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3/2-type criteria for global attractivity of Lotka–Volterra competition system without instantaneous negative feedbacks ☆

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Abstract

This paper deals with a two-species Lotka–Volterra competition model with discrete delays but without instantaneous negative feedbacks. Motivated by Wright's $\frac{3}{2}$ global attractivity result for the delayed scalar logistic equation, we establish some new $\frac{3}{2}$-type criteria for global attractivity of the positive equilibrium of the system. These criteria provide convenient and better (than some existing) estimates for the diagonal delays.

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1. Introduction

Global attractivity of the positive equilibrium of the delayed Lotka–Volterra system has been one of the main concerns of many authors [3–8,10,11,13–26,28,29]. Most of the existing work consider the model

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assuming undelayed intraspecific competitions are present. In such cases, one can take advantage of the instantaneous negative feedbacks and some "diagonally dominant" conditions for the community matrix to construct appropriate Liapunov functionals or to apply comparison theorems, and the resulting criteria are usually independent of the delays in the delayed intraspecific and interspecific competitions. See, e.g., [4–8,10,11,13–20,26,28]. Typically, the positive equilibrium (if any) is a global attractor if the undelayed intraspecific competition dominate the total competition due to delayed intraspecific and interspecific competitions. For example, So and Hofbauer [11] considered the n-species Lotka–Volterra systems with discrete delays

$$\dot{x}_i(t) = x_i(t)\left(r_i + a_{ii}x_i(t) + \sum_{j\neq i}^{n} a_{ij}x_j(t-\tau_{ij})\right), \quad i = 1, \dots, n, \tag{1.1}$$

and established the following nice result.

Theorem 1.1. *Let A be the $n \times n$ community matrix of* (1.1), *i.e.*, $A = (a_{ij})$, *and suppose that there exists a positive equilibrium x^* for* (1.1). *Then x^* is globally asymptotically stable for* (1.1) (*for positive initial conditions*) *for all delays $\tau_{ij} \geqslant 0$ if and only if $a_{ii} < 0$ for $i = 1, \dots, n$,* $\det A \neq 0$ *and A is weakly diagonally dominant, meaning that all the principal minors of $-\hat{A}$ are non-negative, where $\hat{A} = (\hat{a}_{ij})$ with $\hat{a}_{ii} = a_{ii}$ and $\hat{a}_{ij} = |a_{ij}|$ for $i \neq j$.*

But, as pointed out by Kuang [15], in view of the fact that in real situations, instantaneous responses are rare, and thus, more realistic models should consist of delay differential equations without instantaneous negative feedbacks. For such models, detecting the global attractivity of the positive equilibrium becomes a much harder job, if not impossible. Naturally, one would expect and it is a common sense that if the delays in the intraspecific interactions are sufficiently small, then the positive equilibrium should remain globally attractive under the existing "diagonally dominant" condition. Some recent work (e.g. [6,10,15,17,18]) initiated valuable attempts in this direction, which confirm to some extent the above expectation or common sense. From the aforementioned work, it becomes interesting and important to establish better or even the best measurements or estimates for the "sufficient smallness" of the delays in the intraspecific reactions, and this constitutes the aim of this paper.

To be specific and to make statements easy, we consider the following two-species Lotka–Volterra competition system (normalized) with discrete delays:

$$\begin{aligned} \dot{x}_1(t) &= r_1x_1(t)[1 - x_1(t-\tau_{11}) - \mu_1x_2(t-\tau_{12})], \\ \dot{x}_2(t) &= r_2x_2(t)[1 - \mu_2x_1(t-\tau_{21}) - x_2(t-\tau_{22})], \end{aligned} \tag{1.2}$$

and the initial conditions

$$x_i(t) = \phi_i(t) \geqslant 0, \;\; t \in [\tau_i, 0], \;\; \phi_i(0) > 0, \;\; i = 1, 2, \tag{1.3}$$

where $r_i > 0, \mu_i \geqslant 0, \tau_i = \max \tau_{1i}, \tau_{2i}$ for $i = 1, 2$ and $\tau_{ij} \geqslant 0$ for $i, \; j = 1, 2$. It can be easily seen that the non-boundary equilibrium $x^* = (x_1^*, x_2^*)$ is given by

$$x_1^* = \frac{1 - \mu_1}{1 - \mu_1 \mu_2}, \quad x_2^* = \frac{1 - \mu_2}{1 - \mu_1 \mu_2}.$$

Both the positivity of x^* and the "diagonal dominant" condition for (1.2) in the sense of Theorem 1.1 can all be implied by the assumption

(DD) $\quad \mu_1 < 1 \quad$ and $\quad \mu_2 < 1$,

which will be assumed throughout this paper. If $\tau_{11}^2 + \tau_{22}^2 = 0$, Theorem 1.1 tells that (DD) also implies the global attractivity of x^*. When $\tau_{11}^2 + \tau_{22}^2 \neq 0$, Lu and Takeuchi [23] proved that the global attractivity of x^* remains if $r\tau$ ($r = \max\{r_1, r_2\}$ and $\tau = \max\{\tau_{11}, \tau_{22}\}$) is sufficiently small, but they did not give any estimates for the delays. Gopalsamy [6] (also see [7]) and He [10] obtained some criteria for more general systems, and applying these criteria to (1.2) gives some implicit forms for estimates of delays, but it is not trivial to verify these estimates. Kuang [15] also studied the global attractivity of the positive equilibrium of more general n-species Lotka–Volterra system without dominating instantaneous negative feedbacks. Applying one of the main results in [15] (Corollary 3.1) to system (1.2) results in the following convenient criterion.

Theorem 1.2. *Assume that* (DD) *holds. If*

$$r_i \tau_{ii} e^{r_i \tau_{ii}} < \frac{1 - \mu_i}{1 + \mu_i}, \quad i = 1, 2, \tag{1.4}$$

then, x^* *is globally attractive for* (1.2).

Note that system (1.2) is a result of the coupling of two basic delayed logistic equations for single species growth of the form

$$\begin{cases} \dot{x}(t) = r x(t)[1 - x(t - \tau)], \\ x(s) \geqslant 0 \text{ for } s \in [-\tau, 0], \;\; x(0) > 0. \end{cases} \tag{1.5}$$

For (1.5), Wright [32] proved that the positive equilibrium $x^* = 1$ is globally attractive when $r\tau \leqslant \frac{3}{2}$, which is the best result so far obtained for global attractivity of the positive equilibrium of (1.5). Since then, $\frac{3}{2}$-type stability results have been obtained for various scalar equations with delays, see e.g. [1,12,22,27,30,31,33–35]. But, to the best of the authors' knowledge, there is no similar result for system's cases. In this paper, we will employ some new approach (other than Liapunov functionals) to extend Wright's result to system (1.2). More precisely, we will prove the following three theorems.

Theorem 1.3. *Assume that* (DD) *holds. If*

$$r_i\tau_{ii} \leqslant \frac{3(1-\mu)}{2(1+\mu)}, \quad i = 1,2, \tag{1.6}$$

where $\mu = \max\{\mu_1, \mu_2\}$, *then the positive equilibrium* x^* *of* (1.2) *is a global attractor.*

Theorem 1.4. *Assume that* (DD) *holds. If*

$$r_i\tau_{ii}e^{r_i\tau_{ii}} < \begin{cases} \frac{3-\mu_i}{2(1+\mu_i)}, & \mu_i \leqslant \frac{1}{3}, \\ & \qquad i = 1,2, \\ \sqrt{\frac{2(1-\mu_i)}{1+\mu_i}}, & \mu_i > \frac{1}{3}, \end{cases} \tag{1.7}$$

then the positive equilibrium x^* *of* (1.2) *is a global attractor.*

Theorem 1.5. *Assume that* (DD) *holds. If there exists a positive constant* δ *such that*

$$\delta\mu_1 < 1 \text{ and } \delta^{-1}\mu_2 < 1, \tag{1.8}$$

$$r_1\tau_{11}e^{r_1\tau_{11}} < \begin{cases} \frac{3-\delta\mu_1}{2(1+\delta\mu_1)}, & \delta\mu_1 \leqslant \frac{1}{3}, \\ \sqrt{\frac{2(1-\delta\mu_1)}{1+\delta\mu_1}}, & \delta\mu_1 > \frac{1}{3} \end{cases} \tag{1.9}$$

and

$$r_2\tau_{22}e^{r_2\tau_{22}} < \begin{cases} \frac{3-\delta^{-1}\mu_2}{2(1+\delta^{-1}\mu_2)}, & \delta^{-1}\mu_2 \leqslant \frac{1}{3}, \\ \sqrt{\frac{2(1-\delta^{-1}\mu_2)}{1+\delta^{-1}\mu_2}}, & \delta^{-1}\mu_2 > \frac{1}{3}, \end{cases} \tag{1.10}$$

then the positive equilibrium x^* *of* (1.2) *is a global attractor.*

It is worth noting that Theorem 1.3 reproduces Wright's result when $\mu_i = 0,\ i = 1,2$, we also note that (1.6) gives *explicit* estimates for $\tau_{ii},\ i = 1,2$, and (1.7) *improves* (1.4) since when $\mu_i > \frac{1}{3}$,

$$\frac{1-\mu_i}{1+\mu_i} < \sqrt{\frac{1-\mu_i}{1+\mu_i}} < \sqrt{\frac{2(1-\mu_i)}{1+\mu_i}},$$

and when $\mu_i \leqslant \frac{1}{3}$,

$$\frac{3-\mu_i}{2(1+\mu_i)} = \frac{1}{2} + \frac{1-\mu_i}{1+\mu_i} > \frac{1-\mu_i}{1+\mu_i}.$$

The positive number δ in Theorem 1.5 is motivated by the work of Kuang [15], and it plays a role of balancing the estimates for τ_{11} and τ_{22}.

The remainder of the paper is organized as follows. In Section 2, we establish a preliminary lemma and state an a priori estimate result obtained

[23], which will be used in the proof of the main theorems. Section 3 is dedicated to the proofs of Theorems 1.3–1.5. Section 4 is for a discussion of some related topics.

2. Preliminary lemmas

Lemma 2.1. *Let* $0<a,b\leqslant 1, 0<\mu<1$. *The system of inequalities*

$$\begin{cases} y\leqslant (a+\mu x)\exp\left[(1-\mu)x-\frac{(1-\mu)^2}{6(1+\mu)}x^2\right]-a, \\ x\leqslant b-(b-\mu y)\exp\left[-(1-\mu)y-\frac{(1-\mu)^2}{6(1+\mu)}y^2\right] \end{cases} \tag{2.1}$$

has a unique solution: $(x,y)=(0,0)$ *in the region* $D=\{(x,y): 0\leqslant x<1,\ 0\leqslant y<b/\mu\}$.

Proof. Let

$$\varphi(x)=(1-\mu)x-\frac{(1-\mu)^2}{6(1+\mu)}x^2, \quad \psi(y)=(1-\mu)y+\frac{(1-\mu)^2}{6(1+\mu)}y^2.$$

Then (2.1) can be written as

$$\begin{cases} y\leqslant (a+\mu x)e^{\varphi(x)}-a, \\ x\leqslant b-(b-\mu y)e^{-\psi(y)}. \end{cases} \tag{2.2}$$

Assume that (2.2) has another solution in the region D besides $(0,0)$, say (x_0,y_0). Then $0<x_0<1$ and $0<y_0<b/\mu$. Define two curves Γ_1 and Γ_2 as follows:

$$\Gamma_1\colon\ y=(a+\mu x)e^{\varphi(x)}-a, \quad \Gamma_2\colon\ x=b-(b-\mu y)e^{-\psi(y)}. \tag{2.3}$$

By direct calculation, we have for curve Γ_1:

$$\left.\frac{dy}{dx}\right|_{(0,0)}=a+(1-a)\mu<1$$

and for curve Γ_2:

$$\left.\frac{dy}{dx}\right|_{(0,0)}=\frac{1}{b+(1-b)\mu}>1.$$

Hence Γ_2 lies above Γ_1 near (0,0). The existence of (x_0,y_0) implies that the curves Γ_1 and Γ_2 must intersect at a point in the region D besides $(0,0)$. Let (x_1,y_1) be the first such point, i.e. x_1 is smallest. Then the slope of Γ_1 at (x_1,y_1) is not less than the slope of Γ_2 at (x_1,y_1), i.e.

$$[\mu+(a+\mu x_1)\varphi'(x_1)]e^{\varphi(x_1)}\geqslant\frac{1}{\mu+(b-\mu y_1)\psi'(y_1)}e^{\psi(y_1)}$$

or

$$[\mu + (a + \mu x_1)\varphi'(x_1)][\mu + (b - \mu y_1)\psi'(y_1)] \geqslant e^{\psi(y_1) - \varphi(x_1)}. \tag{2.4}$$

From (2.3), we have

$$\begin{aligned} -\ln\left(1 - \frac{x_1}{b}\right) &= -\ln\left(1 - \frac{\mu y_1}{b}\right) + (1-\mu)y_1 + \frac{(1-\mu)^2}{6(1+\mu)} y_1^2 \\ &< \left(\frac{\mu}{b} y_1 + \frac{\mu^2}{2b^2} y_1^2 + \frac{\mu^3}{3b^3} y_1^3 + \cdots\right) + (1-\mu)y_1 + \frac{(1-\mu)^2}{6(1+\mu)} y_1^2 \\ &\leqslant \frac{1}{b} y_1 + \frac{1}{2b^2} y_1^2 + \frac{1}{3b^3} y_1^3 + \cdots \\ &= -\ln\left(1 - \frac{y_1}{b}\right). \end{aligned}$$

This implies that

$$x_1 < y_1. \tag{2.5}$$

Using (2.5), we derive that

$$\begin{aligned} &[\mu + (a + \mu x_1)\varphi'(x_1)][\mu + (b - \mu y_1)\psi'(y_1)] \\ &\quad \leqslant [\mu + (1 + \mu x_1)\varphi'(x_1)][\mu + (1 - \mu y_1)\psi'(y_1)] \\ &\quad = 1 + \left[\frac{(1-\mu)^2}{3(1+\mu)} - \mu(1-\mu)\right](y_1 - x_1) - \left[\frac{(1-\mu)^2}{3(1+\mu)} - \mu(1-\mu)\right]^2 x_1 y_1 \\ &\qquad - \frac{\mu(1-\mu)^2}{3(1+\mu)}(x_1^2 + y_1^2) + \frac{\mu(1-\mu)^3}{3(1+\mu)}\left[\frac{1-\mu}{3(1+\mu)} - \mu\right] x_1 y_1 (y_1 - x_1) \\ &\qquad + \frac{\mu^2(1-\mu)^4}{9(1+\mu)^2} x_1^2 y_1^2 \\ &\quad < 1 + (1-\mu)\left(\frac{1-\mu}{3(1+\mu)} - \mu\right)(y_1 - x_1) - \frac{\mu(1-\mu)^2}{3(1+\mu)}(x_1^2 + y_1^2) \\ &\qquad + \frac{\mu(1-\mu)^4}{9(1+\mu)^2} x_1 y_1 (y_1 - x_1) + \frac{\mu^2(1-\mu)^4}{9(1+\mu)^2} x_1^2 y_1^2 \\ &\quad < 1 + (1-\mu)\left(\frac{1-\mu}{3(1+\mu)} - \mu\right)(y_1 - x_1) \end{aligned}$$

and

$$\begin{aligned} e^{\psi(y_1) - \varphi(x_1)} &= \exp\left[(1-\mu)(y_1 - x_1) + \frac{(1-\mu)^2}{6(1+\mu)}(x_1^2 + y_1^2)\right] \\ &> 1 + (1-\mu)(y_1 - x_1) + \frac{(1-\mu)^2}{6(1+\mu)}(x_1^2 + y_1^2). \end{aligned}$$

It follows that

$$
\begin{aligned}
&e^{\psi(y_1)-\varphi(x_1)} - [\mu + (a+\mu x_1)\varphi'(x_1)][\mu + (b-\mu y_1)\psi'(y_1)] \\
&\quad > \left[1 + (1-\mu)(y_1 - x_1) + \frac{(1-\mu)^2}{6(1+\mu)}(x_1^2 + y_1^2)\right] \\
&\qquad - \left[1 + (1-\mu)\left(\frac{1-\mu}{3(1+\mu)} - \mu\right)(y_1 - x_1)\right] \\
&\quad = (1-\mu)\left[1 + \mu - \frac{1-\mu}{3(1+\mu)}\right](y_1 - x_1) + \frac{(1-\mu)^2}{6(1+\mu)}(x_1^2 + y_1^2) \\
&\quad > 0,
\end{aligned}
$$

which contradicts (2.4). The proof is complete. □

The next lemma is from [23].

Lemma 2.2. *Assume that* (DD) *holds, let* $(x_1(t), x_2(t))$ *be the solution of* (1.2) *and* (1.3). *Then we have eventually*

$$0 < M \leqslant x_i(t) \leqslant e^{r_i \tau_{ii}}, \quad i = 1, 2, \tag{2.6}$$

for some $M > 0$.

3. Proofs of the theorems

Proof of Theorem 1.3. By the transformation

$$\bar{x}_1 = x_1 - x_1^*, \quad \bar{x}_2 = x_2 - x_2^*,$$

Eq. (1.2) becomes

$$
\begin{aligned}
\dot{x}_1(t) &= -r_1(x_1^* + x_1(t))[x_1(t-\tau_{11}) + \mu_1 x_2(t-\tau_{12})], \\
\dot{x}_2(t) &= -r_2(x_2^* + x_2(t))[\mu_2 x_1(t-\tau_{21}) + x_2(t-\tau_{22})],
\end{aligned} \tag{3.1}
$$

here we used $x_i(t)$ instead of $\bar{x}_i(t)$ for $i = 1, 2$. Clearly, the global attractivity of x^* of system (1.2) is equivalent to that for (3.1),

$$\lim_{t\to\infty} (x_1(t), x_2(t)) = (0, 0) \tag{3.2}$$

for all $x(t) = (x_1(t), x_2(t)) > -x^*$ for $t \geqslant 0$. We will prove (3.2) in the following two cases:

Case 1: Both $x_1(t-\tau_{11}) + \mu_1 x_2(t-\tau_{12})$ and $\mu_2 x_1(t-\tau_{21}) + x_2(t-\tau_{22})$ are non-oscillatory. In this case, $\dot{x}_1(t)$ and $\dot{x}_2(t)$ are sign-definite eventually which implies that $x_1(t)$ and $x_2(t)$ are eventually monotone. By the boundedness of $(x_1(t), x_2(t))$ (Lemma 2.2), we have $x_i(t) \to c_i$ as $t \to \infty$ with $c_i > -x_i^*$, for $i = 1, 2$. On the other hand, using the boundedness of $x_1(t)$ and

$x_2(t)$, we can conclude from (3.1) that both $\dot{x}_1(t)$ and $\dot{x}_2(t)$ are bounded on $[0, \infty)$, which implies that $x_1(t)$ and $x_2(t)$ are uniformly continuous on $[0, \infty)$. It follows immediately that $\dot{x}_1(t)$ and $\dot{x}_2(t)$ are also uniformly continuous on $[0, \infty)$. Therefore, by Gopalsamy [7, Lemma 1.2.3], $\dot{x}_i(t) \to 0$ as $t \to \infty$ for $i = 1, 2$. Hence, we obtain

$$c_1 + \mu_1 c_2 = 0, \qquad \mu_2 c_1 + c_2 = 0,$$

which imply that $c_1 = c_2 = 0$, i.e. (3.2) holds.

Case 2: At least one of $x_1(t - \tau_{11}) + \mu_1 x_2(t - \tau_{12})$ and $\mu_2 x_1(t - \tau_{21}) + x_2(t - \tau_{22})$ is oscillatory, say, the former. Then there exist an infinity sequence $\{t_n\}$ such that

$$x_1(t_n - \tau_{11}) + \mu_1 x_2(t_n - \tau_{12}) = 0, \quad n = 1, 2, \ldots \ . \tag{3.3}$$

Set

$$V_i = \liminf_{t \to \infty} x_i(t) \text{ and } U_i = \limsup_{t \to \infty} x_i(t), \quad i = 1, 2.$$

In view of Lemma 2.2,

$$-x_i^* < V_i \leqslant U_i < \infty, \quad i = 1, 2. \tag{3.4}$$

Let

$$-V = \min\{V_1, V_2\} \quad \text{and} \quad U = \max\{U_1, U_2\}.$$

Then from (3.3) and (3.4), we have

$$0 \leqslant V < \max\{x_1^*, x_2^*\} < 1, \quad 0 \leqslant U < \infty. \tag{3.5}$$

In what follows, we show that V and U satisfy the inequalities

$$a + U \leqslant (a + \mu V) \exp\left[(1 - \mu)V - \frac{(1 - \mu)^2}{6(1 + \mu)} V^2\right] \tag{3.6}$$

and

$$b - V \geqslant (b - \mu U) \exp\left[-(1 - \mu)U - \frac{(1 - \mu)^2}{6(1 + \mu)} U^2\right], \tag{3.7}$$

where $a, b = x_1^*$ or x_2^*. Without loss of generality, we may assume that $U = U_1$ and $V = -V_2$. Then $V < x_2^*$. Let $\varepsilon > 0$ be sufficiently small such that $v_1 \equiv V + \varepsilon < \max\{x_1^*, x_2^*\}$. Choose $T > 0$ such that

$$-v_1 < x_i(t) < U + \varepsilon \equiv u_1, \quad t \geqslant T - \max\{t_{ij}\colon i,j = 1, 2\}, \quad i = 1, 2. \tag{3.8}$$

First, we prove that (3.6) holds. If $U \leqslant \mu V$, then (3.6) obviously holds. Therefore, we will prove (3.6) only in the case when $U > \mu V$. For the sake of simplicity, it is harmless assuming $U > \mu v_1$. Set $v_2 = (1 + \mu)v_1$ and $u_2 = (1 + \mu)u_1$. Then from the first equation in (3.1), we have

$$\frac{\dot{x}_1(t)}{a + x_1(t)} \leqslant r_1[-x_1(t - \tau_{11}) + \mu v_1] \leqslant r_1 v_2, \quad t \geqslant T \tag{3.9}$$

and

$$\frac{\dot{x}_2(t)}{b+x_2(t)} \geqslant r_2[-\mu u_1 - x_2(t-\tau_{22})] \geqslant -r_2 u_2, \quad t \geqslant T, \tag{3.10}$$

where $a = x_1^*, b = x_2^*$. Since $U > \mu v_1$, we cannot have $x_1(t) \leqslant \mu v_1$ eventually. On the other hand, if $x_1(t) \geqslant \mu v_1$ eventually, then it follows from the first inequality in (3.9) that $x_1(t)$ is non-increasing and $U = \lim_{t\to\infty} x_1(t) = \mu v_1$. This is also impossible. Therefore, it follows that $x_1(t)$ oscillates about μv_1.

Let $\{p_n\}$ be an increasing sequence such that $p_n \geqslant T + \tau_{11}, \dot{x}_1(p_n) = 0, x_1(p_n) \geqslant \mu v_1, \lim_{n\to\infty} p_n = \infty$ and $\lim_{n\to\infty} x_1(p_n) = U$. By (3.9), $x_1(p_n - \tau_{11}) \leqslant \mu v_1$. Thus, there exists $\xi_n \in [p_n - \tau_{11}, p_n]$ such that $x_1(\xi_n) = \mu v_1$. For $t \in [\xi_n, p_n]$, integrating (3.9) from $t - \tau_{11}$ to ξ_n we get

$$-\ln \frac{a + x_1(t-\tau_{11})}{a + x_1(\xi_n)} \leqslant r_1 v_2(\xi_n + \tau_{11} - t),$$

or

$$x_1(t-\tau_{11}) \geqslant -a + (a + \mu v_1)\exp[-r_1 v_2(\xi_n + \tau_{11} - t)], \ \ \xi_n \leqslant t \leqslant p_n.$$

Substituting this into the first inequality in (3.9), we obtain

$$\frac{\dot{x}_1(t)}{a+x_1(t)} \leqslant r_1(a + \mu v_1)\{1 - \exp[-r_1 v_2(\xi_n + \tau_{11} - t)]\}, \ \ \xi_n \leqslant t \leqslant p_n.$$

Combining this with (3.9), we have

$$\frac{\dot{x}_1(t)}{a+x_1(t)} \leqslant \min\{r_1 v_2, r_1(1 + \mu v_1)\{1 - \exp[-r_1 v_2(\xi_n + \tau_{11} - t)]\}\}, \\ \xi_n \leqslant t \leqslant p_n. \tag{3.11}$$

To prove (3.6), we consider the following two possible subcases.

Case 2.1: $r_1(p_n - \xi_n) \leqslant -\frac{1}{v_2}\ln[1 - (1-\mu)v_1]$. Then by (1.6) and (3.11)

$$\begin{aligned}
\ln \frac{a + x(p_n)}{a + \mu v_1}
&\leqslant r_1(1 + \mu v_1)(p_n - \xi_n) - r_1(1 + \mu v_1)\int_{\xi_n}^{p_n} \exp[-r_1 v_2(\xi_n + \tau_{11} - t)]dt \\
&= (1 + \mu v_1)\left\{ r_1(p_n - \xi_n) - \frac{1}{v_2}\exp[-r_1 v_2(\xi_n + \tau_{11} - p_n)] \right. \\
&\quad \left. \times [1 - \exp(-r_1 v_2(p_n - \xi_n))] \right\} \\
&\leqslant (1 + \mu v_1)\left\{ r_1(p_n - \xi_n) - \frac{1 - \exp(-r_1 v_2(p_n - \xi_n))}{v_2} \right. \\
&\quad \left. \times \exp\left[-v_2\left(\frac{3(1-\mu)}{2(1+\mu)} - r_1(p_n - \xi_n)\right)\right]\right\}.
\end{aligned}$$

If $r_1(p_n - \xi_n) \leqslant -\frac{1}{v_2}\ln[1-(1-\mu)v_1] \leqslant 3(1-\mu)/2(1+\mu)$, then

$$\begin{aligned}
\ln\frac{a+x(p_n)}{a+\mu v_1}
&\leqslant (1+\mu v_1)\left\{-\frac{1}{v_2}\ln[1-(1-\mu)v_1] - \frac{1-\mu}{1+\mu}\right.\\
&\quad \left.\times \exp\left[-v_2\left(\frac{3(1-\mu)}{2(1+\mu)} + \frac{\ln[1-(1-\mu)v_1]}{v_2}\right)\right]\right\}\\
&\leqslant (1+\mu v_1)\left\{-\frac{1}{v_2}\ln[1-(1-\mu)v_1]\right.\\
&\quad \left.-\frac{1-\mu}{1+\mu}\left[1 - v_2\left(\frac{3(1-\mu)}{2(1+\mu)} + \frac{\ln[1-(1-\mu)v_1]}{v_2}\right)\right]\right\}\\
&= \frac{1+\mu v_1}{1+\mu}\left\{-\frac{1}{v_1}\ln[1-(1-\mu)v_1] - (1-\mu)\right.\\
&\quad \left.\times\left[1 - \frac{3(1-\mu)}{2}v_1 - \ln[1-(1-\mu)v_1]\right]\right\}\\
&= \frac{1+\mu v_1}{1+\mu}\left\{-\frac{1}{v_1}[1-(1-\mu)v_1]\ln[1-(1-\mu)v_1] - (1-\mu) + \frac{3(1-\mu)^2}{2}v_1\right\}\\
&\leqslant \frac{1+\mu v_1}{1+\mu}\left[(1-\mu)^2 v_1 - \frac{(1-\mu)^3}{6}v_1^2\right]\\
&< (1-\mu)v_1 - \frac{(1-\mu)^2}{6(1+\mu)}v_1^2.
\end{aligned}$$

In the above third inequality, we have used the following inequality:

$$\begin{aligned}
&[1-(1-\mu)v_1]\ln[1-(1-\mu)v_1]\\
&\quad \geqslant -(1-\mu)v_1 + \frac{(1-\mu)^2}{2}v_1^2 + \frac{(1-\mu)^3}{6}v_1^3.
\end{aligned} \tag{3.12}$$

If $r_1(p_n - \xi_n) \leqslant 3(1-\mu)/2(1+\mu) \leqslant -\frac{1}{v_2}\ln[1-(1-\mu)v_1]$, then

$$\begin{aligned}
\frac{3}{2}(1-\mu) &\leqslant -\frac{1}{v_1}\ln[1-(1-\mu)v_1]\\
&\leqslant \frac{1-\mu}{1-(1-\mu)v_1}\left[1 - \frac{1-\mu}{2}v_1 - \frac{(1-\mu)^2}{6}v_1^2\right],
\end{aligned}$$

which implies that $(1-\mu)v_1 > 1/2$. Hence,

$$\begin{aligned}
&\ln\frac{1+x(p_n)}{1+\mu v_1}\\
&\leqslant (1+\mu v_1)\left\{\frac{3(1-\mu)}{2(1+\mu)} - \frac{1}{v_2}\left[1-\exp\left(-\frac{3}{2}(1-\mu)v_1\right)\right]\right\}\\
&= \frac{1+\mu v_1}{1+\mu}\left[\frac{3}{2}(1-\mu) - \frac{1}{v_1}\left(1-e^{-3(1-\mu)v_1/2}\right)\right]\\
&\leqslant \frac{1+\mu v_1}{1+\mu}\left\{\frac{3}{2}(1-\mu) - \left[\frac{3}{2}(1-\mu) - \frac{9}{8}(1-\mu)^2 v_1 + \frac{9}{16}(1-\mu)^3 v_1^2\right.\right.\\
&\quad \left.\left. - \frac{27}{128}(1-\mu)^4 v_1^3\right]\right\}\\
&= \frac{(1-\mu)(1+\mu v_1)}{1+\mu}\left[\frac{9}{8}(1-\mu)v_1 - \frac{9}{16}(1-\mu)^2 v_1^2 + \frac{27}{128}(1-\mu)^3 v_1^3\right]\\
&\leqslant \frac{(1-\mu)(1+\mu v_1)}{1+\mu}\left[(1-\mu)v_1 - \frac{1}{6}(1-\mu)^2 v_1^2\right]\\
&< (1-\mu)v_1 - \frac{(1-\mu)^2}{6(1+\mu)}v_1^2.
\end{aligned}$$

Case 2.2: $-\frac{1}{v_2}\ln[1-(1-\mu)v_1] < r_1(p_n-\xi_n) \leqslant 3(1-\mu)/2(1+\mu)$. Choose $l_n \in (\xi_n, p_n)$ such that $r_1(p_n-l_n) = -\frac{1}{v_2}\ln[1-(1-\mu)v_1]$. Then by (1.6) and (3.11),

$$\begin{aligned}
&\ln\frac{1+x(p_n)}{1+\mu v_1}\\
&\leqslant r_1 v_2(l_n-\xi_n) + (1+\mu v_1)\left\{r_1(p_n-l_n) - r_1\int_{l_n}^{p_n}\right.\\
&\quad \left.\times\exp[-r_1 v_2(\xi_n+\tau_{11}-t)]dt\right\}\\
&= r_1 v_2(l_n-\xi_n) + (1+\mu v_1)\\
&\quad \times\left\{r_1(p_n-l_n) - \frac{1}{v_2}\exp[-r_1 v_2(\xi_n+\tau_{11}-p_n)][1-\exp(-r_1 v_2(p_n-l_n))]\right\}\\
&= r_1 v_2(l_n-\xi_n) + (1+\mu v_1)\\
&\quad \times\left\{r_1(p_n-l_n) - \frac{1-\mu}{1+\mu}\exp[-r_1 v_2(\xi_n+\tau_{11}-p_n)]\right\}\\
&\leqslant r_1 v_2(l_n-\xi_n) + (1+\mu v_1)\\
&\quad \times\left\{r_1(p_n-l_n) - \frac{1-\mu}{1+\mu} + \frac{1-\mu}{1+\mu}r_1 v_2(\xi_n+\tau_{11}-p_n)\right\}\\
&\leqslant r_1\tau_{11}v_2 + (1-v_1)r_1(p_n-l_n) - \frac{1-\mu}{1+\mu}
\end{aligned}$$

$$= r_1\tau_{11}v_2 - \frac{1}{v_2}(1-v_1)\ln[1-(1-\mu)v_1] - \frac{1-\mu}{1+\mu}$$
$$\leqslant \frac{3}{2}(1-\mu)v_1 - \frac{1}{1+\mu}\left[-(1-\mu) + \frac{(1-\mu)(1+\mu)}{2}v_1 + \frac{(1-\mu)^2(1+2\mu)}{6}v_1^2\right] - \frac{1-\mu}{1+\mu}$$
$$= (1-\mu)v_1 - \frac{(1-\mu)^2(1+2\mu)}{6(1+\mu)}v_1^2$$
$$< (1-\mu)v_1 - \frac{(1-\mu)^2}{6(1+\mu)}v_1^2.$$

In the above fourth inequality, we have used the following inequality:

$$(1-v_1)\ln[1-(1-\mu)v_1] \geqslant -(1-\mu)v_1 + \frac{(1-\mu)(1+\mu)}{2}v_1^2 + \frac{(1-\mu)^2(1+2\mu)}{6}v_1^3.$$

On combining Cases 2.1 and 2.2, we have

$$\ln\frac{a+x(p_n)}{a+\mu v_1} \leqslant (1-\mu)v_1 - \frac{(1-\mu)^2}{6(1+\mu)}v_1^2, \quad n = 1, 2, \ldots\ .$$

Letting $n\to\infty$ and $\varepsilon\to 0$, we have

$$\ln\frac{a+U}{a+\mu V} \leqslant (1-\mu)V - \frac{(1-\mu)^2}{6(1+\mu)}V^2.$$

This shows that (3.6) holds. Next, we will prove that (3.7) holds as well. If $V = 0$, then it follows from (3.6) that $U = 0$. Hence, the proof is complete. In what follows, we assume that $V > 0$. Then from (3.6), we have

$$U < (a+\mu)e^{1-\mu} - a < 2, \quad \mu U < \mu[(1+\mu V)e^{(1-\mu)V} - 1] < V < b. \quad (3.13)$$

Thus we may assume, without loss of generality, that $V > \mu u_1$. In view of this and (3.10), we can show that neither $x_2(t) \geqslant -\mu u_1$ eventually nor $x_2(t) \leqslant -\mu u_1$ eventually. Therefore, $x_2(t)$ oscillates about $-\mu u_1$.

Let $\{q_n\}$ be an increasing sequence such that $q_n \geqslant T + \tau_{22}, \dot{x}_2(q_n) = 0, x_2(q_n) \leqslant -\mu u_1, \lim_{n\to\infty} q_n = \infty$ and $\lim_{n\to\infty} x_2(q_n) = -V$. By (3.10), $x_2(q_n - \tau_{22}) \geqslant -\mu u_1$. Thus, there exists $\eta_n \in [q_n - \tau_{22}, q_n]$ such that $x_2(\eta_n) = -\mu u_1$. For $t \in [\eta_n, q_n]$, integrating (3.10) from $t - \tau_{22}$ to η_n, we have

$$x_2(t-\tau_{22}) \leqslant (b-\mu u_1)\exp[r_2 u_2(\eta_n + \tau_{22} - t)] - b, \quad \eta_n \leqslant t \leqslant q_n.$$

Substituting this into the first inequality in (3.10), we obtain

$$-\frac{\dot{x}_2(t)}{b+x_2(t)} \leqslant r_2(b-\mu u_1)\{\exp[r_2 u_2(\eta_n + \tau_{22} - t)] - 1\}, \quad \eta_n \leqslant t \leqslant q_n.$$

Combining this with (3.10), we have

$$-\frac{\dot{x}_2(t)}{b+x_2(t)}\leqslant \min\{r_2u_2, r_2(1-\mu u_1)\{\exp[r_2u_2(\eta_n+\tau_{22}-t)]-1\}\},$$
$$\eta_n\leqslant t\leqslant q_n. \tag{3.14}$$

There are also two possibilities:

Case 2.3: $r_2(q_n-\eta_n)\leqslant\frac{3(1-\mu)}{2(1+\mu)}-\frac{1}{u_2}\ln[1+(1-\mu)u_1]$. Integrating (3.14) from η_n to q_n and using the inequality

$$\ln[1+(1-\mu)u_1]\geqslant\frac{1}{2}(1-\mu)u_1-\frac{(1-\mu)^2}{6(1+\mu)}u_1^2,$$

we have

$$\begin{aligned}
-\ln\frac{b+x_2(q_n)}{b-\mu u_1}&\leqslant r_2u_2(q_n-\eta_n)\\
&\leqslant u_2\left\{\frac{3(1-\mu)}{2(1+\mu)}-\frac{1}{u_2}\ln[1+(1-\mu)u_1]\right\}\\
&=\frac{3}{2}(1-\mu)u_1-\ln[1+(1-\mu)u_1]\\
&\leqslant (1-\mu)u_1+\frac{(1-\mu)^2}{6(1+\mu)}u_1^2.
\end{aligned}$$

Case 2.4: $r_2(q_n-\eta_n)>\frac{3(1-\mu)}{2(1+\mu)}-\frac{1}{u_2}\ln[1+(1-\mu)u_1]$. Choose $h_n\in(\eta_n,q_n)$ such that

$$r_2(h_n-\eta_n)=\frac{3(1-\mu)}{2(1+\mu)}-\frac{1}{u_2}\ln[1+(1-\mu)u_1].$$

Then by (1.6) and (3.14) we have

$$\begin{aligned}
&-\ln\frac{b+x(q_n)}{b-\mu u_1}\\
&\leqslant r_2u_2(h_n-\eta_n)+(1-\mu u_1)\\
&\quad\times\left\{r_2\int_{h_n}^{q_n}\exp[r_2u_2(\eta_n+\tau_{22}-t)]dt-r_2(q_n-h_n)\right\}\\
&= r_2u_2(h_n-\eta_n)+(1-\mu u_1)\\
&\quad\times\left\{\frac{1}{u_2}[\exp(r_2u_2(\eta_n+\tau_{22}-h_n))\right.\\
&\quad\left.-\exp(r_2u_2(\eta_n+\tau_{22}-q_n))]-r_2(q_n-h_n)\right\}\\
&= r_2u_2(h_n-\eta_n)-r_2(1-\mu u_1)(q_n-h_n)
\end{aligned}$$

$$\begin{aligned}
&+\frac{1-\mu u_1}{u_2}e^{r_2\tau_{22}u_2}\left\{[1+(1-\mu)u_1]\exp\left(-\frac{3(1-\mu)}{2(1+\mu)}u_2\right)-e^{-r_2u_2(q_n-\eta_n)}\right\}\\
&\leqslant r_2u_2(h_n-\eta_n)-r_2(1-\mu u_1)(q_n-h_n)\\
&\quad+\frac{1-\mu u_1}{u_2}\left\{1+(1-\mu)u_1-\exp\left[u_2\left(\frac{3(1-\mu)}{2(1+\mu)}-r_2(q_n-\eta_n)\right)\right]\right\}\\
&\leqslant r_2u_2(h_n-\eta_n)-r_2(1-\mu u_1)(q_n-h_n)\\
&\quad+\frac{1-\mu u_1}{u_2}\left\{(1-\mu)u_1-(1+\mu)u_1\left[\frac{3(1-\mu)}{2(1+\mu)}-r_2(q_n-\eta_n)\right]\right\}\\
&=(1+u_1)r_2(h_n-\eta_n)-\frac{1-\mu}{2(1+\mu)}(1-\mu u_1)\\
&=\frac{3(1-\mu)}{2(1+\mu)}(1+u_1)-\frac{(1+u_1)\ln[1+(1-\mu)u_1]}{(1+\mu)u_1}-\frac{1-\mu}{2(1+\mu)}(1-\mu u_1)\\
&\leqslant\frac{1-\mu}{1+\mu}u_1+\frac{(1-\mu)^2(1+2\mu)}{6(1+\mu)}u_1^2\\
&\leqslant(1-\mu)u_1+\frac{(1-\mu)^2}{6(1+\mu)}u_1^2.
\end{aligned}$$

In the above fourth inequality, we have used the following inequality:

$$\begin{aligned}
&(1+u_1)\ln[1+(1-\mu)u_1]\\
&\quad\geqslant(1-\mu)u_1+\frac{(1-\mu)(1+\mu)}{2}u_1^2-\frac{(1-\mu)^2(1+2\mu)}{6}u_1^3.
\end{aligned}$$

On combining Cases 2.3 and 2.4, we have

$$-\ln\frac{b+x(q_n)}{b-\mu u_1}\leqslant(1-\mu)u_1+\frac{(1-\mu)^2}{6(1+\mu)}u_1^2,\quad n=1,2,\ldots\ .$$

Letting $n\to\infty$ and $\varepsilon\to 0$, we have

$$-\ln\frac{b-V}{b-\mu U}\leqslant(1-\mu)U+\frac{(1-\mu)^2}{6(1+\mu)}U^2,$$

which implies that (3.7) holds. In view of Lemma 2.1, it follows from (3.6) and (3.7) that $U=V=0$. Thus, $\lim_{t\to\infty}(x_1(t),x_2(t))=(0,0)$. The proof is complete. □

Proof of Theorem 1.4. Let $(x_1(t),x_2(t))$ be any solution of (3.1) with $x_i^*+x_i(t)>0$ for $t\geqslant 0$ and $i=1,2$. By Lemma 2.2, there exists $T>0$ such that

$$x_i^*+x_i(t)\leqslant e^{r_i\tau_{ii}},\quad t\geqslant T,\ \ i=1,2. \tag{3.15}$$

In view of the proof of Theorem 1.3, we only need to prove that the solution $(x_1(t),x_2(t))$ satisfies (3.2). To this end, we consider the following two possible cases.

Case 1: At least one of $x_1(t-\tau_{11})+\mu_1x_2(t-\tau_{12})$ and $\mu_2x_1(t-\tau_{21})+x_2(t-\tau_{22})$ is non-oscillatory, say, the former. Then $x_1(t-\tau_{11})+\mu_1x_2(t-$

$\tau_{12}) > 0$ (or <0) for sufficiently t, which implies that $\dot{x}_1(t)$ is monotonous eventually. By the boundedness of $x_1(t)$ (Lemma 2.2), we have $x_1(t) \to c_1$ as $t \to \infty$. On the other hand, using the boundedness of $x_1(t)$ and $x_2(t)$, we can conclude from (3.1) that both $\dot{x}_1(t)$ and $\dot{x}_2(t)$ are bounded on $[0, \infty)$, which implies that $x_1(t)$ and $x_2(t)$ are uniformly continuous on $[0, \infty)$. It follows immediately that $\dot{x}_1(t)$ and $\dot{x}_2(t)$ are also uniformly continuous on $[0, \infty)$. Therefore, $\dot{x}_1(t) \to 0$ as $t \to \infty$. By the first equation in (3.1), we have $x_2(t) \to c_2$ and so $\dot{x}_2(t) \to 0$ as $t \to \infty$. Hence, we obtain

$$c_1 + \mu_1 c_2 = 0 \quad \text{and} \quad \mu_2 c_1 + c_2 = 0.$$

It follows from (DD) that $c_1 = c_2 = 0$.

Case 2: Both $x_1(t - \tau_{11}) + \mu_1 x_2(t - \tau_{12})$ and $\mu_2 x_1(t - \tau_{21}) + x_2(t - \tau_{22})$ are oscillatory. Set $v_i = \limsup_{t \to \infty} |x_i(t)|, i = 1, 2$. It suffices to prove that $v_1 = v_2 = 0$. Without loss of generality, assume that $v_1 = \max\{v_1, v_2\} > 0$. Then for any $\varepsilon \in (0, (1 - \mu_1)v_1/(1 + \mu_1))$, there exist $T_1 > T + \tau_{11} + \tau_{12} + \tau_{21} + \tau_{22}$ and a sequence $\{t_n\}$ with $t_n > T_1$ such that

$$t_n \to \infty, \ |x_1(t_n)| \to v_1 \text{ as } n \to \infty, \ |x_1'(t_n)| = 0, \ |x_1(t_n)| > v_1 - \varepsilon, \quad n = 1, 2, \ldots$$

and

$$|x_1(t)| < v_1 + \varepsilon, \quad |x_2(t)| < v_1 + \varepsilon \text{ for } t \geqslant T_1 - (\tau_{11} + \tau_{12} + \tau_{21} + \tau_{22}).$$

We only consider the case when $|x_1(t_n)| = x_1(t_n)$ (the case when $|x_1(t_n)| = -x_1(t_n)$ is similar). Then from the first equation in (3.1), we have

$$0 = -x_1(t_n - \tau_{11}) - \mu_1 x_2(t_n - \tau_{12}) \leqslant -x_1(t_n - \tau_{11}) + \mu_1(v_1 + \varepsilon)$$

or

$$x_1(t_n - \tau_{11}) \leqslant \mu_1(v_1 + \varepsilon),$$

which, together with the fact $x_1(t_n) > \mu_1(v_1 + \varepsilon)$ implies that there exists a $\xi_n \in [t_n - \tau_{11}, t_n)$ such that $x_1(\xi_n) = \mu_1(v_1 + \varepsilon)$. Hence from the first equation in (3.1) and (3.15), we have

$$\begin{aligned} \dot{x}_1(t) &\leqslant r_1(x^* + x_1(t))[-x_1(t - \tau_{11}) + \mu_1(v_1 + \varepsilon)] \\ &\leqslant r_1 e^{r_1 \tau_{11}}(1 + \mu_1)(v_1 + \varepsilon), \ T_1 \leqslant t \leqslant t_n. \end{aligned} \tag{3.16}$$

For $t \in [\xi_n, t_n)$, integrating (3.16) from $t - \tau_{11}$ to ξ_n, we have

$$\mu_1(v_1 + \varepsilon) - x_1(t - \tau_{11}) < r_1 e^{r_1 \tau_{11}}(1 + \mu_1)(v_1 + \varepsilon)(\xi_n + \tau_{11} - t), \quad \xi_n \leqslant t \leqslant t_n.$$

Substituting this into the first inequality in (3.16), we obtain

$$\dot{x}_1(t) \leqslant (r_1 e^{r_1 \tau_{11}})^2 (1 + \mu_1)(v_1 + \varepsilon)(\xi_n + \tau_{11} - t), \quad \xi_n \leqslant t \leqslant t_n.$$

Combining this and (3.16), we have

$$\dot{x}_1(t) \leqslant r_1 e^{r_1 \tau_{11}}(1 + \mu_1)(v_1 + \varepsilon) \min\{1, r_1 e^{r_1 \tau_{11}}(\xi_n + \tau_{11} - t)\}, \quad \xi_n \leqslant t \leqslant t_n. \tag{3.17}$$

Set

$$\theta = \begin{cases} \max\{r_1\tau_{11}e^{r_1\tau_{11}} - \frac{1}{2}, \frac{1}{2}\}(1+\mu_1), & \mu_1 < \frac{1}{3}, \\ \frac{1}{2}(1+\mu_1)(r_1\tau_{11}e^{r_1\tau_{11}})^2, & \mu_1 \geqslant \frac{1}{3}. \end{cases}$$

Then by (1.7)

$$\theta < 1 - \mu_1. \tag{3.18}$$

We will show that

$$x_1(t_n) - x_1(\xi_n) \leqslant \theta(v_1 + \varepsilon). \tag{3.19}$$

To this end, we consider the following three subcases:

Case 2.1: $\mu_1 < 1/3$ and $r_1e^{r_1\tau_{11}}(t_n - \xi_n) \leqslant 1$. In this case, by (3.17) we have

$$\begin{aligned} & x_1(t_n) - x_1(\xi_n) \\ & \quad \leqslant (r_1e^{r_1\tau_{11}})^2(1+\mu_1)(v_1+\varepsilon)\int_{\xi_n}^{t_n}(\xi_n + \tau_{11} - t)dt \\ & \quad = (r_1e^{r_1\tau_{11}})^2(1+\mu_1)(v_1+\varepsilon)\left[\tau_{11}(t_n-\xi_n) - \frac{1}{2}(t_n-\xi_n)^2\right] \\ & \quad \leqslant (v_1+\varepsilon)(1+\mu_1)\left[(r_1e^{r_1\tau_{11}})^2\tau_{11}(t_n-\xi_n) - \frac{1}{2}(r_1e^{r_1\tau_{11}}(t_n-\xi_n))^2\right] \\ & \quad \leqslant (v_1+\varepsilon)(1+\mu_1)\left[\max\{r_1\tau_{11}e^{r_1\tau_{11}}, 1\} - \frac{1}{2}\right] \\ & \quad = (v_1+\varepsilon)(1+\mu_1)\max\left\{r_1\tau_{11}e^{r_1\tau_{11}} - \frac{1}{2}, \frac{1}{2}\right\} \\ & \quad = \theta(v_1+\varepsilon). \end{aligned}$$

Case 2.2: $\mu_1 < 1/3$ and $r_1e^{r_1\tau_{11}}(t_n - \xi_n) > 1$. In this case, let $r_1e^{r_1\tau_{11}}(t_n - \eta_n) = 1$. Then by (3.17) we have

$$\begin{aligned} & x_1(t_n) - x_1(\xi_n) \\ & \quad \leqslant r_1e^{r_1\tau_{11}}(1+\mu_1)(v_1+\varepsilon)\left[(\eta_n - \xi_n) + r_1e^{r_1\tau_{11}}\int_{\eta_n}^{t_n}(\xi_n+\tau_{11}-t)dt\right] \\ & \quad = (v_1+\varepsilon)(1+\mu_1)\left[(r_1e^{r_1\tau_{11}})^2\tau_{11}(t_n-\eta_n) - \frac{1}{2}\left(r_1e^{r_1\tau_{11}}(t_n-\eta_n)\right)^2\right] \\ & \quad = (v_1+\varepsilon)(1+\mu_1)\left(r_1\tau_{11}e^{r_1\tau_{11}} - \frac{1}{2}\right) \\ & \quad = \theta(v_1+\varepsilon). \end{aligned}$$

Case 2.3: $\mu_1 \geqslant 1/3$. In this case, $t_n - \xi_n \leqslant \tau_{11}$ hence, by (3.17) we have

$$\begin{aligned} & x_1(t_n) - x_1(\xi_n) \\ & \quad \leqslant (r_1 e^{r_1\tau_{11}})^2 (1+\mu_1)(v_1+\varepsilon) \int_{\xi_n}^{t_n} (\xi_n + \tau_{11} - t)dt \\ & \quad = (r_1 e^{r_1\tau_{11}})^2 (1+\mu_1)(v_1+\varepsilon) \left[\tau_{11}(t_n-\xi_n) - \frac{1}{2}(t_n-\xi_n)^2 \right] \\ & \quad \leqslant \frac{1}{2} (r_1\tau_{11} e^{r_1\tau_{11}})^2 (v_1+\varepsilon)(1+\mu_1) \\ & \quad = \theta(v_1+\varepsilon). \end{aligned}$$

Cases 2.1–2.3 show (3.19) holds. Let $\varepsilon \to 0$ in (3.19). Then we conclude that $v_1 < v_1$. This contradiction implies that $v_1 = 0$. The proof is complete. □

Proof of Theorem 1.5. By letting

$$y_1(t) = x_1(t), \text{ and } y_2(t) = \delta^{-1} x_2(t), \quad i = 1, 2 \tag{3.20}$$

one can transform (3.1) to

$$\begin{aligned} \dot{y}_1(t) &= -r_1(x_1^* + x_1(t))[y_1(t-\tau_{11}) + \delta\mu_1 y_2(t-\tau_{12})], \\ \dot{y}_2(t) &= -r_2(x_2^* + x_2(t))[\delta^{-1}\mu_2 y_1(t-\tau_{21}) + y_2(t-\tau_{22})]. \end{aligned} \tag{3.21}$$

Then, we can similarly show that the conclusion of Theorem 1.5 holds.

4. Discussion

For the delayed logistic equation (1.5), it is well known that when $r\tau < \frac{\pi}{2}$ the positive equilibrium $x^* = 1$ is locally asymptotically stable, and when $r\tau$ passes though $\frac{\pi}{2}$ the stability of x^* is lost and Hopf bifurcation occurs. There is still a range $(\frac{3}{2}, \frac{\pi}{2})$ for $r\tau$, for which the global dynamics of (1.5) remains unclear.

Now, we can similarly consider the local stability of the positive equilibrium $x^* = (x_1^*, x_2^*)$ for (1.2). Recall that under the conditions of Theorems 1.1–1.4, the delays in the interspecific interactions have no impact on the stability of x^*. So, in order to avoid complexity, we only focus on the impact of τ_{ii}, $i = 1, 2$, by assuming $\tau_{12} = 0 = \tau_{21}$. In such a case, the linearization of (1.2) at x^* is

$$\begin{aligned} \dot{x}_1(t) &= -c_{11}x_1(t-\tau_{11}) - c_{12}x_2(t), \\ \dot{x}_2(t) &= -c_{21}x_1(t) - c_{22}x_2(t-\tau_{22}), \end{aligned} \tag{4.1}$$

where $c_{11} = r_1x_1^*$, $c_{12} = r_1x_1^*\mu_1$, $c_{21} = r_2x_2^*\mu_2$, $c_{22} = r_2x_2^*$. Thus, the characteristic equation is

$$(\lambda + c_{11}e^{-\lambda\tau_{11}})(\lambda + c_{22}e^{-\lambda\tau_{22}}) = c_{12}c_{21}. \tag{4.2}$$

Analysing (4.2) is not trivial at all, so we only consider a special case when $c_{11} = c_{22} =: c$ and $\tau_{11} = \tau_{22} =: \tau$. Then (4.2) reduces to

$$(\lambda + ce^{-\lambda\tau})^2 = c_{12}c_{21}, \tag{4.3}$$

which can be further rewritten as

$$(z + \tau ce^{-z})^2 = \tau^2 c_{12}c_{21} \tag{4.4}$$

with $z = \tau\lambda$. Taking square root to (4.4), we have

$$z + \tau ce^{-z} = \tau\sqrt{c_{12}c_{21}} \tag{4.5}$$

and

$$z + \tau ce^{-z} = -\tau\sqrt{c_{12}c_{21}}. \tag{4.6}$$

Now by the well-known result for the Hayes' equation (see, e.g. [2, Theorem 13.8] or [9, Theorem A3]), we know that when

$$\tau c < \frac{\xi}{\sin\xi} \tag{4.7}$$

then, x^* is locally asymptotically stable, and Hopf bifurcation occurs at $\tau c = \frac{\xi}{\sin\xi}$, where ξ is the solution of $\xi = \tau\sqrt{c_{12}c_{21}}\tan\xi$ in $\xi \in (0, \frac{\pi}{2})$. Obviously, (4.7) is equivalent to

$$r_i\tau < \frac{\xi}{x_i^* \sin\xi} = \frac{\xi}{\sin\xi}\frac{(1-\mu_1\mu_2)}{(1-\mu_i)}, \quad i = 1, 2. \tag{4.8}$$

Note that (i) $1 < \frac{\xi}{\sin\xi} < \frac{\pi}{2}$ for $\xi \in (0, \frac{\pi}{2})$; and (ii) when $c_{12}c_{21} \to 0$ (i.e. $\mu_1\mu_2 \to 0$), $\xi \to \frac{\pi}{2}$. Comparing Theorem 1.3 and the above observation, we know that for the above simplified case, Hopf bifurcation occurs at some

$$r_i\tau \in \left(\frac{3(1-\mu)}{2(1+\mu)}, \frac{\pi(1-\mu_1\mu_2)}{2(1-\mu_i)}\right]. \tag{4.9}$$

So, further increasing delay does destabilize the system. When $\mu_i \to 0, i = 1, 2$, we reproduce the unclear interval $(\frac{3}{2}, \frac{\pi}{2})$ for the delayed logistic equation.

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