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# Global attractivity of non-autonomous Lotka–Volterra competition system without instantaneous negative feedback

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## Abstract

We consider a non-autonomous Lotka–Volterra competition system with distributed delays but without instantaneous negative feedbacks (i.e., pure delay systems). We establish some  $3/2$ -type and  $M$ -matrix-type criteria for global attractivity of the positive equilibrium of the system, which generalise and improve the existing ones.

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## 1. Introduction

Recently, Hofbauer and So [8] considered the  $n$ -species Lotka–Volterra systems with discrete delays

$$\dot{x}_i(t) = x_i(t) \left( r_i + a_{ii}x_i(t) + \sum_{j \neq i}^n a_{ij}x_j(t - \tau_{ij}) \right), \quad i = 1, 2, \dots, n, \quad (1.1)$$

and established the following nice result.

**Theorem 1.1.** *Let  $A$  be the  $n \times n$  community matrix of (1.1), i.e.,  $A = (a_{ij})$ , and suppose that there exists a positive equilibrium  $x^*$  for (1.1). Then  $x^*$  is globally asymptotically stable for (1.1) (for positive initial conditions) for all delays  $\tau_{ij} \geq 0$  if and only if  $a_{ii} < 0$  for  $i = 1, 2, \dots, n$ ,  $\det A \neq 0$  and  $A$  is weakly diagonally dominant, meaning that all the principal minors of  $-\hat{A}$  are non-negative, where  $\hat{A} = (\hat{a}_{ij})$  with  $\hat{a}_{ii} = a_{ii}$  and  $\hat{a}_{ij} = |a_{ij}|$  for  $i \neq j$ .*

The proof of Theorem 1 is by constructing a Liapunov functional, taking advantage of the fact that there is no delay in the negative feedback terms  $a_{ii}x_i(t)$ ,  $i = 1, 2, \dots, n$  (i.e., the system has instantaneous negative feedbacks). But, as pointed out by Kuang [11], in view of the fact that in real situations, instantaneous responses are rare, and thus, more realistic models should consist of delay differential equations without instantaneous negative feedbacks. A simple but typical such example is the widely studied delay logistic equation

$$\begin{cases} \dot{x}(t) = rx(t)[1 - x(t - \tau)], \\ x(s) \geq 0 \text{ for } s \in [-\tau, 0], \quad x(0) > 0. \end{cases} \quad (1.2)$$

When incorporating delays into the terms  $a_{ii}x_i(t)$ ,  $i = 1, \dots, n$ , in (1.1), the negative feedbacks (assuming  $a_{ii} < 0$ ) are delayed and we have the following system:

$$\dot{x}_i(t) = x_i(t) \left( r_i + a_{ii}x_i(t - \tau_{ii}) + \sum_{j \neq i}^n a_{ij}x_j(t - \tau_{ij}) \right), \quad i = 1, 2, \dots, n. \quad (1.3)$$

For such models as (1.2) and (1.3), detecting the global attractivity of a positive equilibrium becomes a much harder job, since finding a working Liapunov function/functional for such a system is extremely difficult due to the *lack of instantaneous negative feedbacks*. To experience this a little bit, the reader is suggested to work on the simplest case (1.2), and see how frustrating it will be in constructing a Liapunov function/functional for such an equation.

For (1.3), one would naturally expect and it is common sense that if the delays in the intraspecific interactions (i.e.,  $\tau_{ii}s$ ) are sufficiently small, then the positive equilibrium should remain globally attractive under the existing “diagonally dominant” condition. Some recent work (e.g., [4–7, 11–13]) initiated valuable

attempts in this direction, which confirm to some extent the above expectation or common sense, but the estimates for  $\tau_{ij}$ 's obtained in these works are usually implicit and there seem to be a lot of room for improvement. Note that a well-known criterion for the global attractivity of the positive equilibrium  $x^* = 1$  of (1.2) is  $r\tau \leq \frac{3}{2}$ , which was obtained in [21] by a “sandwiching” technique (see also [10]) which is a non-Liapunov approach. Since Wright [21], similar 3/2 type criteria for global attractivity have been obtained for autonomous logistic equations with *multiple* delays [14], for *non-autonomous* delay logistic equation [18], and for *various other types* of scalar equations with delays (see, e.g., [1,9,15,19,22–24]). When it comes to *system* (1.3), which is a result of coupling of several delayed logistic equations of form (1.2), it is reasonable to expect, as pointed out by Kuang [11], some criteria which would reduce to the Wright's 3/2 condition for the scalar logistic equation (1.2) when the coupling disappears. Unfortunately, none of the aforementioned work for *systems* has obtained criteria of this type (which may be called 3/2-type criteria).

Recently, the present authors made an attempt in [20] towards this direction by considering the following *two-species* Lotka–Volterra competition system (normalized) with discrete delays

$$\begin{aligned}\dot{x}_1(t) &= r_1 x_1(t) [1 - x_1(t - \tau_{11}) - \mu_1 x_2(t - \tau_{12})], \\ \dot{x}_2(t) &= r_2 x_2(t) [1 - \mu_2 x_1(t - \tau_{21}) - x_2(t - \tau_{22})],\end{aligned}\tag{1.4}$$

with initial conditions

$$x_i(t) = \phi_i(t) \geq 0, \quad t \in [-\tau, 0]; \quad \phi_i(0) > 0, \quad i = 1, 2,\tag{1.5}$$

where  $\tau = \max\{\tau_{ij}: i, j = 1, 2\}$ ,  $r_i > 0$ ,  $\mu_i \geq 0$  for  $i = 1, 2$  and  $\tau_{ij} \geq 0$  for  $i, j = 1, 2$ . It can be easily seen that the non-boundary equilibrium  $x^* = (x_1^*, x_2^*)$  of (1.4) is given by

$$x_1^* = \frac{1 - \mu_1}{1 - \mu_1 \mu_2} < 1, \quad x_2^* = \frac{1 - \mu_2}{1 - \mu_1 \mu_2} < 1.\tag{1.6}$$

Both the positivity of  $x^*$  and the “diagonal dominant” condition for (1.4) in the sense of Theorem 1.1 can all be implied by the assumption

(DD)  $\mu_1 < 1$  and  $\mu_2 < 1$ .

Among the other criteria in [20] is the following representative one.

**Theorem 1.2.** *Assume that (DD) holds. If*

$$r_i \tau_{ii} \leq \frac{3(1 - \mu)}{2(1 + \mu)}, \quad i = 1, 2,\tag{1.7}$$

where  $\mu = \max\{\mu_1, \mu_2\}$ , then the positive equilibrium  $x^*$  of (1.4) is a global attractor.

One easily sees that (1.7) reproduces Wright’s result when  $\mu_1 = \mu_2 = 0$ . The proof of Theorem 1.2 is by extending Wright’s “sandwiching” technique from scalar equations to systems. As commented by the referee for the paper Tang and Zou [20], such extension is a non-trivial step, and it indeed contains some subtle steps, even though it is only for a system of *two* equations.

Question arise naturally: can the technique be further extended to (i) systems of more than two equations; (ii) to non-autonomous systems; and (iii) to systems with distributed delays, to establish good criteria for global attractivity of the positive equilibrium? The motivation for (i) is the recent work So et al [17] where a *local* stability criterion for (1.3) is established which is similar to Theorem 1.1 but with a 3/2 type estimate for the diagonal delays  $\tau_{ii}$ s. (ii) is motivated by So and Yu [18] where a similar 3/2 condition in integral form is obtained for the *non-autonomous* logistic scalar equation with a single delay. We point out that for such a non-autonomous case, such a 3/2 estimate in the integral form has been proved to be the best possible (see [22]). (iii) is suggested by the work Kuang [11], and Kuang and Smith [12,13] and the references therein where Lotka–Volterra type systems with distributed delays are considered.

The purpose of this paper is to answer the above questions to some extent. More precisely, we consider the following non-autonomous  $n$ -species Lotka–Volterra competition system (normalized) with delays

$$\dot{x}_i(t) = r_i(t)x_i(t) \left[ 1 - \int_{-\tau_{ii}}^0 x_i(t+s) dv_{ii}(s) - \sum_{j \neq i} \mu_{ij} \int_{-\tau_{ij}}^0 x_j(t+s) dv_{ij}(s) \right],$$

$$i = 1, 2, \dots, n, \tag{1.8}$$

with initial conditions

$$x_i(t) = \phi_i(t) \geq 0, \quad t \in [-\tau, 0]; \quad \phi_i(0) > 0, \quad i = 1, 2, \dots, n, \tag{1.9}$$

where  $\tau = \max\{\tau_{ij}: i, j = 1, 2, \dots, n\}$ . Here, we always assume for  $i, j = 1, 2, \dots, n$ ,

(H1)  $r_i \in C([0, \infty), (0, \infty))$ ;

(H2)  $\tau_{ij} \geq 0$  and  $\mu_{ij} \geq 0$ ;

(H3)  $v_{ij}(t)$  is non-decreasing, bounded and satisfies normalization condition:

$$\int_{-\tau_{ij}}^0 dv_{ij}(s) = 1,$$

and the integral is in the Riemann–Stieltjes sense.

The remainder of the paper is organized as follows. In Section 2, we give the main results. In Section 3, we establish some preliminary lemmas, which address the persistence and dissipativity of system (1.8), and therefore, which themselves are of some interest and importance. In Section 4, by combining these lemmas with the “sandwiching” technique and some subtle integration and inequality tricks, we give the proofs of the main theorems.

**2. Main results**

Let  $\mu_i = \sum_{j \neq i} \mu_{ij}$  for  $i = 1, 2, \dots, n$ , and  $\mu = \max\{\mu_1, \mu_2, \dots, \mu_n\}$ . Then, a “diagonal dominant” condition (also a generalization of condition (DD) in Theorem 1.2) for (1.8) is

(DD1)  $\mu < 1$ ,

which implies the existence of a unique positive equilibrium (see [2] or [16, Proposition 7.3, p. 97]). Used in [11] is the following slightly weaker “diagonal dominant” condition

(DD2) there are constants  $\delta_i > 0$ ,  $i = 1, 2, \dots, n$ , such that  $\sum_{j \neq i} \delta_j \mu_{ij} < \delta_i$  for  $i = 1, 2, \dots, n$ ,

but which does not guarantee the existence of a unique positive equilibrium.

Throughout the rest of this paper, as in [8], we always assume the existence of a unique positive equilibrium for (1.8) and denote it by  $x^* = (x_1^*, x_2^*, \dots, x_n^*)$ . We also set  $a = \max\{x_i^* : i = 1, 2, \dots, n\}$ . Obviously,

$$0 < x_i^* \leq a \leq 1 \quad \text{and} \quad a \left( 1 + \sum_{j \neq i} \mu_{ij} \right) \geq 1, \quad i = 1, 2, \dots, n.$$

Our first theorem gives not only a generalization but also an improvement of Theorem 1.2 to the non-autonomous system (1.8) with distributed delay.

**Theorem 2.1.** *Assume that (DD1) holds, and that*

$$\int_0^\infty r_i(s) ds = \infty, \quad i = 1, 2, \dots, n \tag{2.1}$$

and

$$\int_{t-\tau_{ii}}^t r_i(s) ds \leq \frac{3(1-\mu)}{2a(1+\mu_i)} + \frac{(1-\mu)(\mu+\mu_i)}{2a(1+\mu_i)^2}, \quad i = 1, 2, \dots, n. \tag{2.2}$$

*Then the positive equilibrium  $x^* = (x_1^*, \dots, x_n^*)$  of (1.8) is a global attractor.*

It is easily seen that Theorem 2.1 will reproduce the Wright’s 3/2 result for the autonomous delayed logistic equation and the result in [19] for the non-autonomous delayed logistic equation.

Kuang [11] also studied the global attractivity of the positive equilibrium of general  $n$ -species Lotka–Volterra system without dominating instantaneous negative feedbacks. Applying one of the main results in [11, Corollary 3.1] to systems (1.8) gives the following convenient criterion.

**Theorem 2.2.** Assume that  $r_i(t) \equiv r_i$ ,  $i = 1, 2, \dots, n$ , and there are positive constants  $\delta_i$ ,  $i = 1, 2, \dots, n$  such that (DD2) holds. If

$$r_i \tau_{ii} e^{r_i \tau_{ii}} < \frac{1 - \delta_i^{-1} \sum_{j \neq i} \delta_j \mu_{ij}}{1 + \delta_i^{-1} \sum_{j \neq i} \delta_j \mu_{ij}}, \quad i = 1, 2, \dots, n, \tag{2.3}$$

then,  $x^*$  is globally attractive for (1.8).

Along this direction, we can generalize (to the non-autonomous system (1.8)) and improve Theorem 2.2 with the following theorem.

**Theorem 2.3.** Assume that (2.1) holds, and that

$$\int_{t-\tau_{ii}}^t r_i(s) ds \leq d_i, \quad i = 1, 2, \dots, n. \tag{2.4}$$

Suppose that there are positive constants  $\delta_i$ ,  $i = 1, 2, \dots, n$  such that (DD2) holds, and for  $i = 1, 2, \dots, n$

$$d_i \exp(d_i + e^{-d_i} - 1) < \begin{cases} \frac{3 - \delta_i^{-1} \sum_{j \neq i} \delta_j \mu_{ij}}{2(1 + \delta_i^{-1} \sum_{j \neq i} \delta_j \mu_{ij})} & \text{if } \delta_i^{-1} \sum_{j \neq i} \delta_j \mu_{ij} \leq \frac{1}{3}, \\ \sqrt{\frac{2(1 - \delta_i^{-1} \sum_{j \neq i} \delta_j \mu_{ij})}{1 + \delta_i^{-1} \sum_{j \neq i} \delta_j \mu_{ij}}} & \text{if } \delta_i^{-1} \sum_{j \neq i} \delta_j \mu_{ij} > \frac{1}{3}. \end{cases} \tag{2.5}$$

Then the positive equilibrium  $x^*$  of (1.8) is a global attractor.

To see that Theorem 2.3 improves Theorem 2.2, we show that when  $r_i(t) \equiv r_i$ ,  $i = 1, 2, \dots, n$ , (2.5) is weaker than (2.3). Indeed, in this case, we take  $d_i = r_i \tau_{ii}$ , and thus

$$d_i \exp(d_i + e^{-d_i} - 1) \leq d_i \exp(d_i) = r_i \tau_{ii} e^{r_i \tau_{ii}}.$$

On the other hand, when  $\delta_i^{-1} \sum_{j \neq i} \delta_j \mu_{ij} > \frac{1}{3}$ ,

$$\frac{1 - \delta_i^{-1} \sum_{j \neq i} \delta_j \mu_{ij}}{1 + \delta_i^{-1} \sum_{j \neq i} \delta_j \mu_{ij}} < \sqrt{\frac{1 - \delta_i^{-1} \sum_{j \neq i} \delta_j \mu_{ij}}{1 + \delta_i^{-1} \sum_{j \neq i} \delta_j \mu_{ij}}} < \sqrt{\frac{2(1 - \delta_i^{-1} \sum_{j \neq i} \delta_j \mu_{ij})}{1 + \delta_i^{-1} \sum_{j \neq i} \delta_j \mu_{ij}}},$$

and when  $\delta_i^{-1} \sum_{j \neq i} \delta_j \mu_{ij} \leq \frac{1}{3}$ ,

$$\frac{3 - \delta_i^{-1} \sum_{j \neq i} \delta_j \mu_{ij}}{2(1 + \delta_i^{-1} \sum_{j \neq i} \delta_j \mu_{ij})} = \frac{1}{2} + \frac{1 - \delta_i^{-1} \sum_{j \neq i} \delta_j \mu_{ij}}{1 + \delta_i^{-1} \sum_{j \neq i} \delta_j \mu_{ij}} > \frac{1 - \delta_i^{-1} \sum_{j \neq i} \delta_j \mu_{ij}}{1 + \delta_i^{-1} \sum_{j \neq i} \delta_j \mu_{ij}}.$$

Along the direction of Theorem 1.1, where conditions are given in terms of a related matrix, we can also obtain some result. For this purpose, we first recall the concept of non-singular  $M$ -matrix (see [3, p. 114]).

**Definition 1.1.** The matrix  $B = (b_{ij})$  of order  $n$  is called a non-singular  $M$ -matrix if (i)  $b_{ij} \leq 0$ , for all  $i \neq j$ , and (ii) all the principal minors of  $B$  are positive.

There are many equivalent formulations of this concept (cf. [3, Theorem 5.1]). In particular, if  $A$  is an non-singular  $M$ -matrix, then  $A^{-1}$  is a positive matrix. We will use this version in proving the following theorem.

**Theorem 2.4.** Let  $B = (b_{ij})$  be the matrix defined by

$$b_{ii} = 1, \quad i = 1, 2, \dots, n \tag{2.6}$$

and

$$b_{ij} = \begin{cases} -[(2 + d_i^2 D_i^2)/(2 - d_i^2 D_i^2)]\mu_{ij} & \text{if } d_i D_i < 1, \\ -[(1 + 2d_i D_i)(3 - 2d_i D_i)]\mu_{ij} & \text{if } d_i D_i \geq 1, \end{cases} \quad i \neq j, \tag{2.7}$$

where

$$D_i = \exp(d_i + e^{-d_i} - 1), \quad i = 1, 2, \dots, n. \tag{2.8}$$

Assume that (2.1) and (2.4) hold, and that  $B$  is an  $M$ -matrix. Suppose that there exists a positive equilibrium  $x^*$  for (1.8). Then  $x^*$  is a global attractor.

When  $\tau_{ii} = 0, i = 1, \dots, n, d_i = 0, i = 1, \dots, n$  and  $B$  reduces to  $\bar{A} = (\bar{a}_{ij})$ , where  $\bar{a}_{ii} = b_{ii} = 1$  and  $\bar{a}_{ij} = -\mu_{ij}$ . When confining to the competitive case and after normalization in (1.1), the matrix  $\bar{A}$  is exactly the matrix  $-\hat{A}$  in Theorem 1.1. Thus, in such a special case, Theorem 1.1 is slightly less restrictive than Theorem 2.4, with a difference being between “non-negative” and “positive” for the principle minors. However, as stated in the title and in the introduction, dealing with positive  $\tau_{ii}$ s is the primary goal of this work, to which Theorem 1.1 fails.

### 3. Preliminary lemmas

**Lemma 3.1.** Let  $0 < a < 1, 0 < \mu < 1$ . Then system of inequalities

$$\begin{cases} y \leq (a + \mu x) \exp\left[\frac{1-\mu}{a} x - \frac{(1-\mu)^2(1+2\mu)}{6a^2(1+\mu)} x^2\right] - a, \\ x \leq a - (a - \mu y) \exp\left[-\frac{1-\mu}{a} y - \frac{(1-\mu)^2(1+2\mu)}{6a^2(1+\mu)} y^2\right] \end{cases} \tag{3.1}$$

has a unique solution:  $(x, y) = (0, 0)$  in the region  $D = \{(x, y) : 0 \leq x < a, 0 \leq y < a/\mu\}$ .

**Proof.** Let

$$\varphi(x) = \frac{1 - \mu}{a}x - \frac{(1 - \mu)^2(1 + 2\mu)}{6a^2(1 + \mu)}x^2, \quad \psi(y) = \frac{1 - \mu}{a}y + \frac{(1 - \mu)^2(1 + 2\mu)}{6a^2(1 + \mu)}y^2.$$

Then (3.1) can be written as

$$\begin{cases} y \leq (a + \mu x)e^{\varphi(x)} - a, \\ x \leq a - (a - \mu y)e^{-\psi(y)}. \end{cases} \tag{3.2}$$

Assume that (3.2) has another solution in the region  $D$  besides  $(0, 0)$ , say  $(x_0, y_0)$ . Then  $0 < x_0 < a$  and  $0 < y_0 < a/\mu$ . Define two curves  $\Gamma_1$  and  $\Gamma_2$  as follows:

$$\Gamma_1: y = (a + \mu x)e^{\varphi(x)} - a, \quad \Gamma_2: x = a - (a - \mu y)e^{-\psi(y)}. \tag{3.3}$$

By direct calculation, we have for curve  $\Gamma_1$ :

$$\begin{aligned} \left. \frac{dy}{dx} \right|_{(0,0)} &= 1, & \left. \frac{d^2y}{dx^2} \right|_{(0,0)} &= \frac{(1 - \mu)(2 + 5\mu + 5\mu^2)}{3a(1 + \mu)}, \\ \left. \frac{d^3y}{dx^3} \right|_{(0,0)} &= \frac{\mu(1 - \mu)^2(1 + 2\mu)}{a^2(1 + \mu)}. \end{aligned}$$

For  $\Gamma_2$ , we first establish the following:

$$\begin{aligned} \left. \frac{dx}{dy} \right|_{(0,0)} &= 1, & \left. \frac{d^2x}{dy^2} \right|_{(0,0)} &= -\frac{(1 - \mu)(2 + 5\mu + 5\mu^2)}{3a(1 + \mu)}, \\ \left. \frac{d^3x}{dy^3} \right|_{(0,0)} &= \frac{\mu(1 - \mu)^2(1 + 2\mu)}{a^2(1 + \mu)}. \end{aligned}$$

Noting that

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}, \quad \frac{d^2y}{dx^2} = -\frac{\frac{d^2x}{dy^2}}{\left(\frac{dx}{dy}\right)^3}$$

and

$$\frac{d^3y}{dx^3} = \frac{3\left(\frac{d^2x}{dy^2}\right)^2 - \frac{d^3x}{dy^3} \frac{dx}{dy}}{\left(\frac{dx}{dy}\right)^5},$$



we can easily see that  $\Gamma_2$  shares the same  $\frac{dy}{dx}$  and  $\frac{d^2y}{dx^2}$  at  $(0, 0)$  with  $\Gamma_1$ , but has a strictly larger  $\frac{d^3y}{dx^3}$  than  $\Gamma_1$ , as estimated below:

$$\begin{aligned} \frac{d^3y}{dx^3}\Big|_{(0,0)} &= 3\left(\frac{d^2x}{dy^2}\Big|_{(0,0)}\right)^2 - \frac{d^3x}{dy^3}\Big|_{(0,0)} \\ &= \frac{(1-\mu)^2[4+17\mu+36\mu^2+44\mu^3+25\mu^4]}{3a^2(1+\mu)^2} \\ &> \frac{\mu(1-\mu)^2(1+2\mu)}{a^2(1+\mu)}. \end{aligned}$$

Hence  $\Gamma_2$  lies above  $\Gamma_1$  near  $(0,0)$ . The existence of  $(x_0, y_0)$  implies that the curves  $\Gamma_1$  and  $\Gamma_2$  must intersect at a point in the region  $D$  besides  $(0, 0)$ . Let  $(x_1, y_1)$  be the first such point, i.e.  $x_1$  is smallest. Then the slope of  $\Gamma_1$  at  $(x_1, y_1)$  is no less than the slope of  $\Gamma_2$  at  $(x_1, y_1)$ , i.e.

$$[\mu + (a + \mu x_1)\phi'(x_1)]e^{\phi(x_1)} \geq \frac{1}{\mu + (a - \mu y_1)\psi'(y_1)} e^{\psi(y_1)}$$

or

$$[\mu + (a + \mu x_1)\phi'(x_1)][\mu + (a - \mu y_1)\psi'(y_1)] \geq e^{\psi(y_1) - \phi(x_1)}. \tag{3.4}$$

From (3.3), we have

$$\begin{aligned} -\ln\left(1 - \frac{x_1}{a}\right) &= -\ln\left(1 - \frac{\mu y_1}{a}\right) + \frac{1-\mu}{a}y_1 + \frac{(1-\mu)^2(1+2\mu)}{6a^2(1+\mu)}y_1^2 \\ &= \left(\frac{\mu}{a}y_1 + \frac{\mu^2}{2a^2}y_1^2 + \frac{\mu^3}{3a^3}y_1^3 + \dots\right) \\ &\quad + \frac{1-\mu}{a}y_1 + \frac{(1-\mu)^2(1+2\mu)}{6a^2(1+\mu)}y_1^2 \\ &\leq \frac{1}{a}y_1 + \frac{1}{2a^2}y_1^2 + \frac{1}{3a^3}y_1^3 + \dots \\ &= -\ln\left(1 - \frac{y_1}{a}\right). \end{aligned}$$

This implies that

$$x_1 \leq y_1. \tag{3.5}$$

We then calculate and estimate (using (3.5)) as below:

$$\begin{aligned}
 & [\mu + (a + \mu x_1)\varphi'(x_1)][\mu + (a - \mu y_1)\psi'(y_1)] \\
 &= \left\{ 1 - \left[ \frac{(1 - \mu)^2(1 + 2\mu)}{3a(1 + \mu)} - \frac{\mu(1 - \mu)}{a} \right] x_1 - \frac{\mu(1 - \mu)^2(1 + 2\mu)}{3a^2(1 + \mu)} x_1^2 \right\} \\
 &\quad \times \left\{ 1 + \left[ \frac{(1 - \mu)^2(1 + 2\mu)}{3a(1 + \mu)} - \frac{\mu(1 - \mu)}{a} \right] y_1 - \frac{\mu(1 - \mu)^2(1 + 2\mu)}{3a^2(1 + \mu)} y_1^2 \right\} \\
 &= 1 + \left[ \frac{(1 - \mu)^2(1 + 2\mu)}{3a(1 + \mu)} - \frac{\mu(1 - \mu)}{a} \right] (y_1 - x_1) \\
 &\quad - \left[ \frac{(1 - \mu)^2(1 + 2\mu)}{3a(1 + \mu)} - \frac{\mu(1 - \mu)}{a} \right]^2 x_1 y_1 \\
 &\quad - \frac{\mu(1 - \mu)^2(1 + 2\mu)}{3a^2(1 + \mu)} (x_1^2 + y_1^2) + \frac{\mu^2(1 - \mu)^4(1 + 2\mu)^2}{9a^4(1 + \mu)^2} x_1^2 y_1^2 \\
 &\quad + \frac{\mu(1 - \mu)^2(1 + 2\mu)}{3a^2(1 + \mu)} \left[ \frac{(1 - \mu)^2(1 + 2\mu)}{3a(1 + \mu)} - \frac{\mu(1 - \mu)}{a} \right] x_1 y_1 (y_1 - x_1) \\
 &= 1 + \left[ \frac{(1 - \mu)^2(1 + 2\mu)}{3a(1 + \mu)} - \frac{\mu(1 - \mu)}{a} \right] (y_1 - x_1) \\
 &\quad - \left[ \frac{(1 - \mu)^2(1 + 2\mu)}{3a(1 + \mu)} - \frac{\mu(1 - \mu)}{a} \right]^2 x_1 y_1 \\
 &\quad - \frac{\mu(1 - \mu)^2(1 + 2\mu)}{3a^2(1 + \mu)} (x_1^2 + y_1^2) + \frac{\mu^2(1 - \mu)^4(1 + 2\mu)^2}{9a^4(1 + \mu)^2} x_1^2 y_1^2 \\
 &\quad + \frac{\mu(1 - \mu)^3(1 + 2\mu)}{3a^3(1 + \mu)} \left[ \frac{(1 - \mu)(1 + 2\mu)}{3(1 + \mu)} - \mu \right] x_1 y_1 (y_1 - x_1) \\
 &< 1 + (1 - \mu) \left[ \frac{(1 - \mu)(1 + 2\mu)}{3a(1 + \mu)} - \frac{\mu}{a} \right] (y_1 - x_1) - \frac{\mu(1 - \mu)^2(1 + 2\mu)}{3a^2(1 + \mu)} (x_1^2 + y_1^2) \\
 &\quad + \frac{\mu(1 - \mu)^4(1 + 2\mu)^2}{9a^3(1 + \mu)^2} x_1 y_1 (y_1 - x_1) + \frac{\mu^2(1 - \mu)^4(1 + 2\mu)^2}{9a^4(1 + \mu)^2} x_1^2 y_1^2 \\
 &< 1 + (1 - \mu) \left[ \frac{(1 - \mu)(1 + 2\mu)}{3a(1 + \mu)} - \frac{\mu}{a} \right] (y_1 - x_1)
 \end{aligned}$$

and

$$\begin{aligned}
 e^{\psi(y_1) - \varphi(x_1)} &= \exp \left[ \frac{1 - \mu}{a} (y_1 - x_1) + \frac{(1 - \mu)^2(1 + 2\mu)}{6a^2(1 + \mu)} (x_1^2 + y_1^2) \right] \\
 &> 1 + \frac{1 - \mu}{a} (y_1 - x_1) + \frac{(1 - \mu)^2(1 + 2\mu)}{6a^2(1 + \mu)} (x_1^2 + y_1^2).
 \end{aligned}$$

It follows that

$$\begin{aligned}
 & e^{\psi(y_1) - \varphi(x_1)} - [\mu + (a + \mu x_1)\varphi'(x_1)][\mu + (a - \mu y_1)\psi'(y_1)] \\
 & > \left[ 1 + \frac{1 - \mu}{a}(y_1 - x_1) + \frac{(1 - \mu)^2(1 + 2\mu)}{6a^2(1 + \mu)}(x_1^2 + y_1^2) \right] \\
 & \quad - \left[ 1 + (1 - \mu) \left( \frac{(1 - \mu)(1 + 2\mu)}{3a(1 + \mu)} - \frac{\mu}{a} \right) (y_1 - x_1) \right] \\
 & = (1 - \mu) \left[ \frac{1 + \mu}{a} - \frac{(1 - \mu)(1 + 2\mu)}{3a(1 + \mu)} \right] (y_1 - x_1) + \frac{(1 - \mu)^2(1 + 2\mu)}{6a^2(1 + \mu)} (x_1^2 + y_1^2) \\
 & > 0,
 \end{aligned}$$

which contradicts (3.4). The proof is complete.  $\square$

The next two lemmas establish the persistence and dissipativity of system (1.8).

**Lemma 3.2.** *Assume that (2.1) and (2.4) hold and let  $(x_1(t), x_2(t), \dots, x_n(t))$  be the solution of (1.8) and (1.9). Then we have eventually*

$$0 < x_i(t) \leq \exp(d_i + e^{-d_i} - 1), \quad i = 1, 2, \dots, n. \tag{3.6}$$

**Proof.** From (1.8) and (1.9), it is easy to see that  $x_i(t) > 0$  for  $t \geq 0$  and  $i = 1, 2, \dots, n$ . Hence,

$$\dot{x}_i(t) \leq r_i(t)x_i(t) \left[ 1 - \int_{-\tau_{ii}}^0 x_i(t+s) dv_{ii}(s) \right], \quad i = 1, 2, \dots, n. \tag{3.7}$$

If  $x_i(t) \leq 1$  eventually, then (3.6) holds naturally for large  $t$ . If  $x_i(t) > 1$  eventually, then (3.7) implies that  $x_i(t)$  is non-increasing eventually. Let  $\lim_{t \rightarrow \infty} x_i(t) = c_i$ . Then  $x_i(t) \geq c_i \geq 1$  for large  $t$  and from (3.7) we have

$$\dot{x}_i(t) \leq r_i(t)x_i(t)(1 - c_i), \quad i = 1, 2, \dots, n$$

for large  $t$ . Integrating the above from  $t$  to  $\infty$ , we obtain

$$\ln c_i - \ln x_i(t) \leq (1 - c_i) \int_t^\infty r_i(s) ds, \quad \text{for large } t,$$

which, together with (2.1), implies that  $c_i = 1, i = 1, 2, \dots, n$ . Hence, (3.6) holds too. In the sequel, we only consider the case when  $x_i(t) - 1$  oscillates. For this case, let  $t_i^*$  be an arbitrary local left maximum point of  $x_i(t)$  such that  $x_i(t_i^*) > 1$ . Then  $\dot{x}_i(t_i^*) \geq 0$ , it follows from (3.7) that there exists  $\xi_i \in [t_i^* - \tau_{ii}, t_i^*]$  such that  $x_i(\xi_i) = 1$  and  $x_i(t) > 1$

for  $\zeta_i < t \leq t_i^*$ . For  $t \in [\zeta_i, t_i^*]$ , integrating (3.7) from  $t + s$  to  $\zeta_i$  we get

$$-\ln x_i(t + s) \leq \int_{t+s}^{\zeta_i} r_i(u) du \leq \int_{t-\tau_{ii}}^{\zeta_i} r_i(u) du, \quad -\tau_{ii} \leq s \leq \zeta_i - t.$$

Note that  $-\ln x_i(t + s) < 0$  for  $\zeta_i - t < s \leq 0$ . Hence,

$$x_i(t + s) \geq \exp\left(-\int_{t-\tau_{ii}}^{\zeta_i} r_i(u) du\right), \quad \zeta_i \leq t \leq t_i^*, \quad -\tau_{ii} \leq s \leq 0.$$

Substituting this into (3.7), we obtain

$$\frac{\dot{x}_i(t)}{x_i(t)} \leq r_i(t) \left[1 - \exp\left(-\int_{t-\tau_{ii}}^{\zeta_i} r_i(u) du\right)\right], \quad \zeta_i \leq t \leq t_i^*. \tag{3.8}$$

Integrating (3.8) from  $\zeta_i$  to  $t_i^*$  and using (2.4), we have

$$\begin{aligned} \ln x_i(t_i^*) &\leq \int_{\zeta_i}^{t_i^*} r_i(t) \left[1 - \exp\left(-\int_{t-\tau_{ii}}^{\zeta_i} r_i(s) ds\right)\right] dt \\ &\leq \int_{\zeta_i}^{t_i^*} r_i(s) ds - \int_{\zeta_i}^{t_i^*} r_i(t) \exp\left(-d_i + \int_{\zeta_i}^t r_i(s) ds\right) dt \\ &= \int_{\zeta_i}^{t_i^*} r_i(s) ds + e^{-d_i} - \exp\left(-d_i + \int_{\zeta_i}^{t_i^*} r_i(s) ds\right) \\ &\leq d_i + e^{-d_i} - 1, \end{aligned}$$

which implies that

$$x_i(t_i^*) \leq \exp(d_i + e^{-d_i} - 1), \quad i = 1, 2, \dots, n.$$

It follows that for large  $t$

$$x_i(t) \leq \exp(d_i + e^{-d_i} - 1), \quad i = 1, 2, \dots, n.$$

The proof is complete.  $\square$

**Lemma 3.3.** Assume that (2.1) and (2.2) hold, and let  $(x_1(t), x_2(t), \dots, x_n(t))$  be the solution of (1.8) and (1.9). Then

$$0 < \liminf_{t \rightarrow \infty} x_i(t) \leq \limsup_{t \rightarrow \infty} x_i(t) < \infty, \quad i = 1, 2, \dots, n. \tag{3.9}$$

**Proof.** By (2.2) and Lemma 3.2 with  $d_i = \frac{(1-\mu)(3+\mu+4\mu_i)}{2a(1+\mu_i)^2}$ ,  $i = 1, 2, \dots, n$ , we have

$$x_i(t) \leq \exp(d_i + e^{-d_i} - 1) \leq \exp\left(\frac{d_i^2}{2}\right) \leq \exp\left[\frac{(1-\mu)^2(3+\mu+4\mu_i)^2}{8a^2(1+\mu_i)^4}\right],$$

$$i = 1, 2, \dots, n. \tag{3.10}$$

From (3.10) and use that fact that  $a(1+\mu_i) \geq 1$  for  $i = 1, 2, \dots, n$ , we obtain

$$\begin{aligned} \sum_{j \neq i} \mu_{ij} \int_{-\tau_{ij}}^0 x_j(t+s) dv_{ij}(s) &\leq \mu \exp\left[\frac{(1-\mu)^2(3+\mu+4\mu_i)^2}{8a^2(1+\mu_i)^4}\right] \\ &\leq \mu \exp\left[\frac{(1-\mu)^2(3+\mu+4\mu_i)^2}{8(1+\mu_i)^2}\right] \\ &\leq \mu \exp\left[\frac{(1-\mu)^2(3+5\mu)^2}{8(1+\mu)^2}\right] \equiv \alpha, \quad i = 1, 2, \dots, n. \end{aligned}$$

It is easy to see that  $\alpha < 1$ . Substituting these into (1.8), we have

$$\dot{x}_i(t) \geq r_i(t)x_i(t) \left[1 - \alpha - \int_{-\tau_{ii}}^0 x_i(t+s) dv_{ii}(s)\right], \quad i = 1, 2, \dots, n. \tag{3.11}$$

Now (3.9) follows from (3.10) and (3.11) and by a standard comparison argument. The proof is complete.  $\square$

#### 4. Proofs of the main results

**Proof of Theorem 2.1.** By the transformation

$$\bar{x}_i = x_i - x_i^*, \quad i = 1, 2, \dots, n.$$

System (1.8) becomes

$$\dot{x}_i(t) = -r_i(t)(x_i^* + x_i(t)) \left[ \int_{-\tau_{ii}}^0 x_i(t+s) dv_{ii}(s) + \sum_{j \neq i} \mu_{ij} \int_{-\tau_{ij}}^0 x_j(t+s) dv_{ij}(s) \right],$$

$$i = 1, 2, \dots, n. \tag{4.1}$$

Here we used  $x_i(t)$  instead of  $\bar{x}_i(t)$  for  $i = 1, 2, \dots, n$ . Clearly, the global attractivity of  $x^*$  of system (1.8) is equivalent to that for (4.1),

$$\lim_{t \rightarrow \infty} x_i(t) = 0, \quad i = 1, 2, \dots, n, \tag{4.2}$$

for all  $x(t) = (x_1(t), x_2(t), \dots, x_n(t)) > -x^*$  for  $t \geq 0$ . We divide into two cases to prove (4.2).

Case 1:  $\int_{-\tau_{ii}}^0 x_i(t+s) dv_{ii}(s) + \sum_{j \neq i} \mu_{ij} \int_{-\tau_{ij}}^0 x_j(t+s) dv_{ij}(s)$ ,  $i = 1, 2, \dots, n$  are all non-oscillatory. In this case,  $\dot{x}_i(t)$ ,  $i = 1, 2, \dots, n$  are all sign-definite eventually which implies that  $x_i(t)$ ,  $i = 1, 2, \dots, n$  are monotonous eventually. By Lemma 3.3, we have  $x_i(t) \rightarrow c_i$  as  $t \rightarrow \infty$  and  $x_i^* + c_i > 0$  for  $i = 1, 2, \dots, n$ . Hence for large  $T$ , integrating (4.1) from  $T$  to  $\infty$ , we obtain for  $i = 1, 2, \dots, n$

$$\begin{aligned} & \left| \ln \frac{x_i^* + x_i(T)}{x_i^* + c_i} \right| \\ &= \int_T^\infty r_i(t) \left| \int_{-\tau_{ii}}^0 x_i(t+s) dv_{ii}(s) + \sum_{j \neq i} \mu_{ij} \int_{-\tau_{ij}}^0 x_j(t+s) dv_{ij}(s) \right| dt. \end{aligned}$$

Note that

$$\begin{aligned} & \liminf_{t \rightarrow \infty} \left| \int_{-\tau_{ii}}^0 x_i(t+s) dv_{ii}(s) + \sum_{j \neq i} \mu_{ij} \int_{-\tau_{ij}}^0 x_j(t+s) dv_{ij}(s) \right| \\ &= \left| c_i + \sum_{j \neq i} \mu_{ij} c_j \right|, \end{aligned}$$

$i = 1, 2, \dots, n$ . It follows from (2.1) and (DD1) that  $c_1 = c_2 = \dots = c_n = 0$ , i.e., (4.2) holds.

Case 2:  $\int_{-\tau_{ii}}^0 x_i(t+s) dv_{ii}(s) + \sum_{j \neq i} \mu_{ij} \int_{-\tau_{ij}}^0 x_j(t+s) dv_{ij}(s)$  is oscillatory for some  $i = l \in \{1, 2, \dots, n\}$ . Then there exists an infinity sequence  $\{t_k\}$  such that

$$\int_{-\tau_{ll}}^0 x_l(t_k+s) dv_{ll}(s) + \sum_{j \neq l} \mu_{lj} \int_{-\tau_{lj}}^0 x_j(t_k+s) dv_{lj}(s) = 0, \quad k = 1, 2, \dots \quad (4.3)$$

Set

$$V_i = \liminf_{t \rightarrow \infty} x_i(t) \quad \text{and} \quad U_i = \limsup_{t \rightarrow \infty} x_i(t), \quad i = 1, 2, \dots, n.$$

In view of Lemma 3.3,

$$-x_i^* < V_i \leq U_i < \infty, \quad i = 1, 2, \dots, n. \quad (4.4)$$

Let

$$-V = \min\{V_1, V_2, \dots, V_n\} \quad \text{and} \quad U = \max\{U_1, U_2, \dots, U_n\}.$$

Then from (4.3) and (4.4), we have

$$0 \leq V < a = \max\{x_i^*: i = 1, 2, \dots, n\}, \quad 0 \leq U < \infty. \tag{4.5}$$

In what follows, we show that  $V$  and  $U$  satisfy the inequalities

$$a + U \leq (a + \mu V) \exp \left[ \frac{1 - \mu}{a} V - \frac{(1 - \mu)^2(1 + 2\mu)}{6a^2(1 + \mu)} V^2 \right] \tag{4.6}$$

and

$$a - V \geq (a - \mu U) \exp \left[ -\frac{1 - \mu}{a} U - \frac{(1 - \mu)^2(1 + 2\mu)}{6a^2(1 + \mu)} U^2 \right]. \tag{4.7}$$

For the sake of simplicity, we set

$$A_i = \frac{3(1 - \mu)}{2a(1 + \mu_i)} + \frac{(1 - \mu)(\mu + \mu_i)}{2a(1 + \mu_i)^2}, \quad i = 1, 2, \dots, n.$$

Without loss of generality, we may assume that  $U = U_i$  and  $V = -V_j$ . Then  $V < x_j^*$ . Let  $\varepsilon > 0$  be sufficiently small such that  $v_1 \equiv V + \varepsilon < x_j^*$ . Choose  $T > 0$  such that

$$-v_1 < x_i(t) < U + \varepsilon \equiv u_1, \quad t \geq T - \tau, \quad i = 1, 2, \dots, n, \tag{4.8}$$

where  $\tau = \max\{\tau_{ij}: i, j = 1, 2, \dots, n\}$ . First, we prove that (4.6) holds. If  $U \leq \mu_i V$ , then (4.6) obviously holds. Therefore, we will prove (4.6) only in the case when  $U > \mu_i V$ . For the sake of simplicity, it is harmless assuming  $U > \mu_i v_1$ . Set  $v_2 = (1 + \mu_i)v_1$  and  $u_2 = (1 + \mu_j)u_1$ . Then from (4.1), we have

$$\frac{\dot{x}_i(t)}{x_i^* + x_i(t)} \leq r_i(t) \left[ -\int_{-\tau_{ii}}^0 x_i(t+s) dv_{ii}(s) + \mu_i v_1 \right] \leq r_i(t)v_2, \quad t \geq T \tag{4.9}$$

and

$$-\frac{\dot{x}_j(t)}{x_j^* + x_j(t)} \leq r_j(t) \left[ \int_{-\tau_{jj}}^0 x_j(t+s) dv_{jj}(s) + \mu_j u_1 \right] \leq r_j(t)u_2, \quad t \geq T. \tag{4.10}$$

Since  $U > \mu_i v_1$ , we cannot have  $x_i(t) \leq \mu_i v_1$  eventually. On the other hand, if  $x_i(t) \geq \mu_i v_1$  eventually, then it follows from (2.1) and the first inequality in (4.9) that  $x_i(t)$  is non-increasing and  $U = \lim_{t \rightarrow \infty} x_i(t) = \mu_i v_1$ . This is also impossible. Therefore, it follows that  $x_i(t)$  oscillates about  $\mu_i v_1$ .

Let  $\{p_k\}$  be an increasing sequence such that  $p_k \geq T + \tau_{ii}, \dot{x}_i(p_k) = 0, x_i(p_k) \geq \mu_i v_1, \lim_{k \rightarrow \infty} p_k = \infty$  and  $\lim_{k \rightarrow \infty} x_i(p_k) = U$ . By (4.9), there exists  $\check{\xi}_k \in [p_k - \tau_{ii}, p_k]$  such that  $x_i(\check{\xi}_k) = \mu_i v_1$  and  $x_i(t) > \mu_i v_1$  for  $\check{\xi}_k < t \leq p_k$ .

For  $t \in [\xi_k, p_k]$ , integrating (4.9) from  $t + s$  to  $\xi_k$  we get

$$-\ln \frac{x_i^* + x_i(t + s)}{x_i^* + x_i(\xi_k)} \leq v_2 \int_{t+s}^{\xi_k} r_i(s) ds \leq v_2 \int_{t-\tau_{ii}}^{\xi_k} r_i(s) ds, \quad -\tau_{ii} \leq s \leq \xi_k - t.$$

Note that  $-\ln [(x_i^* + x_i(t + s))/(x_i^* + x_i(\xi_k))] < 0$  for  $\xi_k - t < s \leq 0$ . Thus,

$$x_i(t + s) \geq -x_i^* + (x_i^* + \mu_i v_1) \exp\left(-v_2 \int_{t-\tau_{ii}}^{\xi_k} r_i(s) ds\right),$$

$$\xi_k \leq t \leq p_k, \quad -\tau_{ii} \leq s \leq 0.$$

Substituting this into the first inequality in (4.9), we obtain

$$\frac{\dot{x}_i(t)}{x_i^* + x_i(t)} \leq (x_i^* + \mu_i v_1) r_i(t) \left[1 - \exp\left(-v_2 \int_{t-\tau_{ii}}^{\xi_k} r_i(s) ds\right)\right], \quad \xi_k \leq t \leq p_k.$$

Combining this with (4.9), we have

$$\frac{\dot{x}_i(t)}{a + x_i(t)} \leq \min \left\{ v_2 r_i(t), (a + \mu_i v_1) r_i(t) \left[1 - \exp\left(-v_2 \int_{t-\tau_{ii}}^{\xi_k} r_i(s) ds\right)\right] \right\},$$

$$\xi_k \leq t \leq p_k. \tag{4.11}$$

To prove (4.6), we consider the following two possible subcases.

Case 2.1:  $\int_{\xi_k}^{p_k} r_i(s) ds \leq -\frac{1}{v_2} \ln(1 - (1 - \mu)v_1/a)$ . Then by (2.2) and (4.11)

$$\begin{aligned} & \ln \frac{a + x_i(p_k)}{a + \mu v_1} \\ & \leq \ln \frac{a + x_i(p_k)}{a + \mu_i v_1} \\ & \leq (a + \mu_i v_1) \left[ \int_{\xi_k}^{p_k} r_i(s) ds - \int_{\xi_k}^{p_k} r_i(t) \exp\left(-v_2 \int_{t-\tau_{ii}}^{\xi_k} r_i(s) ds\right) dt \right] \\ & \leq (a + \mu_i v_1) \left\{ \int_{\xi_k}^{p_k} r_i(s) ds - \int_{\xi_k}^{p_k} r_i(t) \exp\left[-v_2 \left(A_i - \int_{\xi_k}^t r_i(s) ds\right)\right] dt \right\} \\ & = (a + \mu_i v_1) \left\{ \int_{\xi_k}^{p_k} r_i(s) ds - \frac{1}{v_2} \exp\left[-v_2 \left(A_i - \int_{\xi_k}^{p_k} r_i(s) ds\right)\right] \right. \\ & \quad \left. \times \left[1 - \exp\left(-v_2 \int_{\xi_k}^{p_k} r_i(s) ds\right)\right] \right\}. \end{aligned}$$



If  $\int_{\xi_k}^{p_k} r_i(s) ds \leq -\frac{1}{v_2} \ln(1 - (1 - \mu)v_1/a) \leq A_i$ , then,

$$\begin{aligned} & \ln \frac{a + x_i(p_k)}{a + \mu v_1} \\ & \leq (a + \mu_i v_1) \left\{ -\frac{1}{v_2} \ln(1 - (1 - \mu)v_1/a) \right. \\ & \quad \left. - \frac{1 - \mu}{(1 + \mu_i)a} \exp \left[ -v_2 \left( A_i + \frac{\ln(1 - (1 - \mu)v_1/a)}{v_2} \right) \right] \right\} \\ & \leq (a + \mu_i v_1) \left\{ -\frac{1}{v_2} \ln(1 - (1 - \mu)v_1/a) \right. \\ & \quad \left. - \frac{1 - \mu}{(1 + \mu_i)a} \left[ 1 - v_2 \left( A_i + \frac{\ln(1 - (1 - \mu)v_1/a)}{v_2} \right) \right] \right\} \\ & = \frac{a + \mu_i v_1}{a(1 + \mu_i)} \left\{ -\frac{a - (1 - \mu)v_1}{v_1} \ln \left( 1 - \frac{(1 - \mu)v_1}{a} \right) \right. \\ & \quad \left. - 1 + \mu + A_i(1 - \mu)(1 + \mu_i)v_1 \right\} \\ & \leq \frac{a + \mu_i v_1}{a(1 + \mu_i)} \left\{ \left[ A_i(1 + \mu_i) - \frac{1 - \mu}{2a} \right] (1 - \mu)v_1 - \frac{(1 - \mu)^3}{6a^2} v_1^2 - \frac{(1 - \mu)^4}{12a^3} v_1^3 \right\} \\ & \leq \left[ A_i(1 + \mu_i) - \frac{1 - \mu}{2a} \right] (1 - \mu)v_1 - \frac{(1 - \mu)^3}{6a^2} v_1^2 - \frac{(1 - \mu)^4}{12a^3} v_1^3 \\ & = \left[ \frac{1 - \mu}{a} + \frac{(1 - \mu)(\mu + \mu_i)}{2a(1 + \mu_i)} \right] (1 - \mu)v_1 - \frac{(1 - \mu)^3}{6a^2} v_1^2 - \frac{(1 - \mu)^4}{12a^3} v_1^3 \\ & \leq \frac{1 - \mu}{a} v_1 - \frac{(1 - \mu)^2(1 + 2\mu)}{6a^2(1 + \mu)} v_1^2. \end{aligned}$$

In the above third inequality, we have used the following inequality:

$$(1 - x) \ln(1 - x) \geq -x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{12}x^4, \quad 0 \leq x < 1.$$

If  $\int_{\xi_k}^{p_k} r_i(s) ds \leq A_i \leq -\frac{1}{v_2} \ln(1 - (1 - \mu)v_1/a)$ , then

$$\begin{aligned} \frac{3(1 - \mu)}{2a} & \leq -\frac{1}{v_1} \ln \left( 1 - \frac{(1 - \mu)v_1}{a} \right) \\ & \leq \frac{1 - \mu}{a - (1 - \mu)v_1} \left[ 1 - \frac{1 - \mu}{2a} v_1 - \frac{(1 - \mu)^2}{6a^2} v_1^2 \right], \end{aligned}$$

which implies that  $(1 - \mu)v_1 > a/2$ . Hence,

$$\begin{aligned} & \ln \frac{a + x_i(p_k)}{a + \mu v_1} \\ & \leq (a + \mu_i v_1) \left[ A_i - \frac{1}{v_2} (1 - e^{-A_i v_2}) \right] \\ & \leq (a + \mu_i v_1) \left( \frac{1}{2} A_i^2 v_2 - \frac{1}{6} A_i^3 v_2^2 + \frac{1}{24} A_i^4 v_2^3 \right) \\ & = \frac{(1 - \mu)^2 (a + \mu_i v_1)}{a(1 + \mu_i)} \left[ \frac{(3 + \mu + 4\mu_i)^2}{8a(1 + \mu_i)^2} v_1 - \frac{(1 - \mu)(3 + \mu + 4\mu_i)^3}{48a^2(1 + \mu_i)^3} v_1^2 \right. \\ & \quad \left. + \frac{(1 - \mu)^2 (3 + \mu + 4\mu_i)^4}{384a^3(1 + \mu_i)^4} v_1^3 \right] \\ & \leq (1 - \mu)^2 \left[ \frac{(3 + \mu + 4\mu_i)^2}{8a(1 + \mu_i)^2} v_1 - \frac{(1 - \mu)(3 + \mu + 4\mu_i)^3}{48a^2(1 + \mu_i)^3} v_1^2 \right. \\ & \quad \left. + \frac{(1 - \mu)^2 (3 + \mu + 4\mu_i)^4}{384a^3(1 + \mu_i)^4} v_1^3 \right] \\ & \leq \frac{1 - \mu}{a} v_1 - \frac{(1 - \mu)^2 (1 + 2\mu)}{6a^2(1 + \mu_i)} v_1^2 \\ & \leq \frac{1 - \mu}{a} v_1 - \frac{(1 - \mu)^2 (1 + 2\mu)}{6a^2(1 + \mu)} v_1^2. \end{aligned}$$

Case 2.2:  $-\frac{1}{v_2} \ln(1 - (1 - \mu)v_1/a) < \int_{\xi_k}^{p_k} r_i(s) ds \leq A_i$ . Choose  $l_k \in (\xi_k, p_k)$  such that  $\int_{l_k}^{p_k} r_i(s) ds = -\frac{1}{v_2} \ln(1 - (1 - \mu)v_1/a)$ . Then by (2.2) and (4.11),

$$\begin{aligned} & \ln \frac{a + x_i(p_k)}{a + \mu v_1} \leq \ln \frac{a + x_i(p_k)}{a + \mu_i v_1} \\ & \leq v_2 \int_{\xi_k}^{l_k} r_i(s) ds + (a + \mu_i v_1) \left[ \int_{l_k}^{p_k} r_i(s) ds \right. \\ & \quad \left. - \int_{l_k}^{p_k} r_i(t) \exp\left(-v_2 \int_{t-\tau_{ii}}^{\xi_k} r_i(s) ds\right) dt \right] \\ & \leq v_2 \int_{\xi_k}^{l_k} r_i(s) ds + (a + \mu_i v_1) \left\{ \int_{l_k}^{p_k} r_i(s) ds \right. \\ & \quad \left. - \frac{1}{(1 + \mu_i)v_1} \exp\left[-v_2 \left(A_i - \int_{\xi_k}^{p_k} r_i(s) ds\right)\right] \left[ 1 - \exp\left(-v_2 \int_{l_k}^{p_k} r_i(s) ds\right) \right] \right\} \\ & = v_2 \int_{\xi_k}^{l_k} r_i(s) ds + (a + \mu_i v_1) \\ & \quad \times \left\{ \int_{l_k}^{p_k} r_i(s) ds - \frac{1 - \mu}{a(1 + \mu_i)} \exp\left[-v_2 \left(A_i - \int_{\xi_k}^{p_k} r_i(s) ds\right)\right] \right\} \end{aligned}$$

$$\begin{aligned}
&\leq A_i v_2 + (a - v_1) \int_{I_k}^{p_k} r_i(s) ds - \frac{(1 - \mu)(a + \mu_i v_1)}{a(1 + \mu_i)} \\
&= A_i v_2 - \frac{1}{(1 + \mu_i) v_1} (a - v_1) \ln \left( 1 - \frac{(1 - \mu) v_1}{a} \right) - \frac{(1 - \mu)(a + \mu_i v_1)}{a(1 + \mu_i)} \\
&\leq A_i (1 + \mu_i) v_1 - \frac{1}{1 + \mu_i} \left[ -(1 - \mu) + \frac{1 - \mu^2}{2a} v_1 + \frac{(1 - \mu)^2 (1 + 2\mu)}{6a^2} v_1^2 \right] \\
&\quad - \frac{(1 - \mu)(a + \mu_i v_1)}{a(1 + \mu_i)} \\
&\leq \left[ A_i (1 + \mu_i) - \frac{(1 - \mu)(1 + \mu + 2\mu_i)}{2a(1 + \mu_i)} \right] v_1 - \frac{(1 - \mu)^2 (1 + 2\mu)}{6a^2 (1 + \mu_i)} v_1^2 \\
&\leq \frac{1 - \mu}{a} v_1 - \frac{(1 - \mu)^2 (1 + 2\mu)}{6a^2 (1 + \mu)} v_1^2.
\end{aligned}$$

Combining Cases 2.1 with 2.2, we have prove that

$$\ln \frac{a + x_i(p_k)}{a + \mu v_1} \leq \frac{1 - \mu}{a} v_1 - \frac{(1 - \mu)^2 (1 + 2\mu)}{6a^2 (1 + \mu)} v_1^2, \quad k = 1, 2, \dots$$

Letting  $k \rightarrow \infty$  and  $\varepsilon \rightarrow 0$ , we have

$$\ln \frac{a + U}{a + \mu V} \leq \frac{1 - \mu}{a} V - \frac{(1 - \mu)^2 (1 + 2\mu)}{6a^2 (1 + \mu)} V^2.$$

This shows that (4.6) holds. Next, we will prove that (4.7) holds as well. If  $V = 0$ , then (4.7) holds naturally. In what follows, we assume that  $V > 0$ . Then from (4.6), we have

$$U < a(1 + \mu)e^{1-\mu} - a < 2a, \quad \mu U < \mu[(a + \mu V)e^{(1-\mu)V/a} - a] < V < x_j^* \leq a. \quad (4.12)$$

Thus we may assume, without loss of generality, that  $V > \mu_j u_1$ . In view of this and (4.10), we can show that neither  $x_j(t) \geq -\mu_j u_1$  eventually nor  $x_j(t) \leq -\mu_j u_1$  eventually. Therefore,  $x_j(t)$  oscillates about  $-\mu_j u_1$ .

Let  $\{q_k\}$  be an increasing sequence such that  $q_k \geq T + \tau_{jj}$ ,  $\dot{x}_j(q_k) = 0$ ,  $x_j(q_k) \leq -\mu_j u_1$ ,  $\lim_{k \rightarrow \infty} q_k = \infty$  and  $\lim_{k \rightarrow \infty} x_j(q_k) = -V$ . By (4.10), there exists  $\eta_k \in [q_k - \tau_{jj}, q_k]$  such that  $x_j(\eta_k) = -\mu_j u_1$  and  $x_j(t) < -\mu_j u_1$  for  $\eta_k < t \leq q_k$ . For  $t \in [\eta_k, q_k]$ , by (4.10), we have

$$\begin{aligned}
x_j(t + s) &\leq (x_j^* - \mu_j u_1) \exp \left( u_2 \int_{t-\tau_{jj}}^{\eta_k} r_j(s) ds \right) - x_j^*, \\
\eta_k &\leq t \leq q_k, \quad -\tau_{jj} \leq s \leq 0.
\end{aligned}$$

Substituting this into the first inequality in (4.10), we obtain

$$-\frac{\dot{x}_j(t)}{x_j^* + x_j(t)} \leq (x_j^* - \mu_j u_1) r_j(t) \left[ \exp \left( u_2 \int_{t-\tau_{jj}}^{\eta_k} r_j(s) ds \right) - 1 \right], \quad \eta_k \leq t \leq q_k.$$

Combining this with (4.10), we have

$$-\frac{\dot{x}_j(t)}{a + x_j(t)} \leq \min \left\{ u_2 r_j(t), (a - \mu_j u_1) r_j(t) \left[ \exp \left( u_2 \int_{t-\tau_{jj}}^{\eta_k} r_j(s) ds \right) - 1 \right] \right\}, \quad \eta_k \leq t \leq q_k. \tag{4.13}$$

There are also two possibilities:

Case 2.3:  $\int_{\eta_k}^{q_k} r_j(s) ds \leq A_j - \frac{1}{u_2} \ln(1 + (1 - \mu)u_1/a)$ . It is easy to verify the inequality

$$(a + u_1) \ln \left( 1 + \frac{(1 - \mu)u_1}{a} \right) \geq (1 - \mu)u_1 + \frac{(1 - \mu)(1 + \mu)}{2a} u_1^2 - \frac{(1 - \mu)^2(1 + 2\mu)}{6a^2} u_1^3. \tag{4.14}$$

From (4.14) and use the fact that  $\mu u_1 < a$ , we have

$$\begin{aligned} & \ln \left( 1 + \frac{(1 - \mu)u_1}{a} \right) \\ & \geq \frac{1}{a + u_1} \left[ (1 - \mu)u_1 + \frac{(1 - \mu)(1 + \mu)}{2a} u_1^2 - \frac{(1 - \mu)^2(1 + 2\mu)}{6a^2} u_1^3 \right] \\ & \geq \frac{(1 - \mu)(1 + \mu + 2\mu_j)}{2a(1 + \mu_j)} u_1 - \frac{(1 - \mu)^2(1 + 2\mu)}{6a^2(1 + \mu_j)} u_1^2. \end{aligned}$$

Hence, integrating (4.13) from  $\eta_k$  to  $q_k$  and using the above inequality, we have

$$\begin{aligned} -\ln \frac{a + x_j(q_k)}{a - \mu_j u_1} & \leq u_2 \int_{\eta_k}^{q_k} r_j(s) ds \\ & \leq u_2 \left[ \frac{(1 - \mu)(3 + \mu + 4\mu_j)}{2a(1 + \mu_j)^2} - \frac{1}{u_2} \ln \left( 1 + \frac{(1 - \mu)u_1}{a} \right) \right] \\ & = \frac{(1 - \mu)(3 + \mu + 4\mu_j)}{2a(1 + \mu_j)} u_1 - \ln \left( 1 + \frac{(1 - \mu)u_1}{a} \right) \\ & \leq \frac{1 - \mu}{a} u_1 + \frac{(1 - \mu)^2(1 + 2\mu)}{6a^2(1 + \mu_j)} u_1^2. \end{aligned}$$

Case 2.4:  $\int_{\eta_k}^{q_k} r_j(s) ds > A_j - \frac{1}{u_2} \ln(1 + (1 - \mu)u_1/a)$ . Choose  $h_k \in (\eta_k, q_k)$  such that

$$\int_{\eta_k}^{h_k} r_j(s) ds = A_j - \frac{1}{u_2} \ln\left(1 + \frac{(1 - \mu)u_1}{a}\right).$$

Then by (2.2), (4.13) and (4.14) we have

$$\begin{aligned} & -\ln \frac{a + x_j(q_n)}{a - \mu_j u_1} \\ & \leq u_2 \int_{\eta_k}^{h_k} r_j(s) ds + (a - \mu_j u_1) \left[ \int_{h_k}^{q_k} r_j(t) \exp\left(u_2 \int_{t-\tau_{jj}}^{\eta_k} r_j(s) ds\right) dt - \int_{\eta_k}^{q_k} r_j(s) ds \right] \\ & \leq u_2 \int_{\eta_k}^{h_k} r_j(s) ds + (a - \mu_j u_1) \left\{ \int_{h_k}^{q_k} r_j(t) \exp\left[u_2 \left(A_j - \int_{\eta_k}^t r_j(s) ds\right)\right] dt \right. \\ & \quad \left. - \int_{\eta_k}^{q_k} r_j(s) ds \right\} \\ & = u_2 \int_{\eta_k}^{h_k} r_j(s) ds + (a - \mu_j u_1) \left\{ \frac{1}{u_2} \left[ \exp\left(u_2 \left(A_j - \int_{\eta_k}^{h_k} r_j(s) ds\right)\right) \right. \right. \\ & \quad \left. \left. - \exp\left(u_2 \left(A_j - \int_{\eta_k}^{q_k} r_j(s) ds\right)\right) \right] - \int_{h_k}^{q_k} r_j(s) ds \right\} \\ & = u_2 \int_{\eta_k}^{h_k} r_j(s) ds - (a - \mu_j u_1) \int_{h_k}^{q_k} r_j(s) ds \\ & \quad + \frac{a - \mu_j u_1}{(1 + \mu_j)u_1} \left\{ 1 + \frac{(1 - \mu)u_1}{a} - \exp\left[u_2 \left(A_j - \int_{\eta_k}^{q_k} r_j(s) ds\right)\right] \right\} \\ & \leq u_2 \int_{\eta_k}^{h_k} r_j(s) ds - (a - \mu_j u_1) \int_{h_k}^{q_k} r_j(s) ds \\ & \quad + \frac{a - \mu_j u_1}{(1 + \mu_j)u_1} \left[ \frac{(1 - \mu)u_1}{a} - (1 + \mu_j)u_1 \left(A_j - \int_{\eta_k}^{q_k} r_j(s) ds\right) \right] \\ & = (a + u_1) \int_{\eta_k}^{h_k} r_j(s) ds - A_j(a - \mu_j u_1) + \frac{(1 - \mu)(a - \mu_j u_1)}{a(1 + \mu_j)} \\ & = (1 + \mu_j)A_j u_1 - \frac{a + u_1}{(1 + \mu_j)u_1} \ln\left(1 + \frac{(1 - \mu)u_1}{a}\right) + \frac{(1 - \mu)(a - \mu_j u_1)}{a(1 + \mu_j)} \\ & \leq \left[ (1 + \mu_j)A_j - \frac{(1 - \mu)(1 + \mu + 2\mu_j)}{2a(1 + \mu_j)} \right] u_1 + \frac{(1 - \mu)(1 + 2\mu)}{6a^2(1 + \mu_j)} u_1^2 \\ & = \frac{1 - \mu}{a} u_1 + \frac{(1 - \mu)(1 + 2\mu)}{6a^2(1 + \mu_j)} u_1^2. \end{aligned}$$

Combining Case 2.3 with Case 2.4, we have shown that

$$-\ln \frac{a + x_j(q_k)}{a - \mu_j u_1} \leq \frac{1 - \mu}{a} u_1 + \frac{(1 - \mu)^2(1 + 2\mu)}{6a^2(1 + \mu_j)} u_1^2, \quad k = 1, 2, \dots$$

Letting  $k \rightarrow \infty$  and  $\varepsilon \rightarrow 0$ , we have

$$-\ln \frac{a - V}{a - \mu_j U} \leq \frac{1 - \mu}{a} U + \frac{(1 - \mu)^2(1 + 2\mu)}{6a^2(1 + \mu_j)} U^2.$$

Note that the fact that  $U < 2a$ , we have

$$\begin{aligned} (a - \mu_j U) \exp \left[ -\frac{1 - \mu}{a} U - \frac{(1 - \mu)^2(1 + 2\mu)}{6a^2(1 + \mu_j)} U^2 \right] \\ \geq (a - \mu U) \exp \left[ -\frac{1 - \mu}{a} U - \frac{(1 - \mu)^2(1 + 2\mu)}{6a^2(1 + \mu)} U^2 \right]. \end{aligned}$$

It follows that

$$a - V \geq (a - \mu U) \exp \left[ -\frac{1 - \mu}{a} U - \frac{(1 - \mu)^2(1 + 2\mu)}{6a^2(1 + \mu)} U^2 \right],$$

which implies that (4.7) holds. In view of Lemma 3.1, it follows from (4.6) and (4.7) that  $U = V = 0$ . Thus, (4.2) holds. The proof is complete.  $\square$

**Proof of Theorem 2.3.** Let  $(x_1(t), x_2(t), \dots, x_n(t))$  be any solution of (4.1) with  $x_i^* + x_i(t) > 0$  for  $t \geq 0$  and  $i = 1, 2, \dots, n$ . By Lemma 3.2, there exists  $T > 0$  such that

$$x_i^* + x_i(t) \leq \exp(d_i + e^{-d_i} - 1) \equiv D_i, \quad t \geq T, \quad i = 1, 2, \dots, n. \tag{4.15}$$

In view of the proof of Theorem 2.1, we only need to prove that the solution  $(x_1(t), x_2(t), \dots, x_n(t))$  satisfies (4.2). Let

$$x_i(t) = \delta_i y_i(t), \quad i = 1, 2, \dots, n.$$

Then (4.1) is transformed into

$$\begin{aligned} \dot{y}_i(t) = -r_i(t)(x_i^* + x_i(t)) \left[ \int_{-\tau_{ii}}^0 y_i(t+s) dv_{ii}(s) \right. \\ \left. + \delta_i^{-1} \sum_{j \neq i} \delta_j \mu_{ij} \int_{-\tau_{ij}}^0 y_j(t+s) dv_{ij}(s) \right], \quad i = 1, 2, \dots, n. \tag{4.16} \end{aligned}$$

Set

$$\mu_i = \delta_i^{-1} \sum_{j \neq i} \delta_j \mu_{ij}, \quad i = 1, 2, \dots, n \tag{4.17}$$

and

$$v_i = \limsup_{t \rightarrow \infty} |y_i(t)|, \quad i = 1, 2, \dots, n. \tag{4.18}$$

We shall show that  $v_1 = v_2 = \dots = v_n = 0$ . Without loss of generality, assume that  $v_1 = \max\{v_j: j = 1, 2, \dots, n\} > 0$ . Then there are two possible cases.

*Case 1:*  $\dot{y}_1(t)$  is sign-definite eventually. In this case, the limit  $c_1 \equiv \lim_{t \rightarrow \infty} y_1(t)$  exists and  $c_1 \geq -\delta_1^{-1} x_1^*$ . For the sake of simplicity, it is harmless assuming that  $\dot{y}_1(t) \geq 0, t \geq T$  or  $\dot{y}_1(t) \leq 0, t \geq T$ . If  $c_1 > -\delta_1^{-1} x_1^*$ , then, integrating (4.16) from  $T$  to  $\infty$ , we obtain

$$\begin{aligned} \infty &> \left| \ln \frac{x_1^* + \delta_1 y_1(T)}{x_1^* + c_1 \delta_1} \right| \\ &= \delta_1 \int_T^\infty r_1(t) \left| \int_{-\tau_{11}}^0 y_1(t+s) dv_{11}(s) \right. \\ &\quad \left. + \delta_1^{-1} \sum_{j \neq 1} \delta_j \mu_{1j} \int_{-\tau_{1j}}^0 y_j(t+s) dv_{1j}(s) \right| dt. \end{aligned} \tag{4.19}$$

Note that

$$\begin{aligned} &\liminf_{t \rightarrow \infty} \left| \int_{-\tau_{11}}^0 y_1(t+s) dv_{11}(s) + \delta_1^{-1} \sum_{j \neq 1} \delta_j \mu_{1j} \int_{-\tau_{1j}}^0 y_j(t+s) dv_{1j}(s) \right| \\ &\geq \liminf_{t \rightarrow \infty} \left[ \left| \int_{-\tau_{11}}^0 y_1(t+s) dv_{11}(s) \right| - \delta_1^{-1} \sum_{j \neq 1} \delta_j \mu_{1j} \int_{-\tau_{1j}}^0 |y_j(t+s)| dv_{1j}(s) \right] \\ &\geq \left[ v_1 - \delta_1^{-1} \sum_{j \neq 1} \delta_j \mu_{1j} v_j \right] \\ &\geq (1 - \mu_1) v_1. \end{aligned}$$

It follows from (2.1) and (4.19) that  $v_1 = 0$ , which is a contradiction. If  $c_1 = -\delta_1^{-1} x_1^*$ , then  $\dot{y}_1(t) \leq 0$  for  $t \geq T$  and  $v_1 = -c_1$ . By (DD2), there exists  $\varepsilon > 0$  such that  $(v_1 + \varepsilon)\mu_1 < v_1 - 2\varepsilon$ . For the given  $\varepsilon$ , by (4.18) and  $v_1 = \max\{v_j: j = 1, 2, \dots, n\} > 0$ , we can choose  $T_1 > T$  such that

$$y_1(t - \tau_{11}) < -v_1 + \varepsilon \quad \text{and} \quad y_j(t - \tau_{1j}) < v_1 + \varepsilon, \quad t \geq T_1. \tag{4.20}$$

Hence, from (4.16) and (4.20), we have

$$\begin{aligned} -\dot{y}_1(t) &= r_1(t)(x_1^* + x_1(t)) \left[ \int_{-\tau_{11}}^0 y_1(t+s) dv_{11}(s) \right. \\ &\quad \left. + \delta_1^{-1} \sum_{j \neq 1} \delta_j \mu_{1j} \int_{-\tau_{1j}}^0 y_j(t+s) dv_{1j}(s) \right] \\ &\leq r_1(t)(x_1^* + x_1(t))[-v_1 + \varepsilon + \mu_1(v_1 + \varepsilon)] \\ &\leq -\varepsilon r_1(t)(x_1^* + x_1(t)) \\ &< 0, \quad t \geq T_1. \end{aligned}$$

This contradicts to the fact that  $-\dot{y}_i(t) \geq 0$  for  $t \geq T$ .

Case 2:  $\dot{y}_1(t)$  is oscillatory. For any  $\varepsilon \in (0, (1 - \mu_1)v_1/(1 + \mu_1))$ , there exist  $T_2 > T + \max\{\tau_{ij}: i, j = 1, 2, \dots, n\}$  and a sequence  $\{t_k\}$  with  $t_k > T_2$  such that

$$t_k \rightarrow \infty, |y_1(t_k)| \rightarrow v_1 \text{ as } k \rightarrow \infty, |\dot{y}_1(t_k)| = 0, |y_1(t_k)| > v_1 - \varepsilon, \quad k = 1, 2, \dots$$

and

$$|y_j(t)| < v_1 + \varepsilon \quad \text{for } t \geq T_2 - \max\{\tau_{ij}: i, j = 1, 2, \dots, n\}, \quad j = 1, 2, \dots, n.$$

We only consider the case when  $|y_1(t_k)| = y_1(t_k)$  (the case when  $|y_1(t_k)| = -y_1(t_k)$  is similar by using  $-y_1(t)$  instead of  $y_1(t)$ ). Then from (4.16), we have

$$\begin{aligned} 0 &= - \int_{-\tau_{11}}^0 y_1(t_k + s) dv_{11}(s) - \delta_1^{-1} \sum_{j \neq 1} \delta_j \mu_{1j} \int_{-\tau_{1j}}^0 y_j(t_k + s) dv_{1j}(s) \\ &\leq - \int_{-\tau_{11}}^0 y_1(t_k + s) dv_{11}(s) + \mu_1(v_1 + \varepsilon) \end{aligned}$$

or

$$\int_{-\tau_{11}}^0 y_1(t_k + s) dv_{11}(s) \leq \mu_1(v_1 + \varepsilon),$$

which, together with the fact  $y_1(t_k) > \mu_1(v_1 + \varepsilon)$  implies that there exists a  $\zeta_k \in [t_k - \tau_{11}, t_k)$  such that  $y_1(\zeta_k) = \mu_1(v_1 + \varepsilon)$  and  $y_1(t) > \mu_1(v_1 + \varepsilon)$  for  $\zeta_k < t \leq t_k$ . Hence from (4.15) and (4.16), we have

$$\begin{aligned} \dot{y}_1(t) &\leq r_1(t)(x_1^* + x_1(t)) \left[ - \int_{-\tau_{11}}^0 y_1(t+s) dv_{11}(s) + \mu_1(v_1 + \varepsilon) \right] \\ &\leq r_1(t)D_1(1 + \mu_1)(v_1 + \varepsilon), \quad t \geq T_2. \end{aligned} \tag{4.21}$$



By (4.21) and the fact that  $y_1(t) > \mu_1(v_1 + \varepsilon)$  for  $\xi_k < t \leq t_k$ , we have

$$\begin{aligned} \mu_1(v_1 + \varepsilon) - y_1(t + s) &\leq D_1(1 + \mu_1)(v_1 + \varepsilon) \int_{t-\tau_{11}}^{\xi_k} r_1(u) \, du, \\ \xi_k &\leq t \leq t_k, \quad -\tau_{11} \leq s \leq 0. \end{aligned}$$

Substituting this into the first inequality in (4.21), we obtain

$$y_1(t) \leq D_1^2(1 + \mu_1)(v_1 + \varepsilon)r_1(t) \int_{t-\tau_{11}}^{\xi_k} r_1(s) \, ds, \quad \xi_k \leq t \leq t_k.$$

Combining this and (4.21), we have

$$y_1(t) \leq D_1(1 + \mu_1)(v_1 + \varepsilon)r_1(t) \min \left\{ 1, D_1 \int_{t-\tau_{11}}^{\xi_k} r_1(s) \, ds \right\}, \quad \xi_k \leq t \leq t_k. \tag{4.22}$$

Set

$$\theta = \begin{cases} \max\{d_1 D_1 - \frac{1}{2}, \frac{1}{2}\}(1 + \mu_1), & \mu_1 < \frac{1}{3}, \\ \frac{1}{2}(1 + \mu_1)(d_1 D_1)^2, & \mu_1 \geq \frac{1}{3}. \end{cases}$$

Then by (2.5)

$$\theta < 1 - \mu_1. \tag{4.23}$$

We will show that

$$y_1(t_k) - y_1(\xi_k) \leq \theta(v_1 + \varepsilon). \tag{4.24}$$

To this end, we consider the following three subcases:

Case 2.1:  $\mu_1 < 1/3$  and  $D_1 \int_{\xi_k}^{t_k} r_1(s) \, ds \leq 1$ . In this case, by (4.22) we have

$$\begin{aligned} &y_1(t_k) - y_1(\xi_k) \\ &\leq D_1^2(1 + \mu_1)(v_1 + \varepsilon) \int_{\xi_k}^{t_k} r_1(t) \int_{t-\tau_{11}}^{\xi_k} r_1(s) \, ds \, dt \end{aligned}$$

$$\begin{aligned}
 &\leq D_1^2(1 + \mu_1)(v_1 + \varepsilon) \int_{\check{\xi}_k}^{t_k} r_1(t) \left( d_1 - \int_{\check{\xi}_k}^t r_1(s) ds \right) dt \\
 &= (1 + \mu_1)(v_1 + \varepsilon) \left[ d_1 D_1^2 \int_{\check{\xi}_k}^{t_k} r_1(s) ds - \frac{1}{2} \left( D_1 \int_{\check{\xi}_k}^{t_k} r_1(s) ds \right)^2 \right] \\
 &\leq (1 + \mu_1)(v_1 + \varepsilon) \left( \max\{d_1 D_1, 1\} - \frac{1}{2} \right) \\
 &= (1 + \mu_1)(v_1 + \varepsilon) \max \left\{ d_1 D_1 - \frac{1}{2}, \frac{1}{2} \right\} \\
 &= \theta(v_1 + \varepsilon).
 \end{aligned}$$

Case 2.2:  $\mu_1 < 1/3$  and  $D_1 \int_{\check{\xi}_k}^{t_k} r_1(s) ds > 1$ . In this case, there exists  $\eta_k \in (\check{\xi}_k, t_k)$  such that  $D_1 \int_{\eta_k}^{t_k} r_1(s) ds = 1$ . Then by (4.22) we have

$$\begin{aligned}
 &y_1(t_k) - y_1(\check{\xi}_k) \\
 &\leq D_1(1 + \mu_1)(v_1 + \varepsilon) \left[ \int_{\check{\xi}_k}^{\eta_k} r_1(s) ds + D_1 \int_{\eta_k}^{t_k} r_1(t) \int_{t-\tau_{11}}^{\check{\xi}_k} r_1(s) ds dt \right] \\
 &= D_1^2(1 + \mu_1)(v_1 + \varepsilon) \left[ \int_{\eta_k}^{t_k} r_1(t) dt \int_{\check{\xi}_k}^{\eta_k} r_1(s) ds + \int_{\eta_k}^{t_k} r_1(t) \int_{t-\tau_{11}}^{\check{\xi}_k} r_1(s) ds dt \right] \\
 &= D_1^2(1 + \mu_1)(v_1 + \varepsilon) \int_{\eta_k}^{t_k} r_1(t) \int_{t-\tau_{11}}^{\eta_k} r_1(s) ds dt \\
 &\leq D_1^2(1 + \mu_1)(v_1 + \varepsilon) \int_{\eta_k}^{t_k} r_1(t) \left( d_1 - \int_{\eta_k}^t r_1(s) ds \right) dt \\
 &= (1 + \mu_1)(v_1 + \varepsilon) \left[ d_1 D_1^2 \int_{\eta_k}^{t_k} r_1(s) ds - \frac{1}{2} \left( D_1 \int_{\eta_k}^{t_k} r_1(s) ds \right)^2 \right] \\
 &= (1 + \mu_1)(v_1 + \varepsilon) \left( d_1 D_1 - \frac{1}{2} \right) \\
 &= \theta(v_1 + \varepsilon).
 \end{aligned}$$

Case 2.3:  $\mu_1 \geq 1/3$ . In this case,  $\int_{\check{\xi}_k}^{t_k} r_1(s) ds \leq d_1$  hence, by (4.22) we have

$$\begin{aligned}
 &y_1(t_k) - y_1(\check{\xi}_k) \\
 &\leq D_1^2(1 + \mu_1)(v_1 + \varepsilon) \int_{\check{\xi}_k}^{t_k} r_1(t) \int_{t-\tau_{11}}^{\check{\xi}_k} r_1(s) ds dt \\
 &\leq D_1^2(1 + \mu_1)(v_1 + \varepsilon) \int_{\check{\xi}_k}^{t_k} r_1(t) \left( d_1 - \int_{\check{\xi}_k}^t r_1(s) ds \right) dt
 \end{aligned}$$

$$\begin{aligned} &= (1 + \mu_1)(v_1 + \varepsilon) \left[ d_1 D_1^2 \int_{\xi_k}^{t_k} r_1(s) ds - \frac{1}{2} \left( D_1 \int_{\xi_k}^{t_k} r_1(s) ds \right)^2 \right] \\ &\leq \frac{1}{2} (1 + \mu_1)(v_1 + \varepsilon) (d_1 D_1)^2 \\ &= \theta(v_1 + \varepsilon). \end{aligned}$$

Cases 2.1–2.3 show (4.24) holds. Let  $\varepsilon \rightarrow 0$  in (4.24). Then we conclude that  $v_1 < v_1$ . This is also a contradiction. The proof is complete.  $\square$

**Proof of Theorem 2.4.** Let  $(x_1(t), x_2(t), \dots, x_n(t))$  be any solution of (4.1) with  $x_i^* + x_i(t) > 0$  for  $t \geq 0$ . By Lemma 3.2, there exists  $T > 0$  such that (4.15) holds for  $t \geq T$ . We will divide two cases to prove (4.2) holds.

*Case 1:*  $\int_{-\tau_{ii}}^0 x_i(t+s) dv_{ii}(s) + \sum_{j \neq i} \mu_{ij} \int_{-\tau_{ij}}^0 x_j(t+s) dv_{ij}(s)$ ,  $i = 1, 2, \dots, n$  are all non-oscillatory. In this case,  $\dot{x}_i(t)$ ,  $i = 1, 2, \dots, n$  are all sign-definite eventually which implies that  $x_i(t)$ ,  $i = 1, 2, \dots, n$  are monotonous eventually. For the sake of simplicity, it is harmless assuming that  $\dot{x}_i(t) \geq 0$ ,  $t \geq T$  or  $\dot{x}_i(t) \leq 0$ ,  $t \geq T$ . By Lemma 3.2, we have  $x_i(t) \rightarrow c_i$  as  $t \rightarrow \infty$  and  $x_i^* + c_i \geq 0$  for  $i = 1, 2, \dots, n$ . Set

$$I_1 = \{i: c_i > -x_i^*\} \quad \text{and} \quad I_2 = \{i: c_i = -x_i^*\}.$$

If  $c_i \in I_1$ , then by integrating (4.1) from  $T$  to  $\infty$ , we obtain

$$\begin{aligned} &\left| \ln \frac{x_i^* + x_i(T)}{x_i^* + c_i} \right| \\ &= \int_T^\infty r_i(t) \left| \int_{-\tau_{ii}}^0 x_i(t+s) dv_{ii}(s) + \sum_{j \neq i} \mu_{ij} \int_{-\tau_{ij}}^0 x_j(t+s) dv_{ij}(s) \right| dt \\ &\geq \int_T^\infty r_i(t) \left[ \left| \int_{-\tau_{ii}}^0 x_i(t+s) dv_{ii}(s) \right| - \sum_{j \neq i} \mu_{ij} \int_{-\tau_{ij}}^0 |x_j(t+s)| dv_{ij}(s) \right] dt. \end{aligned}$$

Note that

$$\begin{aligned} &\lim_{t \rightarrow \infty} \left[ \left| \int_{-\tau_{ii}}^0 x_i(t+s) dv_{ii}(s) \right| - \sum_{j \neq i} \mu_{ij} \int_{-\tau_{ij}}^0 |x_j(t+s)| dv_{ij}(s) \right] \\ &= |c_i| - \sum_{j \neq i} \mu_{ij} |c_j|. \end{aligned}$$

It follows from (2.1) that

$$|c_i| - \sum_{j \neq i} \mu_{ij} |c_j| \leq 0, \quad i \in I_1. \tag{4.25}$$

If  $c_i \in I_2$ , then  $\dot{x}_i(t) \leq 0$  for  $t \geq T$ . For any given  $\varepsilon > 0$ , there exists  $T_1 > T$  such that

$$x_j(t - \tau_{ij}) < c_j + \varepsilon, \quad t \geq T_1, \quad j = 1, 2, \dots, n. \tag{4.26}$$

Hence, from (4.1) and (4.26), we have

$$\begin{aligned} & -\dot{x}_i(t) \\ &= r_i(t)(x_i^* + x_i(t)) \left[ \int_{-\tau_{ii}}^0 x_i(t+s) dv_{ii}(s) + \sum_{j \neq i} \mu_{ij} \int_{-\tau_{ij}}^0 x_j(t+s) dv_{ij}(s) \right] \\ &\leq r_i(t)(x_i^* + x_i(t)) [c_i + \varepsilon + \sum_{j \neq i} \mu_{ij}(c_j + \varepsilon)], \quad t \geq T_1. \end{aligned}$$

It follows that

$$c_i + \varepsilon + \sum_{j \neq i} \mu_{ij}(c_j + \varepsilon) \geq 0, \quad i \in I_2$$

or

$$|c_i| - \sum_{j \neq i} \mu_{ij}|c_j| \leq \varepsilon \left( 1 + \sum_{j \neq i} \mu_{ij} \right), \quad i \in I_2.$$

Let  $\varepsilon \rightarrow 0$  in the above. Then we have

$$|c_i| - \sum_{j \neq i} \mu_{ij}|c_j| \leq 0, \quad i \in I_2. \tag{4.27}$$

Combining (4.25) and (4.27), we have

$$|c_i| + \sum_{j \neq i} b_{ij}|c_j| \leq 0, \quad i = 1, 2, \dots, n. \tag{4.28}$$

Since the matrix  $B$  is an  $M$ -matrix, hence, from (4.28), we easy conclude that  $c_1 = c_2 = \dots = c_n = 0$ .

Case 2: At least one of  $\int_{-\tau_{ii}}^0 x_i(t+s) dv_{ii}(s) + \sum_{j \neq i} \mu_{ij} \int_{-\tau_{ij}}^0 x_j(t+s) dv_{ij}(s)$ ,  $i = 1, 2, \dots, n$  is oscillatory. Set

$$U_i = \limsup_{t \rightarrow \infty} |x_i(t)|, \quad i = 1, 2, \dots, n.$$

By Lemma 3.2,  $0 \leq U_i < \infty$ ,  $i = 1, 2, \dots, n$ . It suffices to prove that  $U_1 = U_2 = \dots = U_n = 0$ . Without loss of generality, assume that  $\int_{-\tau_{ii}}^0 x_i(t+s) dv_{ii}(s) + \sum_{j \neq i} \mu_{ij} \int_{-\tau_{ij}}^0 x_j(t+s) dv_{ij}(s)$ ,  $i = 1, 2, \dots, k$  are oscillatory and  $\int_{-\tau_{ii}}^0 x_i(t+s) dv_{ii}(s) + \sum_{j \neq i} \mu_{ij} \int_{-\tau_{ij}}^0 x_j(t+s) dv_{ij}(s)$ ,  $i = k + 1, k + 2, \dots, n$  are non-oscillatory. Then it

follows from (4.1) that  $\dot{x}_i(t)$ ,  $i = 1, 2, \dots, k$  are oscillatory and

$$\dot{x}_i(t) \text{ is nonoscillatory and } \lim_{t \rightarrow \infty} |x_i(t)| = U_i, \quad i = k + 1, \dots, n. \tag{4.29}$$

Hence, for any given sufficiently small  $\varepsilon > 0$ , there exist  $k$  sequences  $\{t_{im}\}$ ,  $i = 1, 2, \dots, k$  with  $t_{im} - \tau_{ij} > T$  such that

$$\begin{cases} t_{im} \rightarrow \infty, & |x_i(t_{im})| \rightarrow U_i \text{ as } m \rightarrow \infty, & |x_i(t_{im})| > U_i - \varepsilon, \\ |\dot{x}_i(t_{im})| = 0, & |x_i(t)| < U_i + \varepsilon \text{ for } t \geq t_1, & i = 1, 2, \dots, k, \end{cases} \tag{4.30}$$

and  $|x_i(t)| < U_i + \varepsilon$  for  $t \geq t_1$ ,  $i = k + 1, \dots, n$ , where  $t_1 = \min\{t_{i1} : i = 1, 2, \dots, k\}$ . We can assume  $|x_i(t_{im})| = x_i(t_{im})$  (if necessary, we use  $-x_i(t)$  instead of  $x_i(t)$  and  $-\mu_{ij}$  instead of  $\mu_{ij}$  for  $j \neq i$ ). Then by (4.1), we have

$$0 = \int_{-\tau_{ii}}^0 x_i(t_{im} + s) dv_{ii}(s) + \sum_{j \neq i} \mu_{ij} \int_{-\tau_{ij}}^0 x_j(t_{im} + s) dv_{ij}(s),$$

which yields

$$\int_{-\tau_{ii}}^0 x_i(t_{im} + s) dv_{ii}(s) \leq \sum_{j \neq i} \mu_{ij} (U_j + \varepsilon), \quad i = 1, 2, \dots, k. \tag{4.31}$$

Set

$$\beta_i = \sum_{j \neq i} \mu_{ij} (U_j + \varepsilon), \quad i = 1, 2, \dots, k. \tag{4.32}$$

In what follows, we show that

$$\begin{aligned} & x_i(t_{im}) + \sum_{j \neq i} b_{ij} (U_j + \varepsilon) \\ & \leq \begin{cases} 2\varepsilon d_i^2 D_i^2 / (2 - d_i^2 D_i^2) & \text{if } d_i D_i < 1, \\ 2\varepsilon (2d_i D_i - 1) / (3 - 2d_i D_i) & \text{if } d_i D_i \geq 1, \end{cases} \quad i = 1, 2, \dots, k. \end{aligned} \tag{4.33}$$

If  $x_i(t_{im}) \leq \beta_i$ , then (4.33) obviously holds. If  $x_i(t_{im}) > \beta_i$ , then by (4.31) there exists  $\xi_{im} \in [t_{im} - \tau_{ii}, t_{im}]$  such that  $x_i(\xi_{im}) = \beta_i$  and  $x_i(t) > \beta_i$  for  $\xi_{im} < t \leq t_{im}$ . From (4.1) we have

$$\begin{aligned} \dot{x}_i(t) & \leq r_i(t) (x_i^* + x_i(t)) \left[ - \int_{-\tau_{ii}}^0 x_i(t + s) dv_{ii}(s) + \beta_i \right] \\ & \leq r_i(t) D_i [(U_i + \varepsilon) + \beta_i], \quad t \geq T_2 = t_1 + \tau, \end{aligned} \tag{4.34}$$

where  $\tau = \{\tau_{ij}: i, j = 1, 2, \dots, n\}$ . By (4.34) and the fact that  $x_i(t) > \beta_i$  for  $\xi_{im} < t \leq t_{im}$ , we have

$$\beta_i - x_i(t + s) < D_i[(U_i + \varepsilon) + \beta_i] \int_{t-\tau_{ii}}^{\xi_{im}} r_i(u) du, \quad \xi_{im} \leq t \leq t_{im}, \quad -\tau_{ii} \leq s \leq 0.$$

Substituting this into the first inequality in (4.34), we obtain

$$\dot{x}_i(t) \leq D_i^2[(U_i + \varepsilon) + \beta_i] r_i(t) \int_{t-\tau_{ii}}^{\xi_{im}} r_i(s) ds, \quad \xi_{im} \leq t \leq t_{im}.$$

Combining this and (4.34), we have

$$\dot{x}_i(t) \leq D_i[(U_i + \varepsilon) + \beta_i] r_i(t) \min \left\{ 1, D_i \int_{t-\tau_{ii}}^{\xi_{im}} r_i(s) ds \right\}, \quad \xi_{im} \leq t \leq t_{im}. \tag{4.35}$$

We consider the following three subcases:

Case 2.1:  $d_i D_i \leq 1$ . In this case, by (2.4) and (4.35) we have

$$\begin{aligned} & x_i(t_{im}) - x_i(\xi_{im}) \\ & \leq [(U_i + \varepsilon) + \beta_i] D_i^2 \int_{\xi_{im}}^{t_{im}} r_i(t) \int_{t-\tau_{ii}}^{\xi_{im}} r_i(s) ds dt \\ & = [(U_i + \varepsilon) + \beta_i] D_i^2 \int_{\xi_{im}}^{t_{im}} r_i(t) \left( \int_{t-\tau_{ii}}^t r_i(s) ds - \int_{\xi_{im}}^t r_i(s) ds \right) dt \\ & \leq [(U_i + \varepsilon) + \beta_i] D_i^2 \left[ d_i \int_{\xi_{im}}^{t_{im}} r_i(s) ds - \int_{\xi_{im}}^{t_{im}} r_i(t) \int_{\xi_{im}}^t r_i(s) ds dt \right] \\ & = [(U_i + \varepsilon) + \beta_i] D_i^2 \left[ d_i \int_{\xi_{im}}^{t_{im}} r_i(s) ds - \frac{1}{2} \left( \int_{\xi_{im}}^{t_{im}} r_i(s) ds \right)^2 \right] \\ & \leq \frac{1}{2} d_i^2 D_i^2 [(U_i + \varepsilon) + \beta_i] \\ & \leq \frac{1}{2} d_i^2 D_i^2 [x_i(t_{im}) + \beta_i + 2\varepsilon]. \end{aligned}$$

Case 2.2:  $d_i D_i > 1$  and  $D_i \int_{\xi_{im}}^{t_{im}} r_i(s) ds \leq 1$ . In this case, by (2.4) and (4.35) we have

$$\begin{aligned} & x_i(t_{im}) - x_i(\xi_{im}) \\ & \leq [(U_i + \varepsilon) + \beta_i] D_i^2 \int_{\xi_{im}}^{t_{im}} r_i(t) \int_{t-\tau_{ii}}^{\xi_{im}} r_i(s) ds dt \\ & \leq [(U_i + \varepsilon) + \beta_i] D_i^2 \left[ d_i \int_{\xi_{im}}^{t_{im}} r_i(s) ds - \frac{1}{2} \left( \int_{\xi_{im}}^{t_{im}} r_i(s) ds \right)^2 \right] \end{aligned}$$

$$\begin{aligned} &\leq \frac{1}{2}(2d_i D_i - 1)[(U_i + \varepsilon) + \beta_i] \\ &\leq \frac{1}{2}(2d_i D_i - 1)[x_i(t_{im}) + \beta_i + 2\varepsilon]. \end{aligned}$$

*Case 2.3:*  $d_i D_i > 1$  and  $D_i \int_{\xi_{im}}^{t_{im}} r_i(s) ds > 1$ . In this case, let  $\eta_{im} \in [\xi_{im}, t_{im}]$  such that  $D_i \int_{\eta_{im}}^{t_{im}} r_i(s) ds = 1$ . Then by (2.4) and (4.35) we have

$$\begin{aligned} &x_i(t_{im}) - x_i(\xi_{im}) \\ &\leq [(U_i + \varepsilon) + \beta_i] D_i \left[ \int_{\xi_{im}}^{\eta_{im}} r_i(s) ds + D_i \int_{\eta_{im}}^{t_{im}} r_i(t) \int_{t-\tau_{ii}}^{\xi_{im}} r_i(s) ds dt \right] \\ &= [(U_i + \varepsilon) + \beta_i] D_i \left[ \left( 1 - D_i \int_{\eta_{im}}^{t_{im}} r_i(s) ds \right) \int_{\xi_{im}}^{\eta_{im}} r_i(s) ds \right. \\ &\quad \left. + D_i \int_{\eta_{im}}^{t_{im}} r_i(t) \int_{t-\tau_{ii}}^{\eta_{im}} r_i(s) ds dt \right] \\ &= [(U_i + \varepsilon) + \beta_i] D_i^2 \int_{\eta_{im}}^{t_{im}} r_i(t) \left( \int_{t-\tau_{ii}}^t r_i(s) ds - \int_{\eta_{im}}^t r_i(s) ds \right) dt \\ &\leq [(U_i + \varepsilon) + \beta_i] D_i^2 \left[ d_i \int_{\eta_{im}}^{t_{im}} r_i(s) ds - \frac{1}{2} \left( \int_{\eta_{im}}^{t_{im}} r_i(s) ds \right)^2 \right] \\ &= \frac{1}{2}(2d_i D_i - 1)[(U_i + \varepsilon) + \beta_i] \\ &\leq \frac{1}{2}(2d_i D_i - 1)[x_i(t_{im}) + \beta_i + 2\varepsilon]. \end{aligned}$$

Combining Cases 2.1–2.3, we have for  $i = 1, 2, \dots, k$

$$x_i(t_{im}) \leq \frac{2 + d_i^2 D_i^2}{2 - d_i^2 D_i^2} \sum_{j \neq i} \mu_{ij}(U_j + \varepsilon) + \frac{2\varepsilon d_i^2 D_i^2}{2 - d_i^2 D_i^2} \quad \text{if } d_i D_i \leq 1$$

or

$$x_i(t_{im}) \leq \frac{1 + 2d_i D_i}{3 - 2d_i D_i} \sum_{j \neq i} \mu_{ij}(U_j + \varepsilon) + \frac{2\varepsilon(2d_i D_i - 1)}{3 - 2d_i D_i} \quad \text{if } d_i D_i \geq 1.$$

This shows (4.33) is true. Let  $m \rightarrow \infty$  and  $\varepsilon \rightarrow 0$  in (4.33), we obtain

$$U_i + \sum_{j \neq i} b_{ij} U_j \leq 0, \quad i = 1, 2, \dots, k. \tag{4.36}$$

On the other hand, from (4.1) and (4.29), we see that the limit  $c_i = \lim_{t \rightarrow \infty} x_i(t)$  exists and  $c_i \geq -x_i^*$  for  $i = k + 1, \dots, n$ . If  $c_i > -x_i^*$ , then for large  $T_3 > T$ , integrating (4.1) from  $T_3$  to  $\infty$ , we obtain

$$\begin{aligned} & \left| \ln \frac{x_i^* + x_i(T_3)}{x_i^* + c_i} \right| \\ &= \int_{T_3}^{\infty} r_i(t) \left| \int_{-\tau_{ii}}^0 x_i(t+s) dv_{ii}(s) + \sum_{j \neq i} \mu_{ij} \int_{-\tau_{ij}}^0 x_j(t+s) dv_{ij}(s) \right| dt \\ &\geq \int_{T_3}^{\infty} r_i(t) \left[ \left| \int_{-\tau_{ii}}^0 x_i(t+s) dv_{ii}(s) \right| - \sum_{j \neq i} \mu_{ij} \int_{-\tau_{ij}}^0 |x_j(t+s)| dv_{ij}(s) \right] dt. \end{aligned}$$

Note that

$$\begin{aligned} & \liminf_{t \rightarrow \infty} \left[ \left| \int_{-\tau_{ii}}^0 x_i(t+s) dv_{ii}(s) \right| - \sum_{j \neq i} \mu_{ij} \int_{-\tau_{ij}}^0 |x_j(t+s)| dv_{ij}(s) \right] \\ &\geq U_i - \sum_{j \neq i} \mu_{ij} U_j. \end{aligned}$$

It follows from (2.1) that

$$U_i - \sum_{j \neq i} \mu_{ij} U_j \leq 0. \tag{4.37}$$

If  $c_i = -x_i^*$ , then  $\dot{x}_i(t) \leq 0$  for  $t \geq T$  and  $c_i = -U_i$ . For any given  $\varepsilon > 0$ , there exists  $T_4 > T$  such that

$$x_i(t - \tau_{ii}) < -U_i + \varepsilon, \quad \text{and} \quad x_j(t - \tau_{ij}) < U_j + \varepsilon, \quad t \geq T_4. \tag{4.38}$$

Hence, from (4.1) and (4.38), we have

$$\begin{aligned} & -\dot{x}_i(t) \\ &= r_i(t)(x_i^* + x_i(t)) \left[ \int_{-\tau_{ii}}^0 x_i(t+s) dv_{ii}(s) + \sum_{j \neq i} \mu_{ij} \int_{-\tau_{ij}}^0 x_j(t+s) dv_{ij}(s) \right] \\ &\leq r_i(t)(x_i^* + x_i(t)) [-U_i + \varepsilon + \sum_{j \neq i} \mu_{ij}(U_j + \varepsilon)], \quad t \geq T_4. \end{aligned}$$

It follows that

$$-U_i + \varepsilon + \sum_{j \neq i} \mu_{ij}(U_j + \varepsilon) \geq 0$$



or

$$U_i - \sum_{j \neq i} \mu_{ij} U_j \leq \varepsilon \left( 1 + \sum_{j \neq i} \mu_{ij} \right).$$

Let  $\varepsilon \rightarrow 0$  in the above. Then we have

$$U_i - \sum_{j \neq i} \mu_{ij} U_j \leq 0. \quad (4.39)$$

Combining (4.37) and (4.39), we have

$$U_i - \sum_{j \neq i} \mu_{ij} U_j \leq 0, \quad i = k + 1, \dots, n$$

and so

$$U_i + \sum_{j \neq i} b_{ij} U_j \leq 0, \quad i = k + 1, \dots, n. \quad (4.40)$$

By (4.36) and (4.40), and by using the fact that  $B$  is an  $M$ -matrix, we have  $U_1 = U_2 = \dots = U_n = 0$ . Hence, the proof is complete.  $\square$

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