x_j, t_j) - w(x_j, t_j - h)}{h} \\
&= \ e^{-M_0 t_j} \left[\underline{\mathrm{D}}_t w(x_j, t_j) - M_0 w(x_j, t_j) \right],
\end{aligned}$$

where $\underline{D}_t u(x,t) = \liminf_{h\to 0} \frac{u(x,t+h)-u(x,t)}{h}$, which yields

$$\underline{D}_t(u^2-u^1)(x_j,t_j) = \underline{D}_t w(x_j,t_j) \geqslant M_0 w(x_j,t_j) > 0.$$

Therefore, it follows from (4.4) that

$$\begin{aligned} 0 \;&\geqslant\; \underline{D}_t w(x_j,t_j) - D[w(x_j+1,t_j) + w(x_j-1,t_j) - 2w(x_j,t_j)] + dw(x_j,t_j) \\ &\quad - b\left(u^2(x_j,t_j-r)\right) + b\left(u^1(x_j,t_j-r)\right) \\ &\geqslant\; (M_0+2D+d)w(x_j,t_j) - D[w(x_j+1,t_j) + w(x_j-1,t_j)] \\ &\quad - b'_{\max}\max\{0,\omega(t_j-r)\} \\ &\geqslant\; (M_0+2D+d)w(x_j,t_j) - 2D\omega(t_j) - b'_{\max}\max\{0,\omega(t_j-r)\}. \end{aligned}$$

Sending $j\to+\infty$ to get

$$\begin{aligned} 0 \;&\geqslant\; (M_0+2D+d)\omega(t_0) - 2D\omega(t_0) - b'_{\max}e^{M_0(t_0-r)}\max\left\{0,\omega(t_0-r)e^{-M_0(t_0-r)}\right\} \\ &\geqslant\; (M_0+d)\omega(t_0) - b'_{\max}e^{M_0(t_0-r)}\omega(t_0)e^{-M_0t_0} \\ &=\; \left[M_0+d-b'_{\max}e^{-M_0r}\right]\omega(t_0). \end{aligned}$$

Recall that $M_0+d-b'_{\max}e^{-M_0r}>0$, we conclude that $\omega(t_0)\leqslant 0$, which contradicts to $\omega(t_0)>0$. This contradiction shows that $w=u^2-u^1\leqslant 0$ on $\mathbb{R}\times(0,+\infty)$ and the proof is complete. $\square$

Lemma 4.3. *Suppose that* $u^1,u^2\in C(\mathbb{R}\times[-r,+\infty),[0,K])$ *satisfies* $u^1_t(x,t)-F[u^1](x,t)\geqslant u^2_t(x,t)-F[u^2](x,t)$ *on* $\mathbb{R}\times(0,+\infty)$, $u^1(x,s)\geqslant u^2(x,s)$ *on* $\mathbb{R}\times[-r,0]$, *and that for any* $x\in\mathbb{R}$ *there exists* $j\in\mathbb{Z}$ *so that* $u^1(x+j,0)>u^2(x+j,0)$. *Then* $u^1(x,t)>u^2(x,t)$ *on* $\mathbb{R}\times(0,+\infty)$.

Proof. Put $w(x,t):=u^1(x,t)-u^2(x,t)$. By virtue of Lemma 4.2, we have $w(x,t)\geqslant 0$ on $\mathbb{R}\times[-r,+\infty)$. So it follows from (4.4) and the monotonicity of $b(\cdot)$ that

$$\begin{aligned} w(x,t)e^{(2D+d)t} \;&\geqslant\; w(x,0) + D\int_0^t e^{(2D+d)\tau}[w(x+1,\tau)+w(x-1,\tau)]\,d\tau \\ &\geqslant\; D\int_0^t e^{(2D+d)\tau}[w(x+1,\tau)+w(x-1,\tau)]\,d\tau \geqslant 0. \end{aligned}$$

Therefore, by using an induction argument, we can show that if $w(x,t)=u^1(x,t)-u^2(x,t)=0$ for some $x\in\mathbb{R}$ and $t>0$, then $w(x+j,\tau)=u^1(x+j,\tau)-u^2(x+j,\tau)=0$ for all $j\in\mathbb{Z}$ and $\tau\in[0,t]$. The assumption on the initial condition then gives $u^1(x,t)>u^2(x,t)$ on $\mathbb{R}\times(0,+\infty)$. This completes the proof. $\square$

Lemma 4.4. *Let* $u^1, u^2 \in C(\mathbb{R}\times[-r,+\infty),[0,K])$ *be any two solutions to* (4.1). *Then*

$$\begin{aligned}&\sup_{x\in\mathbb{R}}\left\{u^1(x,t)-u^2(x,t)\right\}\\&\quad\leqslant \sup_{(x,s)\in\mathbb{R}\times[-r,0]}\left\{\max\left\{u^1(x,s)-u^2(x,s),0\right\}\right\}e^{\bar{K}t}\quad \textit{for all } t\geqslant 0,\end{aligned}\tag{4.7}$$

where $\bar{K}=b'_{\max}e^{(2D+d)r}-d$.

Proof. Let u^3 be the solution to (4.1) with the initial value $u^3(\cdot,s)=\max\{u^1(\cdot,s), u^2(\cdot,s)\}$, $s\in[-r,0]$. Set $w(x,t)=u^3(x,t)-u^2(x,t)$. Then by virtue of Lemma 4.2, we have $K\geqslant w(x,t)\geqslant 0$ on $\mathbb{R}\times[-r,+\infty)$ and for $t\geqslant 0$,

$$\begin{aligned}w(x,t)e^{(2D+d)t} &\leqslant w(x,0)+\int_0^t e^{(2D+d)\tau}\{D[w(x+1,\tau)\\&\quad + w(x-1,\tau)]+b'_{\max}w(x,\tau-r)\}\,d\tau\\&\leqslant w(x,0)+\left(2D+b'_{\max}e^{(2D+d)r}\right)\\&\quad\times\int_0^t \sup\nolimits_{s\in[-r,\tau]}\|w(\cdot,s)\|e^{(2D+d)s}\,d\tau.\end{aligned}$$

Hence,

$$\begin{aligned}\sup_{s\in[-r,t]}\|w(\cdot,s)\|e^{(2D+d)s} &\leqslant \sup_{s\in[-r,0]}\|w(\cdot,s)\|+\left(2D+b'_{\max}e^{(2D+d)r}\right)\\&\quad\times\int_0^t\sup_{s\in[-r,\tau]}\|w(\cdot,s)\|e^{(2D+d)\tau}\,d\tau.\end{aligned}$$

So it follows from the Gronwall's inequality that

$$\sup_{s\in[-r,t]}\|w(\cdot,s)\|e^{(2D+d)s}\leqslant \sup_{s\in[-r,0]}\|w(\cdot,s)\|e^{(2D+b'_{\max}e^{(2D+d)r})t},$$

which implies that

$$u^3(x,t)-u^2(x,t)\leqslant \sup_{(x,s)\in\mathbb{R}\times[-r,0]}\left\{u^3(x,s)-u^2(x,s)\right\}e^{\bar{K}t}\quad\text{for all } t\geqslant 0,$$

from which, the conclusion of the lemma follows. This completes the proof. □

It is convenient in our stability analysis to introduce the following definitions.

Definition 4.1. An absolutely continuous functions $\{\upsilon(x,t)\}$, $x \in \mathbb{R}$, $t \in [-r, b)$, $b > 0$, is called a supersolution (subsolution) of (1.9) on $\mathbb{R} \times [0, b)$ if

$$\upsilon_t(x,t) \geqslant (\leqslant) F[\upsilon](x,t) \tag{4.8}$$

for almost every $x \in \mathbb{R}$ and $t \in [0, b)$.

Finally, we construct a few sub and super solutions for the initial value problem (4.1).

Lemma 4.5. *Suppose an absolutely continuous function* $\phi \in: \mathbb{R} \to [0, K]$ *satisfies* $N_c[\phi](\xi) \geqslant 0$ (*or* $\leqslant 0$) *i.e. on* $\mathbb{R}$. *Then* $w(x,t) = \phi(x+ct)$ *is a supersolution* (*or subsolution*) *to* (4.1).

Proof. The assertion follows immediately from the identity $w_t - F[w] = N_c[\phi]$. □

Lemma 4.6. *Assume that* (H1)–(H4) *hold and* (U, c) *is the travelling wave given in Theorem* 2.1. *Then for each* $\delta \in (0,1)$, *there exist* $\beta_0 > 0$ *and* $\sigma_0 > 0$ *such that for each* $\varepsilon \in (0, \delta]$ *and for any* $\xi^{\pm} \in \mathbb{R}$, *the following functions are a super and a sub solution to* (4.1), *respectively*:

$$w^{\pm}(x,t) := \left(1 \pm \varepsilon e^{-\beta_0 t}\right) U\left(x + ct + \xi^{\pm} \mp \sigma_0 \varepsilon e^{-\beta_0 t}\right). \tag{4.9}$$

Proof. Fix $\delta \in (0,1)$. Since $b'(K) < d$ and $b(u) > du$ for $u \in (0, K)$, we see that

$$\varpi := \sup_{0<s\leqslant(1+\delta)/2} \frac{b(K) - b((1-s)K)}{sK} < d.$$

Hence, we can choose $\beta_0 > 0$ and $\imath > 0$ such that

$$\delta e^{\beta_0 r} \leqslant \frac{1+\delta}{2}, \quad \varpi e^{\beta_0 r} < d \tag{4.10}$$

and

$$(d - \beta_0)e^{-\beta_0 r} - \max\left\{\varpi, b'(K) + \imath\right\} > 0. \tag{4.11}$$

Choose $\kappa > 0$ small enough so that

$$b'(\eta) < b'(K) + \imath \quad \text{for } \eta \in [K - \kappa, K + \kappa], \tag{4.12}$$

$$K[(d - \beta_0)e^{-\beta_0 r} - b'(K) - \imath] > \kappa\left[de^{-\beta_0 r} - b'(K) - \imath\right] \tag{4.13}$$

and

$$K\left[(d-\beta_0)e^{-\beta_0 r}-\varpi\right]>\kappa\left[de^{-\beta_0 r}-\varpi\right]+2LK\kappa^{\nu}. \tag{4.14}$$

Take $M_1>cr$ sufficiently large so that

$$U(\xi)\geqslant K-\kappa/2 \quad \text{for } \xi\geqslant M_1-cr. \tag{4.15}$$

As $\lim_{\xi\to-\infty}U(\xi)e^{-\Lambda_1(c)\xi}=1$ and $\lim_{\xi\to-\infty}U'(\xi)e^{-\Lambda_1(c)\xi}=\Lambda_1(c)$, we can take $M_2>0$ sufficiently large such that

$$\tfrac{1}{2}<U(\xi)e^{-\Lambda_1(c)\xi}<\tfrac{3}{2}, \quad U'(\xi)e^{-\Lambda_1(c)\xi}>\tfrac{1}{2}\Lambda_1(c) \quad \text{for } \xi\leqslant-M_2. \tag{4.16}$$

Denote

$$\varrho:=\min\{U'(\xi);\,-M_2\leqslant\xi\leqslant M_1\}>0.$$

Finally, choose $\sigma_0>0$ sufficiently large so that

$$\begin{aligned}\sigma_0 \geqslant \max\Bigg\{&\frac{3e^{\beta_0 r}}{\beta_0\Lambda_1(c)}\left[\beta_0e^{-\beta_0 r}+\left(b'_{\max}-de^{-\beta_0 r}\right)e^{-\Lambda_1(c)cr}\right],\\ &\frac{e^{\beta_0 r}}{\beta_0\varrho}\left[\beta_0e^{-\beta_0 r}+b'_{\max}-de^{-\beta_0 r}\right]K\Bigg\}\end{aligned} \tag{4.17}$$

and

$$\begin{aligned}\sigma_0 \geqslant \max\Bigg\{&\frac{3e^{\beta_0 r}}{\beta_0(1-\delta)\Lambda_1(c)}\left[\beta_0e^{-\beta_0 r}+2LK^{\nu}e^{-\Lambda_1(c)cr}\right],\\ &\frac{e^{\beta_0 r}}{\beta_0(1-\delta)\varrho}\left[\beta_0e^{-\beta_0 r}+2LK^{1+\nu}\right]\Bigg\}.\end{aligned} \tag{4.18}$$

For any $\varepsilon\in(0,\delta]$, put $\xi=x+ct+\xi^+-\sigma_0\varepsilon e^{-\beta_0 t}$, then for any $t\geqslant0$, we have

$$\begin{aligned}S[w^+](x,t) :=\ & w_t^+(x,t)-D\left[w^+(x+1,t)+w^+(x-1,t)-2w^+(x,t)\right]\\ &+dw^+(x,t)-b\left(w^+(x,t-r)\right)\\ =\ &-\beta_0\varepsilon e^{-\beta_0 t}U(\xi)+\left(c+\sigma_0\beta_0\varepsilon e^{-\beta_0 t}\right)\left(1+\varepsilon e^{-\beta_0 t}\right)U'(\xi)\\ &-D\left(1+\varepsilon e^{-\beta_0 t}\right)[U(\xi+1)+U(\xi-1)-2U(\xi)]+d\left(1+\varepsilon e^{-\beta_0 t}\right)U(\xi)\end{aligned}$$

$$
\begin{aligned}
&-b\left[\left(1+\varepsilon e^{-\beta_0(t-r)}\right)U\left(\xi-cr-\sigma_0\varepsilon e^{-\beta_0 t}(e^{\beta_0 r}-1)\right)\right]\\
\geqslant\ & -\beta_0\varepsilon e^{-\beta_0 t}U(\xi)+\sigma_0\beta_0\varepsilon e^{-\beta_0 t}\left(1+\varepsilon e^{-\beta_0 t}\right)U'(\xi)\\
&+\left(1+\varepsilon e^{-\beta_0 t}\right)b[U(\xi-cr)]-b\left[\left(1+\varepsilon e^{-\beta_0(t-r)}\right)U(\xi-cr)\right],
\end{aligned}
$$

from which we obtain

$$
\begin{aligned}
\varepsilon^{-1}e^{\beta_0(t-r)}S[w^+](x,t)\ \geqslant\ & -\beta_0 e^{-\beta_0 r}U(\xi)+\sigma_0\beta_0 e^{-\beta_0 r}U'(\xi)\\
&+\left(de^{-\beta_0 r}-b'(\eta)\right)U(\xi-cr),
\end{aligned}\tag{4.19}
$$

where $\eta\in(U(\xi-cr),(1+\delta e^{\beta_0 r})U(\xi-cr))$.

We distinguish among three cases.

Case (i): $\xi\geqslant M_1$. In this case, by (4.15), we have $K-\kappa/2\leqslant\eta\leqslant K+\kappa$. Hence, it follows from (4.11)–(4.13), (4.15) and (4.19) that

$$
\varepsilon^{-1}e^{\beta_0(t-r)}S[w^+](x,t)\geqslant-\beta_0 e^{-\beta_0 r}K+\left(de^{-\beta_0 r}-b'(K)-\iota\right)(K-\kappa)>0.
$$

Case (ii): $\xi\leqslant-M_2$. In this case, by (4.16), (4.17) and (4.19), we have

$$
\begin{aligned}
\varepsilon^{-1}e^{\beta_0(t-r)}S[w^+](x,t)e^{-\Lambda_1(c)\xi}\ \geqslant\ & -\beta_0 e^{-\beta_0 r}U(\xi)e^{-\Lambda_1(c)\xi}+\sigma_0\beta_0 e^{-\beta_0 r}U'(\xi)e^{-\Lambda_1(c)\xi}\\
&+\left(de^{-\beta_0 r}-b'_{\max}\right)U(\xi-cr)e^{-\Lambda_1(c)(\xi-cr)}e^{-\Lambda_1(c)cr}\\
\geqslant\ & \tfrac{1}{2}\sigma_0\beta_0 e^{-\beta_0 r}\Lambda_1(c)-\tfrac{3}{2}\Big[\beta_0 e^{-\beta_0 r}+(b'_{\max}-de^{-\beta_0 r})\\
&\times e^{-\Lambda_1(c)cr}\Big]\\
\geqslant\ & 0.
\end{aligned}
$$

Case (iii): $\xi\in[-M_2,M_1]$. In this case, it follows from (4.17) and (4.19) that

$$
\varepsilon^{-1}e^{\beta_0(t-r)}S[w^+](x,t)\geqslant-\beta_0 e^{-\beta_0 r}K+\sigma_0\beta_0 e^{-\beta_0 r}\varrho+\left(de^{-\beta_0 r}-b'_{\max}\right)K\geqslant 0.
$$

Combining cases (i)–(iii), we obtain

$$
\begin{aligned}
&w_t^+(x,t)-D\left[w^+(x+1,t)+w^+(x-1,t)-2w^+(x,t)\right]+dw^+(x,t)\\
&\quad-b(w^+(x,t-r))\geqslant 0
\end{aligned}
$$

for all $x\in\mathbb{R}$ and $t\geqslant 0$. Therefore, $w^+(x,t)$ is a supersolution of (4.1).

Next, we prove that $w^-(x,t)$ is a subsolution of (4.1). For any $\varepsilon \in (0,\delta]$, put $\xi = x + ct + \xi^+\sigma_0\varepsilon e^{-\beta_0 t}$, then for any $t \geqslant 0$, we have

$$\begin{aligned} S[w^-](x,t) := &\ w_t^-(x,t) - D[w^-(x+1,t) + w^-(x-1,t) - 2w^-(x,t)] \\ &+ dw^-(x,t) - b(w^-(x,t-r)) \\ \leqslant &\ \beta_0\varepsilon e^{-\beta_0 t}U(\xi) - \sigma_0\beta_0\varepsilon e^{-\beta_0 t}\left(1-\varepsilon e^{-\beta_0 t}\right)U'(\xi) \\ &+ \left(1-\varepsilon e^{-\beta_0 t}\right)b[U(\xi-cr)] - b[\left(1-\varepsilon e^{-\beta_0(t-r)}\right)U(\xi-cr)] \\ \leqslant &\ \beta_0\varepsilon e^{-\beta_0 t}U(\xi) - \sigma_0\beta_0\varepsilon e^{-\beta_0 t}(1-\delta)U'(\xi) - d\varepsilon e^{-\beta_0 r}U(\xi-cr) \\ &+ b[U(\xi-cr)] - b\left[\left(1-\varepsilon e^{-\beta_0(t-r)}\right)U(\xi-cr)\right]. \end{aligned}$$

For any $0 < \varsigma \leqslant (1+\delta)/2$, we find

$$\begin{aligned} & b[U(\xi-cr)] - b[(1-\varsigma)U(\xi-cr)] \\ &= \int_0^{\varsigma} b'[(1-s)U(\xi-cr)]U(\xi-cr)\,ds \\ &= \int_0^{\varsigma} \{b'[(1-s)U(\xi-cr)] - b'[(1-s)K]\}\,ds\,U(\xi-cr) \\ &\quad + \frac{b'(K) - b'((1-\varsigma)K)}{K}U(\xi-cr) \\ &\leqslant 2L\varsigma[K - U(\xi-cr)]^{\nu}U(\xi-cr) + \varsigma\varpi U(\xi-cr). \end{aligned}$$

Therefore, we have

$$\begin{aligned} \varepsilon^{-1}e^{\beta_0(t-r)}S[w^-](x,t) \leqslant &\ \beta_0 e^{-\beta_0 r}U(\xi) - \sigma_0\beta_0 e^{-\beta_0 r}(1-\delta)U'(\xi) - (de^{-\beta_0 r} - \varpi) \\ &\times U(\xi-cr) + 2L[K - U(\xi-cr)]^{\nu}U(\xi-cr). \end{aligned} \tag{4.20}$$

Again, we distinguish among three cases.

Case (iv): $\xi \geqslant M_1$. In this case, by (4.15), we have $K - \kappa/2 \leqslant \eta \leqslant K + \kappa$. Hence, it follows from (4.14), (4.15) and (4.20) that

$$\varepsilon^{-1}e^{\beta_0(t-r)}S[w^-](x,t) \leqslant \beta_0 e^{-\beta_0 r}K - \left(de^{-\beta_0 r} - \varpi)(K-\kappa\right) + 2LK\kappa^{\nu} < 0.$$

Case (v): $\xi \leqslant -M_2$. In this case, by (4.16), (4.18) and (4.20), we have

$$\begin{aligned}
&\varepsilon^{-1} e^{\beta_0(t-r)} S[w^-](x,t) e^{-\Lambda_1(c)\xi} \\
&\quad \leqslant \beta_0 e^{-\beta_0 r} U(\xi) e^{-\Lambda_1(c)\xi} - \sigma_0 \beta_0 e^{-\beta_0 r}(1-\delta) U'(\xi) e^{-\Lambda_1(c)\xi} \\
&\qquad + 2L[K - U(\xi - cr)]^\nu U(\xi - cr) e^{-\Lambda_1(c)(\xi - cr)} e^{-\Lambda_1(c)cr} \\
&\quad \leqslant -\tfrac{1}{2}\sigma_0 \beta_0 e^{-\beta_0 r}(1-\delta)\Lambda_1(c) + \tfrac{3}{2}\beta_0 e^{-\beta_0 r} + 3LK^\nu e^{-\Lambda_1(c)cr} \\
&\quad \leqslant 0.
\end{aligned}$$

Case (vi): $\xi \in [-M_2, M_1]$. In this case, it follows from (4.18) and (4.20) that

$$\varepsilon^{-1} e^{\beta_0(t-r)} S[w^-](x,t) \leqslant \beta_0 e^{-\beta_0 r} K - \sigma_0 \beta_0 e^{-\beta_0 r}(1-\delta)\varrho + 2LK^{1+\nu} \geqslant 0.$$

Combining cases (iv)–(vi), we obtain

$$\begin{aligned}
&w_t^-(x,t) - D[w^-(x+1,t) + w^-(x-1,t) - 2w^-(x,t)] + dw^-(x,t) \\
&\quad - b(w^-(x,t-r)) \leqslant 0
\end{aligned}$$

for all $x \in \mathbb{R}$ and $t \geqslant 0$. Therefore, $w^-(x,t)$ is a subsolution of (4.1) and this completes the proof. $\square$

5. Asymptotic stability of travelling waves

In this section, for $c > c^*$, we establish the asymptotic stability of the unique travelling wave by using the squeezing technique, which have been used in Chen [6], Chen–Guo [7] and Smith and Zhao [19].

Theorem 5.1. *Assume that* (H1)–(H4) *hold. Let* $c > c^*$ *and* (U, c) *be the travelling wave as given in Theorem* 2.1. *Assume that there exists* $\rho_0 \in (0, +\infty)$ *such that the initial data* $\varphi \in C(\mathbb{R} \times [-r, 0], [0, K])$ *satisfies*

$$\liminf_{x \to +\infty} \varphi(x, 0) > 0$$

and

$$\lim_{x \to -\infty} \max_{s \in [-r, 0]} |\varphi(x,s) e^{-\Lambda_1(c)x} - \rho_0 e^{\Lambda_1(c)cs}| = 0.$$

Then

$$\lim_{t\to+\infty}\sup_{x\in\mathbb{R}}\left|\frac{u(x,t)}{U(x+ct+\xi_0)}-1\right|=0, \tag{5.1}$$

where $\xi_0 = \frac{1}{\Lambda_1(c)}\ln\rho_0$.

Lemma 5.1. *For any* $\varepsilon>0$, *there exists* $\xi_1(\varepsilon)<0$ *such that*

$$\forall\xi\leqslant\xi_1(\varepsilon),\qquad \sup_{t\geqslant -r} u(\xi-2\varepsilon-ct,t)<U(\xi+\xi_0)<\inf_{t\geqslant -r}u(\xi+2\varepsilon-ct,t). \tag{5.2}$$

Proof. At first, we notice that there exists $x_1(\varepsilon)<0$ such that $\varphi(x-\varepsilon,s)<e^{\Lambda_1(c)(x+\xi_0+cs)}<\varphi(x+\varepsilon,s)$ for all $x\leqslant x_1(\varepsilon)$ and $s\in[-r,0]$.

Let $\phi^-(\xi)=\max\{0,e^{\Lambda_1(c)(\xi+\xi_0)}-qe^{\beta\Lambda_1(c)(\xi+\xi_0)}\}$, where $\beta=\frac{1}{2}(1+\min\{1+\nu,\frac{\Lambda_2(c)}{\Lambda_1(c)}\})$ and $q\geqslant\max\{Q(c,\beta),e^{-(\beta-1)\Lambda_1(c)(x_1(\varepsilon)+\xi_0-cr)}\}$. Then by virtue of Lemmas 2.2 and 4.5, $\phi^-(x+ct)$ is a subsolution of (4.1). As $e^{\Lambda_1(c)(x+\xi_0+cs)}-qe^{\beta\Lambda_1(c)(x+\xi_0+cs)}<0$ for all $x>x_1(\varepsilon)$ and $s\in[-r,0]$, we have $\varphi(x+\varepsilon,s)\geqslant\max\{0,e^{\Lambda_1(c)(x+\xi_0+cs)}-qe^{\beta\Lambda_1(c)(x+\xi_0+cs)}\}$ for all $x\in\mathbb{R}$ and $s\in[-r,0]$. The comparison principle then gives

$$u(x+\varepsilon,t)\geqslant e^{\Lambda_1(c)(x+\xi_0+ct)}-qe^{\beta\Lambda_1(c)(x+\xi_0+ct)}\quad\text{for all } x\in\mathbb{R} \text{ and } t\geqslant -r.$$

As $\lim_{\xi\to-\infty}U(\xi)e^{-\Lambda_1(c)\xi}=1$, there exists $x_2(\varepsilon)<0$ such that

$$e^{\Lambda_1(c)(\xi+\xi_0+\varepsilon)}-qe^{\beta\Lambda_1(c)(\xi+\xi_0+\varepsilon)}>U(\xi+\xi_0)\quad\text{for all } \xi\leqslant x_2(\varepsilon).$$

Consequently, for all $\xi\leqslant x_2(\varepsilon)$, we have

$$\inf_{t\geq -r}u(\xi+2\varepsilon-ct,t)\geqslant e^{\Lambda_1(c)(\xi+\xi_0+\varepsilon)}-qe^{\beta\Lambda_1(c)(\xi+\xi_0+\varepsilon)}>U(\xi+\xi_0).$$

Let $\phi^+(\xi)=\min\{K,e^{\Lambda_1(c)(\xi+\xi_0)}+qe^{\beta\Lambda_1(c)(\xi+\xi_0)}\}$. Then by virtue of Lemmas 2.2 and 4.5, $\phi^+(x+ct)$ is a supersolution of (4.1). Since $e^{\Lambda_1(c)\xi}+qe^{\beta\Lambda_1(c)\xi}>K$ for $\xi>-\frac{1}{\beta\Lambda_1(c)}\ln\frac{q}{K}$, we see that we can take q large enough so that $e^{\Lambda_1(c)(x+\xi_0+cs)}+qe^{\beta\Lambda_1(c)(x+\xi_0+cs)}>K$ for all $x>x_1(\varepsilon)$ and $s\in[-r,0]$. As $\varphi(x-\varepsilon,s)<e^{\Lambda_1(c)(x+\xi_0+cs)}<e^{\Lambda_1(c)(x+\xi_0+cs)}+qe^{\beta\Lambda_1(c)(x+\xi_0+cs)}$ for all $x\leqslant x_1(\varepsilon)$ and $s\in[-r,0]$, we have

$$\varphi(x-\varepsilon,s)\leqslant\min\left\{K,e^{\Lambda_1(c)(x+\xi_0+cs)}+qe^{\beta\Lambda_1(c)(x+\xi_0+cs)}\right\}$$
$$\text{for all } x\in\mathbb{R} \text{ and } s\in[-r,0].$$

Consequently, the comparison gives

$$u(x-\varepsilon,t)\leqslant \min\left\{K, e^{\Lambda_1(c)(x+\xi_0+ct)}+qe^{\beta\Lambda_1(c)(x+\xi_0+cs)}\right\}$$
$$\text{for all } x\in\mathbb{R} \text{ and } t\in[-r,+\infty).$$

Since

$$\lim_{\xi\to-\infty}\frac{e^{\Lambda_1(c)(\xi-\varepsilon)}+qe^{\beta\Lambda_1(c)(\xi-\varepsilon)}}{U(\xi)}$$
$$=\lim_{\xi\to-\infty}\frac{e^{-\Lambda_1(c)\varepsilon}+qe^{(\beta-1)\Lambda_1(c)\xi}e^{-\beta\Lambda_1(c)\varepsilon}}{U(\xi)e^{-\Lambda_1(c)\xi}}=e^{-\Lambda_1(c)\varepsilon}<1,$$

there exists $x_3(\varepsilon)<0$ such that $e^{\Lambda_1(c)(\xi+\xi_0-\varepsilon)}+qe^{\beta\Lambda_1(c)(\xi+\xi_0-\varepsilon)}<U(\xi+\xi_0)$ for all $\xi\leqslant x_3(\varepsilon)$. Hence, for all $\xi\leqslant x_3(\varepsilon)$, we have

$$\sup_{t\geqslant -r} u(\xi-2\varepsilon-ct,t)\leqslant e^{\Lambda_1(c)(\xi+\xi_0-\varepsilon)}+qe^{\beta\Lambda_1(c)(\xi+\xi_0-\varepsilon)}<U(\xi+\xi_0).$$

This completes the proof. □

Lemma 5.2. *There exist $\delta\in(0,1)$, $\beta_0>0$ and $z_0>0$ such that for all $\xi\in\mathbb{R}$ and $t\geqslant 1+r$,*

$$1-\delta e^{-\beta_0(t-1-r)}\leqslant \inf_{\mathbb{R}}\frac{u(\cdot-ct,t)}{U(\cdot+\xi_0-z_0)},\quad \sup_{\mathbb{R}}\frac{u(\cdot-ct,t)}{U(\cdot+\xi_0+z_0)}\leqslant 1+\delta e^{-\beta_0 t}. \tag{5.3}$$

Proof. In view of (5.2), $u(x+2-c(1+r+s),1+r+s)\geqslant U(x+\xi_0)$ for all $x\leqslant\xi_1(1)$, and hence, $u(x+2,1+r+s)\geqslant U(x+c(1+r+s)+\xi_0)$ for all $x\leqslant \xi_1(1)-c(1+r)$ and $s\in[-r,0]$.

Since $\liminf_{x\to+\infty}\varphi(x,0)>0$, there exists $\delta_1>0$ and $x_4>0$ such that

$$\varphi(x,0)>\delta_1 \quad \text{for all } x>x_4.$$

Fix a positive integer $N>x_4-[\xi_1(1)-c(1+r)]$. If $x\geqslant\xi_1(1)-c(1+r)$, then $x+N>x_4$, and hence, it follows from Lemma 4.1 that

$$\begin{aligned} u(x+2,1+r+s) &\geqslant D^N\varphi(x+2+N,0)(1+r+s)^N e^{-(2D+d)(1+r+s)}/N! \\ &\geqslant D^N\delta_1(1+r+s)^N e^{-(2D+d)(1+r+s)}/N! \\ &\geqslant D^N\delta_1 e^{-(2D+d)(1+r)}/N! \\ &\geqslant (1-\delta)K \end{aligned}$$

for all $x \geqslant \xi_1(1) - c(1+r), s \in [-r, 0]$ and some $\delta < 1$. Thus, for all $x \in \mathbb{R}$ and $s \in [-r, 0]$, we have

$$\begin{aligned} u(x+2, 1+r+s) &\geqslant (1-\delta)U(x + c(1+r+s) + \xi_0) \\ &\geqslant (1 - \delta e^{-\beta_0 s})U\left(x + c(1+r+s) + \xi_0 - \sigma_0\delta e^{\beta_0 r} + \sigma_0 \delta e^{-\beta_0 s}\right). \end{aligned}$$

The comparison function in (4.9) then gives

$$u(x+2, 1+r+t) \geqslant \left(1 - \delta e^{-\beta_0 t}\right) U\left(x + c(1+r+t) + \xi_0 - \sigma_0\delta e^{\beta_0 r} + \sigma_0\delta e^{-\beta_0 t}\right)$$

and hence,

$$\begin{aligned} u(x - c(1+r+t), 1+r+t) \geqslant \left(1 - \delta e^{-\beta_0 t}\right) U\Big(&x - 2 + \xi_0 \\ &-\sigma_0\delta e^{\beta_0 r} + \sigma_0\delta e^{-\beta_0 t}\Big). \end{aligned} \tag{5.4}$$

Again, in view of (5.2), $\varphi(x-2-cs, s) < U(x+\xi_0)$ for all $x \leqslant \xi_1(1)$, and hence, $\varphi(x-2, s) < U(x + cs + \xi_0)$ for all $x \leqslant \xi_1(1)$ and $s \in [-r, 0]$. Also, for δ given in the lower bound estimate, we have $\varphi(x-2, s) \leqslant K \leqslant (1+\delta)U(x + cs + x_5 + \xi_0)$ for all $x \geqslant \xi_1(1)$ and $s \in [-r, 0]$, if we take large $x_5 > 0$ such that $U(\xi_1(1) - cr + x_5 + \xi_0) \geqslant K/(1+\delta)$. Thus, $\varphi(x-2, s) \leqslant (1+\delta)U(x + cs + x_5 + \xi_0) \leqslant (1 + \delta e^{-\beta_0 s})U(x + cs + x_5 + \xi_0 + \sigma_0\delta e^{\beta_0 r} - \sigma_0\delta e^{-\beta_0 s})$ for all $x \in \mathbb{R}$ and $s \in [-r, 0]$. Using the comparison function in (4.9) then gives

$$\varphi(x-2, t) \leqslant \left(1 + \delta e^{-\beta_0 t}\right) U\left(x + ct + x_5 + \xi_0 + \sigma_0\delta e^{\beta_0 r} - \sigma_0\delta e^{-\beta_0 t}\right)$$

and hence,

$$\varphi(x - ct, t) \leqslant \left(1 + \delta e^{-\beta_0 t}\right) U\left(x + 2 + x_5 + \xi_0 + \sigma_0\delta e^{\beta_0 r} - \sigma_0\delta e^{-\beta_0 t}\right). \tag{5.5}$$

Finally, (5.3) follows from (5.4) and (5.5) by setting $z_0 = 2 + x_5 + \sigma_0\delta e^{\beta_0 r}$. This completes the proof. $\square$

Lemma 5.3. *There exists $M_0 > 0$ such that for all $\varepsilon \in (0, \delta]$ and $\xi \geqslant M_0 - \xi_0$,*

$$(1-\varepsilon)U\left(\xi + 3\varepsilon\sigma_0 e^{\beta_0 r}\right) \leqslant U(\xi) \leqslant (1+\varepsilon)U\left(\xi - 2\varepsilon\sigma_0 e^{\beta_0 r}\right). \tag{5.6}$$

Proof. Notice that

$$\frac{d}{ds}\left\{(1+s)U\left(\xi - 2s\sigma_0 e^{\beta_0 r}\right)\right\} = U\left(\xi - 3s\sigma_0 e^{\beta_0 r}\right) - 3\sigma_0 e^{\beta_0 r}(1+s)U'\left(\xi - 3s\sigma_0 e^{\beta_0 r}\right).$$

Since $U'(\xi) = \frac{1}{c}\{D[U(\xi+1) + U(\xi-1) - 2U(\xi)] - dU(\xi) + b(U(\xi - cr))\} \to 0$ as $\xi \to +\infty$, we see that there exists $M_0 > 0$ such that $U(\xi) - 6\sigma_0 e^{\beta_0 r} U'(\xi) > 0$ for all $\xi \geqslant M_0 - \xi_0 - 3\sigma_0 e^{\beta_0 r}$. Thus, $\frac{d}{ds}\{(1+s)U(\xi - 3s\sigma_0 e^{\beta_0 r})\} > 0$ for all $s \in [-\delta, \delta]$ and $\xi \geqslant M_0 - \xi_0$. The assertion of the lemma thus follows.

Lemma 5.4. *Let z and M_1 be arbitrarily fixed positive constants. Let $w^{\pm}$ be the solution to*

$$w_t(x,t) = D[w(x+1,t) + w(x-1,t) - 2w(x,t)] - dw(x,t) + b(w(x,t-r))$$

on $\mathbb{R} \times (0, +\infty)$, with the initial value

$$\begin{aligned} w^+(x,s) &= U(x+cs+\xi_0+z)\zeta(x+cs+M_1) + U(x+cs+\xi_0+2z) \\ &\quad \times (1 - \zeta(x+cs+M_1)), \end{aligned} \tag{5.7}$$

$$\begin{aligned} w^-(x,s) &= U(x+cs+\xi_0-z)\zeta(x+cs+M_1) + U(x+cs+\xi_0-2z) \\ &\quad \times (1 - \zeta(x+cs+M_1)) \end{aligned} \tag{5.8}$$

for $x \in \mathbb{R}$ and $s \in [-r, 0]$, where $\zeta(y) = \min\{\max\{0, -y\}, 1\}$ for all $y \in \mathbb{R}$. Then there exists an $\varepsilon \in (0, \min\{\delta, ze^{-\beta_0 r}/(3\sigma_0)\})$ such that

$$\begin{aligned} &w^+(x - c(1+r+s), 1+r+s) \leqslant (1+\varepsilon)U\left(x + \xi_0 + 2z - 3\varepsilon\sigma_0 e^{\beta_0 r}\right) \\ &\forall x \in [-M_1, +\infty), \end{aligned} \tag{5.9}$$

$$\begin{aligned} &w^-(x - c(1+r+s), 1+r+s) \geqslant (1-\varepsilon)U\left(x + \xi_0 + 2z + 3\varepsilon\sigma_0 e^{\beta_0 r}\right) \\ &\forall\, x \in [-M_1, +\infty). \end{aligned} \tag{5.10}$$

Proof. We only consider w^+. A similar argument can be used for w^-. Since $w^+(\cdot, s) \leqslant U(\cdot + cs + \xi_0 + 2z)$ on $\mathbb{R}$, and $w^+(\cdot, s) < U(\cdot + cs + \xi_0 + 2z)$ on $(-\infty, -M_1 - 1]$, by Lemma 4.3, we have

$$w^+(\cdot - c(1+r+s), 1+r+s) < U(\cdot + \xi_0 + 2z) \quad \text{for all } x \in \mathbb{R} \text{ and } s \in [-r, 0].$$

As w^+ and U are continuous, there exists $\varepsilon \in (0, \min\{\delta, ze^{-\beta_0 r}/(3\sigma_0)\}]$ such that $w^+(\cdot - c(1+r+s), 1+r+s) \leqslant U(\cdot + \xi_0 + 2z - 3\varepsilon\sigma_0 e^{\beta_0 r})$ on the compact interval $[-M_1, M_0 - 2z]$, where $M_0 > 0$ is as in Lemma 5.3 which asserts that $U(\cdot + \xi_0) \leqslant (1+\varepsilon)U(\cdot + \xi_0 - 3\varepsilon\sigma_0 e^{\beta_0 r})$ on $[M_0, +\infty)$. Hence, we also have $w^+(\cdot - c(1+r+s), 1+r+s) <$

$U(\cdot+\xi_0+2z)\leqslant(1+\varepsilon)U(\cdot+\xi_0+2z-3\varepsilon\sigma_0e^{\beta_0 r})$ on $[M_0-2z,+\infty)$. Therefore, (5.9) holds and the proof is complete.

Proof of Theorem 5.1. We define

$$z^+ := \inf\{z|z\in A^+\},\quad A^+ := \left\{z\geqslant 0 \,\middle|\, \limsup_{t\to+\infty}\sup_{\mathbb{R}} \frac{u(\cdot-ct,t)}{U(\cdot+\xi_0+2z)}\leqslant 1\right\}, \tag{5.11}$$

$$z^- := \inf\{z|z\in A^-\},\quad A^- := \left\{z\geqslant 0 \,\middle|\, \liminf_{t\to+\infty}\inf_{\mathbb{R}} \frac{u(\cdot-ct,t)}{U(\cdot+\xi_0-2z)}\geqslant 1\right\}. \tag{5.12}$$

In view of (5.3), we see that $\frac{1}{2}z_0\in A^{\pm}$. Hence, z^+ and z^- are well defined and $z^{\pm}\in[0,\frac{1}{2}z_0]$. Furthermore, as $\lim_{\xi\to-\infty}U(\xi)e^{-\Lambda_1(c)\xi}=1$ and $\lim_{\xi\to-\infty}U'(\xi)e^{-\Lambda_1(c)\xi}=\Lambda_1(c)$, it can be easily checked that $\lim_{\varepsilon\to 0}\frac{U(\cdot+\varepsilon)}{U(\cdot)}=1$ uniformly on $\mathbb{R}$. So it follows that $z^{\pm}\in A^{\pm}$ and $A^{\pm}=[z^{\pm},+\infty)$.

Thus, to complete the proof, we need only show that $z^+=z^-=0$. First, we prove that $z^+=0$, by a contradiction argument. Suppose for the contrary that $z^+>0$.

We fix $z=z^+$ and $M_1=-\xi_1(z^+/2)$, and denote by ε the resulting constant in Lemma 5.4. Since $z^+\in A^+$, $\limsup_{t\to+\infty}\sup_{\mathbb{R}}\frac{u(\cdot-ct,t)}{U(\cdot+\xi_0+2z^+)}\leqslant 1$. It then follows that there exists $T\geqslant 0$ such that $\sup_{\mathbb{R}}\frac{u(\cdot-c(T+s),T+s)}{U(\cdot+\xi_0+2z^+)}\leqslant 1+\hat{\varepsilon}/K$ for all $s\in[-r,0]$, where $\hat{\varepsilon}=\varepsilon U(-M_1+\xi_0-3\varepsilon\sigma_0e^{\beta_0 r})e^{-\bar{K}(1+r)}$, $\bar{K}=b'_{\max}e^{(2D+d)r}-d$. From (5.7), $w^+(\cdot,s)=U(\cdot+cs+\xi_0+2z^+)$ on $[-M_1-cs,+\infty)$, so that on $[-M_1-cs,+\infty)$, $u(\cdot-cT,T+s)\leqslant U(\cdot+cs+\xi_0+2z^+)+\hat{\varepsilon}=w^+(\cdot,s)+\hat{\varepsilon}$.

On $(-\infty,-M_1-cs]=(-\infty,\xi_1(z^+/2)-cs]$, we have, from (5.2), that $u(\cdot-cT,T+s)\leqslant U(\cdot+cs+\xi_0+z^+)\leqslant w^+(\cdot,s)$ by the definition of $w^+(\cdot,s)$ in (5.7). Thus, for all $s\in[-r,0]$, $u(\cdot-cT,T+s)\leqslant w^+(\cdot,s)+\hat{\varepsilon}$ on $\mathbb{R}$. Therefore, by virtue of Lemma 4.4, we have $u(\cdot-cT,T+1+r+s)\leqslant w^+(\cdot,1+r+s)+\hat{\varepsilon}e^{\bar{K}(1+r)}=w^+(\cdot,1+r+s)+\varepsilon U(-M_1+\xi_0-3\varepsilon\sigma_0e^{\beta_0 r})$ on $\mathbb{R}$. Therefore, it follows from (5.9) that

$$\begin{aligned}
&u(\cdot-c(T+1+r+s),T+1+r+s)\\
&\quad\leqslant w^+(\cdot-c(1+r+s),1+r+s)+\varepsilon U\left(-M_1+\xi_0-3\varepsilon\sigma_0e^{\beta_0 r}\right)\quad\text{on } \mathbb{R}\\
&\quad\leqslant (1+\varepsilon)U\left(\cdot+2z^+-3\varepsilon\sigma_0e^{\beta_0 r}\right)+\varepsilon U\left(-M_1+\xi_0-3\varepsilon\sigma_0e^{\beta_0 r}\right)\quad\text{on } [-M_1,+\infty)\\
&\quad\leqslant (1+2\varepsilon)U\left(\cdot+\xi_0+2z^+-3\varepsilon\sigma_0e^{\beta_0 r}\right)\quad\text{on } [-M_1,+\infty).
\end{aligned}$$

On the other hand, by (5.2), we have $u(\cdot-c(T+1+r+s),T+1+r+s)\leqslant U(\cdot+\xi_0+z^+)$ on $(-\infty,-M_1]$, and $3\varepsilon\sigma_0e^{\beta_0 r}\leqslant z^+$, there holds $u(\cdot-c(T+1+r+s),T+1+r+s)$

$\leqslant U(\cdot + \xi_0 + 2z^+ - 3\varepsilon\sigma_0 e^{\beta_0 r})$ on $(-\infty, -M_1]$. Thus, we have

$$\begin{aligned} & u(\cdot - c(T+1+r+s), T+1+r+s) \\ & \quad \leqslant (1+2\varepsilon) U\left(\cdot + \xi_0 + 2z^+ - 3\varepsilon\sigma_0 e^{\beta_0 r}\right) \\ & \quad \leqslant \left(1 + 2\varepsilon e^{-\beta_0 s}\right) U\left(\cdot + \xi_0 + 2z^+ - \varepsilon\sigma_0 - 2\varepsilon\sigma_0 e^{-\beta_0 s}\right) \quad \text{on } \mathbb{R}. \end{aligned}$$

A comparison then shows that

$$\begin{aligned} & u(x - c(T+1+r+t), T+1+r+t) \leqslant \left(1 + 2\varepsilon e^{-\beta_0 t}\right) \\ & \quad \times U\left(x + \xi_0 + 2z^+ - \varepsilon\sigma_0 - 2\varepsilon\sigma_0 e^{-\beta_0 t}\right) \end{aligned}$$

for all $x \in \mathbb{R}$ and $t \geqslant 0$. This implies that

$$\limsup_{t \to +\infty} \sup_{\mathbb{R}} \frac{u(\cdot - ct, t)}{U(\cdot + \xi_0 + 2z^+ - \varepsilon\sigma_0)} \leqslant 1.$$

That is, $z^+ - \varepsilon\sigma_0 \in A^+$. But this contradicts the definition of z^+. This contradiction shows that $z^+ = 0$.

In a similar manner, we can show that $z^- = 0$, and thereby completing the proof of Theorem 5.1. $\square$

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