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Propagation and its failure in a lattice delayed differential equation with global interaction☆

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Abstract

We study the existence, uniqueness, global asymptotic stability and propagation failure of traveling wave fronts in a lattice delayed differential equation with global interaction for a single species population with two age classes and a fixed maturation period living in a spatially unbounded environment. In the bistable case, under realistic assumptions on the birth function, we prove that the equation admits a strictly monotone increasing traveling wave front. Moreover, if the wave speed does not vanish, then the wave front is unique (up to a translation) and globally asymptotic stable with phase shift. Of particular interest is the phenomenon of "propagation failure" or "pinning" (that is, wave speed $c = 0$), we also give some criteria for pinning in this paper.

MSC: 34K30; 35B40; 35R10; 58D25

Keywords: Lattice delayed differential equation; Global interaction; Traveling wave front; Existence; Uniqueness; Asymptotic stability; Propagation failure; Pinning

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1. Introduction

In recent years, spatially non-local differential equations have attracted the interest of more and more researchers. In the context of population biology, So et al. [22] recently derived the following delayed reaction–equation model:

$$w_t = Dw_{xx} - dw + \int_{-\infty}^{\infty} b(w(t-r, y)) f(x-y)\, dy, \tag{1.1}$$

which describes the evolution of the adult population of a single species population with two age classes and moves around in a unbounded one-dimensional spatial domain. Here $D > 0$ and $d > 0$ denote the diffusion rate and death rate of the adult population, respectively, $r \geqslant 0$ is the maturation time for the species, b is related to the birth function, and the kernel function f describes the diffusion pattern of the immature population during the maturation process, and hence, depends also on the maturation delay. We refer to So et al. [22] for more details and some specific forms of f, obtained from integrating along characteristic of a structured population model, an idea from the work of Smith and Thieme [20]. See also [23] for a similar model and [11] for a survey on the history and the current status of the study of reaction–diffusion equations with non-local delayed interactions. Also explored in [22] is the existence of traveling wave fronts of (1.1) when the reaction term is of monostable type. When the reaction term is of bistable type, Ma and Wu [16] investigated the existence, uniqueness and stability of a traveling wave front of (1.1).

More recently, Weng et al. [24] also derived a discrete analog of (1.1) for a single species in one-dimensional patchy environment with infinite number of patches connected locally by diffusion. This lattice equation has the form

$$u_n' = D[u_{n+1} + u_{n-1} - 2u_n] - du_n + \sum_{i=-\infty}^{\infty} J(i) b(u_{n-i}(t-r)). \tag{1.2}$$

In this paper, we always assume that $J(i) = J(-i) \geqslant 0$, $\sum_i J(i) = 1$ and $\sum_i |i| J(i) < +\infty$, here and in what follows, $\sum_i$ denotes the sum over $i \in \mathbb{Z}$. We also assume that the birth function $b \in C^1(\mathbb{R})$ and there exists a constant $K > 0$ such that

$$b(0) = dK - b(K) = 0.$$

Therefore, (1.2) has at least two spatially homogeneous equilibria 0 and K.

We point out that non-local discrete equations also arise from other fields. For example, in studying the phase transition phenomena, discrete convolutions equations are used in, e.g., Bates et al. [1] and Bates and Chmaj [2] and the references therein. We point out that the non-local terms in the models of [1,2] are linear, while the non-local term in (1.2) is nonlinear.

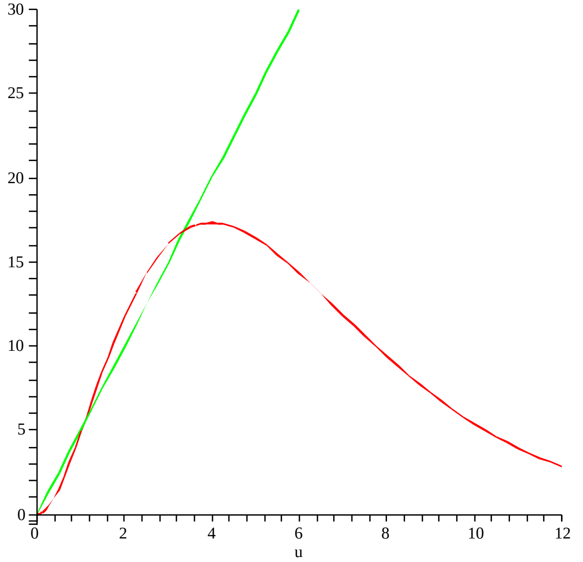

Fig. 1. The relation of $b(u)$ and $d\,u$.

We notice that if $J(0) = 1$ and $J(i) = 0$ for $i \neq 0$, then Eq. (1.2) reduces to the following local equation:

$$u_n' = D[u_{n+1} + u_{n-1} - 2u_n] - du_n + b(u_n(t-r)). \tag{1.3}$$

Traveling wave fronts in such local lattice differential equations have been intensively studied in recent years, see, e.g., [3–10,12–15,17–19,25–30].

From an earlier work of Keener [14], one knows that as far as traveling wave fronts are concerned, a discrete model could behave totally different from its continuous version. It is such an essential difference that drives us to investigate, in this paper, the existence, uniqueness, asymptotic stability and propagation failure of traveling wave fronts of (1.2). To this end, we make the following assumptions:

(H1) $b'(\eta) > 0$, for $\eta \in (0, K)$;

(H2) $d > \max\{b'(0), b'(K)\}$;

(H3) $u^* := \sup\{u \in [0, K);\ du = b(u)\} = \inf\{u \in (0, K];\ du = b(u)\}$ and $b'(u^*) > d$.

A specific function which has been widely used in the mathematical biology literature given by $b(u) = pu^2e^{-\alpha u}$ with $p > 0$ and $\alpha > 0$ does satisfy the above conditions for a wide range of parameters p, α. Fig. 1 illustrates the relation of $b(u)$ and du.

Under assumptions (H1)–(H3), by using the comparison and squeezing technique, Ma and Wu [16] have proved that Eq. (1.1) has exactly one non-decreasing traveling wave front (up to a translation) which is monotonically increasing and globally asymptotic stable with phase shift. The technique used in [16] has also been used previously by several authors [6,21] for some continuum models. To our knowledge, this technique has not been used for discrete equations, and hence, it is not clear if this technique can also be used to prove similar results for the lattice equation (1.2).

Motivated by Bates et al. [1,2], our approach for the existence of traveling wave fronts is to "approximate" the wave equation of (1.2) by a sequence of equations with smooth kernel functions, and then obtain a solution of the wave equation of (1.2) by taking the limit in a sequence of solutions with some desired properties of the latter equations, which can be obtained by using the continuation technique and a result established previously by Ma and Wu [16]. The above setting enables us to obtain the existence, uniqueness and asymptotic stability of traveling wave fronts for the lattice equation (1.2) by using the comparison and squeezing technique. In contrast to the results obtained in [16], we succeed in determining the sign of the wave speed in this paper.

An important qualitative difference between traveling wave solutions of the two systems (1.1) and (1.2) is the occurrence of "propagation failure" or "pinning" (that is, wave speed c = 0) in the discrete system (1.2). Such a phenomenon has been observed by several authors, see, e.g. [4,10,14] and references therein, in different contexts, where the authors proposed a crucial assumption on the reaction functions, that is, the reaction functions are piecewise linear. Such an assumption allows the authors to make straightforward use of the Fourier transform, which played a central role in the above-mentioned papers.

In this paper, we employ a new method for studying the pinning phenomenon for bistable lattice equations. That is, by proving asymptotic stability of traveling wave fronts with non-zero speed, we can reduce the problem to an easier one. The significant feature of this method is that it is applicable to lattice equations with general reaction functions.

Our main results can be summarized by the following two theorems.

Theorem 1.1. *Assume that* (H1)–(H3) *hold. Then* (1.2) *admits a strictly monotone traveling wave front $U(n-ct)$ satisfying $U(-\infty)=0$ and $U(+\infty)=K$. Moreover, if*

$$\sum_{i\neq 0} J(i) < \max\left\{ \frac{2\int_0^K [b(u)-du]\,du}{\int_0^K b(u)\,du},\ \frac{2\int_0^K [du-b(u)]\,du}{dK^2-\int_0^K b(u)\,du} \right\},$$

then there exists $D_0>0$ such that $c=|c|\operatorname{sgn}\int_0^K [du-b(u)]\,du \neq 0$ for all $D\geqslant D_0$. If $c\neq 0$, then the traveling wave front $U(n-ct)$ is unique (*up to a translation*) *and globally asymptotically stable with phase shift in the sense that there exists $\gamma>0$ such that for any $\varphi(s)=\{\varphi_n(s)\}_{n\in\mathbb{Z}}$ with $\varphi_n\in C([-r,0],[0,K])$ and*

$$\liminf_{n\to+\infty}\min_{s\in[-r,0]}\varphi_n(s)>u^*,\quad \limsup_{n\to-\infty}\max_{s\in[-r,0]}\varphi_n(s)<u^*,$$

the solution $u_n(t,\varphi)$ of (1.2), *with $u_n(s,\varphi)=\varphi_n(s)$ for $s\in[-r,0]$ and $n\in\mathbb{Z}$, satisfies*

$$|u_n(t,\varphi)-U(n-ct+\xi_0)|\leqslant Me^{-\gamma t},\quad t\geqslant 0,\ n\in\mathbb{Z},$$

for some $M=M(\varphi)>0$ and $\xi_0=\xi_0(\varphi)\in\mathbb{R}$.

Theorem 1.2. *Assume that* (H1)–(H3) *hold. Then* (1.2) *admits pinning if and only if it has a stationary solution* $u = \{u_n\}_{n\in\mathbb{Z}}$ *with* $u_n \in [0, K]$ *for all* $n \in \mathbb{Z}$ *satisfying* $\limsup_{n\to-\infty} u_n < u^*$ *and* $\liminf_{n\to+\infty} u_n > u^*$. *In particular, pinning occurs if* $\sum_{i\neq 0} J(i) = 0$ *and* $\int_0^K [du - b(u)]\, du = 0$, *or*

$$D \leqslant \frac{1}{2} \min \left\{ \max_{u\in[0,u^*]} \left\{ \left[\frac{du - b(u)}{dK - b(u)} - \sum_{i\neq 0} J(i) \right] \frac{dK - b(u)}{K - u} \right\}, \right.$$

$$\left. \max_{u\in[u^*,K]} \left\{ \left[\frac{b(u) - du}{b(u)} - \sum_{i\neq 0} J(i) \right] \frac{b(u)}{u} \right\} \right\}$$

and

$$d > b'_D := \sup\{b'(u);\ \ u \in [0, u^-) \cup (u^+, K]\},$$

where

$$u^- := \inf\{u \in (0, K];\ \ 2D(K - u) + [dK - b(u)] \sum_{i\neq 0} J(i) \leqslant du - b(u)\}$$

and

$$u^+ := \sup\{u \in [0, K);\ \ 2Du + b(u) \sum_{i\neq 0} J(i) \leqslant b(u) - du\}.$$

We remark that these two theorems are direct implications of the main results obtained in the later sections. More precisely, Theorem 1.1 is a direct consequence of Theorems 2.1, 3.1 and 4.1, and Theorem 1.2 is a direct consequence of Theorems 2.1, 5.1, Remark 5.1 and Corollary 5.1. Throughout this paper, by "pinning" we mean that all the waves connecting 0 and K have speed $c = 0$.

Under assumptions (H1) and (H2), we can choose a positive constant $\delta_0 > 0$ such that

$$du < b(u) < 0, \quad \text{for } u \in [-\delta_0, 0) \tag{1.4}$$

and

$$du > b(u) > 0, \quad \text{for } u \in (K, K + \delta_0]. \tag{1.5}$$

We also assume that b is strictly increasing in $[-\delta_0, K + \delta_0]$. By (H1), this can be achieved by modifying (if necessary) the definition of b outside the closed interval $[0, K]$ to a new C^1-smooth function and apply our results to the new function b.

The rest of this paper is organized as follows. In Section 2, we establish the existence of traveling wave fronts of (1.2). Section 3 is devoted to the uniqueness of the obtained traveling waves with non-zero speed and the technique is similar to that used in [1,2]. A comparison result and the asymptotic stability of the unique traveling wave front are obtained in Section 4. The arguments in the proof of the stability is very much similar to those in [4, 12, 15], and for the reader's convenience, the details will be given in this section. In the last section, a necessary and sufficient condition and some sufficient conditions for propagation failure are given.

2. Existence of traveling waves

In this section, we study the existence of monotone traveling wave solutions of (1.2).

Define $J_\delta(x) = \sum_i J(i)\delta(x-i)$, where $\delta(\cdot)$ is the Dirac delta function, $J(i) = J(-i) \geqslant 0$ for all $i \in \mathbb{Z}$, $\sum_i J(i) = 1$ and $\sum_i |i| J(i) < +\infty$. Clearly, for any bounded continuous function υ, we have

$$J_\delta * \upsilon(x) = \int_{\mathbb{R}} J_\delta(x-y)\upsilon(y)\,dy = \sum_i J(i) \int_{\mathbb{R}} \delta(x-y-i)\upsilon(y)\,dy = \sum_i J(i)\upsilon(x-i).$$

We consider the equation

$$\begin{aligned} &cU'(x) + D[U(x+1) + U(x-1) - 2U(x)] - dU(x) \\ &\quad + \int_{\mathbb{R}} b(U(x+cr-y))J_\delta(y)\,dy = 0, \end{aligned} \tag{2.1}$$

and the boundary conditions

$$U(-\infty) = 0, \quad U(+\infty) = K. \tag{2.2}$$

Clearly, solutions (U, c) of (2.1) and (2.2) give traveling wave fronts for (1.2), by setting $u_n(t) = U(n - ct)$.

Our approach for the existence of traveling wave fronts is to combine continuation technique used in [1,2] and a result obtained previously in [16] (see Lemma 2.1) to establish firstly the existence of a strictly monotone solutions to the equation

$$\begin{aligned} &cU'(x) + D[U(x+1) + U(x-1) - 2U(x)] - dU(x) \\ &\quad + \int_{\mathbb{R}} b(U(x+cr-y))\tilde{J}(y)\,dy = 0, \end{aligned} \tag{2.3}$$

and then, "approximate" the wave equation (2.1) by a sequence of equations

$$cU'(x) + D[U(x+1) + U(x-1) - 2U(x)] - dU(x)$$

$$+\int_{\mathbb{R}} b(U(x+cr-y))J_m(y)\,dy = 0, \tag{2.4}$$

where $\tilde{J}$ and J_m are smooth kernel functions to be specified later.

Firstly, we establish the existence of monotone solutions to Eq. (2.3) subject to the boundary conditions (2.2). We assume that

$$\begin{aligned}&\tilde{J}(x) \in C^{\infty}(\mathbb{R}), \quad \tilde{J}(x) = \tilde{J}(-x) \geqslant 0, \quad \int_{\mathbb{R}} \tilde{J}(x)\,dx = 1,\\ &\int_{\mathbb{R}} |x|\tilde{J}(x)\,dx < +\infty, \quad \int_{\mathbb{R}} |\tilde{J}'(x)|\,dx < +\infty, \quad \int_{\mathbb{R}} |\tilde{J}''(x)|\,dx < +\infty.\end{aligned} \tag{2.5}$$

Let $\theta \in [0, 1]$, we consider first the following equation:

$$\begin{aligned}&cU'(x) + \theta D[U(x+1) + U(x-1) - 2U(x)] + (1-\theta)DU''(x)\\ &\quad - dU(x) + \int_{\mathbb{R}} b(U(x+cr-y))\tilde{J}(y)\,dy = 0,\end{aligned} \tag{2.6}$$

In a recent paper [16], the following result has been obtained.

Lemma 2.1. *Assume that* (H1)–(H3) *hold. Then for* $\theta = 0$, (2.6) *and* (2.2) *admit a unique solution*(U, c) *satisfying* $0 < U' \leqslant \frac{b(K)}{2\sqrt{Dd}}$ *on* $\mathbb{R}$.

Lemma 2.2. *Let* $\theta \in [0, 1)$ *and let* U *satisfy* (2.6) *and* (2.2). *Then* $U(x) \in (0, K)$ *for all* $x \in \mathbb{R}$.

Proof. If $\theta = 0$, then the conclusion follows from Lemma 2.1. In what follows, we assume $\theta \in (0, 1)$.

First, it is clear that any L^∞ solution of (2.6) is of class C^3. If U has a global maximum at x_0 with $U(x_0) \geqslant K$, then $U'(x_0) = 0, U''(x_0) \leqslant 0$ and $U(x) \leqslant U(x_0)$ for all $x \in \mathbb{R}$. So it follows from (2.6) that

$$\theta D[U(x_0+1) + U(x_0-1) - 2U(x_0)] - dU(x_0) + \int_{\mathbb{R}} b(U(x_0+cr-y))\tilde{J}(y)\,dy \geqslant 0.$$

Since

$$\int_{\mathbb{R}} b(U(x_0+cr-y))\tilde{J}(y)\,dy \leqslant \int_{\mathbb{R}} b(U(x_0))\tilde{J}(y)\,dy = b(U(x_0)) \leqslant dU(x_0),$$

it follows that

$$U(x_0+1) + U(x_0-1) - 2U(x_0) \geqslant 0,$$

which together with the fact that $U(x_0 \pm 1) \leqslant U(x_0)$ yields

$$U(x_0 + 1) = U(x_0 - 1) = U(x_0).$$

Therefore, $U(x_0 + n) = U(x_0) \geqslant K$ for all $n \in \mathbb{Z}$, contradicting to $U(-\infty) = 0$.

A similar argument shows that $U(x) > 0$ for all $x \in \mathbb{R}$. The proof is complete. $\square$

Now suppose that (U_0, c_0) is a solution to (2.6) and (2.2) for some $\theta_0 \in [0, 1)$ and suppose that $U_0' > 0$ on $\mathbb{R}$. We will use the Implicit Function Theorem to obtain a solution for $\theta > \theta_0$.

We take perturbations in the space

$$X_0 = \{\text{uniformly continuous functions on } \mathbb{R} \text{ which vanish at } \pm\infty\}.$$

Let $L = L(U_0, c_0; \theta_0)$ be the linear operator defined in X_0 by

$$\begin{aligned} \operatorname{dom} L &= X_2 \equiv \{u \in X_0; u'' \in X_0\}, \\ Lu &= c_0 u' + \theta_0 D[u(\cdot + 1) + u(\cdot - 1) - 2u] + (1 - \theta_0) D u'' \\ &\quad - du + \int_{\mathbb{R}} b'(U_0(\cdot + c_0 r - y)) u(\cdot + c_0 r - y) \tilde{J}(y)\, dy. \end{aligned} \tag{2.7}$$

Lemma 2.3. *L has 0 as a simple eigenvalue.*

Proof. Clearly, $p \equiv U_0'$ is an eigenfunction of L with corresponding eigenvalue 0. So the only question is simplicity. Suppose that ϕ is another eigenfunction with eigenvalue 0 and assume ϕ takes on positive values at some points. We shall show that p and ϕ are linearly dependent by considering the family of eigenfunctions

$$\phi_\beta \equiv p + \beta\phi, \quad \beta \in \mathbb{R}.$$

Let

$$\bar{\beta} = \sup\{\beta < 0; \phi_\beta(x) < 0 \text{ for some } x\}.$$

Then $\bar{\beta}$ is well defined since ϕ is positive at some points. Recall that $p > 0$ on $\mathbb{R}$. For $\beta < \bar{\beta}$, let x_β be a point where ϕ_β achieves its minimum. So $\phi_\beta''(x_\beta) \geqslant 0 = \phi_\beta'(x_\beta)$ and it follows that

$$\begin{aligned} &\theta_0 D[\phi_\beta(x_\beta + 1) + \phi_\beta(x_\beta - 1) - 2\phi_\beta(x_\beta)] + (1 - \theta_0) D \phi_\beta''(x_\beta) \\ &\quad - d\phi_\beta(x_\beta) + \int_{\mathbb{R}} b'(U_0(x_\beta + c_0 r - y)) \phi_\beta(x_\beta + c_0 r - y) \tilde{J}(y)\, dy = 0. \end{aligned}$$

Hence

$$
\begin{aligned}
0 > d\phi_\beta(x_\beta) &\geqslant \int_{\mathbb{R}} b'(U_0(x_\beta + c_0 r - y))\phi_\beta(x_\beta + c_0 r - y)\tilde{J}(y)\,dy \\
&\geqslant \phi_\beta(x_\beta)\int_{\mathbb{R}} b'(U_0(x_\beta + c_0 r - y))\tilde{J}(y)\,dy.
\end{aligned}
$$

Therefore, we have

$$
\int_{\mathbb{R}} b'(U_0(x_\beta + c_0 r - y))\tilde{J}(y)\,dy \geqslant d.
$$

Suppose that there exists a sequence $\{\beta_n\}$ with $\beta_n < \bar{\beta}$ such that $|x_{\beta_n}| \to +\infty$ as $n \to \infty$. Without loss of generality, we assume $x_{\beta_n} \to -\infty$ as $n \to \infty$, then it follows from the Dominated Convergence Theorem that $b'(0) \geqslant d$, a contradiction. Therefore, $\{x_\beta\}_{\beta<\bar{\beta}}$ is bounded.

Now take the limit $\beta \nearrow \bar{\beta}$ along a sequence such that x_β converges to some $\bar{x}$, and observe that $\phi_{\bar{\beta}}(\bar{x}) = 0 \leqslant \phi_{\bar{\beta}}(x)$ for all $x \in \mathbb{R}$ and $\phi'_{\bar{\beta}}(\bar{x}) = 0$, it follows that

$$
\begin{aligned}
&\theta_0 D[\phi_{\bar{\beta}}(\bar{x}+1) + \phi_{\bar{\beta}}(\bar{x}-1) - 2\phi_{\bar{\beta}}(\bar{x})] + (1-\theta_0)D\phi''_{\bar{\beta}}(\bar{x}) \\
&\quad + \int_{\mathbb{R}} b'(U_0(\bar{x} + c_0 r - y))\phi_{\bar{\beta}}(\bar{x} + c_0 r - y)\tilde{J}(y)\,dy = 0.
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
0 &\geqslant \int_{\mathbb{R}} b'(U_0(\bar{x} + c_0 r - y))\phi_{\bar{\beta}}(\bar{x} + c_0 r - y)\tilde{J}(y)\,dy \\
&\geqslant \phi_{\bar{\beta}}(\bar{x})\int_{\mathbb{R}} b'(U_0(\bar{x} + c_0 r - y))\tilde{J}(y)\,dy = 0,
\end{aligned}
$$

that is,

$$
\int_{\mathbb{R}} b'(U_0(\bar{x} + c_0 r - y))\phi_{\bar{\beta}}(\bar{x} + c_0 r - y)\tilde{J}(y)\,dy = 0.
$$

By Lemma 2.2, $U_0(x) \in (0, K)$, and hence $b'(U_0(x)) > 0$. Therefore, if $[a, b] \subset \operatorname{supp}(\tilde{J})$, then $\phi_{\bar{\beta}}(x) = 0$ for $x \in [\bar{x} + c_0 r - b, \bar{x} + c_0 r - a] \cup [\bar{x} + c_0 r + a, \bar{x} + c_0 r + b]$, and then an induction argument shows that $\phi_{\bar{\beta}}(x) = 0$ for all $x \in \mathbb{R}$. Hence, p and ϕ are linearly dependent. This completes the proof.

The formal adjoint of L is given by

$$L^*u = -c_0u' + \theta_0 D[u(\cdot+1) + u(\cdot-1) - 2u] + (1-\theta_0)Du''$$
$$- du + \int_{\mathbb{R}} b'(U_0(\cdot + c_0r - y))u(\cdot + c_0r - y)\tilde{J}(y)\,dy.$$

It is easy to show that 0 is also a simple eigenvalue of L^*. Moreover, it can be shown that 0 is an isolated eigenvalue. Let ϕ^* be the corresponding eigenfunction, then by the Fredholm Alternative, for $g \in X_0, Lu = g$ has a solution in X_2 if and only if $\int_{\mathbb{R}} g\phi^* = 0$.

We now give the continuation result.

Lemma 2.4. *With θ_0, U_0 and c_0 as above, there exists $\eta > 0$ such that for $\theta \in [\theta_0, \theta_0 + \eta)$, problem* (2.6), (2.2) *has a solution* (U, c).

Proof. Without loss of generality, we may assume $U_0(0) = u^*$. For $(u, c) \in X_2 \times \mathbb{R}$ and $\theta \in \mathbb{R}$, define

$$\begin{aligned} &G(u, c, \theta)\\ &\quad= \Big((c_0 + c)(U_0 + u)' + \theta D[(U_0 + u)(\cdot+1) + (U_0 + u)(\cdot-1) - 2(U_0 + u)]\\ &\qquad + (1-\theta)D(U_0 + u)'' - d(U_0 + u)\\ &\qquad + \int_{\mathbb{R}} b((U_0 + u)(\cdot + (c_0 + c)r - y))\tilde{J}(y)\,dy, u(0)\Big), \end{aligned}$$

so that $G : X_2 \times \mathbb{R}^2 \to X_0 \times \mathbb{R}$ is of class C^1. We have $G(0, 0, \theta_0) = (0, 0)$ and

$$\begin{aligned} DG &\equiv \frac{\partial G}{\partial(u, c)}(0, 0, \theta_0)\\ &= \begin{pmatrix} L & U_0' + r\int_{\mathbb{R}} b'(U_0(\cdot + c_0r - y))U_0'(\cdot + c_0r - y)\tilde{J}(y)\,dy \\ \delta & 0 \end{pmatrix}, \end{aligned}$$

where $\delta u \equiv u(0)$.

If we can show that $DG : X_2 \times \mathbb{R} \to X_0 \times \mathbb{R}$ is invertible, then the lemma would follow from the Implicit Function Theorem. To this end, let $h \in X_0$ and $b \in \mathbb{R}$, and

$$DG(u, c) = (h, b),$$

that is,

$$Lu + cU_0' + cr\int_{\mathbb{R}} b'(U_0(\cdot + c_0r - y))U_0'(\cdot + c_0r - y)\tilde{J}(y)\,dy = h, \tag{2.8}$$

$$u(0) = b. \tag{2.9}$$

As we observed above, (2.8) is solvable if and only if

$$c \int_{\mathbb{R}} \left(U_0' + r \int_{\mathbb{R}} b'(U_0(\cdot + c_0 r - y)) U_0'(\cdot + c_0 r - y) \tilde{J}(y)\, dy \right) \phi^* = \int_{\mathbb{R}} h \phi^*. \tag{2.10}$$

We claim that the integral on the left of (2.10) is not zero. Suppose for the contrary that this is not true, then there exists $u_0 \in X_2$ such that

$$Lu_0 = U_0' + r \int_{\mathbb{R}} b'(U_0(\cdot + c_0 r - y)) U_0'(\cdot + c_0 r - y) \tilde{J}(y)\, dy. \tag{2.11}$$

Multiplying (2.11) by U_0' and integrating over $\mathbb{R}$ to get

$$0 = \int_{\mathbb{R}} U_0' L u_0 = \int_{\mathbb{R}} U_0' \{ U_0' + r \int_{\mathbb{R}} b'(U_0(\cdot + c_0 r - y)) U_0'(\cdot + c_0 r - y) \tilde{J}(y)\, dy \} > 0,$$

which leads to a contradiction and establish the assertion.

So (2.10) determines c. With this value of c, the solution to (2.8) is determined up to an additive term $\gamma U_0'$, where $\gamma \in \mathbb{R}$. Now (2.9) is satisfied by a unique choice of γ since $U_0'(0) > 0$. Thus, DG is invertible and the lemma is proved.

Lemma 2.5. *Let $\theta \in [\theta_0, \theta_0 + \eta)$ and (U_θ, c_θ) be the solution given above. Then $U_\theta'(x) > 0$ for all $x \in \mathbb{R}$.*

Proof. Let

$$\bar{\theta} = \sup\{\theta_0 \leqslant \theta < \theta_0 + \eta;\ U_\theta' > 0 \text{ on } \mathbb{R}\}.$$

If $\bar{\theta} = \theta_0 + \eta$, then we are done. So we assume that $\theta_0 < \bar{\theta} < \theta_0 + \eta$. Then there exists $\bar{x}$ such that $U_{\bar{\theta}}'(\bar{x}) = 0 \leqslant U_{\bar{\theta}}'(x)$ for all $x \in \mathbb{R}$. Hence $U_{\bar{\theta}}''(\bar{x}) = 0, U_{\bar{\theta}}'''(\bar{x}) \geqslant 0$. Differentiating (2.6) with $\theta = \bar{\theta}, U = U_{\bar{\theta}}$ and evaluated at $\bar{x}$, we obtain

$$\bar{\theta} D[U_{\bar{\theta}}'(\bar{x}+1) + U_{\bar{\theta}}'(\bar{x}-1) - 2U_{\bar{\theta}}'(\bar{x})] + (1-\bar{\theta}) D U_{\bar{\theta}}'''(\bar{x})$$
$$+ \int_{\mathbb{R}} b'(U_{\bar{\theta}}(\bar{x} + c_{\bar{\theta}} r - y)) U_{\bar{\theta}}'(\bar{x} + c_{\bar{\theta}} r - y) \tilde{J}(y)\, dy = 0.$$

Hence, we have

$$\int_{\mathbb{R}} b'(U_{\bar{\theta}}(\bar{x} + c_{\bar{\theta}} r - y)) U_{\bar{\theta}}'(\bar{x} + c_{\bar{\theta}} r - y) \tilde{J}(y)\, dy = 0.$$

Notice that $U_{\bar{\theta}}(x) \in (0, K)$ and hence $b'(U_{\bar{\theta}}(x)) > 0$, we conclude that $U'_{\bar{\theta}}(x) \equiv 0$, that is, $U_{\bar{\theta}}(x) \equiv$ const., a contradiction. This completes the proof. □

Lemma 2.6. *Suppose that for $\theta \in [0, \bar{\theta})$, there exists a solution (U_θ, c_θ) to* (2.6) *and* (2.2). *Then $\{c_\theta; \theta \in [0, \bar{\theta})\}$ is bounded.*

Proof. Suppose, on the contrary, that this set is unbounded. Then there would exist a sequence $\{\theta_n\}$ with $c_n \equiv c_{\theta_n} \to \pm\infty$ as $n \to \infty$. Writing $U_n \equiv U_{\theta_n}$. Since $U'_n(x) \to 0$ as $|x| \to +\infty$, $|U'_n(x)|$ has a maximum value at some point x_n. Clearly, $U''_n(x_n) = 0$ and hence

$$\begin{aligned}\|c_n U'_n\|_\infty &= |c_n U'_n(x_n)| \\ &= \bigg|\theta_n D[U_n(x_n+1) + U_n(x_n-1) - 2U_n(x_n)] \\ &\quad -dU_n(x_n) + \int_{\mathbb{R}} b(U_n(x_n + c_n r - y))\tilde{J}(y)\,dy\bigg| \\ &\leqslant 2DK + dK + b(K) = 2(D+d)K.\end{aligned}$$

Therefore, we have

$$\|U'_n\|_\infty \to 0 \quad \text{as } n \to \infty. \tag{2.12}$$

We now assert that for any $\varepsilon > 0$ and any closed interval $S \subset (0, K)$ of positive length there exists x_n such that $U_n(x_n) \in S$ and $|U''_n(x_n)| < \varepsilon$. If this were not the case, there would exist such an interval S and a number $\varepsilon > 0$ such that $|U''_n| \geqslant \varepsilon$ on the interval $[a_n, b_n]$, where $U_n([a_n, b_n]) = S$. Then

$$2\|U'_n\|_\infty \geqslant |U'_n(b_n) - U'_n(a_n)| \geqslant \varepsilon(b_n - a_n),$$

and by the Mean Value Theorem, the length of S, is

$$|S| = U_n(\bar{b}_n) - U_n(\bar{a}_n) \leqslant \|U'_n\|_\infty |\bar{b}_n - \bar{a}_n| \leqslant \|U'_n\|_\infty (b_n - a_n),$$

where $\bar{a}_n, \bar{b}_n \in [a_n, b_n]$ with $U_n(\bar{a}_n) = \min_{x\in[a_n,b_n]} U_n(x)$, $U_n(\bar{b}_n) = \max_{x\in[a_n,b_n]} U_n(x)$. It follows that $2\|U'_n\|^2 \geqslant \varepsilon|S|$, contradicting to the fact that $\|U'_n\|_\infty \to 0$ as $n \to \infty$, thus establishing the assertion.

Now take $\eta > 0$ small and let S be such that

$$du - b(u) \leqslant -\eta, \quad \text{for all } u \in S \tag{2.13}$$

in the case that $c_n \to +\infty$, and such that

$$du - b(u) \geqslant \eta, \quad \text{for all } u \in S \tag{2.14}$$

in the case that $c_n \to -\infty$. Take $\varepsilon = \frac{\eta}{2D}$ and take $\{x_n\}$ to be the sequence given by the assertion above. Without loss of generality, we assume that $c_n \to +\infty$, then (2.6) with $\theta = \theta_n, c = c_n$ and $U = U_n$ evaluated at x_n gives

$$\begin{aligned}
\eta &\leqslant c_n U_n'(x_n) - dU_n(x_n) + b(U_n(x_n)) \\
&\leqslant c_n U_n'(x_n) - dU_n(x_n) + b(U_n(x_n + c_n r)) \\
&= -\theta_n D[U_n(x_n+1) + U_n(x_n-1) - 2U_n(x_n)] - (1-\theta_n) DU_n''(x_n) \\
&\quad - \int_{\mathbb{R}} [b(U_n(x_n + c_n r - y)) - b(U_n(x_n + c_n r))] \tilde{J}(y)\, dy \\
&\leqslant D|U_n(x_n+1) + U_n(x_n-1) - 2U_n(x_n)| + D|U_n''(x_n)| \\
&\quad + b'_{\max} \|U_n'\|_\infty \int_{\mathbb{R}} |y| \tilde{J}(y)\, dy \\
&\leqslant 2D\|U_n'\|_\infty + D\varepsilon + b'_{\max} \|U_n'\|_\infty \int_{\mathbb{R}} |y| \tilde{J}(y)\, dy,
\end{aligned}$$

Sending $n \to \infty$, by (2.11), we then get $\eta \leqslant D\varepsilon = \eta/2$, a contradiction. This completes the proof.

Lemma 2.7. *Suppose that for $\theta \in [0, \bar{\theta})$, there exists a solution (U_θ, c_θ) to* (2.6) *and* (2.2). *Then $\{U_\theta; \theta \in [0, \bar{\theta})\}$ is bounded in C^3.*

Proof. Notice that

$$\begin{aligned}
&\int_{\mathbb{R}} b'(U_\theta(x + c_\theta r - y)) U_\theta'(x + c_\theta r - y) \tilde{J}(y)\, dy \\
&\quad = -b(U_\theta(x + c_\theta r - y)) \tilde{J}(y)|_{-\infty}^{+\infty} + \int_{\mathbb{R}} b(U_\theta(x + c_\theta r - y)) \tilde{J}'(y)\, dy \\
&\quad = \int_{\mathbb{R}} b(U_\theta(x + c_\theta r - y)) \tilde{J}'(y)\, dy,
\end{aligned}$$

it follows from (2.6) that $p_\theta \equiv U_\theta'$ satisfies the variational equation

$$\begin{aligned}
&c_\theta p_\theta' + \theta D[p_\theta(\cdot + 1) + p_\theta(\cdot - 1) - 2p_\theta] + (1-\theta) Dp_\theta'' \\
&\quad - dp_\theta + \int_{\mathbb{R}} b(U_\theta(\cdot + c_\theta r - y)) \tilde{J}'(y)\, dy = 0.
\end{aligned} \tag{2.15}$$

Therefore, at the point x_θ where p_θ achieves its positive maximum, we have

$$0 < dp_\theta(x_\theta) \leqslant \int_{\mathbb{R}} b(U_\theta(x_\theta + c_\theta r - y))\tilde{J}'(y)\,dy \leqslant b(K)\int_{\mathbb{R}} |\tilde{J}'(y)|\,dy,$$

and hence

$$\|p_\theta\|_\infty = p_\theta(x_\theta) \leqslant \frac{1}{d} b(K)\int_{\mathbb{R}} |\tilde{J}'(y)|\,dy.$$

Notice that

$$\begin{aligned}&\int_{\mathbb{R}} b'(U_\theta(x + c_\theta r - y))U'_\theta(x + c_\theta r - y)\tilde{J}'(y)\,dy\\ &\quad = -b(U_\theta(x + c_\theta r - y))\tilde{J}'(y)|_{-\infty}^{+\infty} + \int_{\mathbb{R}} b(U_\theta(x + c_\theta r - y))\tilde{J}''(y)\,dy\\ &\quad = \int_{\mathbb{R}} b(U_\theta(x + c_\theta r - y))\tilde{J}''(y)\,dy,\end{aligned}$$

by differentiating (2.15), we obtain

$$\begin{aligned}&c_\theta p''_\theta + \theta D[p'_\theta(\cdot + 1) + p'_\theta(\cdot - 1) - 2p'_\theta] + (1-\theta)Dp'''_\theta\\ &\quad - dp'_\theta + \int_{\mathbb{R}} b(U_\theta(\cdot + c_\theta r - y))\tilde{J}''(y)\,dy = 0.\end{aligned} \tag{2.16}$$

Assume that $|p'_\theta(x)|$ achieves its maximum at the point x_θ. Without loss of generality, we also assume $p'_\theta(x_\theta) \geqslant 0$, then $p''_\theta(x_\theta) = 0$, $p'''_\theta(x_\theta) \leqslant 0$, and hence

$$dp'_\theta(x_\theta) \leqslant \int_{\mathbb{R}} b(U_\theta(x_\theta + c_\theta r - y))\tilde{J}''(y)\,dy \leqslant b(K)\int_{\mathbb{R}} |\tilde{J}''(y)|\,dy.$$

Therefore, we obtain

$$\|p'_\theta\|_\infty = p'_\theta(x_\theta) \leqslant \frac{1}{d} b(K)\int_{\mathbb{R}} |\tilde{J}''(y)|\,dy.$$

Differentiating (2.16), we get

$$\begin{aligned}&c_\theta p'''_\theta + \theta D[p''_\theta(\cdot + 1) + p''_\theta(\cdot - 1) - 2p''_\theta] + (1-\theta)Dp^{(4)}_\theta\\ &\quad - dp''_\theta + \int_{\mathbb{R}} b'(U_\theta(\cdot + c_\theta r - y))p_\theta(\cdot + c_\theta r - y)\tilde{J}''(y)\,dy = 0.\end{aligned}$$

A similar argument shows that

$$\|p''_\theta\|_\infty \leqslant \frac{1}{d} b'_{\max} \|p_\theta\|_\infty \int_{\mathbb{R}} |\tilde{J}''(y)|\, dy.$$

The proof is complete. □

Lemma 2.8. *There exists a solution* (U, c) to (2.3) *satisfying* (2.2).

Proof. Lemma 2.4 gives a solution, U_θ, to (2.6) and (2.2) for each $\theta \in [0, \bar{\theta})$ for some $\bar{\theta} \in (0, 1]$. Furthermore, $U'_\theta > 0$ on $\mathbb{R}$ by Lemma 2.5.

Along a sequence $\theta_n \nearrow \bar{\theta}$, by Lemma 2.6 and 2.6, we may pass to the limit in (2.6), thereby obtaining a smooth solution $(\bar{U}, \bar{c})$ to (2.6) for $\theta = \bar{\theta}$. Clearly, this solution satisfies $\bar{U}' \geqslant 0$. Therefore, if $\bar{\theta} < 1$ and $\bar{U}$ satisfies (2.2), then by Lemma 2.2, $\bar{U}(x) \in (0, K)$, and hence the proof of Lemma 2.5 again shows that $\bar{U}' > 0$. So if $\bar{\theta} < 1$ and $\bar{U}$ satisfies (2.2), Lemma 2.4 again may be applied, showing that solutions exist for $\theta \in [0, 1)$. Thus, by passing to the limit in (2.6) along a sequence $\theta_n \nearrow 1$, we obtain a solution (U, c) to (2.3).

We now show that $\bar{U}$ satisfies (2.2). The same argument holds for either of the case $\bar{\theta} < 1$ or $\bar{\theta} = 1$. Because $\bar{U}$ is bounded and monotone, it has limits as $x \to \pm\infty$, and using the Dominated Convergence Theorem, we see from (2.6) that these limits are zeros of the function $du - b(u), u \in [0, K]$.

Suppose that $\bar{c} \geqslant 0$. Recall the intermediate zero u^* of $du - b(u)$. Take $\bar{u} \in (u^*, K)$ and translate U_θ so that $U_\theta(0) = \bar{u}$ for each θ. We still may take a sequence of $\theta \nearrow \bar{\theta}$, a subsequence of the original one, so that U_θ converges pointwise to some $\bar{U}$. Since c is independent of translations, we still have $c_\theta \to \bar{c}$. Then $\lim_{x\to+\infty} \bar{U}(x) = K$ and $\lim_{x\to-\infty} \bar{U}(x) \in \{u^*, 0\}$. If $\lim_{x\to-\infty} \bar{U}(x) = 0$, then we are done. So from now on, assume $\lim_{x\to-\infty} \bar{U}(x) = u^*$. Therefore, we have $d\bar{U}(x) - b(\bar{U}(x)) < 0$ on $\mathbb{R}$.

By the above discussion, we see that $\bar{U}$ is of class C^2 and satisfies (2.6). So

$$\begin{aligned}
0 &> \int_{-R}^{R} \{d\bar{U}(x) - b(\bar{U}(x))\}\, dx \\
&\geqslant \int_{-R}^{R} \{d\bar{U}(x) - b(\bar{U}(x + \bar{c}r))\}\, dx \\
&= \int_{-R}^{R} \{\bar{c}\bar{U}'(x) + \bar{\theta} D[\bar{U}(x+1) + \bar{U}(x-1) - 2\bar{U}(x)] + (1-\bar{\theta}) D\bar{U}''(x) \\
&\quad + \int_{\mathbb{R}} [b(\bar{U}(x + \bar{c}r - y)) - b(\bar{U}(x + \bar{c}r))] \tilde{J}(y)\, dy\}\, dx \\
&\geqslant \bar{\theta} D \int_{-R}^{R} \int_0^1 [\bar{U}'(x+t) - \bar{U}'(x-t)]\, dt dx + (1-\bar{\theta}) D \int_{-R}^{R} \bar{U}''(x)\, dx
\end{aligned}$$

$$
\begin{aligned}
&-\int_{-R}^{R}\int_{\mathbb{R}}\int_0^1 b'(\bar U(x+\bar c r-ty))\bar U'(x+\bar c r-ty)\,dty\tilde J(y)\,dydx\\
&= \bar\theta D\int_0^1[\bar U(R+t)-\bar U(-R+t)-\bar U(R-t)+U(-R-t)]\,dt\\
&\quad+(1-\bar\theta)D(\bar U'(R)-\bar U'(-R))\\
&\quad-\int_{\mathbb{R}}\int_0^1[b(\bar U(R+\bar c r-ty))-b(\bar U(-R+\bar c r-ty))]\,dty\tilde J(y)\,dy.
\end{aligned}
$$

Sending $R\to+\infty$ in the above inequality, it follows from Fubini's Theorem, Lebesgue's Theorem and the evenness of J that

$$0>\int_{\mathbb{R}}[d\bar U(x)-b(\bar U(x))]\,dx\geqslant-(b(K)-b(u^*))\int_{\mathbb{R}}y\tilde J(y)\,dy=0,$$

which is a contradiction.

If $\bar c<0$, a similar argument is used taking $\bar u\in(0,u^*)$. This completes the proof. $\square$

In our analysis, we need the following lemma.

Lemma 2.9 (*Dominated Convergence Theorem*). *Let* $\{f_{i,j}\}, i\in\mathbb{Z}, j\geqslant 1$, *be a double sequence of summable functions* (*i.e.*, $\sum_i f_{i,j}<+\infty$), *such that* $f_{i,j}\to f_i$ *as* $j\to+\infty$ *for all* $i\in\mathbb{Z}$. *If there exists a summable sequence* $\{g_i\}$ *such that* $|f_{i,j}|\leqslant g_i$ *for all* $i, j's$, *then*

$$\sum_i f_{i,j}\to\sum_i f_i,\quad as\ \ j\to+\infty.$$

The proof is similar to that of Lebesgue's dominated convergence theorem and is omitted.

Theorem 2.1. *Assume that* (H1)–(H3) *hold. Then* (1.2) *admits a strictly monotone traveling wave front* $u_n(t)=U(n-ct)$ *satisfying* $U(-\infty)=0$ and $U(+\infty)=K$. *Moreover, if*

$$\sum_{i\neq 0}J(i)<\max\left\{\frac{2\int_0^K[b(u)-du]\,du}{\int_0^K b(u)\,du},\ \frac{2\int_0^K[du-b(u)]\,du}{dK^2-\int_0^K b(u)\,du}\right\},\tag{2.17}$$

then there exists $D_0>0$ *such that for each* $D\geqslant D_0$, $c=|c|\mathrm{sgn}\int_0^K[du-b(u)]\,du\neq 0$. *In particular, if* $\sum_{i\neq 0}J(i)=0$, *then*

(i) $c=0$, *if* $\int_0^K[du-b(u)]\,du=0$;

(ii) $\operatorname{sgn}\{c\} = \operatorname{sgn} \int_0^K [du - b(u)]\, du$, *if* $c \neq 0$.

Remark 2.1. It follows from (2.17) that the following statements hold true:

(a) if $\int_0^K [du - b(u)]\, du < 0$, then

$$\frac{1}{2}\sum_{i\neq 0} J(i) \int_0^K b(u)\, du < \int_0^K [b(u) - du]\, du; \tag{2.18}$$

(b) if $\int_0^K [du - b(u)]\, du > 0$, then

$$\frac{1}{2}\sum_{i\neq 0} J(i)[dK^2 - \int_0^K b(u)\, du] < \int_0^K [du - b(u)]\, du. \tag{2.19}$$

Proof of Theorem 2.1. Let $\psi \in C_0^\infty(\mathbb{R})$ be such that $\psi(x) = \psi(-x) \geqslant 0$ and $\int_{\mathbb{R}} \psi(x)\, dx = 1$. Clearly, since ψ has compact support, we also have

$$\int_{\mathbb{R}} |x|\psi(x)\, dx < +\infty, \quad \int_{\mathbb{R}} |\psi'(x)|\, dx < +\infty, \quad \int_{\mathbb{R}} |\psi''(x)|\, dx < +\infty. \tag{2.20}$$

Set $\delta_m(x) = m\psi(mx), m \geqslant 1$, then $\delta_m(x) = \delta_m(-x) \geqslant 0$, $\int_{\mathbb{R}} \delta_m(x)\, dx = 1$. Moreover, we have

$$\operatorname{meas}\{x; \delta_m(x) \neq 0\} = \frac{1}{m}\operatorname{meas}\{x; \psi(x) \neq 0\},$$

where $\operatorname{meas}(E)$ denotes the Lebesgue's measure of E.

For any $m \geqslant 1$, define

$$J_m(x) = \frac{1}{W_m} \sum_{|i|\leqslant m} J(i)\delta_m(x - i), \tag{2.21}$$

where $W_m = \sum_{|i|\leqslant m} J(i)$. Then $J_m(x) = J_m(-x) \geqslant 0$ and $\int_{\mathbb{R}} J_m(x)\, dx = 1$, and it follows from (2.18) that

$$\int_{\mathbb{R}} |x| J_m(x)\, dx < +\infty, \quad \int_{\mathbb{R}} |J_m'(x)|\, dx < +\infty, \quad \int_{\mathbb{R}} |J_m''(x)|\, dx < +\infty. \tag{2.22}$$

Set $E_m = \{x \in \mathbb{R}; J_m(x) \neq 0\}$, then it is easily seen that

$$\operatorname{meas}(E_m) \leqslant \sum_{|i|\leqslant m} \operatorname{meas}\{x; \delta_m(x - i) \neq 0\} \leqslant 3\operatorname{meas}\{x; \psi(x) \neq 0\} < +\infty. \tag{2.23}$$

Furthermore, we assert that for any $\phi \in C_0^\infty(\mathbb{R})$ and $\{y_m\}$ satisfying $y_m \to 0$,

$$J_m * \phi(y + y_m) \to \sum_i J(i)\phi(y - i), \quad \text{as} \quad m \to \infty, \tag{2.24}$$

uniformly on $y \in \mathbb{R}$. In fact, let $M > 0$ be such that $\psi = 0$ for $|x| > M$. Then by the Mean Value Theorem, we have

$$\begin{aligned}
&|J_m * \phi(y + y_m) - \sum_i J(i)\phi(y - i)| \\
&\quad \leqslant \frac{1}{W_m} \sum_{|i| \leqslant m} J(i) \int_{\mathbb{R}} \delta_m(x)|\phi(y + y_m - i - x) - \phi(y - i)|\, dx \\
&\qquad + \frac{1}{W_m} \sum_{|i| \leqslant m} J(i)[1 - W_m]|\phi(y - i)| + \sum_{|i| > m} J(i)|\phi(y - i)| \\
&\quad \leqslant \frac{1}{W_m} \sum_{|i| \leqslant m} J(i) \int_{-M}^{M} \psi(x)|\phi(y + y_m - i - x/m) - \phi(y - i)|\, dx \\
&\qquad + \frac{1}{W_m} \sum_{|i| \leqslant m} J(i)[1 - W_m]|\phi(y - i)| + \sum_{|i| > m} J(i)|\phi(y - i)| \\
&\quad \leqslant \|\phi'\|_\infty[|y_m| + M/m] + \frac{1}{W_m} \sum_{|i| \leqslant m} J(i)[1 - W_m]\|\phi\|_\infty \\
&\qquad + \sum_{|i| > m} J(i)\|\phi\|_\infty \\
&\quad \to 0,
\end{aligned}$$

as $m \to \infty$ and the assertion follows. □

By virtue of Lemma 2.8, for each J_m, there exists a monotone solution (U_m, c_m) with $U_m(-\infty) = 0$ and $U_m(+\infty) = K$ to (2.4). The solutions (U_m, c_m) are of course also weak solutions of (2.4), i.e., for any $\phi \in C_0^\infty(\mathbb{R})$, they satisfy

$$\begin{aligned}
&-c \int_{\mathbb{R}} U\phi' + D \int_{\mathbb{R}} [U(\cdot + 1) + U(\cdot - 1) - 2U]\phi - d \int_{\mathbb{R}} U\phi \\
&\quad + \int_{\mathbb{R}} \int_{\mathbb{R}} b(U(\cdot + cr - y)) J_m(y)\, dy \phi = 0.
\end{aligned} \tag{2.25}$$

Consider first the case $c_m \geqslant 0$. Take $u \in (u^*, K)$ and translate each U_m so that $U_m(0) = u$. By Helly's Theorem, there exists a subsequence of U_m, which we still denote by

U_m, converging pointwise to a monotone function U as $m \to \infty$. Moreover, the c_m' are uniformly bounded, as can be seen from the following argument.

Assume on the contrary, that there is a sequence $c_m \to \infty$ as $m \to \infty$. From (2.4) we see that $\|c_m U_m'\|_\infty \leqslant 2(D+d)K$, from which we get $\|U_m'\|_\infty \to 0$ as $m \to \infty$. This implies that $U \equiv u$. It is easy to see that

$$\begin{aligned}
& dU_m(x) - b(U_m(x)) \\
& \quad \geqslant dU_m(x) - b(U_m(x + c_m r)) \\
& \quad = c_m U_m'(x) + D[U_m(x+1) + U_m(x-1) - 2U_m(x)] \\
& \qquad + \int_{\mathbb{R}} [b(U_m(x + c_m r - y)) - b(U_m(x + c_m r))] J_m(y)\, dy \\
& \quad = c_m U_m'(x) + D[U_m(x+1) + U_m(x-1) - 2U_m(x)] \\
& \qquad - \int_{\mathbb{R}} \int_0^1 b'(U_m(x + c_m r - ty)) U_m'(x + c_m r - ty)\, dt \cdot y J_m(y)\, dy \\
& \quad \geqslant D[U_m(x+1) + U_m(x-1) - 2U_m(x)] - b'_{\max} \|U_m'\|_\infty \int_{\mathbb{R}} |y| J_m(y)\, dy \\
& \quad \geqslant D[U_m(x+1) + U_m(x-1) - 2U_m(x)] \\
& \qquad - b'_{\max} \|U_m'\|_\infty \left\{ \frac{1}{m} \int_{\mathbb{R}} |y| \psi(y)\, dy + \frac{1}{W_m} \sum_i |i| J(i) \right\}.
\end{aligned}$$

Sending $m \to \infty$, from the above inequality, we get $0 > du - b(u) \geqslant 0$, a contradiction.

Thus by passing to another subsequence, we also have $c_m \to c$, for some c, as $m \to \infty$.

We now show that U solves (2.1) and (2.2). Firstly, by Fubini's Theorem, we find

$$\begin{aligned}
& \int_{\mathbb{R}} \int_{\mathbb{R}} b(U_m(x + c_m r - y)) J_m(y)\, dy \phi(x)\, dx \\
& \quad = \int_{\mathbb{R}} \int_{\mathbb{R}} b(U_m(x + cr - y)) J_m(y)\, dy \phi(x + (c - c_m) r)\, dx \\
& \quad = \int_{\mathbb{R}} \int_{\mathbb{R}} b(U_m(y + cr)) J_m(x - y)\, dy \phi(x + (c - c_m) r)\, dx \\
& \quad = \int_{\mathbb{R}} \int_{\mathbb{R}} \phi(x + (c - c_m) r) J_m(x - y)\, dx b(U_m(y + cr))\, dy \\
& \quad = \int_{\mathbb{R}} \int_{\mathbb{R}} \phi(y + (c - c_m) r - x) J_m(x)\, dx b(U_m(y + cr))\, dy \\
& \quad = \int_{\mathbb{R}} (J_m * \phi)(y + (c - c_m) r) b(U_m(y + cr))\, dy. \qquad (2.26)
\end{aligned}$$

Since ϕ has compact support, it is easily seen from (2.23) that

$$\text{meas}\{y \in \mathbb{R};\, J_m * \phi(y + (c - c_m)r) \neq 0\} < +\infty. \tag{2.27}$$

So it follows that

$$\begin{aligned}
&\lim_{m\to+\infty} \int_{\mathbb{R}} (J_m * \phi)(y + (c - c_m)r) b(U_m(y + cr))\, dy \\
&= \int_{\mathbb{R}} \sum_i J(i)\phi(y - i) b(U(y + cr)\, dy \\
&= \lim_{k\to+\infty} \int_{-k}^{k} \sum_i J(i)\phi(y - i) b(U(y + cr)\, dy \\
&= \lim_{k\to+\infty} \sum_i J(i) \int_{-k}^{k} \phi(y - i) b(U(y + cr))\, dy \\
&= \sum_i J(i) \int_{\mathbb{R}} \phi(y - i) b(U(y + cr)\, dy \\
&= \sum_i J(i) \int_{\mathbb{R}} b(U(y + cr - i)\phi(y)\, dy \\
&= \int_{\mathbb{R}} \sum_i J(i) b(U(y + cr - i)\phi(y)\, dy \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} b(U(y + cr - x) J_\delta(x)\, dx \phi(y)\, dy,
\end{aligned} \tag{2.28}$$

where the first equality follows from (2.24) and (2.27) and the Lebesgue's Dominated Convergence Theorem, the third equality follows because ϕ has compact support and thus the sum is finite, and the fourth equality follows from Lemma 2.9 and the fact that

$$\left| \int_{-k}^{k} \phi(y - i) b(U(y + cr))\, dy \right| \leqslant b(K) \int_{\mathbb{R}} |\phi(y - i)|\, dy = \text{const}.$$

By passing to the limit $m \to \infty$ in (2.25), it follows from (2.26), (2.28) and the Lebesgue's Dominated Convergence Theorem that U is a weak solution of (2.1), i.e., it satisfies

$$\begin{aligned}
&-c \int_{\mathbb{R}} U\phi' + D \int_{\mathbb{R}} [U(\cdot + 1) + U(\cdot - 1) - 2U]\phi - d \int_{\mathbb{R}} U\phi \\
&\quad + \int_{\mathbb{R}} \int_{\mathbb{R}} b(U(\cdot + cr - y)) J_\delta(y)\, dy \phi = 0.
\end{aligned} \tag{2.29}$$

for $\phi \in C_0^\infty(\mathbb{R})$.

If $c \neq 0$, then it follows that $U \in W^{1,\infty}(\mathbb{R})$. A bootstrap argument then shows that U is of class C^1 (and actually C^2) and thus a solution of (2.1).

If $c = 0$, then U need not be continuous, so $J_\delta * b(U)(n)$ need not equal $\sum_i J(i)b(U(n-i))$. However, U is monotone, and so the set of jump discontinuities is at most countable. Thus we can find a sequence $\{s_k\}$ such that $s_k \searrow 0$ as $k \to \infty$ and U is continuous at $n + s_k$ for all $n \in \mathbb{Z}$ and $k > 0$. The equation (2.1) implies that

$$D[U(n+1+s_k) + U(n-1+s_k) - 2U(n+s_k)] - dU(n+s_k) + \sum_i J(i)b(U(n+s_k-i)) = 0,$$

for all $n \in \mathbb{Z}$ and $k > 0$. It is then easily seen that the sequence U_n defined by

$$U_n \equiv \lim_{k\to\infty} U(n+s_k), \quad n \in \mathbb{Z},$$

satisfies

$$D[U_{n+1} + U_{n-1} - 2U_n] - dU_n + \sum_i J(i)b(U_{n-i}) = 0, \tag{2.30}$$

so is a stationary wave solution of (1.2).

We now show that $U(-\infty) = 0$ and $U(+\infty) = K$. From the monotonicity of U, we easily see that $dU(\pm\infty) = b(U(\pm\infty))$. Since $U(0) = u$, we have $U(+\infty) = K$ and $U(-\infty) \in \{u^*, 0\}$. If $U(-\infty) = 0$, we are done. So assume otherwise, that $U(-\infty) = u^*$. Then $dU(x) < b(U(x))$ on $\mathbb{R}$.

Consider first the case $c > 0$. Integrate the equation (2.1) over $(-N, N)$ to get

$$\begin{aligned}
&c\int_{-N}^{N} U'(x)\,dx + D\int_{-N}^{N}[U(x+1) + U(x-1) - 2U(x)]\,dx \\
&= d\int_{-N}^{N} U(x)\,dx - \sum_i J(i)\int_{-N}^{N} b(U(x+cr-i))\,dx \\
&\leqslant \int_{-N}^{N}[dU(x) - b(U(x))]\,dx - \sum_i J(i)\int_{-N}^{N}[b(U(x+cr-i)) \\
&\quad - b(U(x+cr))]\,dx \\
&= \int_{-N}^{N}[dU(x) - b(U(x))]\,dx \\
&\quad + \sum_i iJ(i)\int_{-N}^{N}\int_0^1 b'(U(x+cr-it))U'(x+cr-it)\,dt dx \\
&= \int_{-N}^{N}[dU(x) - b(U(x))]\,dx + \sum_i iJ(i)\int_0^1 [b(U(N+cr-it)) \\
&\quad - b(U(-N+cr-it))]\,dt.
\end{aligned}$$

Sending $N \to +\infty$, we obtain

$$\begin{aligned} c(K-u^*) &\leqslant \int_{\mathbb{R}} [dU(x)-b(U(x))]\,dx + \sum_i iJ(i)[b(K)-b(u^*)] \\ &= \int_{\mathbb{R}} [dU(x)-b(U(x))]\,dx < 0, \end{aligned}$$

which is a contradiction. In the above calculation, we have used the Lebesgue' Dominated Convergence Theorem, Fubini' Theorem, the evenness of J, and observe that

$$\begin{aligned} &\int_{-N}^{N} [U(x+1)+U(x-1)-2U(x)]\,dx \\ &\quad = \int_{-N}^{N}\int_0^1 [U'(x+t)-U'(x-t)]\,dt dx \\ &\quad = \int_0^1 [U(N+t)-U(-N+t)-U(N-t)+U(-N-t)]\,dt \\ &\quad \to 0, \end{aligned}$$

as $N \to +\infty$.

Next, assume that $c=0$. Then, using an argument similar to the above, the sequence $\{U_n\}$ defined above is a stationary solution of (1.2), i.e., it satisfies (2.30). Without loss of generality, we assume $\lim_{n\to+\infty} U_n = K$ and $\lim_{n\to-\infty} U_n = u^*$. Then we have

$$\begin{aligned} &D \sum_{|n|\leqslant N} [U_{n+1}+U_{n-1}-2U_n] + \sum_{|n|\leqslant N}\sum_i J(i)[b(U_{n-i})-b(U_n)] \\ &\quad = \sum_{|n|\leqslant N} [dU_n - b(U_n)] < 0, \end{aligned}$$

and hence

$$\begin{aligned} &D[U_{N+1}+U_{-N-1}-U_N-U_{-N}] \\ &\quad + \sum_{i>0} J(i)[\sum_{-N-i\leqslant n\leqslant -N-1} b(U_n) - \sum_{N-i+1\leqslant n\leqslant N} b(U_n)] \\ &\quad + \sum_{i<0} J(i)[\sum_{N+1\leqslant n\leqslant N-i} b(U_n) - \sum_{-N\leqslant n\leqslant -N-i-1} b(U_n)] \\ &\quad = \sum_{|n|\leqslant N} [dU_n - b(U_n)] < 0. \end{aligned}$$

Sending $N \to +\infty$, by the evenness of J, we get

$$0 = \sum_i iJ(i)[b(u^*)-b(K)] \leqslant \sum_{n\in\mathbb{Z}} [dU_n - b(U_n)] < 0,$$

which is a contradiction.

Finally, in the case $c_m \leqslant 0$, a similar argument is used taking $u \in (0, u^*)$.

To show strict monotonicity, we consider first the case $c = 0$. We argue by contradiction. Suppose that $U_{n_0+1} = U_{n_0}$ for some $n_0 \in \mathbb{Z}$. We then have

$$D[U_{n_0-1} - U_{n_0}] - dU_{n_0} + \sum_i J(i)b(U_{n_0-i}) = 0 \tag{2.31}$$

and

$$D[U_{n_0+2} - U_{n_0+1}] - dU_{n_0+1} + \sum_i J(i)b(U_{n_0+1-i}) = 0. \tag{2.32}$$

Hence,

$$D[U_{n_0+2} - U_{n_0-1}] + \sum_i J(i)[b(U_{n_0+1-i}) - b(U_{n_0-i})] = 0,$$

which together with the monotonicity of U_n yields

$$U_{n_0+2} = U_{n_0-1} = U_{n_0+1} = U_{n_0}.$$

By induction, it follows that $U_n \equiv$ const., a contradiction.

Let $c \neq 0$. Suppose that $U'(x_0) = 0$ for some x_0. Since $U'(x) \geqslant 0$ for all $x \in \mathbb{R}$, $U''(x_0) = 0$. Therefore,

$$\begin{aligned} 0 = -cU''(x_0) &= D[U'(x_0+1) + U'(x_0-1) - 2U'(x_0)] - dU'(x_0) \\ &\quad + \sum_i J(i)b'(U(x_0+cr-i))U'(x_0+cr-i) \\ &= D[U'(x_0+1) + U'(x_0-1)] \\ &\quad + \sum_i J(i)b'(U(x_0+cr-i))U'(x_0+cr-i). \end{aligned}$$

Hence, we obtain

$$U'(x_0+1) = U'(x_0-1) = U'(x_0) = 0,$$

and

$$b'(U(x_0+cr-i))U'(x_0+cr-i) = 0, \quad \text{for } i \in \mathbb{Z} \text{ with } J(i) > 0.$$

By induction, we conclude that $U'(x_0+n+mcr) = 0$ for all $n \in \mathbb{Z}$ and $m \geqslant 0$. Define $w_{n,m}(t) = U(x_0+n+mcr-ct)$, $n \in \mathbb{Z}$, $m \geqslant 0$, then $w_{n,m}(t)$ satisfies the initial value problem

$$\begin{aligned} w'_{n,m}(t) &= D[w_{n+1,m}(t) + w_{n-1,m}(t) - 2w_{n,m}(t)] - dw_{n,m}(t) \\ &\quad + \sum_i J(i)b(w_{n-i,m+1}(t)), \\ w_{n,m}(0) &= U(x_0 + n + mcr), \quad n \in \mathbb{Z},\ m \geqslant 0. \end{aligned}$$

Since $U'(x_0+n+mcr) = 0$ for all $n \in \mathbb{Z},\ m \geqslant 0$, the constant $w_{n,m}(t) \equiv U(x_0+n+mcr)$ also solves the initial value problem, contradicting the uniqueness of solutions of the initial value problem.

Now, we suppose that $c = 0$ for all $D > 0$ whenever (2.18) or (2.19) holds. View D as a parameter and let $u^D \equiv U$ be the monotone stationary solution of (1.2) satisfying $u^D(-\infty) = 0$ and $u^D(+\infty) = K$, i.e.,

$$D[u^D_{n+1} + u^D_{n-1} - 2u^D_n] = du^D_n - \sum_i J(i)b(u^D_{n-i}), \quad n \in \mathbb{Z}. \tag{2.33}$$

Clearly, we have $\sum_{n\in\mathbb{Z}}(u^D_{n+1} - u^D_n) = K$. Multiply (2.33) by $u^D_{n+1} - u^D_n$ and $u^D_n - u^D_{n-1}$, respectively, and then sum over $|n| \leqslant N$ the obtained equalities, we get

$$\begin{aligned} &D[u^D_{N+1} - u^D_N]^2 - D[u^D_{-N} - u^D_{-N-1}]^2 \\ &\quad = \sum_{|n|\leqslant N} [(du^D_n - J(0)b(u^D_n))(u^D_{n+1} - u^D_n) \\ &\quad\quad + (du^D_n - J(0)b(u^D_n))(u^D_n - u^D_{n-1})] \\ &\quad\quad - \sum_{i\neq 0} J(i) \sum_{|n|\leqslant N} [b(u^D_{n-i})(u^D_{n+1} - u^D_n) \\ &\quad\quad + b(u^D_{n-i})(u^D_n - u^D_{n-1})], \end{aligned}$$

by sending $N \to +\infty$, which together with the monotonicity of u^D implies that

$$\begin{aligned} &\sum_{n\in\mathbb{Z}}[(du^D_n - J(0)b(u^D_n))(u^D_{n+1} - u^D_n) + (du^D_n - J(0)b(u^D_n))(u^D_n - u^D_{n-1})] \\ &\quad = \sum_{i\neq 0} J(i) \sum_{n\in\mathbb{Z}}[b(u^D_{n-i})(u^D_{n+1} - u^D_n) + b(u^D_{n-i})(u^D_n - u^D_{n-1})] \\ &\quad \geqslant \sum_{i<0} J(i) \sum_{n\in\mathbb{Z}}[b(u^D_n)(u^D_{n+1} - u^D_n) + b(u^D_n)(u^D_n - u^D_{n-1})] \end{aligned} \tag{2.34}$$

and

$$\sum_{n\in\mathbb{Z}}[(du^D_n - J(0)b(u^D_n))(u^D_{n+1} - u^D_n) + (du^D_n - J(0)b(u^D_n))(u^D_n - u^D_{n-1})]$$

$$
\begin{aligned}
&= \sum_{i\neq 0} J(i) \sum_{n\in\mathbb{Z}} [b(u^D_{n-i})(u^D_{n+1} - u^D_n) + b(u^D_{n-i})(u^D_n - u^D_{n-1})] \\
&\leqslant \sum_{i>0} J(i) \sum_{n\in\mathbb{Z}} [b(u^D_n)(u^D_{n+1} - u^D_n) + b(u^D_n)(u^D_n - u^D_{n-1})] \\
&\quad + 2dK^2 \sum_{i<0} J(i). \qquad (2.35)
\end{aligned}
$$

We assert that

$$\max_{n\in\mathbb{Z}} (u^D_{n+1} - u^D_n) \to 0, \quad \text{as } D \to +\infty. \qquad (2.36)$$

Suppose that there exist $\varepsilon > 0$ and a sequence $\{D_k\}$ converging to $+\infty$ so that

$$\max_{n\in\mathbb{Z}} (u^{D_k}_{n+1} - u^{D_k}_n) = u^{D_k}_{n_k+1} - u^{D_k}_{n_k} \geqslant \varepsilon,$$

for k large enough. Through translation, we can assume $n_k = 0$. Since every solution u^{D_k} is monotone, by Helly's Theorem, there exists a subsequence of u^{D_k} which converges to some monotone $\bar{u}$. It is easily seen from (2.33) that $\bar{u}$ satisfies

$$\bar{u}_{n+1} + \bar{u}_{n-1} - 2\bar{u}_n = 0, \qquad (2.37)$$

for all $n \in \mathbb{Z}$. That is, $\bar{u}_{n+1} - \bar{u}_n = \bar{u}_n - \bar{u}_{n-1}$. Since $0 \leqslant \bar{u}_n \leqslant K$ for all $n \in \mathbb{Z}$, the number of integers n for which $\bar{u}_{n+1} - \bar{u}_n \geqslant \varepsilon$ is finite, which is a contradiction and establish the assertion above.

Statements (2.36) implies that the two sides of (2.34) are Riemann sums. If $\int_0^K [du - b(u)]\, du < 0$, then by sending $D \to +\infty$, (2.34) and the evenness of J gives

$$\int_0^K [du - J(0)b(u)]\, du \geqslant \frac{1}{2} \sum_{i\neq 0} J(i) \int_0^K b(u)\, du,$$

which contradicts to (2.18). This contradiction shows that there exists a sufficiently large number $D_0 > 0$ such that for each $D \geqslant D_0$, $c \neq 0$ holds. In a same way, we can show, by using (2.19) and (2.35c), that such a conclusion also valid if $\int_0^K [du - b(u)]\, du > 0$.

In what follows, we show that $\text{sgn}\{c\} = |c| \text{sgn} \int_0^K [du - b(u)]\, du$ if $c \neq 0$. Without loss of generality, we assume that $\int_0^K [du - b(u)]\, du < 0$. It suffices to show that $c < 0$. Suppose for the contrary that $c > 0$, then we have

$$
\begin{aligned}
&cU'(x) + D[U(x+1) + U(x-1) - 2U(x)] - dU(x) \\
&\quad + [J(0) + \tfrac{1}{2} \sum_{i\neq 0} J(i)] b(U(x)) \\
&\qquad \leqslant cU'(x) + D[U(x+1) + U(x-1) - 2U(x)] - dU(x)
\end{aligned}
$$

$$+J(0)b(U(x+cr)) + \sum_{i\neq 0} J(i)b(U(x+cr-i))$$

$$= 0.$$

Multiply the above inequality by U' and integrate over $\mathbb{R}$, we get

$$\begin{aligned} c\int_{\mathbb{R}}[U'(x)]^2\,dx &+ D\int_{\mathbb{R}}[U(x+1)+U(x-1)-2U(x)]U'(x)\,dx \\ &\leqslant \int_{\mathbb{R}}\{dU(x) - [J(0) + \tfrac{1}{2}\sum_{i\neq 0} J(i)]b(U(x))\}U'(x)\,dx \\ &= \int_0^K [du - b(u)]\,du + \tfrac{1}{2}\sum_{i\neq 0} J(i)\int_0^K b(u)\,du. \end{aligned}$$

However, since

$$\begin{aligned} \int_{\mathbb{R}} U(x+1)U'(x)\,dx &= \int_{\mathbb{R}} U(y)U'(y-1)\,dy \\ &= U(y)U(y-1)|_{-\infty}^{+\infty} - \int_{\mathbb{R}} U(y-1)U'(y)\,dy \\ &= K^2 - \int_{\mathbb{R}} U(y-1)U'(y)\,dy, \end{aligned}$$

we have

$$\int_{\mathbb{R}}[U(x+1)+U(x-1)-2U(x)]U'(x)\,dx = 0.$$

Hence, we obtain

$$c\int_{\mathbb{R}}[U'(x)]^2\,dx \leqslant \int_0^K [du - b(u)]\,du + \frac{1}{2}\sum_{i\neq 0} J(i)\int_0^K b(u)\,du, \tag{2.38}$$

which together with (2.18) yields $c < 0$, a contradiction.

We have proved that if $c > 0$, then (2.38) holds. If $c < 0$, a similar argument shows that

$$c\int_{\mathbb{R}}[U'(x)]^2\,dx \geqslant \int_0^K [du - b(u)]\,du - \frac{1}{2}\sum_{i\neq 0} J(i)[dK^2 - \int_0^K b(u)\,du]. \tag{2.39}$$

Finally, we assume that $\sum_{i\neq 0} J(i) = 0$. Then $J(0) = 1$ and $J(i) = 0$ for $i \neq 0$. Then by (2.38) and (2.39), we obtain

$$c\int_{\mathbb{R}}[U'(x)]^2\,dx \leqslant \int_0^K [du - b(u)]\,du, \quad \text{if } c > 0$$

and

$$c\int_{\mathbb{R}}[U'(x)]^2\,dx \geqslant \int_0^K [du - b(u)]\,du, \quad \text{if } c < 0.$$

If $\int_0^K [du - b(u)]\,du = 0$, then the inequalities given above gives a contradiction. This contradiction shows that (i) holds.

If $c \neq 0$, then the inequalities given above imply

$$\text{sgn}\{c\} = \text{sgn}\int_0^K [du - b(u)]\,du.$$

This prove that (ii) also holds.

3. Uniqueness of traveling waves

In this section, we study the uniqueness of the traveling waves and establish the following main result.

Theorem 3.1. *Assume that* (H1)–(H3) *hold. Let* (U, c) *be a solution to* (2.1) *and* (2.2) *as given in Theorem* 2.1, *such that* $c \neq 0$. *Let* $(\hat{U}, \hat{c})$ *be another solution to* (2.1) *and* (2.2). *Then* $\hat{c} = c$ *and, up to a translation,* $\hat{U} = U$.

Proof. Firstly, we observe that if $(\hat{U}, \hat{c})$ with $\hat{c} \neq 0$ is a solution to (2.1) and (2.2), then

$$0 \leqslant \hat{U} \leqslant K. \tag{3.1}$$

Suppose otherwise, i.e., let x_0 be such that $\hat{U}(x_0) > K$ and $\hat{U}(x) \leqslant \hat{U}(x_0)$ for all $x \in \mathbb{R}$. Then we have $\hat{U}'(x_0) = 0$ and so

$$\begin{aligned} 0 &\geqslant \hat{c}\hat{U}'(x_0) + D[\hat{U}(x_0+1) + \hat{U}(x_0-1) - 2\hat{U}(x_0)] \\ &= d\hat{U}(x_0) - \sum_i J(i)b(\hat{U}(x_0 + \hat{c}r - i)) \\ &\geqslant d\hat{U}(x_0) - b(\hat{U}(x_0)) > 0, \end{aligned}$$

which is a contradiction. Similarly, we can show that $\hat{U} \geqslant 0$.

If $\{\hat{U}_n\}_{n\in\mathbb{Z}}$ is a stationary solution of (1.2) with $\lim_{n\to-\infty}\hat{U}_n = 0$ and $\lim_{n\to+\infty}\hat{U}_n = K$, then

$$0 < \hat{U}_n < K. \tag{3.2}$$

Suppose that there exists $n_0 \in \mathbb{Z}$ satisfying $\hat{U}_{n_0} \geqslant K$ and $\hat{U}_{n_0} \geqslant \hat{U}_n$ for all $n \in \mathbb{Z}$. We can choose n_0 so that $\hat{U}_{n_0+1} + \hat{U}_{n_0-1} < 2\hat{U}_{n_0}$. Otherwise, if $\hat{U}_{n_0+1} + \hat{U}_{n_0-1} \geqslant \hat{U}_{n_0}$, then $\hat{U}_{n_0+1} = \hat{U}_{n_0-1} = \hat{U}_{n_0}$, and then by an induction argument it can be shown that $\hat{U}_n \equiv \hat{U}_{n_0} \geqslant K$, which contradicts to $\lim_{n\to-\infty}\hat{U}_n = 0$. Therefore, we have

$$0 < D[\hat{U}_{n_0+1} + \hat{U}_{n_0-1} - 2\hat{U}_{n_0}] = -d\hat{U}_{n_0} + \sum_i J(i)b(\hat{U}_{n_0-i}) \leqslant -d\hat{U}_{n_0} + b(\hat{U}_{n_0}) \leqslant 0,$$

which leads to a contradiction and prove that $\hat{U}_n < K$ for all $n \in \mathbb{Z}$. In a similar way, we can show that $\hat{U}_n > 0$ for all $n \in \mathbb{Z}$.

Since $\max\{b'(0), b'(K)\} < d$, we can choose $\varpi > 0$, $\alpha > 0$ and $N > 0$ such that

$$\alpha + \sum_{|i|>N} J(i)b'_{\max} + \varpi < d - \max\{b'(0), b'(K)\}. \tag{3.3}$$

Take $\kappa > 0$ sufficiently small, so that

$$b'(\eta) < \max\{b'(0), b'(K)\} + \varpi/2, \quad \text{for } \eta \in [-\kappa, \kappa] \cup [K-\kappa, K+\kappa]. \tag{3.4}$$

Take $M > |\hat{c}|r + N$ sufficiently large so that

$$U(\xi) \geqslant K - \kappa/2, \quad \text{for } \xi \geqslant M - |\hat{c}|r - N \tag{3.5}$$

and

$$U(\xi) \leqslant \kappa/2, \quad \text{for } \xi \leqslant -M + |\hat{c}|r + N. \tag{3.6}$$

Denote

$$\varrho := \min\{U'(\xi); |\xi| \leqslant M\} > 0.$$

Let $\mu \in (0, \kappa/2)$ and define

$$B = \frac{\mu}{\alpha\varrho} b'_{\max}. \tag{3.7}$$

First consider the case where $c \geqslant \hat{c}$. If $\hat{c} \neq 0$, (so that $\hat{U}$ is of class C^2), we define

$$w(x,t) = U(x + z + (\hat{c} - c)t + B(1 - e^{-\alpha t})) + \mu e^{-\alpha t} - \hat{U}(x), \tag{3.8}$$

where by (3.1), z can be chosen so that

$$w(x,0) = U(x+z) + \mu - \hat{U}(x) > 0.$$

We claim that $w(x,t) > 0$ for all $x \in \mathbb{R}$ and $t \geqslant 0$. To see this, suppose that there exists (x_0, t_0) such that

$$\begin{aligned} w(x_0,t_0) &= U(P_0) + \mu e^{-\alpha t_0} - \hat{U}(x_0) \\ &= 0 \leqslant w(x,t), \quad \text{for all } x \in \mathbb{R} \text{ and } 0 \in [0,t_0], \end{aligned} \tag{3.9}$$

where

$$P_0 = x_0 + z + (\hat{c} - c)t_0 + B(1 - e^{-\alpha t_0}).$$

Clearly, we have

$$w_x(x_0,t_0) = U'(P_0) - \hat{U}'(x_0) = 0. \tag{3.10}$$

By using (2.1), (3.9) and (3.10), it is easily seen that

$$\begin{aligned} 0 &\geqslant w_t(x_0,t_0) - D[w(x_0+1,t_0) + w(x_0-1,t_0) - 2w(x_0,t_0)] \\ &= -\alpha\mu e^{-\alpha t_0} + \alpha B U'(P_0) + (\hat{c} - c)U'(P_0) \\ &\quad - D[U(P_0+1) + U(P_0-1) - 2U(P_0)] + D[\hat{U}(x_0+1) \\ &\quad + \hat{U}(x_0-1) - 2\hat{U}(x_0)] \\ &= (-\alpha\mu + \alpha B U'(P_0))e^{-\alpha t_0} - d(U(P_0) - \hat{U}(x_0)) \\ &\quad + \sum_i J(i) b(U(P_0 + cr - i)) - \sum_i J(i) b(\hat{U}(x_0 + \hat{c}r - i)) \\ &= (d\mu - \alpha\mu + \alpha B U'(P_0))e^{-\alpha t_0} + \sum_i J(i)[b(U(P_0 + cr - i)) \\ &\quad - b(\hat{U}(x_0 + \hat{c}r - i))] \\ &\geqslant (d\mu - \alpha\mu + \alpha B U'(P_0))e^{-\alpha t_0} + \sum_i J(i)[b(U(P_0 + \hat{c}r - i)) \\ &\quad - b(U(P_0 + \hat{c}r - i) - w(x_0 + \hat{c}r - i, t_0) + \mu e^{-\alpha t_0})] \\ &\geqslant (d\mu - \alpha\mu + \alpha B U'(P_0))e^{-\alpha t_0} \end{aligned}$$

$$
\begin{aligned}
&+\sum_{i} J(i)[b(U(P_0+\hat{c}r-i))-b(U(P_0+\hat{c}r-i)+\mu e^{-\alpha t_0})]\\
\geqslant\ &[d-\alpha+\frac{\alpha B}{\mu}U'(P_0)-\sum_{|i|>N} J(i)b'_{\max}\\
&-\sum_{|i|\leqslant N} J(i)b'(\eta_i)]\mu e^{-\alpha t_0},
\end{aligned}
\tag{3.11}
$$

where $\eta_i \in (U(P_0+\hat{c}r-i), U(P_0+\hat{c}r-i)+\mu e^{-\alpha t_0})$.

If $|P_0|\leqslant M$, then (3.7) implies that

$$
\frac{\alpha B}{\mu}U'(P_0)-\sum_{|i|>N} J(i)b'_{\max}-\sum_{|i|\leqslant N} J(i)b'(\eta_i)\geqslant 0,
$$

and hence, the right-hand side of (3.11) is strictly greater than 0, which leads to a contradiction.

If $|P_0|\geqslant M$, then $|P_0+\hat{c}r-i|\geqslant M-|\hat{c}|r-N$ for $|i|\leqslant N$. Therefore, by (3.5) and (3.6), we have

$$
U(P_0+\hat{c}r-i)\geqslant K-\kappa/2, \quad \text{or } U(P_0+\hat{c}r-i)\leqslant \kappa/2,
$$

which together with (3.4) implies

$$
b'(\eta_i) < \max\{b'(0), b'(K)\}+\varpi/2, \quad \text{for } |i|\leqslant N.
$$

Thus, by (3.3), the right-hand side of (3.11) is positive, also giving a contradiction and establishing the claim that $w(x,t)>0$ for all $x\in\mathbb{R}$ and $t\geqslant 0$.

If $\hat{c}=0$, assume that $\hat{U}_n$, $n\in\mathbb{Z}$, is the corresponding stationary wave solution. Let

$$
w_n(t)=U(n+z-ct+B(1-e^{-\alpha t}))+\mu e^{-\alpha t}-\hat{U}_n, \tag{3.12}
$$

where z is chosen so that

$$
w_n(0)=U(n+z)+\mu-\hat{U}_n>0.
$$

In this case, we also have that $w_n(t)>0$ for all $n\in\mathbb{Z}$ and $t\geqslant 0$. In fact, suppose that there exists (n_0,t_0) such that

$$
w_{n_0}(t_0)=U(Q_0)+\mu e^{-\alpha t_0}-\hat{U}_{n_0}=0\leqslant w_n(t), \quad \text{for all } n\in\mathbb{Z} \text{ and } t\in[0,t_0], \tag{3.13}
$$

where

$$
Q_0=n_0+z-ct_0+B(1-e^{-\alpha t_0}).
$$

Then $w'_{n_0}(t_0) \leqslant 0$ and so

$$\begin{aligned} 0 &\geqslant w'_{n_0}(t_0) - D[w_{n_0+1}(t_0) + w_{n_0-1}(t_0) - 2w_{n_0}(t_0)] \\ &= \alpha B U'(Q_0)e^{-\alpha t_0} - \alpha\mu e^{-\alpha t_0} - cU'(Q_0) - D[U(Q_0+1) \\ &\quad + U(Q_0-1) - 2U(Q_0)] - D[\hat{U}_{n_0+1} + \hat{U}_{n_0-1} - 2\hat{U}_{n_0}] \\ &= \alpha B U'(Q_0)e^{-\alpha t_0} - \alpha\mu e^{-\alpha t_0} + d(\hat{U}_{n_0} - U(Q_0)) \\ &\quad + \sum_i J(i)[b(U(Q_0 + cr - i)) - b(\hat{U}_{n_0-i})] \\ &\geqslant [d\mu - \alpha\mu + \alpha B U'(Q_0)]e^{-\alpha t_0} + \sum_i J(i)[b(U(Q_0 - i)) \\ &\quad - b(U(Q_0 - i) + \mu e^{-\alpha t_0})] \\ &= [d\mu - \alpha\mu + \alpha B U'(Q_0)]e^{-\alpha t_0} - \sum_i J(i) b'(\eta_i)\mu e^{-\alpha t_0} \\ &\geqslant [d - \alpha + \frac{\alpha B}{\mu} U'(Q_0) - \sum_{|i|>N} J(i) b'_{\max} - \sum_{|i|\leqslant N} J(i) b'(\eta_i)]\mu e^{-\alpha t_0}, \end{aligned}$$

where $\eta_i \in (U(Q_0 - i), U(Q_0 - i) + \mu e^{-\alpha t_0})$. Therefore, a similar argument as above shows that $w_n(t) > 0$ for all $n \in \mathbb{Z}$ and $t \geqslant 0$.

Suppose that $c > \hat{c}$. Fix $\bar{x}$ such that $\hat{U}(\bar{x}) > 0$ and then it is easily seen that $w(\bar{x}, t) \to -\hat{U}(\bar{x})$ as $t \to +\infty$, which contradicts the positivity of w.

In the case where $c < \hat{c}$, a similar analysis as before leads to a contradiction too. Thus we have $\hat{c} = c$.

Next, we show that, up to a translation, $\hat{U} = U$. Taking the limit $t \to +\infty$ in (3.8), we get

$$U(x + z + B) \geqslant \hat{U}(x), \quad \text{for all } x \in \mathbb{R}.$$

Thus there exists a minimal $\hat{z}$ such that

$$U(x) \geqslant \hat{U}(x - z), \quad \text{for all } x \in \mathbb{R} \text{ and } z > \hat{z}.$$

We assert that if $U(x) \neq \hat{U}(x - \hat{z})$ for some x, then $U(x) > \hat{U}(x - \hat{z})$ for all $x \in \mathbb{R}$. Suppose otherwise that for some x_0, $U(x_0) = \hat{U}(x_0 - \hat{z})$. Let $w(x) = U(x) - \hat{U}(x - \hat{z})$. Then we have $w'(x_0) = 0$ and $w(x) \geqslant w(x_0) = 0$ for all $x \in \mathbb{R}$, and hence

$$\begin{aligned} 0 &\leqslant D[w(x_0+1) + w(x_0-1) - 2w(x_0)] \\ &= \hat{c}w'(x_0) + D[w(x_0+1) + w(x_0-1) - 2w(x_0)] \\ &= \hat{c}U'(x_0) + D[U(x_0+1) + U(x_0-1) - 2U(x_0)] \\ &\qquad \times \hat{c}\hat{U}'(x_0 - \hat{z}) + D[\hat{U}(x_0 + 1 - \hat{z}) + \hat{U}(x_0 - 1 - \hat{z}) - 2\hat{U}(x_0 - \hat{z})] \end{aligned}$$

$$\begin{aligned}
&= dU'(x_0) - \sum_i J(i)b(U(x_0 + \hat{c}r - i)) \\
&\quad - d\hat{U}'(x_0 - \hat{z}) + \sum_i J(i)b(\hat{U}(x_0 - \hat{z} + \hat{c}r - i)) \\
&= -\sum_i J(i)b'(\eta_i)[U(x_0 + \hat{c}r - i) - \hat{U}(x_0 - \hat{z} + \hat{c}r - i)] \\
&\leqslant 0,
\end{aligned}$$

where $\eta_i \in (0, K)$. Hence, we have $w(x_0 + 1) = w(x_0 - 1) = w(x_0) = 0$ and $w(x_0 + \hat{c}r - i) = U(x_0 + \hat{c}r - i) - \hat{U}(x_0 - \hat{z} + \hat{c}r - i) = 0$ for all $i \in \mathbb{Z}$ with $J(i) \neq 0$. From which, by an induction argument, we can show that

$$w(x_0 + m\hat{c}r + n) = 0, \quad \text{for all } n, m \in \mathbb{Z} \text{ with } m \geqslant 0. \tag{3.14}$$

Let $v_{n,m}(t) = w(x_0 + m\hat{c}r + n - \hat{c}t), n \in \mathbb{Z}, m \geqslant 0$, then by the Mean Value Theorem, it is easily seen that $v_{n,m}$ satisfies the initial value problem

$$v'_{n,m}(t) = D[v_{n+1,m} + v_{n-1,m} - 2v_{n,m}] - dv_{n,m} + \sum_i J(i)P_{n-i,m+1}v_{n-1,m+1},$$

$$v_{n,m}(0) = 0,$$

where $n \in \mathbb{Z}, m \geqslant 0$ and

$$\begin{aligned}
P_{n,m}(t) = \int_0^1 & b'[U(x_0 + m\hat{c}r + n - \hat{c}t) + \alpha(\hat{U}(x_0 + m\hat{c}r + n - \hat{z} - \hat{c}t) \\
& - U(x_0 + m\hat{c}r + n - \hat{c}t))]\, d\alpha.
\end{aligned}$$

By the uniqueness of solutions to the initial value problem, we conclude that $v_{n,m}(t) \equiv 0$, and hence $w(x) \equiv 0$, which leads to a contradiction and establish the assertion.

For $\eta > 0$, define

$$z(\eta) = \inf\{z; U(x) \geqslant \hat{U}(x - z) - \eta \quad \text{for all } x \in \mathbb{R}\}.$$

Notice that $z(\eta) < \hat{z}$ since U' is bounded and $\lim_{\eta \searrow 0} z(\eta) = \hat{z}$ by the minimality of $\hat{z}$.

Fix $N > 0$. We claim that there exists $\eta_N > 0$ such that for all $\eta \in (0, \eta_N]$,

$$U(x) > \hat{U}(x - z(\eta)) - \eta, \quad \text{for } |x| \leqslant N. \tag{3.15}$$

If not, there exist $\eta_n \searrow 0, x_n \to x_0 \in [-N, N]$ with

$$U(x_n) = \hat{U}(x_n - z(\eta_n)) - \eta_n.$$

Taking the limit as $n \to \infty$ then gives $U(x_0) = \hat{U}(x_0 - \hat{z})$, a contradiction to our previously established assertion.

Let

$$\hat{w}(x,t) = U(x) - \hat{U}(x - (\hat{z} - \varepsilon)) + \mu e^{-\alpha t},$$

where $\mu < \eta_M$, M is from (3.5) and (3.6), α is as in (3.3), and $\varepsilon > 0$ is taken so that $2\varepsilon < \hat{z} - z(\mu)$. Then $\hat{w}(x,0) > 0$ for all $x \in \mathbb{R}$. In fact, since $U(x) - \hat{U}(x - z(\mu)) + \mu \geqslant 0$, we have $U(x) - \hat{U}(x - (\hat{z} - \varepsilon)) + \mu \geqslant 0$. Suppose that there exists x_0 such that $\hat{w}(x_0, 0) = U(x_0) - \hat{U}(x_0 - (\hat{z} - \varepsilon)) + \mu = 0$. Then it follows that $z(\mu) = \hat{z} - \varepsilon > z(\mu)$, a contradiction.

If for some $t_0 > 0$ and $x_0 \in \mathbb{R}$ such that $\hat{w}(x_0, t_0) = 0 < w(x,t)$ for all $t \in [0, t_0)$ and $x \in \mathbb{R}$. Then $\hat{w}_x(x_0, t_0) = U'(x_0) - \hat{U}'(x_0 - (\hat{z} - \varepsilon)) = 0$, and hence

$$\begin{aligned}
0 &\geqslant -D[\hat{w}(x_0+1, t_0) + \hat{w}(x_0-1, t_0) - 2\hat{w}(x_0, t_0)] \\
&= -D[U(x_0+1) + U(x_0-1) - 2U(x_0)] \\
&\quad + D[\hat{U}(x_0 + 1 - (\hat{z} - \varepsilon)) + \hat{U}(x_0 - 1 - (\hat{z} - \varepsilon)) - 2U(x_0 - (\hat{z} - \varepsilon))] \\
&= -d[U(x_0) - \hat{U}(x_0 - (\hat{z} - \varepsilon))] + \hat{c}[U'(x_0) - \hat{U}'(x_0 - (\hat{z} - \varepsilon))] \\
&\quad + \sum_i J(i)[b(U(x_0 + \hat{c}r - i)) - b(\hat{U}(x_0 - (\hat{z} - \varepsilon) + \hat{c}r - i))] \\
&= d\mu e^{-\alpha t_0} + \sum_i J(i)[b(U(x_0 + \hat{c}r - i)) - b(\hat{U}(x_0 - (\hat{z} - \varepsilon) + \hat{c}r - i))] \\
&\geqslant d\mu e^{-\alpha t_0} + \sum_i J(i)[b(U(x_0 + \hat{c}r - i)) - b(U(x_0 + \hat{c}r - i) + \mu e^{-\alpha t_0})] \\
&\geqslant [d - b'_{\max} \sum_{|i|>N} J(i) - \sum_{|i| \leqslant N} J(i) b'(\eta_i)] \mu e^{-\alpha t_0},
\end{aligned} \tag{3.16}$$

where $\eta_i \in (U(x_0 + \hat{c}r - i), U(x_0 + \hat{c}r - i) + \mu e^{-\alpha t_0})$ and N is given by (3.3). Since $U(x_0) = \hat{U}(x_0 - (\hat{z} - \varepsilon)) - \mu e^{-\alpha t_0}$, it follows that $z(\mu e^{-\alpha t_0}) = \hat{z} - \varepsilon$, and because $\mu e^{-\alpha t_0} < \mu < \eta_M$, (3.15) implies that $|x_0| > M$. By (3.5) and (3.6), $\eta_i \geqslant K - \kappa/2$ or $\eta_i \leqslant \kappa/2$ for all $|i| \leqslant N$, and hence, by (3.4), $b'(\eta_i) < \max\{b'(0), b'(K)\} + \varpi/2$ for all $|i| \leqslant N$. Therefore, it follows from (3.3) that

$$0 \geqslant [d - b'_{\max} \sum_{|i|>N} J(i) - \sum_{|i| \leqslant N} J(i) b'(\eta_i)] \mu e^{-\alpha t_0} > 0,$$

which is a contradiction. Thus, $\hat{w}(x,t) > 0$ for all $x \in \mathbb{R}$ and $t \geqslant 0$. Taking the limit as $t \to +\infty$ gives

$$U(x) \geqslant \hat{U}(x - (\hat{z} - \varepsilon)), \quad \text{for all } x \in \mathbb{R},$$

contradicting the minimality of $\hat{z}$ and proving that $U \equiv \hat{U}$. The proof is complete. □

4. Asymptotic stability of traveling waves

In this section, we shall establish the asymptotic stability of traveling waves with non-zero speed. To do this, we shall construct various pairs of super and subsolutions and utilize the comparison and squeezing technique, which has been used previously in continuum cases by several authors (e.g., [6,7,16,17,21,25,30]) for various local equations.

Definition 4.1. A sequence of continuous functions $\{v_n(t)\}_{n\in\mathbb{Z}}, t \in [-r, b), b > 0$, is called a supersolution (subsolution) of (1.2) on $[0, b)$ if

$$v_n'(t) \geqslant (\leqslant) D[v_{n+1}(t) + v_{n-1}(t) - 2v_n(t)] - dv_n(t) + \sum_i J(i)b(v_{n-i}(t-r)) \quad (4.1)$$

for all $t \in [0, b)$.

At first, we establish the following existence and comparison result.

Lemma 4.1. *For any $\delta \in [0, \delta_0]$ and any $\varphi = \{\varphi_n\}_{n\in\mathbb{Z}}$ with $\varphi_n \in C([-r, 0], [-\delta, K + \delta])$, (1.2) admits a unique solution $u(t) = \{u_n(t)\}_{n\in\mathbb{Z}}$ on $[0, +\infty)$ satisfying $u_n(s) = \varphi_n(s)$ and $-\delta \leqslant u_n(t) \leqslant K + \delta$ for $s \in [-r, 0]$, $t \in [-r, +\infty)$ and $n \in \mathbb{Z}$. For any pair of supersolution $w_n^+(t)$ and subsolution $w_n^-(t)$ of (1.2) on $[0, +\infty)$ with $-\delta_0 \leqslant w_n^-(t) \leqslant K$, $0 \leqslant w_n^+(t) \leqslant K + \delta_0$ for $t \in [-r, +\infty), n \in \mathbb{Z}$, and $w_n^+(s) \geqslant w_n^-(s)$ for $s \in [-r, 0], n \in \mathbb{Z}$, there holds $w_n^+(t) \geqslant w_n^-(t)$ for $t \geqslant 0, n \in \mathbb{Z}$, and*

$$w_n^+(t) - w_n^-(t) \geqslant e^{-(2D+d)(t-t_0)} \sum_{k\in\mathbb{Z}} (w_k^+(t_0) - w_k^-(t_0)) \times \sum_{j=0}^{+\infty} C_{2j+|n-k|}^j \frac{[D(t-t_0)]^{2j+|n-k|}}{(2j+|n-k|)!}, \quad (4.2)$$

for any $n \in \mathbb{Z}$ and $t > t_0 \geqslant 0$.

Proof. Clearly, (1.2) is equivalent to

$$u_n(t) = \varphi_n(0)e^{-(2D+d)t} + \int_0^t e^{(2D+d)(\tau-t)} H[u_n](\tau)\, d\tau,$$

where $H[u_n](\tau) = D[u_{n+1}(\tau) + u_{n-1}(\tau)] + \sum_i J(i)b(u_{n-i}(\tau - r))$.

For $u = \{u_n\}_{n\in Z}$ with $u_n \in C([-r, +\infty), [-\delta, K + \delta])$ and $u_n(t) = \varphi_n(t)$ for $t \in [-r, 0]$, define

$$F_n[u](t) = \begin{cases} \varphi_n(0)e^{-(2D+d)t} + \int_0^t e^{(2D+d)(\tau-t)} H[u_n](\tau)\, d\tau & \text{for } n \in \mathbb{Z} \text{ and } t > 0, \\ \varphi_n(t) & \text{for } n \in \mathbb{Z} \text{ and } t \in [-r, 0]. \end{cases}$$

Then for $t > 0$, we have

$$F_n[u](t) \leqslant (K+\delta)e^{-(2D+d)t} + (K+\delta)(2D+d)\int_0^t e^{(2D+d)(\tau-t)}\,d\tau = K+\delta$$

and

$$F_n[u](t) \geqslant -\delta e^{-(2D+d)t} - \delta(2D+d)\int_0^t e^{(2D+d)(\tau-t)}\,d\tau = -\delta,$$

and hence, $F = \{F_n\}_{n\in\mathbb{Z}} : S \to S$ is well defined, where

$$\begin{aligned} S := \{u = \{u_n\}_{n\in\mathbb{Z}};\ & u_n \in C([-r,+\infty),[-\delta, K+\delta]),\\ & u_n(t) = \varphi_n(t) \quad \text{for } t \in [-r,0]\}. \end{aligned}$$

For $\lambda > 0$, let

$$X_\lambda := \{u = \{u_n\}_{n\in\mathbb{Z}};\ u_n \in C([-r,+\infty),\mathbb{R}), \quad \sup_{t\geqslant -r, n\in\mathbb{Z}} |u_n(t)|e^{-\lambda t} < +\infty\},$$

$$\|u\|_\lambda := \sup_{t\geqslant -r, n\in\mathbb{Z}} |u_n(t)|e^{-\lambda t} < +\infty.$$

Then $(X_\lambda, \|\cdot\|_\lambda)$ is a Banach space and $S \subset X_\lambda$ is a closed subset of X_λ.

For any $u, \bar{u} \in S$, let $w = \{w_n\}_{n\in\mathbb{Z}}$, $w_n(t) = u_n(t) - \bar{u}_n(t)$ for $n \in \mathbb{Z}$, then for $t > 0$, we have

$$\begin{aligned} &|F_n[u](t) - F_n[\bar{u}](t)|e^{-\lambda t} \\ &\quad \leqslant e^{-(2D+d+\lambda)t}\int_0^t e^{(2D+d)\tau}|H[u_n](\tau) - H[\bar{u}_n](\tau)|\,d\tau \\ &\quad \leqslant \int_0^t e^{(2D+d+\lambda)(\tau-t)}\{D[|w_{n+1}(\tau)| + w_{n-1}(\tau)|]e^{-\lambda\tau} \\ &\qquad + b'_{\max}e^{-\lambda r}|w_{n-i}(\tau-r)|e^{-\lambda(\tau-r)}\}\,d\tau \\ &\quad \leqslant \|w\|_\lambda(2D + b'_{\max}e^{-\lambda r})\int_0^t e^{(2D+d+\lambda)(\tau-t)}\,d\tau \\ &\quad \leqslant \frac{2D + b'_{\max}e^{-\lambda r}}{2D+d+\lambda}\|w\|_\lambda. \end{aligned}$$

Therefore, we can choose $\lambda > 0$ large enough so that $F : S \to S$ is a contracting map. Clearly, the unique fixed point $u \in S$ is a solution of (2.1) on $[0,+\infty)$.

In what follows, we show the comparison result. Put $w_n(t) := w_n^-(t) - w_n^+(t), n \in \mathbb{Z}, t \in [-r, +\infty)$, then $w_n(t)$ is continuous and bounded from above by K, $\omega(t) := \sup_{n\in\mathbb{Z}} w_n(t)$ is continuous on $[-r, +\infty)$. Suppose the assertion is not true. Let $M_0 > 0$ be such that $M_0 + d - b'_{\max} e^{-M_0 r} > 0$, then there exists $t_0 > 0$ such that $\omega(t_0) > 0$ and

$$\omega(t_0)e^{-M_0 t_0} = \sup_{t\geqslant -r} \{\omega(t)e^{-M_0 t}\} > \omega(\tau)e^{-M_0\tau}, \quad \text{for all } \tau \in [-r, t_0). \tag{4.3}$$

Let $\{n_j\}_{j=1}^{\infty}$ be a sequence so that $w_{n_j}(t_0) > 0$ for all $j \geqslant 1$ and $\lim_{j\to+\infty} w_{n_j}(t_0) = \omega(t_0)$. Let $\{t_j\}_{j=1}^{\infty}$ be a sequence in $(0, t_0]$ so that

$$e^{-M_0 t_j} w_{n_j}(t_j) = \max_{t\in[0,t_0]} \{e^{-M_0 t} w_{n_j}(t)\}. \tag{4.4}$$

It follows from (4.3) that $\lim_{j\to+\infty} t_j = t_0$. Since

$$e^{-M_0 t_0} w_{n_j}(t_0) \leqslant e^{-M_0 t_j} w_{n_j}(t_j) \leqslant e^{-M_0 t_j} \omega(t_j) \leqslant e^{-M_0 t_0} \omega(t_0),$$

we have

$$e^{-M_0(t_0-t_j)} w_{n_j}(t_0) \leqslant w_{n_j}(t_j) \leqslant e^{-M_0(t_0-t_j)} \omega(t_0),$$

which yields $\lim_{j\to+\infty} w_{n_j}(t_j) = \omega(t_0)$.

In view of (4.4), for each $j \geqslant 1$, we obtain

$$0 \leqslant \frac{d}{dt}\{e^{-M_0 t} w_{n_j}(t)\}|_{t=t_j-} = e^{-M_0 t_j}[w'_{n_j}(t_j) - M_0 w_{n_j}(t_j)],$$

and hence,

$$w'_{n_j}(t_j) \geqslant M_0 w_{n_j}(t_j).$$

Therefore, it follows from (4.1) that

$$\begin{aligned}
0 \ &\geqslant \ w'_{n_j}(t_j) - D[w_{n_j+1}(t_j) + w_{n_j-1}(t_j) - 2w_{n_j}(t_j)] + d w_{n_j}(t_j) \\
&\quad - \sum_i J(i)[b(w^-_{n_j-i}(t_j - r)) - b(w^+_{n_j}(t_j - r))] \\
&\geqslant \ (M_0 + 2D + d) w_{n_j}(t_j) - D[w_{n_j+1}(t_j) + w_{n_j-1}(t_j)] - b'_{\max} \max\{0, \omega(t_j - r)\} \\
&\geqslant \ (M_0 + 2D + d) w_{n_j}(t_j) - 2D\omega(t_j) - b'_{\max} \max\{0, \omega(t_j - r)\}.
\end{aligned}$$

Sending $j \to +\infty$ to get

$$\begin{aligned} 0 &\geqslant (M_0 + 2D + d)\omega(t_0) - 2D\omega(t_0) - b'_{\max} e^{M_0(t_0-r)} \max\{0, \omega(t_0 - r)e^{-M_0(t_0-r)}\} \\ &\geqslant (M_0 + d)\omega(t_0) - b'_{\max} e^{M_0(t_0-r)} \omega(t_0) e^{-M_0 t_0} \\ &= [M_0 + d - b'_{\max} e^{-M_0 r}]\omega(t_0). \end{aligned}$$

Recall that $M_0 + d - b'_{\max} e^{-M_0 r} > 0$, we conclude that $\omega(t_0) \leqslant 0$, which contradicts to $\omega(t_0) > 0$. This contradiction shows that $w_n(t) = w_n^-(t) - w_n^+(t) \leqslant 0$ for all $n \in \mathbb{Z}$ and $t \in (0, +\infty)$.

Since $w_n^+(t) \geqslant w_n^-(t)$ for all $n \in \mathbb{Z}$ and $t \geqslant -r$, it follows from (4.1) that

$$\begin{aligned} w_n^+(t) - w_n^-(t) &\geqslant e^{-(2D+d)(t-t_0)}(w_n^+(t_0) - w_n^-(t_0)) \\ &\quad + \int_{t_0}^{t} e^{(2D+d)(s-t)}\{D[w_{n+1}^+(s) - w_{n+1}^-(s) + w_{n-1}^+(s) - w_{n-1}^-(s)] \\ &\quad + \sum_i J(i)[b(w_{n-i}^+(s-r)) - b(w_{n-i}^-(s-r))]\}\, ds \\ &\geqslant e^{-(2D+d)(t-t_0)}(w_n^+(t_0) - w_n^-(t_0)) \\ &\quad + D\int_{t_0}^{t} e^{(2D+d)(s-t)}[w_{n+1}^+(s) - w_{n+1}^-(s) + w_{n-1}^+(s) \\ &\quad - w_{n-1}^-(s)]\, ds. \end{aligned}$$

Then (4.2) follows from a straightforward and tedious calculation. The proof is complete. □

Remark 4.1. In particular, (4.2) yields

$$w_n^+(t) - w_n^-(t) > e^{-(2D+d)(t-t_0)} \sum_{k \in \mathbb{Z}} (w_k^+(t_0) - w_k^-(t_0)) \frac{[D(t-t_0)]^{|n-k|}}{|n-k|!}, \tag{4.5}$$

for any $n \in \mathbb{Z}$ and $t > t_0 \geqslant 0$.

Let $\zeta \in C^\infty(\mathbb{R})$ be a fixed function with the following properties:

$$\zeta(s) = 0, \quad \text{if } s \leqslant -2; \qquad \zeta(s) = 1, \quad \text{if } s \geqslant 2;$$
$$0 < \zeta'(s) < 1, \quad \text{if } s \in (-2, 2).$$

Then we have the following result.

Lemma 4.2. *Assume that* (H1)–(H3) *hold. Then for any* $\delta \in (0, \delta_0)$, *there exist two positive numbers* $\varepsilon = \varepsilon(\delta)$ *and* $C = C(\delta)$ *such that for every* $\xi^{\pm} \in \mathbb{R}$, *the functions* $\upsilon_n^{\pm}(t)$ *defined by*

$$\upsilon_n^+(t) := (K + \delta) - [K - (u^* - 2\delta)e^{-\varepsilon t}]\zeta(-\varepsilon(n - \xi^+ + Ct)),$$
$$\upsilon_n^-(t) := -\delta + [K - (K - u^* - 2\delta)e^{-\varepsilon t}]\zeta(\varepsilon(n - \xi^- - Ct))$$

are a supersolution and a subsolution of (1.2) *on* $[0, +\infty)$, *respectively.*

Proof. Since $du - b(u) > 0, u \in (0, u^*) \cup (K, K + \delta_0)$, we have

$$M_1 = M_1(\delta) = \min\{du - b(u); u \in [\delta, u^* - \delta/2]\} > 0,$$

$$M_2 = M_2(\delta) = \min\{du - b(u); u \in [K + \delta/2, K + \delta]\} > 0.$$

Take $\varepsilon = \varepsilon(\delta) > 0$ small enough such that $u^* e^{\varepsilon r} < K$ and

$$\varepsilon u^* + 2\varepsilon DK + \varepsilon r u^* b'_{\max} e^{\varepsilon r} + \varepsilon K b'_{\max} \sum_i |i| J(i) < \min\{M_1, M_2\}. \tag{4.6}$$

Let $\varepsilon^* = \varepsilon^*(\delta) > 0$ be such that $K\varepsilon^* < \delta$, and let $\kappa = \kappa(\delta) \in (0, 1)$ be such that

$$0 \leqslant \zeta(s) < \varepsilon^*/2, \quad \text{if } s < -2 + \kappa, \tag{4.7}$$

$$1 \geqslant \zeta(s) > 1 - \varepsilon^*/2, \quad \text{if } s > 2 - \kappa. \tag{4.8}$$

Define

$$\sigma := \min\{\zeta'(s); -2 + \kappa/2 \leqslant s \leqslant 2 - \kappa/2\} > 0.$$

Then take $C = C(\delta) > 0$ sufficiently large so that

$$\begin{aligned} \varepsilon C\sigma(K - u^*) > \; & \varepsilon u^* + 2\varepsilon DK + \varepsilon r u^* b'_{\max} e^{\varepsilon r} + \varepsilon K b'_{\max} \sum_i |i| J(i) \\ & + \max\{|du - b(u)|; u \in [-\delta, K + \delta]\}. \end{aligned} \tag{4.9}$$

Set $\xi = n - \xi^+ + Ct$. Then for $t \geqslant 0$, we have

$$S(\upsilon_n^+)(t) : = \frac{d}{dt}\upsilon_n^+(t) - D[\upsilon_{n+1}^+(t) + \upsilon_{n-1}^+(t) - 2\upsilon_n^+(t)]$$

$$
\begin{aligned}
&+ d\upsilon_n^+(t) - \sum_i J(i) b(\upsilon_{n-i}^+(t-r)) \\
= \; & -\varepsilon(u^* - 2\delta)e^{-\varepsilon t}\zeta(-\varepsilon\xi) + \varepsilon C[K - (u^* - 2\delta)e^{-\varepsilon t}]\zeta'(-\varepsilon\xi) \\
&+ D[K - (u^* - 2\delta)e^{-\varepsilon t}]\{\zeta(-\varepsilon(\xi+1)) + \zeta(-\varepsilon(\xi-1)) - 2\zeta(-\varepsilon\xi)\} \\
&+ d\upsilon_n^+(t) - \sum_i J(i) b(\upsilon_{n-i}^+(t-r)) \\
\geqslant \; & -\varepsilon u^* + \varepsilon C(K - u^*)\zeta'(-\varepsilon\xi) \\
&+ D[K - (u^* - 2\delta)e^{-\varepsilon t}]\{\zeta(-\varepsilon(\xi+1)) + \zeta(-\varepsilon(\xi-1)) - 2\zeta(-\varepsilon\xi)\} \\
&+ d\upsilon_n^+(t) - b(\upsilon_n^+(t-r)) - \sum_i J(i)[b(\upsilon_{n-i}^+(t-r)) \\
&- b(\upsilon_n^+(t-r))].
\end{aligned} \tag{4.10}
$$

By the Mean Value Theorem, it is easily seen that

$$
|\zeta(-\varepsilon(\xi+1)) + \zeta(-\varepsilon(\xi-1)) - 2\zeta(-\varepsilon\xi)| \leqslant 2\varepsilon,
$$

$$
\begin{aligned}
|b(\upsilon_{n-i}^+(t-r)) - b(\upsilon_n^+(t-r))| \; &\leqslant \; b'_{\max}|\upsilon_{n-i}^+(t-r) - \upsilon_n^+(t-r)| \\
&\leqslant \; K b'_{\max}|\zeta(-\varepsilon(\xi - i - Cr)) - \zeta(-\varepsilon(\xi - Cr))| \\
&\leqslant \; \varepsilon|i|K b'_{\max}
\end{aligned}
$$

and

$$
\begin{aligned}
b(\upsilon_n^+(t)) - b(\upsilon_n^+(t-r)) \; &= \; r b'(\eta)\frac{d}{dt}\upsilon_n^+(\bar{t}) \\
&= \; r b'(\eta)\{-\varepsilon(u^* - 2\delta)e^{-\varepsilon\bar{t}}\zeta(-\varepsilon(\xi + C\bar{t} - Ct)) \\
&\quad + \varepsilon C[K - (u^* - 2\delta)e^{-\varepsilon\bar{t}}]\zeta'(-\varepsilon(\xi + C\bar{t} - Ct))\} \\
&\geqslant \; -\varepsilon r u^* b'_{\max} e^{\varepsilon r},
\end{aligned}
$$

where $\eta \in [\upsilon_n^+(t), \upsilon_n^+(t-r)]$ and $\bar{t} \in [t-r, t]$. Hence, (4.10) implies that

$$
\begin{aligned}
S(\upsilon_n^+)(t) \; \geqslant \; & -\varepsilon u^* - 2\varepsilon DK - \varepsilon r u^* b'_{\max} e^{\varepsilon r} - \varepsilon K b'_{\max} \sum_i |i| J(i) \\
& + \varepsilon C(K - u^*)\zeta'(-\varepsilon\xi) + d\upsilon_n^+(t) - b(\upsilon_n^+(t)).
\end{aligned} \tag{4.11}
$$

We distinguish among three cases:

Case (i): $-\varepsilon\xi \leqslant -2+\kappa/2$. In this case, $-\varepsilon\xi \leqslant -2+\kappa, 0 \leqslant \zeta(-\varepsilon\xi) \leqslant \varepsilon^*/2$. Recall that $K\varepsilon^* < \delta$, we then find

$$\begin{aligned} K+\delta \geqslant \upsilon_n^+(t) &\geqslant (K+\delta) - [K-(u^*-2\delta)e^{-\varepsilon t}]\varepsilon^*/2 \\ &\geqslant K+\delta - K\varepsilon^*/2 \\ &\geqslant K+\delta/2, \end{aligned}$$

for all $t \geqslant 0$. It then follows from (4.6) and (4.11) that

$$S(\upsilon_n^+)(t) \geqslant -\varepsilon u^* - 2\varepsilon DK - \varepsilon r u^* b'_{\max} e^{\varepsilon r} - \varepsilon K b'_{\max} \sum_i |i| J(i) + M_2 > 0.$$

Case (ii): $-\varepsilon\xi \geqslant 2-\kappa/2$. In this case, $-\varepsilon\xi \geqslant 2-\kappa$, $1-\varepsilon^*/2 \leqslant \zeta(-\varepsilon\xi) \leqslant 1$. It then follows that

$$\begin{aligned} \delta \leqslant \upsilon_n^+(t) &\leqslant (K+\delta) - [K-(u^*-2\delta)e^{-\varepsilon t}](1-\varepsilon^*/2) \\ &\leqslant (K+\delta) - [K-(u^*-2\delta)](1-\varepsilon^*/2) \\ &= u^* - \delta + [K-(u^*-2\delta)]\varepsilon^*/2 \\ &\leqslant u^* - \delta + K\varepsilon^*/2 \\ &\leqslant u^* - \delta/2, \end{aligned}$$

for all $t \geqslant 0$. Therefore, by (4.6) and (4.11), we also have

$$S(\upsilon_n^+)(t) \geqslant -\varepsilon u^* - 2\varepsilon DK - \varepsilon r u^* b'_{\max} e^{\varepsilon r} - \varepsilon K b'_{\max} \sum_i |i| J(i) + M_1 > 0.$$

Case(iii): $-2+\kappa/2 \leqslant -\varepsilon\xi \leqslant 2-\kappa/2$. In this case, by (4.9) and (4.11), we also have

$$\begin{aligned} S(\upsilon_n^+)(t) &\geqslant -\varepsilon u^* - 2\varepsilon DK - \varepsilon r u^* b'_{\max} e^{\varepsilon r} - \varepsilon K b'_{\max} \sum_i |i| J(i) \\ &\quad + \varepsilon C(K-u^*)\sigma - \max\{|du-b(u)|; u \in [-\delta, K+\delta]\} \\ &> 0. \end{aligned}$$

Combining cases (i)–(iii), we obtain

$$\frac{d}{dt}\upsilon_n^+(t) - D[\upsilon_{n+1}^+(t) + \upsilon_{n-1}^+(t) - 2\upsilon_n^+(t)] + d\upsilon_n^+(t) - \sum_i J(i) b(\upsilon_{n-i}^+(t-r)) \geqslant 0,$$

for all $t \geqslant 0$ and $n \in \mathbb{Z}$. Thus $\upsilon_n^+(t)$ is a supersolution of (1.2) on $[0,+\infty)$. In a similar way, we can prove that $\upsilon_n^-(t)$ is a subsolution of (1.2) on $[0,+\infty)$. The proof is complete. □

Remark 4.2. Clearly, the functions υ_n^+ and υ_n^- have the following properties:

$$\begin{cases} \upsilon_n^+(s) = K+\delta, & \text{if } s\in[-r,0], \text{ and } n\geqslant \xi^+ - Cs + 2\varepsilon^{-1}, \\ \upsilon_n^+(s) \geqslant u^*-\delta, & \text{for all } s\in[-r,0], \text{ and } n\in\mathbb{Z}, \\ \upsilon_n^+(t) = \delta+(u^*-2\delta)e^{-\varepsilon t}, & \text{for all } t\geqslant -r \text{ and } n\leqslant \xi^+ - Ct - 2\varepsilon^{-1}, \\ \upsilon_n^-(s) = -\delta, & \text{if } s\in[-r,0], \text{ and } n\leqslant \xi^- + Cs - 2\varepsilon^{-1}, \\ \upsilon_n^-(s) \leqslant u^*+\delta, & \text{for all } s\in[-r,0], \text{ and } n\in\mathbb{Z}, \\ \upsilon_n^-(t) = K-\delta-(K-u^*-2\delta)e^{-\varepsilon t}, & \text{for all } t\geqslant -r \text{ and } n\geqslant \xi^- + Ct + 2\varepsilon^{-1}. \end{cases}$$

Lemma 4.3. *Assume that* (H1)–(H3) *hold. Let* (U,c) *be the solution to* (2.1) *and* (2.2) *as given in Theorem* 2.1, *such that* $c\neq 0$. *Then there exist three positive numbers* β_0 (*which is independent of* U), σ_0 *and* $\bar\delta$ *such that for any* $\delta\in(0,\bar\delta]$ *and every* $\hat\xi\in\mathbb{R}$, *the functions* $w_n^\pm(t)$ *defined by*

$$w_n^\pm(t) := U(n-ct+\hat\xi \pm \sigma_0\delta(e^{\beta_0 r}-e^{-\beta_0 t})) \pm \delta e^{-\beta_0 t}$$

are a supersolution and a subsolution of (1.2) *on* $[0,+\infty)$, *respectively.*

Proof. Since $d>\max\{b'(0),b'(K)\}$, we can choose $\beta_0>0$ and $\varepsilon^*>0$ such that

$$d > \beta_0 + e^{\beta_0 r}(\max\{b'(0),b'(K)\}+\varepsilon^*). \tag{4.12}$$

By (1.4) and (1.5), there exists a $\delta^*>0$ such that

$$0\leqslant b'(\eta)\leqslant b'(0)+\varepsilon^*, \quad \text{for all } \eta\in[-\delta^*,\delta^*], \tag{4.13}$$

$$0\leqslant b'(\eta)\leqslant b'(K)+\varepsilon^*, \quad \text{for all } \eta\in[K-\delta^*,K+\delta^*]. \tag{4.14}$$

Let $c_0=|c|r+(e^{\beta_0 r}-1)$. Then there exists a constant $N_0=N_0(U,\beta_0,\varepsilon^*,\delta^*)>0$ such that

$$U(\xi)\leqslant\delta^*, \quad \text{for all } \xi\leqslant -N_0/2+c_0, \tag{4.15}$$

$$U(\xi)\geqslant K-\delta^*, \quad \text{for all } \xi\geqslant N_0/2-c_0 \tag{4.16}$$

and

$$d > \beta_0 + e^{\beta_0 r}(\max\{b'(0),b'(K)\}+\varepsilon^*) + e^{\beta_0 r}b'_{\max}\sum_{|i|>N_0/2} J(i). \tag{4.17}$$

Denote

$$m_0=m_0(U,\beta_0,\varepsilon^*,\delta^*)=\min\{U'(\xi);\,|\xi|\leqslant N_0\}>0,$$

and define

$$\sigma_0 := \frac{1}{\beta_0 m_0}[(e^{\beta_0 r} b'_{\max} - d) + \beta_0] > 0 \tag{4.18}$$

and

$$\bar{\delta} = \min\left\{\frac{1}{\sigma_0}, \delta^* e^{-\beta_0 r}\right\}.$$

For any given $\delta \in (0, \bar{\delta}]$, let $\xi = n - ct + \hat{\xi} + \sigma_0\delta(e^{\beta_0 r} - e^{-\beta_0 t})$. Then for all $t \geqslant 0$, we have

$$\begin{aligned}
S(w_n^+)(t) :=\; & \frac{d}{dt} w_n^+(t) - D[w_{n+1}^+(t) + w_{n-1}^+ - 2w_n^+(t)] + dw_n^+(t) \\
& - \sum_i J(i) b(w_{n-i}^+(t-r)) \\
=\; & (-c + \sigma_0\delta\beta_0 e^{-\beta_0 t})U'(\xi) - \delta\beta_0 e^{-\beta_0 t} \\
& - D[U(\xi+1) + U(\xi-1) - 2U(\xi)] + dU(\xi) + d\delta e^{-\beta_0 t} \\
& - \sum_i J(i) b(U(\xi + cr - i + \sigma_0\delta(1 - e^{\beta_0 r})e^{-\beta_0 t}) + \delta e^{-\beta_0 (t-r)}) \\
=\; & \sigma_0\delta\beta_0 e^{-\beta_0 t} U'(\xi) - \delta\beta_0 e^{-\beta_0 t} + d\delta e^{-\beta_0 t} + \sum_i J(i) b(U(\xi + cr - i)) \\
& - \sum_i J(i) b(U(\xi + cr - i + \sigma_0\delta(1 - e^{\beta_0 r})e^{-\beta_0 t}) + \delta e^{-\beta_0 (t-r)}) \\
=\; & [\sigma_0\beta_0 U'(\xi) - \beta_0 + d]\delta e^{-\beta_0 t} \\
& - \sum_i J(i) b'(\eta_i)[U'(\xi_i)\sigma_0\delta(1 - e^{\beta_0 r})e^{-\beta_0 t} + \delta e^{-\beta_0 (t-r)}] \\
=\; & \{\sigma_0\beta_0 U'(\xi) - \beta_0 + d + \sum_i J(i) b'(\eta_i)[U'(\xi_i)\sigma_0(e^{\beta_0 r} - 1) \\
& - e^{\beta_0 r}]\}\delta e^{-\beta_0 t},
\end{aligned}$$

where $\xi_i = \xi + cr - i + \theta\sigma_0\delta(1 - e^{\beta_0 r})e^{-\beta_0 t}$ and

$$\begin{aligned}
\eta_i =\; & \theta U(\xi + cr - i + \sigma_0\delta(1 - e^{\beta_0 r})e^{-\beta_0 t}) \\
& + \theta\delta e^{-\beta_0 (t-r)} + (1-\theta)U(\xi + cr - i).
\end{aligned} \tag{4.19}$$

Clearly, $0 \leqslant \eta_i \leqslant K + \delta e^{\beta_0 r} \leqslant K + \delta^*$. Therefore, $b'(\eta_i) \geqslant 0$, and hence

$$S(w_n^+)(t) \geqslant \{\sigma_0 \beta_0 U'(\xi) - \beta_0 + d - e^{\beta_0 r} \sum_i J(i) b'(\eta_i)\} \delta e^{-\beta_0 t}. \tag{4.20}$$

We distinguish among three cases:

Case (i): $|\xi| \leqslant N_0$. In this case, we have

$$S(w_n^+)(t) \geqslant \{\sigma_0 \beta_0 m_0 - \beta_0 + d - e^{\beta_0 r} b'_{\max}\} \delta e^{-\beta_0 t} = 0.$$

Case (ii): $\xi \geqslant N_0$. For $i \in [-\xi/2, \xi/2]$, we have

$$\tfrac{1}{2} N_0 \leqslant \tfrac{1}{2} \xi \leqslant \xi - i \leqslant \tfrac{3}{2} \xi.$$

By the choice of $\bar{\delta}$, for any $\delta \in (0, \bar{\delta}]$, we have $\sigma_0 \delta \leqslant 1$, and hence

$$\xi + cr - i + \sigma_0 \delta (1 - e^{\beta_0 r}) e^{-\beta_0 t} \geqslant \tfrac{1}{2} N_0 + cr + \sigma_0 \delta (1 - e^{\beta_0 r}) \geqslant \tfrac{1}{2} N_0 - c_0$$

and

$$\xi + cr - i \geqslant \tfrac{1}{2} N_0 + cr \geqslant \tfrac{1}{2} N_0 - c_0.$$

Therefore, it follows from (4.14) and (4.16) that

$$K + \delta^* \geqslant K + \delta e^{\beta_0 r} \geqslant \eta_i \geqslant K - \delta^*,$$

and hence

$$b'(\eta_i) \leqslant b'(K) + \varepsilon^*.$$

Therefore, by (4.17) and (4.20), we have

$$\begin{aligned} S(w_n^+)(t) &\geqslant \{-\beta_0 + d - e^{\beta_0 r} b'_{\max} \sum_{|i| > \xi/2} J(i) - e^{\beta_0 r} \sum_{|i| \leqslant \xi/2} J(i) b'(\eta_i)\} \delta e^{-\beta_0 t} \\ &\geqslant \{-\beta_0 + d - e^{\beta_0 r} b'_{\max} \sum_{|i| > N_0/2} J(i) - e^{\beta_0 r} (b'(K) + \varepsilon^*)\} \delta e^{-\beta_0 t} \\ &\geqslant 0. \end{aligned}$$

Case (iii): $\xi \leqslant -N_0$. In this case, the proof is similar to the case (ii) and thus is omitted. This completes the proof.

Let (U, c) be the solution given by Theorem 3.1, and let $c \neq 0$. We define the following functions:

$$w^{\pm}(n, \eta, \delta)(t) = U(n - ct + \eta \pm \sigma_0 \delta(e^{\beta_0 r} - e^{-\beta_0 t})) \pm \delta e^{-\beta_0 t}, \tag{4.21}$$

where σ_0 and β_0 are as in Lemma 4.3. By the proof of Lemma 4.3, we can choose $\beta_0 > 0$ as small as we wish.

Lemma 4.4. *Assume that* (H1)–(H3) *hold. Let* (U, c) *be the solution given by Theorem* 2.1, *and let* $c \neq 0$. *Let* $\varphi = \{\varphi_n\}_{n \in \mathbb{Z}}$ *with* $\varphi_n \in C([-r, 0], [0, K])$ *be such that*

$$\liminf_{n\to+\infty} \min_{s\in[-r,0]} \varphi_n(s) > u^*, \qquad \limsup_{n\to-\infty} \max_{s\in[-r,0]} \varphi_n(s) < u^*.$$

Then for any $\delta > 0$, *there exist* $T = T(\varphi, \delta) > 0$, $\xi = \xi(\varphi, \delta) \in \mathbb{R}$ *and* $h = h(\varphi, \delta) > 0$ *such that*

$$w_0^-(n, -cT + \xi, \delta)(s) < (u_n)_T(s) < w_0^+(n, -cT + \xi + h, \delta)(s), \quad s \in [-r, 0], \ n \in \mathbb{Z}.$$

Proof. By Lemma 4.1, $u_n(t, \varphi)$ exists globally for all $t \in [0, \infty)$ and $0 \leqslant u_n(t, \varphi) \leqslant K$ for all $t \geqslant 0$ and $n \in \mathbb{Z}$. For any $\delta > 0$, we can choose a positive constant $\delta_1 = \delta_1(\delta, \varphi) < \min\{\delta, \bar{\delta}\}$ such that

$$\liminf_{n\to+\infty} \min_{s\in[-r,0]} \varphi_n(s) > u^* + \delta_1$$

and

$$\limsup_{n\to-\infty} \max_{s\in[-r,0]} \varphi_n(s) < u^* - \delta_1.$$

Hence, there exists a constant $M_3 = M_3(\delta_1, \varphi) > 0$ such that

$$\varphi_n(s) < u^* - \delta_1, \quad \text{for all } s \in [-r, 0], \ n \leqslant -M_3, \tag{4.22}$$

$$\varphi_n(s) > u^* + \delta_1, \quad \text{for all } s \in [-r, 0], \ n \geqslant M_3. \tag{4.23}$$

Let $\varepsilon = \varepsilon(\delta_1)$ and $C = C(\delta_1)$ be defined in Lemma 4.2 with δ replaced by δ_1. Define $\xi^+ = -M_3 - Cr - 2\varepsilon^{-1}$ and $\xi^- = M_3 + Cr + 2\varepsilon^{-1}$, and let $\upsilon_n^{\pm}(t)$ be defined in Lemma 4.2. By (4.22), (4.23) and Remark 4.2, it follows that for all $s \in [-r, 0]$,

$$\varphi_n(s) < u^* - \delta_1 \leqslant \upsilon_n^+(s), \quad \text{for } n \leqslant -M_3,$$
$$\varphi_n(s) \leqslant K < K + \delta_1 \leqslant \upsilon_n^+(s), \quad \text{for } n \geqslant \xi^+ + Cr + 2\varepsilon^{-1} = -M_3$$

and

$$\varphi_n(s) > u^* + \delta_1 \geqslant \upsilon_n^-(s), \quad \text{for } n \geqslant M_3,$$
$$\varphi_n(s) \geqslant 0 > -\delta_1 \geqslant \upsilon_n^-(s), \quad \text{for } n \leqslant \xi^- - Cr - 2\varepsilon^{-1} = M_3.$$

Therefore, we have

$$v_n^-(s) < \varphi_n(s) < v_n^+(s), \quad s \in [-r, 0], \ n \in \mathbb{Z}. \tag{4.24}$$

By Lemma 4.2 and the comparison, it follows that

$$v_n^-(t) < u_n(t, \varphi) < v_n^+(t), \quad t \geqslant 0, \ n \in \mathbb{Z}. \tag{4.25}$$

Since $\delta_1 < \delta$, we can choose a sufficiently large constant $T > r$ such that, for all $t \geqslant T - r$,

$$\delta_1 + (u^* - 2\delta_1)e^{-\varepsilon t} < \delta, \quad \text{and} \quad K - \delta_1 - (K - u^* - 2\delta_1)e^{-\varepsilon t} > K - \delta,$$

and hence, again by Remark 4.2, we find that for $t \geqslant T$,

$$u_n(t, \varphi) < v_n^+(t) < \delta, \quad \text{for } n \leqslant \xi^+ - Ct - 2\varepsilon^{-1}, \tag{4.26}$$

and

$$u_n(t, \varphi) > v_n^-(t) > K - \delta, \quad \text{for } n \geqslant \xi^- + Ct + 2\varepsilon^{-1}. \tag{4.27}$$

Let $x^- = \xi^+ - CT - 2\varepsilon^{-1}$ and $x^+ = \xi^- + CT + 2\varepsilon^{-1}$. By (4.26) and (4.27), it follows that, for all $t \in [T - r, T]$,

$$u_n(t, \varphi) < \delta, \quad \text{for } n \leqslant x^-, \quad u_n(t, \varphi) > K - \delta, \quad \text{for } n \geqslant x^+. \tag{4.28}$$

Take a large constant $M_4 > 0$ so that

$$U(n) < \delta, \quad \text{for } n \leqslant -M_4, \quad U(n) > K - \delta, \quad \text{for } n \geqslant M_4. \tag{4.29}$$

Let $\xi = -M_4 - x^+ - |c|(T + r)$, then for $n \geqslant x^+$ and $s \in [-r, 0]$, by (4.28), we get

$$U(n - cs - cT + \xi) - \delta \leqslant K - \delta < (u_n)_T(s, \varphi),$$

and for $n \leqslant x^+$ and $s \in [-r, 0]$, by (5.25), we get

$$\begin{aligned} U(n - cs - cT + \xi) - \delta &\leqslant U(x^+ + |c|(T + r) + \xi) - \delta \\ &= U(-M_4) - \delta < 0 \leqslant (u_n)_T(s, \varphi). \end{aligned}$$

Therefore, we have

$$U(n - cs - cT + \xi) - \delta < (u_n)_T(s, \varphi), \quad \text{for } s \in [-r, 0], n \in \mathbb{Z}. \tag{4.30}$$

Let $h = M_4 - x^- + |c|(T+r) - \xi = 2(M_3 + M_4) + 2(C + |c|)(T + r) + 8\varepsilon^{-1} > 0$. Then for $n \leqslant x^-$ and $s \in [-r, 0]$, by (4.28), we get

$$U(n - cs - cT + \xi + h) + \delta \geqslant \delta > (u_n)_T(s, \varphi),$$

and for $n \geqslant x^-$ and $s \in [-r, 0]$, by (4.29), we get

$$\begin{aligned} U(n - cs - cT + \xi + h) + \delta &\geqslant U(x^- - |c|(T + r) + \xi + h) + \delta \\ &= U(M_4) + \delta \geqslant K > (u_n)_T(s, \varphi). \end{aligned}$$

Therefore, we have

$$U(n - cs - cT + \xi + h) + \delta > (u_n)_T(s, \varphi), \quad \text{for } s \in [-r, 0], n \in \mathbb{Z}. \tag{4.31}$$

Thus, it follows from (4.30) and (4.31) that

$$\begin{aligned} &U(n - cs - cT + \xi - \sigma_0\delta(e^{\beta_0 r} - e^{-\beta_0 s})) - \delta e^{-\beta_0 s} \\ &\quad < (u_n)_T(s, \varphi) \\ &\quad < U(n - cs - cT + \xi + h + \sigma_0\delta(e^{\beta_0 r} - e^{-\beta_0 s})) + \delta e^{-\beta_0 s}, \\ &\qquad \text{for all } s \in [-r, 0] \quad \text{and } n \in \mathbb{Z}. \end{aligned}$$

This completes the proof. □

Lemma 4.5. *Assume that* (H1)–(H3) *hold. Let $U(n - ct)$ be the traveling wave front of* (1.2) *and $c \neq 0$. Then there exists a positive number ε^* such that if $u_n(t)$ is a solution of* (1.2) *on $[0, +\infty)$ with $0 \leqslant u_n(t) \leqslant K$ for $t \in [0, +\infty)$ and $n \in \mathbb{Z}$, and for some $\xi \in \mathbb{R}, h > 0, \delta > 0$ and $T \geqslant 0$, there holds*

$$w_0^-(n, -cT + \xi, \delta)(s) < (u_n)_T(s) < w_0^+(n, -cT + \xi + h, \delta)(s), \quad s \in [-r, 0], \ n \in \mathbb{Z},$$

then for any $t \geqslant T + r + 1$, there exist $\hat{\xi}(t), \hat{\delta}(t)$ and $\hat{h}(t)$ such that

$$\begin{aligned} &w_0^-(n, -ct + \hat{\xi}(t), \hat{\delta}(t))(s) \\ &\quad < (u_n)_t(s) \\ &\quad < w_0^+(n, -ct + \hat{\xi}(t) + \hat{h}(t), \hat{\delta}(t))(s), \quad s \in [-r, 0], \ n \in \mathbb{Z}, \end{aligned}$$

with $\hat{\xi}(t), \hat{\delta}(t)$ and $\hat{h}(t)$ satisfying

$$\hat{\xi}(t) \in [\xi - \sigma_0(2\delta + \varepsilon^* \min\{1, h\})e^{\beta_0 r}, \xi + h - \sigma_0(2\delta + \varepsilon^* \min\{1, h\})e^{\beta_0 r}];$$

$$\hat{\delta}(t) = (\delta + \varepsilon^* \min\{1, h\})e^{-\beta_0[t-T-r-1]};$$

$$\begin{aligned}\hat{h}(t) &= h - 2\sigma_0\varepsilon^* \min\{1, h\} + \sigma_0(3\delta + \varepsilon^* \min\{1, h\})e^{\beta_0 r}\\ &= h - \sigma_0[\varepsilon^* \min\{1, h\}(2 - e^{\beta_0 r}) - 3\delta e^{\beta_0 r}] > 0.\end{aligned}$$

Proof. By virtue of Lemma 4.3, $w^+(n, -cT+\xi+h, \delta)(t)$ and $w^-(n, -cT+\xi, \delta)(t)$ are a supersolution and a subsolution of (1.2), respectively. Clearly, $\upsilon_n(t) = u_n(T+t),\ t \geqslant 0$, is also a solution of (1.2) with $(\upsilon_n)_0(s) = (u_n)_T(s),\ s \in [-r, 0], n \in \mathbb{Z}$. Then the comparison implies that

$$w^-(n, -cT + \xi, \delta)(t) < u_n(T + t) < w^+(n, -cT + \xi + h, \delta)(t), \quad t \geqslant 0,\ n \in \mathbb{Z}.$$

That is,

$$\begin{aligned}&U[n - c(T + t) + \xi - \sigma_0\delta(e^{\beta_0 r} - e^{-\beta_0 t})] - \delta e^{-\beta_0 t}\\ &\quad < u_n(T + t)\\ &\quad < U[n - c(T + t) + \xi + h + \sigma_0\delta(e^{\beta_0 r} - e^{-\beta_0 t})] + \delta e^{-\beta_0 t},\\ &\qquad t \geqslant 0, \quad n \in \mathbb{Z}.\end{aligned} \tag{4.32}$$

Let $m \in \mathbb{Z}$ be such that $m - 1 < cT - \xi \leqslant m$. Then it follows from Lemma 4.1 that for all $t > 0$,

$$\begin{aligned}&u_n(T + t) - w^-(n, -cT + \xi, \delta)(t)\\ &\quad > e^{-(2D+d)t}\frac{(Dt)^{|n-m|}}{|n - m|!}[u_m(T) - w^-(m, -cT + \xi, \delta)(0)]\\ &\quad = e^{-(2D+d)t}\frac{(Dt)^{|n-m|}}{|n - m|!}[u_m(T) - U(m - cT + \xi - \sigma_0\delta(e^{\beta_0 r} - 1)) + \delta]\\ &\quad \geqslant e^{-(2D+d)t}\frac{(Dt)^{|n-m|}}{|n - m|!}[u_m(T) - U(m - cT + \xi)].\end{aligned} \tag{4.33}$$

Since $\lim_{|\eta|\to+\infty} U'(\eta) = 0$, we can fix a positive number $M_5 > 0$ such that

$$U'(\eta) \leqslant \frac{1}{2\sigma_0}, \quad \text{for all } |\eta| \geqslant M_5. \tag{4.34}$$

Let $J = M_5 + |c|(1 + r) + 3, \bar{h} = \min\{1, h\}$, and

$$\varepsilon_1 = \tfrac{1}{2}\min\{U'(\eta); 0 \leqslant \eta \leqslant 2\} > 0.$$

Since $0 \leqslant m - cT + \xi \leqslant m - cT + \xi + \bar{h} \leqslant 2$, by the Mean Value Theorem, it follows that

$$U(m - cT + \xi + \bar{h}) - U(m - cT + \xi) \geqslant 2\varepsilon_1 \bar{h},$$

and hence, one of the following must holds:

(i) $u_m(T) - U(m - cT + \xi) \geqslant \varepsilon_1 \bar{h}$,
(ii) $U(m - cT + \xi + \bar{h}) - u_m(T) \geqslant \varepsilon_1 \bar{h}$.

In what follows, we consider only case (i). Case (ii) is similar and thus the proof is omitted. □

For any $s \in [-r, 0]$, $|n - m| \leqslant J$, letting $t = 1 + r + s \geqslant 1$ in (5.29), we get

$$\begin{aligned}
&u_n(T + 1 + r + s)\\
&\quad > U[n - c(T + 1 + r + s) + \xi - \sigma_0\delta(e^{\beta_0 r} - e^{-\beta_0(1+r+s)})] - \delta e^{-\beta_0(1+r+s)}\\
&\qquad + e^{-(2D+d)(1+r+s)} \frac{[D(1 + r + s)]^{|n-m|}}{|n - m|!} [u_m(T) - U(m - cT + \xi)]\\
&\quad \geqslant U[n - c(T + 1 + r + s) + \xi - \sigma_0\delta(e^{\beta_0 r} - e^{-\beta_0(1+r+s)})] - \delta e^{-\beta_0(1+r+s)}\\
&\qquad + \varepsilon_1 \bar{h} e^{-(2D+d)(1+r)} \frac{D^{|n-m|}}{|n - m|!}\\
&\quad \geqslant U[n - c(T + 1 + r + s) + \xi - \sigma_0\delta(e^{\beta_0 r} - e^{-\beta_0(1+r+s)})] - \delta e^{-\beta_0(1+r+s)}\\
&\qquad + F_0(J)\varepsilon_1 \bar{h},
\end{aligned} \tag{4.35}$$

where $F_0(J) = \min_{0 \leqslant j \leqslant J} e^{-(2D+d)(1+r)} D^j / j!$. Let $J_1 = J + |c|(1+r) + 3$, and choose a positive constant $\varepsilon^* > 0$ such that

$$\varepsilon^* \leqslant \min\left\{ \min_{|\eta| \leqslant J_1} \frac{F_0(J)\varepsilon_1}{2\sigma_0 U'(\eta)}, \frac{1}{3\sigma_0} \right\}, \tag{4.36}$$

then

$$\begin{aligned}
&U[n - c(T + 1 + r + s) + \xi + 2\sigma_0\varepsilon^* \bar{h} - \sigma_0\delta(e^{\beta_0 r} - e^{-\beta_0(1+r+s)})]\\
&\quad - U[n - c(T + 1 + r + s) + \xi - \sigma_0\delta(e^{\beta_0 r} - e^{-\beta_0(1+r+s)})]\\
&\quad = U'(\eta_1) 2\sigma_0\varepsilon^* \bar{h} \leqslant F_0(J)\varepsilon_1 \bar{h},
\end{aligned} \tag{4.37}$$

where $\eta_1 = n - c(T + 1 + r + s) + \xi - \sigma_0\delta(e^{\beta_0 r} - e^{-\beta_0(1+r+s)}) + \theta \cdot 2\sigma_0\varepsilon^* \bar{h}$, $\theta \in (0, 1)$, and in the last inequality, we have used (4.36) and the estimate

$$|\eta_1| \leqslant |n - m| + |m - cT + \xi| + |c|(1 + r) + \sigma_0\delta e^{\beta_0 r} + 2\sigma_0\varepsilon^* \leqslant J_1.$$

Hence, (4.35) and (4.37) imply that

$$\begin{aligned} u_n(T+1+r+s) > \; & U[n-c(T+1+r+s)+\xi+2\sigma_0\varepsilon^*\bar{h} \\ & -\sigma_0\delta(e^{\beta_0 r}-e^{-\beta_0(1+r+s)})]-\delta e^{-\beta_0(1+r+s)}. \end{aligned} \tag{4.38}$$

For $s\in[-r,0]$ and $|n-m|\geqslant J$, it follows that

$$\begin{aligned} & U[n-c(T+1+r+s)+\xi-\sigma_0\delta(e^{\beta_0 r}-e^{-\beta_0(1+r+s)})] \\ & \quad -U[n-c(T+1+r+s)+\xi+2\sigma_0\varepsilon^*\bar{h}-\sigma_0\delta(e^{\beta_0 r}-e^{-\beta_0(1+r+s)})] \\ & \quad = -U'(\eta_2)2\sigma_0\varepsilon^*\bar{h}\geqslant-\varepsilon^*\bar{h}, \end{aligned} \tag{4.39}$$

where $\eta_2=n-c(T+1+r+s)+\xi-\sigma_0\delta(e^{\beta_0 r}-e^{-\beta_0(1+r+s)})+\theta\cdot 2\sigma_0\varepsilon^*\bar{h}$, $\theta\in(0,1)$, and in the last inequality, we have used (4.34) and the estimate

$$|\eta_2|\geqslant|n-m|-\{|m-cT+\xi|+|c|(1+r)+\sigma_0\delta e^{\beta_0 r}+2\sigma_0\varepsilon^*\}\geqslant M_5.$$

Therefore, it follows from (4.39) and (4.32) that for all $s\in[-r,0]$ and $|n-m|\geqslant J$,

$$\begin{aligned} u_n(T+1+r+s) > \; & U[n-c(T+1+r+s)+\xi+2\sigma_0\varepsilon^*\bar{h}-\sigma_0\delta(e^{\beta_0 r} \\ & -e^{-\beta_0(1+r+s)})]-\delta e^{-\beta_0(1+r+s)}-\varepsilon^*\bar{h}. \end{aligned} \tag{4.40}$$

Combining (4.38) and (4.40), we find that for all $s\in[-r,0]$ and $n\in\mathbb{Z}$,

$$\begin{aligned} (u_n)_{T+1+r}(s) > \; & U[n-c(T+1+r+s)+\xi+2\sigma_0\varepsilon^*\bar{h}-\sigma_0\delta(e^{\beta_0 r}-e^{-\beta_0(1+r+s)})] \\ & -\delta e^{-\beta_0(1+r+s)}-\varepsilon^*\bar{h} \\ \geqslant \; & U[x+cs+c(T+1+r)+\xi+2\sigma_0\varepsilon^*\bar{h}-\sigma_0\delta e^{\beta_0 r} \\ & -\sigma_0(\delta+\varepsilon^*\bar{h})(e^{\beta_0 r}-e^{-\beta_0 s})]-(\delta+\varepsilon^*\bar{h})e^{-\beta_0 s} \\ = \; & w_0^-(x,\eta,\delta+\varepsilon^*\bar{h})(s), \quad s\in[-r,0], x\in\mathbb{R}, \end{aligned} \tag{4.41}$$

where $\eta=-c(T+1+r)+\xi+2\sigma_0\varepsilon^*\bar{h}-\sigma_0\delta e^{\beta_0 r}$.

Therefore, by the comparison, it follows that for $t \geqslant T+1+r$,

$$\begin{aligned}(u_n)_t(s) &> w^-(n,\eta,\delta+\varepsilon^*\bar{h})(t-(T+1+r)+s)\\ &\geqslant U[n-cs-ct+c(T+1+r)+\eta-\sigma_0(\delta+\varepsilon^*\bar{h})e^{\beta_0 r}]\\ &\quad -(\delta+\varepsilon^*\bar{h})e^{-\beta_0[t-(T+1+r)]}\cdot e^{-\beta_0 s}\\ &\geqslant U[n-cs-ct+c(T+1+r)+\eta-\sigma_0(\delta+\varepsilon^*\bar{h})e^{\beta_0 r}\\ &\quad -\sigma_0\hat{\delta}(t)(e^{\beta_0 r}-e^{-\beta_0 s})]-\hat{\delta}(t)e^{-\beta_0 s}\\ &= w_0^-(n,-ct+\hat{\xi}(t),\hat{\delta}(t))(s), \qquad s\in[-r,0], n\in\mathbb{Z}, \end{aligned} \tag{4.42}$$

where

$$\hat{\delta}(t)=(\delta+\varepsilon^*\bar{h})e^{-\beta_0[t-(T+1+r)]} \tag{4.43}$$

and

$$\begin{aligned}\hat{\xi}(t) &= -c(T+1+r)+\eta-\sigma_0(\delta+\varepsilon^*\bar{h})e^{\beta_0 r}\\ &= \xi+2\sigma_0\varepsilon^*\bar{h}-\sigma_0(2\delta+\varepsilon^*\bar{h})e^{\beta_0 r}.\end{aligned} \tag{4.44}$$

Clearly, by (4.36), it is easily seen that

$$\hat{\xi}(t)\in[\xi-\sigma_0(2\delta+\varepsilon^*\bar{h})e^{\beta_0 r}, \xi+h-\sigma_0(2\delta+\varepsilon^*\bar{h})e^{\beta_0 r}]. \tag{4.45}$$

On the other hand, for $t\geqslant T$, by (4.32), we have

$$u_n(t)<U[n-ct+\xi+h+\sigma_0\delta(e^{\beta_0 r}-e^{-\beta_0(t-T)})]+\delta e^{-\beta_0(t-T)},$$

which implies, for all $t\geqslant T+1+r$, that

$$\begin{aligned}(u_n)_t(s) &< U[n-c(t+s)+\xi+h+\sigma_0\delta(e^{\beta_0 r}-e^{-\beta_0(t+s-T)})]+\delta e^{-\beta_0(t+s-T)}\\ &\leqslant U[n-cs-ct+\xi+h+\sigma_0\delta e^{\beta_0 r}+\sigma_0\hat{\delta}(t)(e^{\beta_0 r}-e^{-\beta_0 s})]+\hat{\delta}(t)e^{-\beta_0 s}\\ &= U[n-cs-ct+\hat{\xi}(t)+(h-2\sigma_0\varepsilon^*\bar{h}+\sigma_0(3\delta+\varepsilon^*\bar{h})e^{\beta_0 r})\\ &\quad +\sigma_0\hat{\delta}(t)(e^{\beta_0 r}-e^{-\beta_0 s})]+\hat{\delta}(t)e^{-\beta_0 s}, \quad s\in[-r,0], n\in\mathbb{Z}.\end{aligned}$$

Therefore, for $t\geqslant T+1+r$, we have

$$(u_n)_t(s)<w_0^+(n,-ct+\hat{\xi}(t)+\hat{h}(t),\hat{\delta}(t))(s), \quad s\in[-r,0], n\in\mathbb{Z}, \tag{4.46}$$

where

$$\hat{h}(t) = h - 2\sigma_0\varepsilon^*\bar{h} + \sigma_0(3\delta + \varepsilon^*\bar{h})e^{\beta_0 r}$$
$$= h - \sigma_0[\varepsilon^*\bar{h}(2 - e^{\beta_0 r}) - 3\delta e^{\beta_0 r}] > 0, \tag{4.47}$$

and in (4.47), we have used (4.36) and the estimate

$$h - 2\sigma_0\varepsilon^*\bar{h} > h - 3\sigma_0\varepsilon^*\bar{h} \geqslant h - \bar{h} \geqslant 0.$$

Now the conclusions of the lemma follow from (4.42), (4.43), and (4.45), (4.46) and (4.47).

Theorem 4.1. *Assume that* (H1)–(H3) *hold. Let* $U(n-ct)$ *with* $c \neq 0$ *be the traveling wave front of* (1.2) *as given in Theorem* 2.1. *Then* $U(n-ct)$ *is globally asymptotically stable with phase shift in the sense that there exists* $\gamma > 0$ *such that for any* $\varphi = \{\varphi_n\}_{n\in\mathbb{Z}}$ *with* $\varphi_n \in C([-r,0],[0,K])$ *satisfying*

$$\liminf_{n\to+\infty} \min_{s\in[-r,0]} \varphi_n(s) > u^*, \quad \limsup_{n\to-\infty} \max_{s\in[-r,0]} \varphi_n(s,x) < u^*,$$

the solution $u_n(t,\varphi)$ *of* (1.2) *satisfies*

$$|u_n(t,\varphi) - U(n-ct+\xi_0)| \leqslant Me^{-\gamma t}, \quad t \geqslant 0, \ n \in \mathbb{Z},$$

for some $M = M(\varphi) > 0$ *and* $\xi_0 = \xi_0(\varphi) \in \mathbb{R}$.

Proof. Let $\beta_0, \sigma_0, \bar{\delta}$ be as in Lemma 4.3 with $\beta_0 > 0$ chosen so that $e^{\beta_0 r} < 2$, and let ε^* be as in Lemma 4.5 with $\varepsilon^* > 0$ chosen so that $\sigma_0\varepsilon^*(2 - e^{\beta_0 r}) < 1$. We further choose a $0 < \delta^* < \min\{\frac{\delta_0}{2}, \bar{\delta}, \frac{1}{\sigma_0}\}$ such that

$$1 > k^* := \sigma_0[\varepsilon^*(2 - e^{\beta_0 r}) - 3\delta^* e^{\beta_0 r}] > 0$$

and then fix a $t^* \geqslant r+1$ such that

$$e^{-\beta_0(t^*-r-1)}(1 + \varepsilon^*/\delta^*) < 1 - k^*.$$

We first prove the following two claims.

Claim 1. *There exist* $T^* = T^*(\varphi) > 0, \xi^* = \xi^*(\varphi) \in \mathbb{R}$ *such that*

$$w_0^-(n, -cT^* + \xi^*, \delta^*)(s) < (u_n)_{T^*}(s,\varphi) < w_0^+(n, -cT^* + \xi^* + 1, \delta^*)(s),$$
$$s \in [-r,0], \ n \in \mathbb{Z}. \tag{4.48}$$

Indeed, by Lemma 4.4, *there exist* $T = T(\varphi) > 0, \xi = \xi(\varphi) \in \mathbb{R}$ *and* $h = h(\varphi) > 0$ *such that*

$$\begin{aligned} &w_0^-(n, -cT + \xi, \delta^*)(s) < (u_n)_T(s, \varphi) < w_0^+(n, -cT + \xi + h, \delta^*)(s), \\ &s \in [-r, 0], \ \ n \in \mathbb{Z}. \end{aligned} \tag{4.49}$$

If $h \leqslant 1$, then the Claim 1 follows from the monotonicity of $U(\cdot)$. In what follows, we assume that $h > 1$, and let

$$N = \max\{m; m \text{ is a nonnegative integer and } mk^* < h\}.$$

Since $0 < k^* < 1$ and $h > 1$, we have $N \geqslant 1$, $Nk^* < h \leqslant (N+1)k^*$, and hence, $0 < h - Nk^* \leqslant k^* < 1$. Clearly, $\bar{h} = \min\{1, h\} = 1$. By (4.49), the choice of k^* and t^*, and Lemma 4.5, we have

$$\begin{aligned} &w_0^-(n, -c(T + t^*) + \hat{\xi}(T + t^*), \hat{\delta}(T + t^*))(s) \\ &\quad < (u_n)_{T+t^*}(s, \varphi) \\ &\quad < w_0^+(n, -c(T + t^*) + \hat{\xi}(T + t^*) + \hat{h}(T + t^*), \hat{\delta}(T + t^*))(s), \\ &\quad\quad s \in [-r, 0], \ \ n \in \mathbb{Z}, \end{aligned} \tag{4.50}$$

where

$$\hat{\xi}(T + t^*) \in [\xi - \sigma_0(2\delta^* + \varepsilon^*)e^{\beta_0 r}, \xi + h - \sigma_0(2\delta^* + \varepsilon^*)e^{\beta_0 r}],$$

$$\hat{\delta}(T + t^*) = (\delta^* + \varepsilon^*)e^{-\beta_0[t^* - r - 1]} < \delta^*(1 - k^*),$$

$$0 \leqslant \hat{h}(T + t^*) \leqslant h - \sigma_0[\varepsilon^*(2 - e^{\beta_0 r}) - 3\delta^* e^{\beta_0 r}] = h - k^*.$$

Repeating the same process N times, we then have that (4.50), with $T + t^*$ replaced by $T + Nt^*$, holds for some $\hat{\xi} \in \mathbb{R}$, $0 < \hat{\delta} \leqslant \delta^*(1 - k^*)^N$, and $0 \leqslant \hat{h} \leqslant h - Nk^* < 1$. Let $T^* = T + Nt^*$, $\xi^* = \hat{\xi}$. Again by the monotonicity of $U(\cdot)$, (4.48) then follows.

Claim 2. *Let* $p = 1 + \sigma_0(2\delta^* + \varepsilon^*)e^{\beta_0 r}, T_m = T^* + mt^*, \delta_m^* = (1 - k^*)^m \delta^*$ *and* $h_m = (1 - k^*)^m < 1, m \geqslant 0$. *Then there exists a sequence* $\{\hat{\xi}_m\}_{m=0}^{\infty}$ *with* $\hat{\xi}_0 = \xi^*$ *such that*

$$|\hat{\xi}_{m+1} - \hat{\xi}_m| \leqslant ph_m, \quad m \geqslant 0 \tag{4.51}$$

and

$$w_0^-(n, -cT_m + \hat{\xi}_m, \delta_m)(s) < (u_n)_{T_m}(s, \varphi) < w_0^+(n, -cT_m + \hat{\xi}_m + h_m, \delta_m^*)(s), \quad s \in [-r, 0], \quad n \in \mathbb{Z}, \quad m \geqslant 0. \tag{4.52}$$

In fact, Claim 1 implies that (4.52) holds for $m = 0$. Now suppose that (4.52) holds for some $m = \ell \geqslant 0$. By Lemma 4.5, with $T = T_\ell, \xi = \hat{\xi}_\ell, h = h_\ell, \delta = \delta_\ell^*$ and $t = T_\ell + t^* = T_{\ell+1} \geqslant T_\ell + r + 1$, we then have

$$\begin{aligned} w_0^-(n, -cT_{\ell+1} + \hat{\xi}, \hat{\delta})(s) &< (u_n)_{T_{\ell+1}}(s, \varphi) \\ &< w_0^+(n, -cT_{\ell+1} + \hat{\xi} + \hat{h}, \hat{\delta})(s), \quad s \in [-r, 0], \ \ n \in \mathbb{Z}, \end{aligned}$$

where

$$\hat{\xi} \in [\hat{\xi}_\ell - \sigma_0(2\delta_\ell^* + \varepsilon^* h_\ell)e^{\beta_0 r}, \hat{\xi}_\ell + h_\ell - \sigma_0(2\delta_\ell^* + \varepsilon^* h_\ell)e^{\beta_0 r}],$$

$$\begin{aligned} \hat{\delta} &= (\delta_\ell^* + \varepsilon^* h_\ell)e^{-\beta_0[T_{\ell+1} - T_\ell - r - 1]} \\ &= (1 - k^*)^\ell(\delta^* + \varepsilon^*)e^{-\beta_0[t^* - r - 1]} \\ &\leqslant (1 - k^*)^\ell \cdot \delta^*(1 - k^*) \\ &= (1 - k^*)^{\ell+1} \cdot \delta^* = \delta_{\ell+1}^*, \end{aligned}$$

$$\begin{aligned} \hat{h} &= h_\ell - \sigma_0[\varepsilon^* h_\ell(2 - e^{\beta_0 r}) - 3\delta_\ell^* e^{\beta_0 r}] \\ &= (1 - k^*)^\ell\{1 - \sigma_0[\varepsilon^*(2 - e^{\beta_0 r}) - 3\delta^* e^{\beta_0 r}]\} \\ &= (1 - k^*)^{\ell+1} = h_{\ell+1}. \end{aligned}$$

We choose $\hat{\xi}_{\ell+1} = \hat{\xi}$. Then

$$\begin{aligned} |\hat{\xi}_{\ell+1} - \hat{\xi}_\ell| &\leqslant h_\ell + \sigma_0(2\delta_\ell^* + \varepsilon^* h_\ell)e^{\beta_0 r} \\ &= [1 + \sigma_0(2\delta^* + \varepsilon^*)e^{\beta_0 r}]h_\ell \\ &= p h_\ell. \end{aligned}$$

It follows that (4.51) holds for $m = \ell$, and (4.52) holds for $m = \ell + 1$. By induction, (4.51) and (4.52) holds for all $m \geqslant 0$.

For every $m \geqslant 0$, by (4.52) and the comparison, it follows that for all $t \geqslant T_m$, $n \in \mathbb{Z}$,

$$\begin{aligned} &U(n - ct + \hat{\xi}_m - \sigma_0 \delta_m^*(e^{\beta_0 r} - e^{-\beta_0(t-T_m)})) - \delta_m^* e^{-\beta_0(t-T_m)} \\ &\quad < u_n(t,\varphi) \\ &\quad < U(n - ct + \hat{\xi}_m + h_m + \sigma_0 \delta_m^*(e^{\beta_0 r} - e^{-\beta_0(t-T_m)})) + \delta_m^* e^{-\beta_0(t-T_m)}. \quad (4.53) \end{aligned}$$

For any $t \geqslant T^*$, let $m = \left[\frac{t-T^*}{t^*}\right] \geqslant 0$ be the largest integer not greater than $\frac{t-T^*}{t^*}$, and define $\delta(t) = \delta_m^*$, $\xi(t) = \hat{\xi}_m - \sigma_0 \delta_m^* e^{\beta_0 r}$, and $h(t) = h_m + 2\sigma_0 \delta_m^* e^{\beta_0 r}$, then we have $T_m = T^* + mt^* \leqslant t < T^* + (m+1)t^* = T_{m+1}$. By (4.53), it follows that for all $t \geqslant T^*$ and $n \in \mathbb{Z}$,

$$U(n - ct + \xi(t)) - \delta(t) < u_n(t,\varphi) < U(n - ct + \xi(t) + h(t)) + \delta(t). \tag{4.54}$$

Set $\gamma := -\frac{1}{t^*}\ln(1-k^*) > 0$ and $q = \exp\{-(1 + T^*/t^*)\ln(1-k^*)\}$. Since $0 \leqslant m \leqslant \frac{t-T^*}{t^*} < m+1$, we have

$$(1-k^*)^m < (1-k^*)^{\frac{t-T^*}{t^*}-1} = \exp\left\{\left(\frac{t-T^*}{t^*} - 1\right)\ln(1-k^*)\right\} = qe^{-\gamma t}.$$

Therefore, for any $t \geqslant T^*$, we have

$$\delta(t) = \delta_m^* = (1-k^*)^m \delta^* \leqslant \delta^* q e^{-\gamma t}, \tag{4.55}$$

$$h(t) = h_m + 2\sigma_0 \delta_m^* e^{\beta_0 r} = (1 + 2\sigma_0 \delta^* e^{\beta_0 r})(1-k^*)^m \leqslant (1 + 2\sigma_0 \delta^* e^{\beta_0 r}) q e^{-\gamma t}, \tag{4.56}$$

and for any $t' \geqslant t \geqslant T^*$, by (4.51), we have

$$\begin{aligned} |\xi(t') - \xi(t)| &= |\hat{\xi}_n - \sigma_0 \delta_m^* e^{\beta_0 r} - (\hat{\xi}_m - \sigma_0 \delta_m^* e^{\beta_0 r})| \\ &\leqslant |\hat{\xi}_n - \hat{\xi}_m| + \sigma_0 |\delta_n^* - \delta_m^*| e^{\beta_0 r} \\ &\leqslant \sum_{\ell=m}^{n-1} p h_\ell + \sigma_0 \delta_m^* e^{\beta_0 r} \\ &= \left[\frac{p}{\delta^*} \sum_{\ell=0}^{n-m-1} (1-k^*)^\ell + \sigma_0 e^{\beta_0 r}\right] \delta_m^* \\ &\leqslant \left(\frac{p}{k^* \delta^*} + \sigma_0 e^{\beta_0 r}\right) \delta(t) \\ &\leqslant \left(\frac{p}{k^*} + \sigma_0 \delta^* e^{\beta_0 r}\right) q e^{-\gamma t}, \end{aligned} \tag{4.57}$$

where $n = \left[\frac{t'-T^*}{t^*}\right] \geqslant m = \left[\frac{t-T^*}{t^*}\right]$. Therefore, it follows from (4.57) that $\xi_0 := \lim_{t\to+\infty} \xi(t)$ exists, and for $t \geqslant T^*$, we have

$$|\xi_0 - \xi(t)| \leqslant \left(\frac{p}{k^*} + \sigma_0\delta^* e^{\beta_0 r}\right) q e^{-\gamma t}. \tag{4.58}$$

Set

$$M := \max\left\{U'_{\max}\left[\frac{p}{k^*} + 3\sigma_0\delta^* e^{\beta_0 r} + 1\right] q + \delta^* q, 2K e^{\gamma T^*}\right\}.$$

it then follows from (4.54)–(4.56) and (4.58) that for all $t \geqslant T^*$,

$$\begin{aligned} |u_n(t,\varphi) - U(n - ct + \xi_0)| &\leqslant U'_{\max}[|\xi(t) - \xi_0| + h(t)] + \delta(t) \\ &\leqslant \left\{U'_{\max}\left[\frac{p}{k^*} + 3\sigma_0\delta^* e^{\beta_0 r} + 1\right] q + \delta^* q\right\} e^{-\gamma t}, \end{aligned}$$

which together with the fact that $|u_n(t,\varphi) - U(n - ct + \xi_0)| \leqslant 2K,\ t \in [0, T^*]$ yields $|u_n(t,\varphi) - U(n - ct + \xi_0)| \leqslant M e^{-\gamma t}$ for all $t \geqslant 0$. The proof is complete. □

5. Propagation failure of traveling waves

An important qualitative difference between traveling wave solutions of the two systems (1.1) and (1.2) is the occurrence of "propagation failure" or "pinning" in the discrete system (1.2). In this section, we shall find some criteria for pinning of traveling waves for the equation (1.2).

The following theorem is an easy consequence of Theorem 2.1 and Theorem 4.1.

Theorem 5.1. *Assume that* (H1)–(H3) *hold. Then* (1.2) *admits pinning if and only if one of the following statements holds true*:

(i) *the equation*

$$D[u_{n+1} + u_{n-1} - 2u_n] - du_n + \sum_i J(i)b(u_{n-i}) = 0 \tag{5.1}$$

has a solution $u = \{u_n\}_{n\in\mathbb{Z}}$ *with* $u_n \in [0, K]$ *for all* $n \in \mathbb{Z}$ *satisfying* $\limsup_{n\to-\infty} u_n < u^*$ *and* $\liminf_{n\to+\infty} u_n > u^*$;

(ii) *Eq.* (5.1) *has a strictly monotone solution* $u = \{u_n\}_{n\in\mathbb{Z}}$ *satisfying* $\lim_{n\to-\infty} u_n = 0$ *and* $\lim_{n\to+\infty} u_n = K$.

Consider the one-parameter family of equations

$$\lambda D[u_{n+1}+u_{n-1}-2u_n]-du_n+(1-\lambda\sum_{i\neq 0}J(i))b(u_n)$$
$$+\lambda\sum_{i\neq 0}J(i)b(u_{n-i})=0, \quad \lambda\in[0,1]. \tag{5.2}$$

Lemma 5.1. *Any bounded non-constant solution* $u=\{u_n\}_{n\in\mathbb{Z}}$ *of* (5.2) *satisfies* $0\leqslant u_n \leqslant K$. *If* $\lambda\in(0,1]$, *then the strict inequality*

$$0<u_n<K, \qquad for\ all\ n\in\mathbb{Z} \tag{5.3}$$

holds true.

Proof. Set $M^-=\inf_{n\in\mathbb{Z}}u_n$, $M^+=\sup_{n\in\mathbb{Z}}u_n$. Let $\{n_j^-\}$ be a sequence in $\mathbb{Z}$ such that $u_{n_j^-}\to M^-$ as $j\to\infty$, and $\{n_j^+\}$ a sequence such that $u_{n_j^+}\to M^+$ as $j\to\infty$. If M^- and M^+ are achieved at some points n^- or $n^+\in\mathbb{Z}$, then the corresponding sequence $\{n_j^-\}$ or $\{n_j^+\}$ is defined as $n_j^-\equiv n^-$ or $n_j^+\equiv n^+$. We have

$$\begin{aligned}(2\lambda D+d)u_{n_j^-} &= \lambda D[u_{n_j^-+1}+u_{n_j^--1}]+(1-\lambda\sum_{i\neq 0}J(i))b(u_{n_j^-})\\ &\quad+\lambda\sum_{i\neq 0}J(i)b(u_{n_j^--i})\\ &\geqslant 2\lambda DM^-+b(M^-).\end{aligned}$$

Passing to the limit as $j\to\infty$ in the last inequality, we get

$$dM^-\geqslant b(M^-), \tag{5.4}$$

from which it follows that $M^-\geqslant 0$. A similar argument can be used to show that $M^+\leqslant K$.

Next, we show that (5.3) holds if $\lambda\in(0,1]$. Without loss of generality, we suppose that $u_{n_0}=K$ and $u_{n_0-1}<K$ for some $n_0\in\mathbb{Z}$, then

$$\begin{aligned}K=u_{n_0} &= \frac{1}{2\lambda D+d}\{\lambda D[u_{n_0+1}+u_{n_0-1}]+(1-\lambda\sum_{i\neq 0}J(i))b(u_{n_0})\\ &\quad+\lambda\sum_{i\neq 0}J(i)b(u_{n_0-i})\}\end{aligned}$$

$$< \frac{1}{2\lambda D + d}\{2\lambda DK + b(K)\} = K,$$

a contradiction. This contradiction shows that $u_n < K$ for all $n \in \mathbb{Z}$. Similarly, we can show that $u_n > 0$ for all $n \in \mathbb{Z}$ and thus completes the proof. □

Lemma 5.2. *For* $\lambda \in (0, 1]$, *any bounded non-constant solution* $u = \{u_n\}_{n\in\mathbb{Z}}$ *of* (5.2) *satisfies*

$$2D(K - u_n) + (dK - b(u_n))\sum_{i\neq 0} J(i) > du_n - b(u_n)$$

and

$$2Du_n + b(u_n)\sum_{i\neq 0} J(i) > b(u_n) - du_n.$$

Proof. By virtue of Lemma 5.1, we have $0 < u_n < K$, and hence

$$\begin{aligned} 0 &< (2\lambda D + d)u_n - (1 - \lambda \sum_{i\neq 0} J(i))b(u_n) \\ &= \lambda D[u_{n+1} + u_{n-1}] + \lambda \sum_{i\neq 0} J(i)b(u_{n-i}) \\ &< 2\lambda DK + \lambda dK \sum_{i\neq 0} J(i), \end{aligned}$$

from which the conclusion follows. The proof is complete. □

In what follows, we shall give some sufficient conditions for the existence of non-constant solutions to (5.1).

Theorem 5.2. *Assume that* (H1)–(H3) *hold. Assume that*

$$\sum_{i\neq 0} J(i) < \min\left\{\max_{u\in[0,u^*]}\left\{\frac{du - b(u)}{dK - b(u)}\right\}, \quad \max_{u\in[u^*,K]}\left\{\frac{b(u) - du}{b(u)}\right\}\right\} \tag{5.5}$$

and

$$D \leqslant \frac{1}{2} \min \left\{ \max_{u\in[0,u^*]} \left\{ \left[\frac{du-b(u)}{dK-b(u)} - \sum_{i\neq 0} J(i) \right] \frac{dK-b(u)}{K-u} \right\}, \max_{u\in[u^*,K]} \left\{ \left[\frac{b(u)-du}{b(u)} - \sum_{i\neq 0} J(i) \right] \frac{b(u)}{u} \right\} \right\}. \tag{5.6}$$

Let

$$u^- := \inf\{u \in (0, K]|\ 2D(K-u) + [dK - b(u)] \sum_{i\neq 0} J(i) \leqslant du - b(u)\}$$

and

$$u^+ := \sup\{u \in [0, K)|\ 2Du + b(u) \sum_{i\neq 0} J(i) \leqslant b(u) - du\}.$$

Suppose that $d > b'_D := \sup\{b'(u)|\ u \in [0, u^-) \cup (u^+, K]\}$. *Then for any two disjoint subsets* S^- *and* S^+ *of* $\mathbb{Z}$ *with* $S^- \cup S^+ = \mathbb{Z}$, (5.1) *admits a unique solution* $u = \{u_n\}_{n\in\mathbb{Z}}$ *satisfying* $u_n \in [0, u^-)$ *for* $n \in S^-$ *and* $u_n \in (u^+, K]$ *for* $n \in S^+$.

Remark 5.1. In fact, since $D > 0$, it is easily seen that (5.5) holds if (5.6) holds.

Proof of Theorem 5.2. By (5.5) and (5.6), it is easily seen that there exist $u_1 \in (0, u^*)$ and $u_2 \in (u^*, K)$ such that

$$2D(K - u_1) + (dK - b(u_1)) \sum_{i\neq 0} J(i) \leqslant du_1 - b(u_1) \tag{5.7}$$

and

$$2Du_2 + b(u_2) \sum_{i\neq 0} J(i) \leqslant b(u_2) - du_2. \tag{5.8}$$

Therefore, by the definition, we have $u^- \in [0, u_1]$ and $u^+ \in [u_2, K]$.

Consider $G(u, \lambda) = \{G_n(u, \lambda)\}_{n\in\mathbb{Z}}$, $u = \{u_n\}_{n\in\mathbb{Z}}$, defined by

$$G_n(u, \lambda) := \lambda D[u_{n+1} + u_{n-1} - 2u_n] - du_n + (1 - \lambda \sum_{i\neq 0} J(i))b(u_n) + \lambda \sum_{i\neq 0} J(i) b(u_{n-i}). \tag{5.9}$$

Then $u^0 = \{u^0_n\}_{n\in\mathbb{Z}}$ with $u^0_n = 0$ for $n \in S^-$ and K for $n \in S^+$ satisfies $G(u^0, 0) = 0$. It is easily seen that the Frechet derivative $D_u G(u^0, 0)$ of G at $(u^0, 0)$ is given by

$$[D_u G(u^0, 0)v]_n = -[d - b'(u^0_n)]v_n, \quad \text{for } v = \{v_n\} \in l^\infty.$$

Since $D_u G(u^0, 0)$ is invertible in l^∞, by the Implicit Function Theorem, there exist some $\lambda_0 > 0$ and a unique continuous map $u(\lambda)$ from $[0, \lambda_0]$ to l^∞ such that $u(0) = u^0$ and $G(u(\lambda), \lambda) = 0$ for $\lambda \in [0, \lambda_0]$. Moreover, by (5.7), (5.8) and Lemma 5.2, it is easy to see that $u_n(\lambda) \in [0, u^-)$ for $n \in S^-$ and $u_n(\lambda) \in (u^+, K]$ for $n \in S^+$. We continue this solution to the interval $\lambda \in [0, 1]$ in the following way.

Suppose that for some $\lambda_1 \in [\lambda_0, 1)$, such a solution $u = u(\lambda_1)$ exists to the equation $G(u, \lambda_1) = 0$. First, we show that there exists $\varepsilon > 0$ such that for $\lambda \in [\lambda_1, \lambda_1 + \varepsilon)$, $G(u, \lambda) = 0$ has a solution with the above described property.

By the Implicit Function Theorem, it suffices to show that $D_u G(u(\lambda_1), \lambda_1)$ is invertible. It is easy to see that for any $v \in l^\infty$,

$$\begin{aligned}
&[D_u G(u(\lambda_1), \lambda_1)v]_n \\
&\quad = \lambda_1 D[v_{n+1} + v_{n-1} - v_n] - dv_n + (1 - \lambda_1 \sum_{i\neq 0} J(i))b'(u_n(\lambda_1))v_n \\
&\qquad +\lambda_1 \sum_{i\neq 0} J(i)b'(u_{n-i}(\lambda_1))v_{n-i} \\
&\quad = (2\lambda_1 D + d)\left\{\frac{1}{2\lambda_1 D + d}[\lambda_1 D(v_{n+1} + v_{n-1}) + (1 - \lambda_1 \sum_{i\neq 0} J(i))b'(u_n(\lambda))v_n \right. \\
&\qquad \left. +\lambda_1 \sum_{i\neq 0} J(i)b'(u_{n-i}(\lambda_1))v_{n-i}] - v_n\right\}. \qquad (5.10)
\end{aligned}$$

Since $u_n(\lambda_1) \in [0, u^-) \cup (u^*, K]$, we have $d \geqslant b'(u_n(\lambda_1))$ if $\sum_{i\neq 0} J(i) > 0$ and $d > b'(u_n(\lambda_1)$ if $\sum_{i\neq 0} J(i) = 0$. Therefore, it follows from the fact that $\lambda_1 < 1$ that for $v \in l^\infty$ with $|v|_{l^\infty} > 0$,

$$\begin{aligned}
&|\lambda_1 D(v_{n+1}+v_{n-1})+(1 - \lambda_1 \sum_{i\neq 0} J(i))b'(u_n(\lambda_1))v_n+\lambda_1\sum_{i\neq 0}J(i)b'(u_{n-i}(\lambda_1))v_{n-i}| \\
&\quad \leqslant \left(2\lambda_1 D + b'_D\right)|v|_{l^\infty} \\
&\quad < (2\lambda_1 D + d)|v|_{l^\infty},
\end{aligned}$$

which together with (5.10) implies that $D_u G(u(\lambda_1), \lambda_1)$ is invertible.

To show that we can continue the solution to $\lambda \in [\lambda_0, 1]$, we argue by contradiction. Suppose that there is some $\bar{\lambda} \in [\lambda_0, 1]$ such that a solution $u(\lambda) = \{u_n(\lambda)\}$, with $u_n(\lambda) \in [0, u^-)$ for $n \in S^-$ and $u_n(\lambda) \in (u^+, K]$ for $n \in S^+$, exists for $\lambda \in [\lambda_0, \bar{\lambda})$, but not for $\lambda = \bar{\lambda}$. Choose a sequence $\lambda_j \to \bar{\lambda}$ as $j \to \infty$. By a diagonal argument, there exists a subsequence, which we also denote by λ_j, such that $u_n(\lambda_j) \to u_n(\bar{\lambda})$ for all $n \in \mathbb{Z}$, as $j \to \infty$. Continuity and the Dominated Convergence Theorem implies that $u = u(\bar{\lambda})$ is a solution of $G(u, \bar{\lambda}) = 0$. By (5.7), (5.8) and Lemma 5.2, we find that $u_n(\bar{\lambda}) \in [0, u^-)$ for $n \in S^-$ and $u_n(\bar{\lambda}) \in (u^+, K]$ for $n \in S^+$. This completes the existence proof.

Finally, suppose that there are two distinct solutions u^1 and u^2 of (5.1), such that $u_n^1, u_n^2 \in [0, u^-)$ for $n \in S^-$ and $u_n^1, u_n^2 \in (u^+, K]$ for $n \in S^+$. Then

$$\begin{aligned} |u_n^1 - u_n^2| &\leqslant \frac{1}{2D+d}\left[2D + \sum_i J(i) b'(\theta_i u_{n-i}^1 + (1-\theta_i) u_{n-i}^2)\right] |u^1 - u^2|_{l^\infty} \\ &\leqslant \frac{2D + b'_D}{2D+d} |u^1 - u^2|_{l^\infty} \\ &< |u^1 - u^2|_{l^\infty}, \end{aligned}$$

where $\theta_i \in (0,1)$ for $i \in \mathbb{Z}$, which is a contradiction. This contradiction establish the statement for uniqueness and completes the proof. $\square$

In particular, taking $S^- = \mathbb{Z} \setminus \mathbb{N}$ and $S^+ = \mathbb{N}$ in Theorem 5.2, we then get the following

Corollary 5.1. *Under the conditions given in Theorem* 5.2, (1.2) *admits pinning*. *In particular*, *pinning occurs provided that* $D > 0$ *and* $\sum_{i \neq 0} J(i) \geqslant 0$ *are small enough.*

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