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Spatial-Temporal Dynamics of a Diffusive Lotka–Volterra Competition Model with a Shifting Habitat II: Case of Faster Diffuser Being a Weaker Competitor

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Abstract
We study a Lotka–Volterra competition–diffusion model that describes the growth, spread and competition of two species in a shifting habitat. Some results have been obtained previously for some cases for the diffusion rates and competitions rates, and in this paper we continue to explore the remaining complementary case for the spatial dynamics of the system. Our main result in this paper reveals an essential difference between the case of *faster diffuser being weak competitor* and the case of *faster diffuser being strong competitor*: with the severe habitat worsening with constant speed, for the former the two competing species can co-persist by spreading, whereas for the latter, co-persistence is impossible.

Keywords Competition · Reaction–diffusion · Lotka–Volterra model · Coexistence · Shifting habitat · Spreading speed

Mathematics Subject Classification 35K57 · 92D40 · 92D25 · 93C10

1 Introduction

In this paper, we are devoted to the study of the spatial-temporal dynamics of the following two-species competition–diffusion model of Lotka-Volterra type

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$$\begin{cases} \dfrac{\partial u_1}{\partial t} = d_1 \dfrac{\partial^2 u_1}{\partial x^2} + u_1[r(x-ct) - u_1 - a_1 u_2], \\ \dfrac{\partial u_2}{\partial t} = d_2 \dfrac{\partial^2 u_2}{\partial x^2} + u_2[r(x-ct) - a_2 u_1 - u_2], \end{cases} \quad t > 0, \quad x \in \Omega \subset \mathbb{R}. \tag{1.1}$$

In model (1.1), $u_1(t, x)$ and $u_2(t, x)$ denote the population densities of two competing species at time t and location x respectively; the constants d_1 and d_2 are the diffusion rates of two competing species respectively and $d_1 \neq d_2$; and the constant $a_1 > 0$ ($a_2 > 0$) is competition strength of species 2 (species 1) against species 1 (species 2). Here, the way the growth rate function $r(x - ct)$ depends on the time t and position x is through a moving pattern with a constant speed c, and $r(\cdot)$ is assumed to satisfy

(A) $r(x)$ is continuous, nondecreasing, bounded and piecewise continuously differentiable for all $x \in \mathbb{R}$ with $0 < r(\infty) < \infty$ and $-\infty < r(-\infty) < 0$.

The non-decreasing property of $r(x)$ assumes that the environment gets worse as time goes, and the negativity of $r(-\infty)$ explains a scenario that the environment is shifting to a very severe level.

It is well known that spatial heterogeneity and diffusion play an important role when considering the interaction of biological species that can diffuse in the real world (see, e.g., [3–5,11,19,27,28]). To understand the effects of the spatial heterogeneity and diffusion on the spatial-temporal dynamics of species, Hastings [10] and Dockery et al [7] discussed a special case of (1.1), with $c = 0$ and $a_1 = a_2 = 1$, and they found that in this special case, if the habitat Ω is a bounded domain with no flux boundary condition (i.e., homogeneous Neumann boundary condition), then the species with slower diffusion rate will win the competition, that is, if $d_1 < d_2$, then all positive solutions to (1.1) converge to $(u_1^*(x), 0)$, where $u_1^*(x)$ is the unique positive solution of the boundary value problem

$$\begin{cases} d_1 \dfrac{\partial^2 u_1}{\partial x^2} + u_1[r(x) - u_1] = 0, \quad x \in \Omega, \\ \dfrac{\partial u_1}{\partial n} = 0, \quad x \in \partial\Omega. \end{cases}$$

Relevantly, Carrère [6], Lin and Li [21], and Girardin and Lam [9] considered the case of constant $r(\cdot)$ in (1.1), and they respectively obtained some spreading properties of (1.1) for the monostable case ($a_1 < 1$, $a_2 < 1$), the bistable case ($a_1 > 1$, $a_2 > 1$) and the case of slower diffuser being stronger competitor ($d_1 < d_2$, $a_1 < 1 < a_2$).

Meanwhile, Lou et al. [7,10,12–14,16–18,24,25] analyzed a slightly more general model (than [6,9,21]), that is, the special case of $c = 0$ but $r(\cdot)$ being non-constant in (1.1). For this special case, they obtained some very surprising and interesting results on the long time behaviors of solutions to (1.1), including the possibility of a globally asymptotically stable positive steady state (co-persistent). For details, see [7,10,12–14,16–18,24,25] and the references therein.

Recently, the climate change has been a major concern of the scientific community, including ecologists and applied mathematicians, see, e.g., [1,2,8,15,20,22,23,26,30–36,42]. A simple climate change pattern is the shifting of environment quality with a constant speed. For the Lotka-Volterra competition model, adopting such a shifting pattern leads to (1.1) with $c > 0$ representing the constant shifting speed. For (1.1) with $\Omega = \mathbb{R}$, Zhang et al [41] and Yuan et al [40] investigated the spatial temporal dynamics of (1.1) for the following cases:

(I) $d_1 < d_2$, $a_1 < 1$ and $a_2 < 1$, i.e., the case of weak competition;

(II) $d_1 < d_2$, $a_1 \geq 1$ and $a_2 \geq 1$, i.e., the case of strong competition;
(III) $d_1 < d_2$ and $a_1 \geq 1$ but $a_2 < 1$, i.e., the case of *faster diffuser being stronger competitor* (hence, the other species is slower diffuser and weaker competitor).

The results in [40,41] show that

(P1) in the weak competition case (I), species i can persist if and only if its intrinsic spreading speed $c_i^*(\infty) = 2\sqrt{d_i r(\infty)}$ is larger than the environmental worsening speed c (hence, co-persistence happens if $c < c_1^*(\infty)$);
(P2) in the cases (II) and (III), since species 2, being faster diffuser ($d_2 > d_1$) and strong competitor ($a_1 \geq 1$), has advantages both in dispersion and competition, species 1 will go to extinction no matter if it is a weak competitor ($a_2 < 1$) or a strong competition ($a_2 > 1$); in the mean time, species 2 will go extinct provided that $c_2^*(\infty) < c$ and will persist provided that $c_2^*(\infty) > c$. This is in contrast to the results in [10] and [7] and such a difference is attributed to the severe (or extreme) habitat's worsening with constant speed ($r(-\infty) < 0$).

We note that [40,41] do not present any essential theoretical results for the following case that is complementary to (I)-(II)-(IV):

(IV) $d_1 < d_2$ and $a_1 < 1 \leq a_2$, i.e., the case of *faster diffuser being weaker competitor*.

One then naturally wonders what happens in this remaining case. For this case, can the two competing species *co-persist* by spreading to the right, and moreover, what would be the spatial-temporal dynamics of species described by (1.1)? Note that model (1.1) is heterogeneous in space and time. The heterogeneity described by $r(x - ct)$ makes the existing theory on spreading properties not applicable to (1.1). In order to obtain their main results, the authors of [40,41] used a fluctuation method, which was developed by Li et al [20]. Unfortunately, since (IV) is a mixed case, the fluctuation method does not apply, at least directly, making the study more challenging and subtle. This may explain (at least partially) why Zhang et al [41] and Yuan et al [40] only presented some simulation results and conjectures for case (IV).

In this paper, we continue to explore the spatial dynamics of model (1.1) with $\Omega = \mathbb{R}$ and case (IV). That is, we consider the following model system

$$\begin{cases} \dfrac{\partial u_1}{\partial t} = d_1 \dfrac{\partial^2 u_1}{\partial x^2} + u_1[r(x - ct) - u_1 - a_1 u_2], \\ \dfrac{\partial u_2}{\partial t} = d_2 \dfrac{\partial^2 u_2}{\partial x^2} + u_2[r(x - ct) - a_2 u_1 - u_2], \end{cases} \quad t > 0, \quad x \in \mathbb{R}, \tag{1.2}$$

where $c > 0$, $r(x)$ satisfies Assumption (**A**), and d_i and a_i, $i = 1, 2$ satisfy (IV). As mentioned before, the heterogeneity described by $r(x - ct)$ prevents us from applying those nice results in the existing theory on spreading properties. To overcome this difficulty, we will modify the method developed in [40] to obtain the spatial dynamics of the model (1.2). More specifically, we construct an auxiliary sequence of functions $\left\{u_1^{(n)}, u_2^{(n)}\right\}_{n=0}^{\infty}$ that satisfies some required properties. We point out that the iteration generating this sequence is similar to but different from that in Sub-section 2.2 in [40] so that the different scenario for the case (IV) can be accommodated. Then by carefully analyzing this sequence, and with the help of Egorov's Theorem, we show that the sequence converges to a limit function $\left(u_1^*, u_2^*\right)$, which is the solution to (1.2) and also satisfies the required properties (see Theorem 2.4 in Section 2). Our results indicate that the spatial-temporal dynamics of (1.2) mainly depend on c and $c_i^*(\infty) = 2\sqrt{d_i r(\infty)}$, $i = 1, 2$ which are defined by the parameters in (1.2). We will show that (i) if $0 < c < c_1^*(\infty)$, then two competing species will *coexist* by spreading to the right; (ii) if $c_1^*(\infty) < c < c_2^*(\infty)$, then species 1 will go to extinction in the habitat and species 2 will persist by spreading to the right (thus, confirming that the being a stronger competitor does not help species 1 survive); (iii) if $c > c_2^*(\infty)$, then two competing species will both go extinct in the habitat. Observe that while the conclusion in (ii) and (iii) above remain the same as those for the cases (I), (II) and (III), the conclusion in (i) reveals that there is an essential difference between the case of *faster diffuser being weak competitor* (Case (IV)), and the case of *faster diffusor being also strong competition* (Cases (II) and (III)). These results complement those in [41] and [40], helping us have a better understanding on the spatial-temporal dynamics of (1.2).

The rest of this paper is organized as follows. In Sect. 2, we present the main mathematical results regarding the spatial-temporal dynamics of (1.2). In Sect. 3 we give some numerical simulation results that help illustrate the results from Sect. 2. To make the reading smoother, we leave the proof of Lemmas 2.2 and 2.5 to the Appendix.

2 Mathematical Results

We first introduce some notations that are consistent with those used in [40]. Let $\mathbb{R}$ and $\mathbb{R}_+$ be the sets of all real numbers and nonnegative real numbers, respectively. For any $u = (u_1, u_2)$, $v = (v_1, v_2) \in \mathbb{R}^2$, we write $u \le v$ if and only if $u_1 \le v_1$ and $u_2 \le v_2$, and $u < v$ means that $u \le v$ but $u \ne v$. For every pair $u, v \in \mathbb{R}^2$ satisfying $u \le v$, the set

$$[u, v] = \left\{ w \in \mathbb{R}^2 \middle| u \le w \le v \right\}$$

is called the order interval between u and v. Clearly, $[u, v]$ is nonempty if and only if $u \le v$. For any constant l, we denote by $\vec{l}$ the vector (l, l). Define $f_i : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+^2 \to \mathbb{R}$, $i = 1, 2$, by

$$f_1(t, x, u) = u_1[r(x - ct) - u_1 - a_1 u_2]$$

and

$$f_2(t, x, u) = u_2[r(x - ct) - u_2 - a_2 u_1]$$

for all $(t, x, u) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+^2$. Then we can rewrite (1.2) as the following more convenient form with given nonnegative initial function

$$\begin{cases} \dfrac{\partial u_1}{\partial t} = d_1 \dfrac{\partial^2 u_1}{\partial x^2} + f_1(t, x, u), \quad t > 0, \quad x \in \mathbb{R}, \\ \dfrac{\partial u_2}{\partial t} = d_2 \dfrac{\partial^2 u_2}{\partial x^2} + f_2(t, x, u), \quad t > 0, \quad x \in \mathbb{R}, \\ u(0, x) = \phi(x) \ge \vec{0}, \quad x \in \mathbb{R}. \end{cases} \tag{2.1}$$

Throughout this section, we always assume that $0 < d_1 < d_2$, $c > 0$, $0 < a_1 < 1 \le a_2$ (e.g., case (IV)) and the function r satisfies Assumption (**A**).

For any $t \in \mathbb{R}_+$, $x \in \mathbb{R}$ and $u, v \in [\vec{0}, \vec{r}(\infty)]$, where $u = (u_1, u_2)$, $v = (v_1, v_2)$, $\vec{0} = (0, 0)$ and $\vec{r}(\infty) = (r(\infty), r(\infty))$, one can easily verify the following Lipschitz condition

$$|f_i(t, x, u) - f_i(t, x, v)| \le 4r(\infty) \left(|u_1 - v_1| + |u_2 - v_2|\right)$$

for $i = 1, 2$. Thus, if $\vec{0} \le \phi(x) \le \vec{r}(\infty)$ for all $x \in \mathbb{R}$, then $\vec{0}$ and $\vec{r}(\infty)$ are coupled upper and lower solutions to (2.1) (see, e.g., [29]). The theory on the existence and uniqueness of solutions for reaction–diffusion systems has been well established (see, e.g., Theorem 2.1 in [29]), by which, it is known that the initial value problem (2.1) with $\vec{0} \le \phi(x) \le \vec{r}(\infty)$ has a unique classical solution $u(t, x, \phi)$ with $\vec{0} \le u(t, x, \phi) \le \vec{r}(\infty)$.

Since the work in this paper can be considered as a continuation of the work of [40,41], it will be natural and convenient to adopt those notations and concepts used in [40,41], as proceeded below. For $r(x) > 0$, define

$$c_i^*(x) = 2\sqrt{d_i r(x)}, \quad i = 1, 2.$$

It is easily seen that

$$c_i^*(x) = \inf_{\mu > 0} g_i(x; \mu),$$

where

$$g_i(x; \mu) = \frac{d_i \mu^2 + r(x)}{\mu}, \quad i = 1, 2.$$

The infimums occur at $\mu_i^*(x) = \sqrt{r(x)/d_i}$, $i = 1, 2$. The function

$$\psi_i(\mu) = 2d_i\mu, \quad i = 1, 2$$

is useful. It is easily seen that $g_i(x; \mu) > \psi_i(\mu)$ for all $0 < \mu < \mu_i^*(x)$ and $g_i(x; \mu_i^*(x)) = \psi_i(\mu_i^*(x))$, $i = 1, 2$. By [20], $c_i^*(\infty)$ is nothing but the asymptotic spread speed for species i in the absence of species j $(j \neq i)$ and $0 < c_1^*(\infty) < c_2^*(\infty)$. In what follows, we consider three generic cases: $c > c_2^*(\infty)$, $c_1^*(\infty) < c < c_2^*(\infty)$ and $c < c_1^*(\infty)$.

For convenience, we sometime use c_i^* to denote the constant $c_i^*(\infty)$, $i = 1, 2$. By the same arguments as in the proofs of [40] (Theorems 2.1 and 2.2), we have the following results.

Theorem 2.1 *Assume that (**A**) and (IV) hold, and suppose $c > c_2^*(\infty)$. Let $u(t, x, \phi)$ be the solution of (2.1) with $\vec{0} \le \phi(x) \le \vec{r}(\infty)$. If $\phi(x) \equiv \vec{0}$ for all sufficiently large x, then for any $\varepsilon > 0$ there exists $T > 0$ such that $u(t, x, \phi) \le \vec{\varepsilon}$ for all $(t, x) \in [T, +\infty) \times \mathbb{R}$, where $\vec{\varepsilon} = (\varepsilon, \varepsilon)$.*

Theorem 2.2 *Assume that (**A**) and (IV) hold, and suppose $c_1^*(\infty) < c < c_2^*(\infty)$. Let $u(t, x, \phi)$ be the solution of (2.1) with $\vec{0} \le \phi(x) \le \vec{r}(\infty)$. Then the following four statements hold.*

- **(i)** *If $\phi_1(x) \equiv 0$ for all sufficiently large x, then for any $\varepsilon > 0$ there exits $T > 0$ such that $u_1(t, x, \phi) \le \varepsilon$ for all $(t, x) \in [T, +\infty) \times \mathbb{R}$;*
- **(ii)** *For any $\varepsilon > 0$,*

$$\lim_{t \to +\infty} \left[\sup_{x \le t(c-\varepsilon)} u_2(t, x, \phi) \right] = 0;$$

- **(iii)** *If $\phi_2(x) \equiv 0$ for all sufficiently large x, then for any $\varepsilon > 0$,*

$$\lim_{t \to +\infty} \left[\sup_{x \ge t(c_2^*+\varepsilon)} u_2(t, x, \phi) \right] = 0;$$

- **(iv)** *If $\phi_1(x) \equiv 0$ for all sufficiently large x and $\phi_2(x) > 0$ on a closed interval, then for any ε with $0 < \varepsilon < (c_2^* - c)/2$,*

$$\lim_{t \to +\infty} \left[\sup_{t(c+\varepsilon) \le x \le t(c_2^*-\varepsilon)} |r(\infty) - u_2(t, x, \phi)| \right] = 0.$$

Remark 2.1 Theorem 2.1 shows that if two competing species initially live only on a bounded domain and their intrinsic spreading speeds are less than the habitat's worsening speed c, then both species will go to extinction. Theorem 2.2 indicates that if one of the species spreads with a speed faster than the habitat's worsening speed, then that species is able to persist in the spreading sense.

Now we deal with the third case: $c < c_1^*(\infty)$. For this case, by employing the comparison principle and results in [20], we first have the the following theorem.

Theorem 2.3 *Assume that* (**A**) *holds and* $c < c_1^*(\infty)$. *Let* $u(t, x, \phi)$ *be the solution of* (2.1) *with* $\vec{0} \leq \phi(x) \leq \vec{r}(\infty)$. *Then the following two statements hold.*

(i) *For any* $\varepsilon > 0$,

$$\lim_{t\to+\infty}\left[\sup_{x\leq t(c-\varepsilon)} u_i(t,x,\phi)\right] = 0, \quad i = 1,2;$$

(ii) *If* $\phi_i(x) \equiv 0$ *for all sufficiently large* x, *then for any* $\varepsilon > 0$,

$$\lim_{t\to+\infty}\left[\sup_{x\geq t(c_i^*+\varepsilon)} u_i(t,x,\phi)\right] = 0, \quad i = 1,2.$$

Theorem 2.3 shows that if the intrinsic spreading speeds c_i^* for the two competing species are both larger than the habitat's worsening speed c, then an observer moving toward the right direction with a speed less than the habitat's worsening speed c, or moving with a speed faster than c_i^* but with the two competing species initially living only in a bounded domain, will not be able see individuals of the two competing species as $t \to \infty$. In such a case, it is very natural and interesting to ask if the conclusion (iv) in Theorem 2.2 remains true for species 1 as well, meaning that species 1 can also persist in the moving mode as stated in Theorem 2.2-(iv) for species 2. This problem turns out to be *very challenging* due to the presence of competition between the two species. As mentioned in the introduction, Zhang et al [41] and Yuan et al [40] have attacked this question for three cases (I)–(II)–(III), and they found that in case (I), under some stronger condition, the answer to above question is *affirmative*, with the persistence levels for each species modified to reflect the effect of competition; and for cases (II) and (III), the answer to the above question is *negative*, as species 1 may become extinct in the habitat whereas species 2 persists by spreading to the right. However, for the case of *faster diffuser being weak competitor,* (i.e., case (IV)), the authors of [40,41] could not obtain any essential theoretical results. In the sequel, we shall show that in case (IV), the answer to above question is *affirmative*, with the two competing species co-persisting and by spreading to the right at their respective intrinsic spreading speeds. Indeed, our approach will allow us to prove the following main theorem of this paper.

Theorem 2.4 *Assume that* (**A**) *and (IV) hold, and suppose* $0 < c < c_1^*(\infty)$. *Let* $u(t, x, \phi) = (u_1(t, x, \phi), u_2(t, x, \phi))$ *be the solution of* (2.1) *with the initial function* ϕ *satisfying* $\vec{0} \leq \phi(x) \leq \vec{r}(\infty)$. *Then the following statements hold.*

(i) *For any* $\varepsilon > 0$,

$$\lim_{t\to\infty}\left[\sup_{x\leq t(c-\varepsilon)} u_1(t,x,\phi)\right] = 0;$$

(ii) *If $\phi_i(x) \equiv 0$ for all sufficiently large x, then for any $\varepsilon > 0$,*

$$\lim_{t\to\infty}\left[\sup_{x\geq t(c_i^*(\infty)+\varepsilon)} u_i(t,x,\phi)\right] = 0, \quad i = 1, 2;$$

(iii) *If $\phi_1(x) > 0$ on a closed interval, then for every ε with $0 < \varepsilon < (c_1^*(\infty) - c)/2$,*

$$\lim_{t\to\infty}\left[\sup_{t(c+\varepsilon)\leq x\leq t(c_1^*(\infty)-\varepsilon)} |r(\infty) - u_1(t,x,\phi)|\right] = 0$$

and

$$\lim_{t\to\infty}\left[\sup_{x\leq t(c_1^*(\infty)-\varepsilon)} u_2(t,x,\phi)\right] = 0;$$

(iv) *If $\phi_i(x) > 0$ on a closed interval for $i = 1, 2$, then for every ε with $0 < \varepsilon < (c_2^*(\infty) - c_1^*(\infty))/2$,*

$$\lim_{t\to\infty}\left[\sup_{t(c_1^*(\infty)+\varepsilon)\leq x\leq t(c_2^*(\infty)-\varepsilon)} |r(\infty) - u_2(t,x,\phi)|\right] = 0.$$

Note that (i) and (ii) in Theorem 2.4 are already included in Theorem 2.3, but we repeat them in this theorem because their proofs together with the proofs of (iii)–(iv) will be carried out under the same framework. However, the result in Theorem 2.3-(i) for $i = 2$ cannot cannot be obtained in our proof of Theorem 2.4, and it needs some modifications.

To proceed toward the goal of proving this theorem, we will use a technique developed in [40] to overcome the difficulty of the heterogeneity described by $r(x - ct)$. Firstly, we consider the following auxiliary system

$$\begin{cases} \dfrac{\partial u_1}{\partial t} = d_1\dfrac{\partial^2 u_1}{\partial x^2} + u_1[r(x-ct) - u_1], \quad t > 0,\ x \in \mathbb{R}, \\ \dfrac{\partial u_2}{\partial t} = d_2\dfrac{\partial^2 u_2}{\partial x^2} + u_2[r(x-ct) - u_2 - a_2u_1], \quad t > 0,\ x \in \mathbb{R}, \\ u(0,x) = \phi(x), \quad x \in \mathbb{R}, \end{cases} \tag{2.2}$$

where $\vec{0} \leq \phi(x) \leq \vec{r}(\infty)$, $\phi(x) \equiv \vec{0}$ for all sufficiently large x and $\phi(x) > \vec{0}$ on a closed interval. We denote by $(u_1^{(0)}(t,x,\phi), u_2^{(0)}(t,x,\phi))$ the solution of the system (2.2). Then by [20], we obtain that for any $\varepsilon > 0$, $u_1^{(0)}(t,x,\phi)$ satisfies

$$\lim_{t\to\infty}\left[\sup_{x\leq t(c-\varepsilon)} u_1^{(0)}(t,x,\phi)\right] = 0,$$
$$\lim_{t\to\infty}\left[\sup_{x\geq t(c_1^*(\infty)+\varepsilon)} u_1^{(0)}(t,x,\phi)\right] = 0$$

and

$$\lim_{t\to\infty}\left[\sup_{t(c+\varepsilon)\leq x\leq t(c_1^*(\infty)-\varepsilon)} \left|r(\infty) - u_1^{(0)}(t,x,\phi)\right|\right] = 0.$$

Therefore, for any given $\varepsilon^{(0)}$ with $0 < \varepsilon^{(0)} < (c_2^*(\infty) - c_1^*(\infty))/3$, we choose a number $\delta^{(0)} \in (0, r(\infty))$ such that

$$c_1^*(\infty) + \varepsilon^{(0)} < 2\sqrt{d_2(r(\infty) - \delta^{(0)})} < c_2^*(\infty) - \varepsilon^{(0)}. \tag{2.3}$$

Then there exists $T_0 > 0$ such that

$$u_1^{(0)}(t, x, \phi) < \delta^{(0)}/a_2, \quad \forall\, (t, x) \in \left\{(s, y)\,\middle|\,(s, y) \in \mathbb{R}^2, s \geq T_0, y \geq (c_1^*(\infty) + \varepsilon^{(0)})s\right\}. \tag{2.4}$$

Obviously, if $\varepsilon^{(0)}$ is sufficiently small, then we can also choose a sufficiently small $\delta^{(0)}$ such that (2.3) holds true.

For the above given $T_0 > 0$, we consider the following equation

$$\begin{cases} \dfrac{\partial u_2}{\partial t} = d_2 \dfrac{\partial^2 u_2}{\partial x^2} + u_2[R^{(0)} - u_2], \quad t > T_0,\ x \in \mathbb{R}, \\ u_2(T_0, x) = u_2^{(0)}(T_0, x, \phi), \quad x \in \mathbb{R}, \end{cases} \tag{2.5}$$

where $R^{(0)}(t, x) = r(x - ct) - a_2 u_1^{(0)}(t, x, \phi)$. It is clear from (2.2) that $u_2^{(0)}(t, x, \phi)$ satisfies (2.5).

For above given $\delta^{(0)} > 0$, let $r^{(0)}(x) = r(x) - \delta^{(0)}$. Then $r^{(0)}(\infty) > 0$. Therefore, for $r^{(0)}(x) > 0$, define

$$c_{i,0}^*(x) = 2\sqrt{d_i r^{(0)}(x)}, \quad i = 1, 2.$$

It is easily seen that

$$c_{i,0}^*(x) = \inf_{\mu > 0} g_{i,0}(x; \mu),$$

where

$$g_{i,0}(x; \mu) = \frac{d_i \mu^2 + r^{(0)}(x)}{\mu}, \quad i = 1, 2.$$

The infimums occur at $\mu_{i,0}^*(x) = \sqrt{r^{(0)}(x)/d_i}$, $i = 1, 2$. It is easily seen that $g_{i,0}(x; \mu) > \psi_i(\mu)$ for all $0 < \mu < \mu_{i,0}^*(x)$ and $g_{i,0}(x; \mu_{i,0}^*(x)) = \psi_i(\mu_{i,0}^*(x))$, $i = 1, 2$. By (2.3), we have

$$c_1^*(\infty) + \varepsilon^{(0)} < c_{2,0}^*(\infty) < c_2^*(\infty) - \varepsilon^{(0)}. \tag{2.6}$$

Definition 2.1 We call a function u_2 a continuous weak lower solution of Eq. (2.5) if u_2 is continuous on $[T_0, T] \times \mathbb{R}$, $u_2(T_0, x) \leq u_2^{(0)}(T_0, x, \phi)$ and

$$\frac{\partial u_2}{\partial t} \leq d_2 \frac{\partial^2 u_2}{\partial x^2} + u_2[R^{(0)} - u_2]$$

in the distributional sense, i.e., for any $\eta \in C^{1,2}([T_0, T] \times \mathbb{R})$ with $\eta \geq 0$ and $\text{supp}\eta(t, \cdot) \Subset \mathbb{R}$ (meaning that $\text{supp}\eta(t, \cdot)$ is bounded interval in $\mathbb{R}$) for all $t \in [T_0, T]$, there holds

$$\begin{aligned} &\int_{-\infty}^{+\infty} u_2(t, x)\eta(t, x)dx \,\Big|_{t=T_0}^{t=T_\sigma} \\ &\quad \leq \int_{T_0}^{T_\sigma} \int_{-\infty}^{+\infty} \left[u_2(s, x)\,(d_2\eta_{xx} + \eta_t)\,(s, x) + \eta(s, x)u_2(s, x)(R^{(0)}(s, x) - u_2(s, x))\right] dxds, \end{aligned}$$

for $T_\sigma \in [T_0, T]$, where $\eta_{xx}(s,x) = \frac{\partial^2 \eta(t,x)}{\partial x^2}\big|_{(t,x)=(s,x)}$ and $\eta_t(s,x) = \frac{\partial \eta(t,x)}{\partial t}\big|_{(t,x)=(s,x)}$.

For fixed $\gamma > 0$, consider the following function of x on the interval $[0, \pi/\gamma]$ parameterized by $\mu > 0$:

$$\varphi(\mu; x) = \begin{cases} e^{-\mu x} sin(\gamma x), & \text{if } 0 \le x \le \pi/\gamma, \\ 0, & \text{elsewhere.} \end{cases} \tag{2.7}$$

Such function was used in [37,38] in studying reaction-diffusion systems, in addition to [20]. Obviously, $\varphi(\mu; x)$ is continuous in x and its second order derivative in x exists and is continuous at $x \neq 0, \pi/\gamma$. The maximum of $\varphi(\mu; x)$ occurs at $\sigma(\mu) = \gamma^{-1}\tan^{-1}(\mu^{-1}\gamma)$ and $\sigma(\mu)$ is strictly decreasing in μ. Therefore, by (2.6), we have the following useful lemmas.

Lemma 2.1 *Assume that* $0 < c < c_1^*(\infty) < c_2^*(\infty)$. *For any* ϵ *with* $0 < \epsilon < \frac{c_{2,0}^*(\infty) - c_1^*(\infty) - \varepsilon^{(0)}}{3}$, *let* ℓ *be a number such that* $c_{2,0}^*(\ell) = c_{2,0}^*(\infty) - \epsilon$. *Let* $0 < \mu_1 < \mu_2 < \mu_{2,0}^*(\ell)$ *with* $\psi_2(\mu_1) = c_1^*(\infty) + \varepsilon^{(0)} + \epsilon$ *and* $\psi_2(\mu_2) = c_{2,0}^*(\infty) - 2\epsilon$. *Then for any* $\mu \in [\mu_1, \mu_2]$ *and for the above given* T_0, *there exists sufficiently small* $\beta > 0$ *and* $\gamma > 0$ *such that* $\beta\varphi(\mu; x - \ell - \psi_2(\mu)t)$ *with* φ *given (2.7) is a continuous weak lower solution of (2.5). Furthermore, if* $u_2^{(0)}(T_0, x, \phi) \ge \beta\varphi(\mu; x - \ell)$, *then* $u_2^{(0)}(t, x, \phi) \ge \beta\varphi(\mu; x - \ell - \psi_2(\mu)(t - T_0))$ *for all* $t > T_0$.

Proof For any $T > T_0$, let $T_\sigma \in [T_0, T]$ and $\eta \in C^{1,2}([T_0, T] \times \mathbb{R})$ with $\eta \ge 0$ and $\text{supp}\eta(t, \cdot) \Subset \mathbb{R}$ for all $t \in [T_0, T]$. Then by employing the definition of φ and integration by parts, we obtain that

$$\begin{aligned} &\int_{T_0}^{T_\sigma}\int_{-\infty}^{+\infty} \varphi(\mu; x - \ell - ls)\eta_{xx}(s,x)dxds \\ &= \int_{T_0}^{T_\sigma}\int_{\ell+ls}^{\ell+ls+\pi/\gamma} \varphi(\mu; x - \ell - ls)\eta_{xx}(s,x)dxds \\ &= \int_{T_0}^{T_\sigma}\int_{\ell+ls}^{\ell+ls+\pi/\gamma} \varphi_{xx}(\mu; x - \ell - ls)\eta(s,x)dxds \\ &\quad + \gamma\int_{T_0}^{T_\sigma}\left[\eta(s, \ell + ls + \pi/\gamma)e^{-\pi\mu/\gamma} + \eta(s, \ell + ls)\right]ds \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} &\int_{T_0}^{T_\sigma}\int_{-\infty}^{+\infty} \varphi(\mu; x - \ell - ls)\eta_t(s,x)dxds \\ &= \int_{-\infty}^{+\infty}\int_{T_0}^{T_\sigma} \varphi(\mu; x - \ell - ls)\eta_t(s,x)dsdx \\ &= \int_{-\infty}^{+\infty}\left[\eta(s,x)\varphi(\mu; x - \ell - ls)\Big|_{s=T_0}^{s=T_\sigma} - \int_{T_0}^{T_\sigma}\varphi_t(\mu; x - \ell - ls)\eta(s,x)ds\right]dx \\ &= \int_{-\infty}^{+\infty}\varphi(\mu; x - \ell - lt)\eta(t,x)dx\Big|_{t=T_0}^{t=T_\sigma} - \int_{T_0}^{T_\sigma}\int_{-\infty}^{+\infty}\eta(s,x)\varphi_t(\mu; x - \ell - ls)dxds, \end{aligned} \tag{2.9}$$

where

$$\varphi_{xx}(\mu; x - \ell - ls) = \frac{\partial^2\varphi(\mu; x - \ell - lt)}{\partial x^2}\Big|_{(t,x)=(s,x)},$$

$$\varphi_t(\mu; x-\ell-ls) = \frac{\partial \varphi(\mu; x-\ell-lt)}{\partial t}\Big|_{(t,x)=(s,x)}$$

and

$$l = \psi_2(\mu). \tag{2.10}$$

Direct calculations show that

$$(\varphi_t - d_2\varphi_{xx})(\mu; x-\ell-lt) = d_2(\mu^2+\gamma^2)\varphi(\mu; x-\ell-lt) \tag{2.11}$$

for all $\mu \in [\mu_1, \mu_2]$, $x \neq \ell + lt$ and $x \neq \ell + lt + \pi/\gamma$, where l is given in (2.10). Therefore, we have

$$\begin{aligned} r(\ell) - \delta^{(0)} - d_2\mu_2^2 &= r^{(0)}(\ell) - d_2\mu_2^2 \\ &= \frac{1}{4d_2}\left[\left(c_{2,0}^*(\ell)\right)^2 - \left(c_{2,0}^*(\infty) - 2\epsilon\right)^2\right] \\ &= \frac{1}{4d_2}\left[\left(c_{2,0}^*(\infty) - \epsilon\right)^2 - \left(c_{2,0}^*(\infty) - 2\epsilon\right)^2\right] \\ &= \frac{\epsilon}{2d_2}\left(c_{2,0}^*(\infty) - 1.5\epsilon\right) \\ &\geq \frac{\epsilon}{2d_2}\left(c_1^*(\infty) + \varepsilon^{(0)} + \epsilon\right). \end{aligned} \tag{2.12}$$

Let $H_2(t,x,u_2) = u_2(R^{(0)} - u_2)$. It follows from (2.8), (2.9), (2.11) and (2.12) that for any $\mu \in [\mu_1, \mu_2]$ and for sufficiently small $\beta > 0$ and $\gamma > 0$, $\widehat{u}_2(\mu; t, x) = \beta\varphi(\mu; x - \ell - \psi_2(\mu)t)$ satisfies

$$\begin{aligned} &\int_{T_0}^{T_\sigma}\int_{-\infty}^{+\infty} \left[\widehat{u}_2(\mu; s, x)\left(d_2\eta_{xx} + \eta_t\right)(s,x) + \eta(s,x)H_2(s,x,\widehat{u}_2(\mu; s,x))\right] dxds \\ &\quad - \int_{-\infty}^{+\infty} \widehat{u}_2(\mu; t, x)\eta(t,x)dx\Big|_{t=T_0}^{t=T_\sigma} \\ &= \int_{T_0}^{T_\sigma}\int_{\ell+ls}^{\ell+ls+\pi/\gamma} \left[-\beta d_2(\mu^2+\gamma^2)\varphi(\mu; x-\ell-ls) + H_2(s,x,\widehat{u}_2(\mu; s,x))\right]\eta(s,x)dxds \\ &\quad + d_2\beta\gamma\int_{T_0}^{T_\sigma}\left[\eta(s,\ell+ls+\pi/\gamma)e^{-\pi\mu/\gamma} + \eta(s,\ell+ls)\right]ds \\ &\quad - \beta\int_{T_0}^{T_\sigma}\int_{\Omega(s)} \eta(s,x)\varphi_t(\mu; x-\ell-ls)dxds \\ &= \beta\int_{T_0}^{T_\sigma}\int_{\ell+ls}^{\ell+ls+\pi/\gamma}\left[r(x-cs) - a_2u_1^{(0)}(s,x,\phi) - d_2(\mu^2+\gamma^2) - \beta\varphi(\mu; x-\ell-ls)\right] \\ &\quad \times\varphi(\mu; x-\ell-ls)\eta(s,x)dxds + d_2\beta\gamma\int_{T_0}^{T_\sigma}\left[\eta(s,\ell+ls+\pi/\gamma)e^{-\pi\mu/\gamma} + \eta(s,\ell+ls)\right]ds \\ &\geq \beta\int_{T_0}^{T_\sigma}\int_{\ell+ls}^{\ell+ls+\pi/\gamma}\left[r(\ell) - \delta^{(0)} - d_2(\mu_2^2+\gamma^2) - \beta\varphi(\mu; x-\ell-ls)\right] \\ &\quad \times\varphi(\mu; x-\ell-ls)\eta(s,x)dxds + d_2\beta\gamma\int_{T_0}^{T_\sigma}\left[\eta(s,\ell+ls+\pi/\gamma)e^{-\pi\mu/\gamma} + \eta(s,\ell+ls)\right]ds \\ &\geq \beta\int_{T_0}^{T_\sigma}\int_{\ell+ls}^{\ell+ls+\pi/\gamma}\left[\frac{\epsilon}{2d_2}\left(c_1^*(\infty) + \varepsilon^{(0)} + \epsilon\right) - d_2\gamma^2 - \beta\varphi(\mu; x-\ell-ls)\right] \end{aligned}$$

$$\times\varphi(\mu; x-\ell-ls)\eta(s,x)dxds + d_2\beta\gamma\int_{T_0}^{T_\sigma}\left[\eta(s,\ell+ls+\pi/\gamma)e^{-\pi\mu/\gamma}+\eta(s,\ell+ls)\right]ds$$
$$\geq 0,$$

where $\Omega(s) = \{x|x\in\mathbb{R}, x\leq \ell+ls\}\bigcup\{x|x\in\mathbb{R}, x\geq \ell+ls+\pi/\gamma\}$ and l is given in (2.10). It follows from Definition 2.1 that for any $\mu\in[\mu_1,\mu_2]$ and for sufficiently small $\beta>0$ and $\gamma>0$, $\beta\varphi(\mu; x-\ell-\psi_2(\mu)t)$ is a continuous weak lower solution of (2.5).

If $u_2^{(0)}(T_0,x,\phi)\geq\beta\varphi(\mu; x-\ell)$, then it follows from Lemma 1.2 in Wang [37] that $u_2^{(0)}(t,x,\phi)\geq\beta\varphi(\mu; x-\ell-\psi_2(\mu)(t-T_0))$ for all $t>T_0$. The proof is completed. □

Lemma 2.2 *Assume that (A) holds and $0<c<c_1^*(\infty)<c_2^*(\infty)$. If $0\leq\phi_2(x)\leq r(\infty)$, and $\phi_2(x)>0$ on a closed interval, then for every ε with $0<\varepsilon<(c_2^*(\infty)-c_1^*(\infty))/2$,*

$$\lim_{t\to\infty}\left[\sup_{t(c_1^*(\infty)+\varepsilon)\leq x\leq t(c_2^*(\infty)-\varepsilon)}\left|r(\infty)-u_2^{(0)}(t,x,\phi)\right|\right]=0.$$

The proof is similar to that of Theorem 2.2 in [20] with some minor modifications, and is given in the appendix for reader's convenience.

Lemma 2.3 *Assume that (**A**) holds and $0<c<c_1^*(\infty)<c_2^*(\infty)$. Then the following statements are valid.*

(i) *If $0\leq\phi_i(x)\leq r(\infty)$ for $i=1,2$, and $\phi_1(x)>0$ on a closed interval, then for any $\varepsilon>0$,*

$$\lim_{t\to\infty}\left[\sup_{x\leq t(c_1^*(\infty)-\varepsilon)}u_2^{(0)}(t,x,\phi)\right]=0;$$

(ii) *If $0\leq\phi_2(x)\leq r(\infty)$ and $\phi_2(x)\equiv 0$ for all sufficiently large x, then for any $\varepsilon>0$,*

$$\lim_{t\to\infty}\left[\sup_{x\geq t(c_2^*(\infty)+\varepsilon)}u_2^{(0)}(t,x,\phi)\right]=0.$$

Proof The verification of statement (ii) is straightforward and is thus omitted. We now prove statement (i). Clearly, for any given $\epsilon>0$, $u_2^{(0)}(t,x,\phi)$ satisfies the integral equation

$$\begin{aligned}u_2^{(0)}(t,x,\phi)=&\int_{-\infty}^{+\infty}e^{-\rho t}k_{21}(t,x-y)\phi_2(y)dy\\&+\int_0^t e^{-\rho s}\int_{-\infty}^{+\infty}k_{21}(s,y)h_2^{(0)}(\rho,t-s,x-y)dyds,\end{aligned}\tag{2.13}$$

where $\rho=\frac{\epsilon}{30}$,

$$k_{21}(s,y)=\frac{1}{\sqrt{4\pi d_2 s}}e^{-\frac{y^2}{4d_2 s}}\tag{2.14}$$

and

$$h_2^{(0)}(\rho,s,y)=u_2^{(0)}(s,y,\phi)\left[\rho+r(y-cs)-u_2^{(0)}(s,y,\phi)-a_2u_1^{(0)}(s,y,\phi)\right].\tag{2.15}$$

Note that $\int_{-\infty}^{+\infty}k_{21}(s,y)dy=1$. Since $0\leq u_i^{(0)}(t,x,\phi)\leq r(\infty)$ for all $(t,x)\in\mathbb{R}_+\times\mathbb{R}$ and $i=1,2$, we obtain that

$$h_2^{(0)}(\rho,t,x)\leq r(\infty)(\rho+2r(\infty)+a_2r(\infty)),\quad\forall(t,x)\in\mathbb{R}_+\times\mathbb{R}.\tag{2.16}$$

On the other hand, due to the fact that $\int_0^{+\infty} e^{-\rho s} ds$ is convergent, for above given $\epsilon > 0$, there exist $\eta > 0$ and $A > \eta$ such that

$$\int_0^{\eta} e^{-\rho s} \int_{-\infty}^{+\infty} k_{21}(s, y) h_2^{(0)}(\rho, t-s, x-y) dy ds < \frac{\epsilon}{10} \tag{2.17}$$

and

$$\int_A^{+\infty} e^{-\rho s} \int_{-\infty}^{+\infty} k_{21}(s, y) h_2^{(0)}(\rho, t-s, x-y) dy ds < \frac{\epsilon}{10}. \tag{2.18}$$

By (2.2) and [20], for above give $\epsilon > 0$ and for any ε with $0 < \varepsilon < (c_1^* - c)/5$, there exists $T > 0$ such that

$$\int_{-\infty}^{+\infty} e^{-\rho t} k_{21}(t, x-y) \phi_2(y) dy < \epsilon/5, \;\; \forall (t, x) \in [T, +\infty) \times \mathbb{R} \tag{2.19}$$

and

$$r(y - cs) - a_2 u_1^{(0)}(s, y, \phi) < \frac{\epsilon}{30}, \;\; \forall (s, y) \in \left\{(s, y) \,\middle|\, s \geq T, s(c+\varepsilon) \leq y \leq s(c_1^* - \varepsilon)\right\}.$$

We claim that

$$h_2^{(0)}(\rho, s, y) < \frac{\epsilon^2}{150}, \;\; \forall (s, y) \in \left\{(s, y) \,\middle|\, s \geq T, s(c+\varepsilon) \leq y \leq s(c_1^* - \varepsilon)\right\}. \tag{2.20}$$

In fact, if $u_2^{(0)}(s, y, \phi) < \frac{\epsilon}{10}$ and $(s, y) \in \left\{(s, y) \,\middle|\, s \geq T, s(c+\varepsilon) \leq y \leq s(c_1^* - \varepsilon)\right\}$, then

$$h_2^{(0)}(\rho, s, y) < \frac{\epsilon}{10}(\rho + \frac{\epsilon}{30}) = \frac{\epsilon^2}{150}.$$

And if $u_2^{(0)}(s, y, \phi) \geq \frac{\epsilon}{10}$ and $(s, y) \in \left\{(s, y) \,\middle|\, s \geq T, s(c+\varepsilon) \leq y \leq s(c_1^* - \varepsilon)\right\}$, then

$$h_2^{(0)}(\rho, s, y) \leq u_2^{(0)}(s, y, \phi)(\frac{\epsilon}{30} + \frac{\epsilon}{30} - \frac{\epsilon}{10}) \leq 0 < \frac{\epsilon^2}{150}.$$

Thus, (2.20) holds true.

For above given $\varepsilon > 0$, we write

$$\int_{\eta}^{A} e^{-\rho s} \int_{-\infty}^{+\infty} k_{21}(s, y) h_2^{(0)}(\rho, t-s, x-y) dy ds = \sum_{i=1}^{3} I_i(\varepsilon, t, x), \tag{2.21}$$

where

$$I_1(\varepsilon, t, x) = \int_{\eta}^{A} e^{-\rho s} \int_{-\infty}^{x-(c_1^*-\varepsilon)(t-s)} k_{21}(s, y) h_2^{(0)}(\rho, t-s, x-y) dy ds,$$

$$I_2(\varepsilon, t, x) = \int_{\eta}^{A} e^{-\rho s} \int_{x-(c_1^*-\varepsilon)(t-s)}^{x-(c+\varepsilon)(t-s)} k_{21}(s, y) h_2^{(0)}(\rho, t-s, x-y) dy ds,$$

and

$$I_3(\varepsilon, t, x) = \int_{\eta}^{A} e^{-\rho s} \int_{x-(c+\varepsilon)(t-s)}^{+\infty} k_{21}(s, y) h_2^{(0)}(\rho, t-s, x-y) dy ds.$$

It follows from $x \leq (c_1^* - 2\varepsilon)t$ and $y \leq x - (c_1^* - \varepsilon)(t - s)$ that

$$y \leq (c_1^* - 2\varepsilon)t - (c_1^* - \varepsilon)(t - s) = -\varepsilon t + (c_1^* - \varepsilon)s.$$

Therefore, by (2.14) and (2.16), for all $x \le (c_1^* - 2\varepsilon)t$ and $t > Ac_1^*/\varepsilon$, we have

$$\begin{aligned} I_1(\varepsilon, t, x) &\le r(\infty)(\rho + 2r(\infty) + a_2 r(\infty)) \int_\eta^A e^{-\rho s} \int_{-\infty}^{x-(c_1^*-\varepsilon)(t-s)} k_{21}(s, y)dyds \\ &\le r(\infty)(\rho + 2r(\infty) + a_2 r(\infty)) \int_\eta^A e^{-\rho s} \int_{-\infty}^{-\varepsilon t+(c_1^*-\varepsilon)s} k_{21}(s, y)dyds \\ &\le r(\infty)(\rho + 2r(\infty) + a_2 r(\infty)) \int_\eta^A e^{-\rho s} \int_{-\infty}^{-\varepsilon t+(c_1^*-\varepsilon)A} k_{21}(s, y)dyds \qquad (2.22) \\ &\le r(\infty)(\rho + 2r(\infty) + a_2 r(\infty)) \int_\eta^A e^{-\rho s} \int_{-\infty}^{[(c_1^*-\varepsilon)A-\varepsilon t]/\sqrt{4d_2 A}} \frac{1}{\sqrt{\pi}} e^{-z^2} dzds \\ &\le r(\infty)(\rho + 2r(\infty) + a_2 r(\infty)) \frac{1}{\rho\sqrt{\pi}} \int_{-\infty}^{[(c_1^*-\varepsilon)A-\varepsilon t]/\sqrt{4d_2 A}} e^{-z^2} dz. \end{aligned}$$

Obviously,

$$\lim_{t\to+\infty} \frac{(c_1^* - \varepsilon)A - \varepsilon t}{\sqrt{4d_2 A}} = -\infty.$$

Thus, for above given $\epsilon > 0$, there exists $t_1 > \max\left\{T, Ac_1^*/\varepsilon\right\}$ such that

$$I_1(\varepsilon, t, x) < \epsilon/5 \quad \text{for all } x \le (c_1^* - 2\varepsilon)t \text{ and } t > t_1. \qquad (2.23)$$

If

$$x - (c_1^* - \varepsilon)(t - s) \le y \le x - (c + \varepsilon)(t - s),$$

then

$$(c + \varepsilon)(t - s) \le x - y \le (c_1^* - \varepsilon)(t - s).$$

Therefore, it is clear from (2.20) that

$$\begin{aligned} h_2^{(0)}(\rho, t - s, x - y) < \frac{\epsilon^2}{150} \quad &\text{for all } t \ge T \text{ and} \\ x - (c_1^* - \varepsilon)(t - s) \le y \le x - (c + \varepsilon)(t - s), \end{aligned} \qquad (2.24)$$

where $s \in [\eta, A]$. Therefore, by (2.24), we obtain that

$$\begin{aligned} I_2(\varepsilon, t, x) &< \frac{\epsilon^2}{150} \int_\eta^A e^{-\rho s} \int_{x-(c_1^*-\varepsilon)(t-s)}^{x-(c+\varepsilon)(t-s)} k_{21}(s, y)dyds \\ &< \frac{\epsilon^2}{150\rho} = \frac{\epsilon}{5} \quad \text{for all } (t, x) \in [T, +\infty) \times \mathbb{R}. \end{aligned} \qquad (2.25)$$

It follows from $x \ge t(c + 2\varepsilon)$ and $y \ge x - (c + \varepsilon)(t - s)$ that

$$y \ge t(c + 2\varepsilon) - (c + \varepsilon)(t - s) = \varepsilon t + (c + \varepsilon)s. \qquad (2.26)$$

Therefore, by (2.14) and (2.16), for all $x \geq (c+2\varepsilon)t$, we have

$$\begin{aligned} I_3(\varepsilon,t,x) &\leq r(\infty)(\rho+2r(\infty)+a_2r(\infty))\int_\eta^A e^{-\rho s}\int_{x-(c+\varepsilon)(t-s)}^{+\infty} k_{21}(s,y)dyds \\ &\leq r(\infty)(\rho+2r(\infty)+a_2r(\infty))\int_\eta^A e^{-\rho s}\int_{\varepsilon t+(c+\varepsilon)s}^{+\infty} k_{21}(s,y)dyds \\ &\leq r(\infty)(\rho+2r(\infty)+a_2r(\infty))\int_\eta^A e^{-\rho s}\int_{\varepsilon t+(c+\varepsilon)\eta}^{+\infty} k_{21}(s,y)dyds \qquad (2.27) \\ &\leq r(\infty)(\rho+2r(\infty)+a_2r(\infty))\int_\eta^A e^{-\rho s}\int_{[\varepsilon t+(c+\varepsilon)\eta]/\sqrt{4d_2A}}^{+\infty} \frac{1}{\sqrt{\pi}}e^{-z^2}dzds \\ &\leq r(\infty)(\rho+2r(\infty)+a_2r(\infty))\frac{1}{\rho\sqrt{\pi}}\int_{[\varepsilon t+(c+\varepsilon)\eta]/\sqrt{4d_2A}}^{+\infty} e^{-z^2}dz. \end{aligned}$$

Since

$$\lim_{t\to+\infty}\frac{\varepsilon t+(c+\varepsilon)\eta}{\sqrt{4d_2A}}=+\infty,$$

(2.27) implies that for above $\epsilon>0$, there exists $t_2>t_1$ such that

$$I_3(\varepsilon,t,x)<\epsilon/5 \quad \text{for all } x\geq(c+2\varepsilon)t \text{ and } t>t_2. \tag{2.28}$$

Thus, it follows from (2.13), (2.17), (2.18), (2.19), (2.21), (2.23), (2.25) and (2.28) that

$$u_2^{(0)}(t,x,\phi)<\epsilon \quad \text{for all } (t,x)\in[t_2,+\infty)\times((c+2\varepsilon)t,(c_1^*-2\varepsilon)t),$$

which implies

$$\lim_{t\to\infty}\left[\sup_{(c+2\varepsilon)t\leq x\leq t(c_1^*-2\varepsilon)} u_2^{(0)}(t,x,\phi)\right]=0.$$

By [20] and the comparison principle, for the above $\varepsilon>0$,

$$\lim_{t\to\infty}\left[\sup_{x\leq(c-\varepsilon)t} u_2^{(0)}(t,x,\phi)\right]=0.$$

Because $\varepsilon>0$ is arbitrary, we have actually shown that for any $\varepsilon>0$,

$$\lim_{t\to\infty}\left[\sup_{x\leq(c_1^*-\varepsilon)t} u_2^{(0)}(t,x,\phi)\right]=0.$$

The proof is completed. □

Similarly, we consider the system

$$\begin{cases} \dfrac{\partial u_1}{\partial t}=d_1\dfrac{\partial^2 u_1}{\partial x^2}+u_1[r(x-ct)-u_1-a_1u_2^{(0)}], & t>0,\ x\in\mathbb{R}, \\ \dfrac{\partial u_2}{\partial t}=d_2\dfrac{\partial^2 u_2}{\partial x^2}+u_2[r(x-ct)-u_2-a_2u_1], & t>0,\ x\in\mathbb{R}, \\ u(0,x)=\phi(x), & x\in\mathbb{R}. \end{cases} \tag{2.29}$$

Let $(u_1^{(1)}(t,x,\phi), u_2^{(1)}(t,x,\phi))$ be the solution of the system (2.29). Then by the comparison principle and $u_2^{(0)} \geq 0$, we obtain that $u_1^{(1)} \leq u_1^{(0)}$ and $u_2^{(1)} \geq u_2^{(0)}$. And it is easy to see from [20] that for any $\varepsilon > 0$,

$$\lim_{t\to+\infty}\left[\sup_{x\leq t(c-\varepsilon)} u_1^{(1)}(t,x,\phi)\right] = 0 \tag{2.30}$$

and

$$\lim_{t\to+\infty}\left[\sup_{x\geq t(c_1^*+\varepsilon)} u_1^{(1)}(t,x,\phi)\right] = 0. \tag{2.31}$$

For any given $\varepsilon^{(1)}$ with $0 < \varepsilon^{(1)} < (c_1^*(\infty) - c)/3$, we choose a real number $\delta^{(1)} \in (0, r(\infty))$ such that

$$c + \varepsilon^{(1)} < 2\sqrt{d_1(r(\infty) - \delta^{(1)})} < c_1^*(\infty) - \varepsilon^{(1)}. \tag{2.32}$$

Then by Lemma 2.3, for above $\delta^{(1)} > 0$, there exists $T_1 > 0$ such that

$$u_2^{(0)}(t,x,\phi) < \delta^{(1)}/a_1 \tag{2.33}$$

for all $x \leq (c_1^*(\infty) - \varepsilon^{(1)})t$ and $t > T_1$. Obviously, if $\varepsilon^{(1)}$ is sufficiently small, then we can also choose a sufficiently small $\delta^{(1)}$ such that (2.32) holds true.

For the above given $T_1 > 0$, we consider the following equation

$$\begin{cases} \dfrac{\partial u_1}{\partial t} = d_1\dfrac{\partial^2 u_1}{\partial x^2} + u_1[R^{(1)} - u_1], \quad t > T_1, \ x \in \mathbb{R}, \\ u_1(T_1, x) = u_1^{(1)}(T_1, x, \phi), \quad x \in \mathbb{R}, \end{cases} \tag{2.34}$$

where $R^{(1)}(t,x) = r(x - ct) - a_1 u_2^{(0)}(t,x,\phi)$. It is easily seen that $u_1^{(1)}(t,x,\phi)$ satisfies (2.34).

For the above given $\delta^{(1)} > 0$, let $r^{(1)}(x) = r(x) - \delta^{(1)}$. Then $r^{(1)}(\infty) > 0$. Therefore, for $r^{(1)}(x) > 0$, define

$$c_{i,1}^*(x) = 2\sqrt{d_i r^{(1)}(x)}, \quad i = 1, 2.$$

It is easily seen that

$$c_{i,1}^*(x) = \inf_{\mu>0} g_{i,1}(x;\mu),$$

where

$$g_{i,1}(x;\mu) = \frac{d_i\mu^2 + r^{(1)}(x)}{\mu}, \quad i = 1, 2.$$

The infima occur at $\mu_{i,1}^*(x) = \sqrt{r^{(1)}(x)/d_i}, i = 1, 2$. It is easily seen that $g_{i,1}(x;\mu) > \psi_i(\mu)$ for all $0 < \mu < \mu_{i,1}^*(x)$ and $g_{i,1}(x;\mu_{i,1}^*(x)) = \psi_i(\mu_{i,1}^*(x)), i = 1, 2$. By (2.32), we have

$$c + \varepsilon^{(1)} < c_{1,1}^*(\infty) < c_1^*(\infty) - \varepsilon^{(1)}. \tag{2.35}$$

Definition 2.2 We call a function u_1 a continuous weak lower solution of Eq. (2.34) if u_1 is continuous on $[T_1, T] \times \mathbb{R}$, $u_1(T_1, x) \leq u_1^{(1)}(T_1, x, \phi)$ and

$$\frac{\partial u_1}{\partial t} \leq d_1\frac{\partial^2 u_1}{\partial x^2} + u_1[R^{(1)} - u_1]$$

in the distributional sense, i.e., for any $\eta \in C^{1,2}([T_1, T] \times \mathbb{R})$ with $\eta \geq 0$ and supp$\eta(t,\cdot) \Subset \mathbb{R}$ for all $t \in [T_1, T]$, there holds

$$\int_{-\infty}^{+\infty} u_1(t,x)\eta(t,x)dx\,\Big|_{t=T_1}^{t=T_\sigma}$$
$$\leq \int_{T_1}^{T_\sigma}\int_{-\infty}^{+\infty}\Big[u_1(s,x)\,(d_1\eta_{xx}+\eta_t)\,(s,x)+\eta(s,x)u_1(s,x)(R^{(1)}(s,x)-u_1(s,x))\Big]dxds,$$

for $T_\sigma \in [T_1, T]$.

Similar to Lemma 2.1, we obtain the following lemma for (2.34).

Lemma 2.4 *Assume that $0 < c < c_1^*(\infty)$. For any ϵ with $0 < \epsilon < \frac{c_{1,1}^*(\infty)-c-\varepsilon^{(1)}}{3}$, let ℓ be a number such that $c_{1,1}^*(\ell) = c_{1,1}^*(\infty) - \epsilon$. Let $0 < \mu_1 < \mu_2 < \mu_{1,1}^*(\ell)$ with $\psi_1(\mu_1) = c + \varepsilon^{(1)} + \epsilon$ and $\psi_1(\mu_2) = c_{1,1}^*(\infty) - 2\epsilon$. Then for any $\mu \in [\mu_1, \mu_2]$ and for the above given T_1, there exists sufficiently small $\beta > 0$ and $\gamma > 0$ such that $\beta\varphi(\mu; x - \ell - \psi_1(\mu)t)$ with φ given (2.7) is a continuous weak lower solution of (2.34). Furthermore, if $u_1^{(1)}(T_1, x, \phi) \geq \beta\varphi(\mu; x - \ell)$, then $u_1^{(1)}(t, x, \phi) \geq \beta\varphi(\mu; x - \ell - \psi_1(\mu)(t - T_1))$ for all $t > T_1$.*

Proof For any $T > T_1$, choose $T_\sigma \in [T_1, T]$ and $\eta \in C^{1,2}([T_1, T] \times \mathbb{R})$ with $\eta \geq 0$ and supp$\eta(t,\cdot) \Subset \mathbb{R}$ for all $t \in [T_1, T]$. By the definition of φ and integration by parts, we have

$$\begin{aligned}
&\int_{T_1}^{T_\sigma}\int_{-\infty}^{+\infty}\varphi(\mu; x-\ell-ls)\eta_{xx}(s,x)dxds\\
&\quad=\int_{T_1}^{T_\sigma}\int_{\ell+ls}^{\ell+ls+\pi/\gamma}\varphi(\mu; x-\ell-ls)\eta_{xx}(s,x)dxds\\
&\quad=\int_{T_1}^{T_\sigma}\int_{\ell+ls}^{\ell+ls+\pi/\gamma}\varphi_{xx}(\mu; x-\ell-ls)\eta(s,x)dxds\\
&\qquad+\gamma\int_{T_1}^{T_\sigma}\big[\eta(s,\ell+ls+\pi/\gamma)e^{-\pi\mu/\gamma}+\eta(s,\ell+ls)\big]ds
\end{aligned}\tag{2.36}$$

and

$$\begin{aligned}
&\int_{T_1}^{T_\sigma}\int_{-\infty}^{+\infty}\varphi(\mu; x-\ell-ls)\eta_t(s,x)dxds\\
&=\int_{-\infty}^{+\infty}\int_{T_1}^{T_\sigma}\varphi(\mu; x-\ell-ls)\eta_t(s,x)dsdx\\
&=\int_{-\infty}^{+\infty}\left[\eta(s,x)\varphi(\mu; x-\ell-ls)\,\Big|_{s=T_1}^{s=T_\sigma}-\int_{T_1}^{T_\sigma}\varphi_t(\mu; x-\ell-ls)\eta(s,x)ds\right]dx\\
&=\int_{-\infty}^{+\infty}\varphi(\mu; x-\ell-lt)\eta(t,x)dx\,\Big|_{t=T_1}^{t=T_\sigma}-\int_{T_1}^{T_\sigma}\int_{-\infty}^{+\infty}\eta(s,x)\varphi_t(\mu; x-\ell-ls)dxds,
\end{aligned}\tag{2.37}$$

where $l = \psi_1(\mu)$. It follows from the direct calculations that

$$(\varphi_t - d_1\varphi_{xx})(\mu; x-\ell-lt) = d_1(\mu^2+\gamma^2)\varphi(\mu; x-\ell-lt)\tag{2.38}$$

for all $\mu \in [\mu_1, \mu_2]$, $x \neq \ell + lt$ and $x \neq \ell + lt + \pi/\gamma$, where $l = \psi_1(\mu)$. Therefore, we have

$$\begin{aligned} r(\ell) - \delta^{(1)} - d_1\mu_2^2 &= r^{(1)}(\ell) - d_1\mu_2^2 \\ &= \frac{1}{4d_1}\left[\left(c^*_{1,1}(\ell)\right)^2 - (\psi_1(\mu_2))^2\right] \\ &= \frac{1}{4d_1}\left[\left(c^*_{1,1}(\infty) - \epsilon\right)^2 - \left(c^*_{1,1}(\infty) - 2\epsilon\right)^2\right] \\ &= \frac{\epsilon}{2d_1}\left(c^*_{1,1}(\infty) - 1.5\epsilon\right) \\ &\geq \frac{\epsilon}{2d_1}\left(c + \varepsilon^{(1)} + 1.5\epsilon\right). \end{aligned} \tag{2.39}$$

Let $H_1(t, x, u_1) = u_1(R^{(1)} - u_1)$. It follows from (2.36), (2.37), (2.38) and (2.39) that for any $\mu \in [\mu_1, \mu_2]$ and for sufficiently small $\beta > 0$ and $\gamma > 0$, $\widehat{u}_1(\mu; t, x) = \beta\varphi(\mu; x - \ell - \psi_1(\mu)t)$ satisfies

$$\begin{aligned} &\int_{T_1}^{T_\sigma}\int_{-\infty}^{+\infty} \left[\widehat{u}_1(\mu; s, x)\,(d_1\eta_{xx} + \eta_t)\,(s, x) + \eta(s, x)H_1(s, x, \widehat{u}_1(\mu; s, x))\right] dxds \\ &\quad - \int_{-\infty}^{+\infty} \widehat{u}_1(\mu; t, x)\eta(t, x)dx\,\Big|_{t=T_1}^{t=T_\sigma} \\ &= \int_{T_1}^{T_\sigma}\int_{\ell+ls}^{\ell+ls+\pi/\gamma} \left[-\beta d_1(\mu^2 + \gamma^2)\varphi(\mu; x - \ell - ls) + H_1(s, x, \widehat{u}_1(\mu; s, x))\right]\eta(s, x)dxds \\ &\quad + d_1\beta\gamma\int_{T_1}^{T_\sigma}\left[\eta(s, \ell + ls + \pi/\gamma)e^{-\pi\mu/\gamma} + \eta(s, \ell + ls)\right] ds \\ &\quad - \beta\int_{T_1}^{T_\sigma}\int_{\Omega(s)} \eta(s, x)\varphi_t(\mu; x - \ell - ls)dxds \\ &= \beta\int_{T_1}^{T_\sigma}\int_{\ell+ls}^{\ell+ls+\pi/\gamma} \left[r(x - cs) - a_1u_2^{(0)}(s, x, \phi) - d_1(\mu^2 + \gamma^2) - \beta\varphi(\mu; x - \ell - ls)\right] \\ &\quad \times \varphi(\mu; x - \ell - ls)\eta(s, x)dxds + d_1\beta\gamma\int_{T_1}^{T_\sigma}\left[\eta(s, \ell + ls + \pi/\gamma)e^{-\pi\mu/\gamma} + \eta(s, \ell + ls)\right] ds \\ &\geq \beta\int_{T_1}^{T_\sigma}\int_{\ell+ls}^{\ell+ls+\pi/\gamma} \left[r(\ell) - \delta^{(1)} - d_1(\mu_2^2 + \gamma^2) - \beta\varphi(\mu; x - \ell - ls)\right] \\ &\quad \times \varphi(\mu; x - \ell - ls)\eta(s, x)dxds + d_1\beta\gamma\int_{T_1}^{T_\sigma}\left[\eta(s, \ell + ls + \pi/\gamma)e^{-\pi\mu/\gamma} + \eta(s, \ell + ls)\right] ds \\ &\geq \beta\int_{T_1}^{T_\sigma}\int_{\ell+ls}^{\ell+ls+\pi/\gamma} \left[\frac{\epsilon}{2d_1}\left(c + \varepsilon^{(1)} + 1.5\epsilon\right) - d_1\gamma^2 - \beta\varphi(\mu; x - \ell - ls)\right] \\ &\quad \times \varphi(\mu; x - \ell - ls)\eta(s, x)dxds + d_1\beta\gamma\int_{T_1}^{T_\sigma}\left[\eta(s, \ell + ls + \pi/\gamma)e^{-\pi\mu/\gamma} + \eta(s, \ell + ls)\right] ds \\ &\geq 0, \end{aligned}$$

where $\Omega(s) = \{x | x \in \mathbb{R}, x \leq \ell + ls\} \bigcup \{x | x \in \mathbb{R}, x \geq \ell + ls + \pi/\gamma\}$ and $l = \psi_1(\mu)$. It follows from Definition 2.2 that for any $\mu \in [\mu_1, \mu_2]$ and for sufficiently small $\beta > 0$ and $\gamma > 0$, $\beta\varphi(\mu; x - \ell - \psi_1(\mu)t)$ is a continuous weak lower solution of (2.34).

If $u_1^{(1)}(T_1, x, \phi) \geq \beta\varphi(\mu; x - \ell)$, then it follows from Lemma 1.2 in Wang [37] that $u_1^{(1)}(t, x, \phi) \geq \beta\varphi(\mu; x - \ell - \psi_1(\mu)(t - T_1))$ for all $t > T_1$. The proof is completed. □

By Lemma 2.4, we obtain the following lemma.

Lemma 2.5 *Assume that* **(A)** *holds and* $0 < c < c_1^*$. *If* $0 \le \phi_1(x) \le r(\infty)$, *and* $\phi_1(x) > 0$ *on a closed interval, then for every* ε *with* $0 < \varepsilon < (c_1^* - c)/2$,

$$\lim_{t\to+\infty}\left[\sup_{t(c+\varepsilon)\le x\le t(c_1^*-\varepsilon)}\left|r(\infty)-u_1^{(1)}(t,x,\phi)\right|\right]=0.$$

The proof is similar to that of Theorem 2.2 in [20] with some minor modifications, and is given in the appendix for reader's convenience.

Obviously, by employing (2.29)–(2.31), Lemma 2.5 and the same arguments as in the proof of Lemmas 2.2 and 2.3, we obtain the following lemma.

Lemma 2.6 *Assume that (***A***) holds and* $0 < c < c_1^*(\infty) < c_2^*(\infty)$. *Then the following statements are valid.*

(i) *If* $0 \le \phi_i(x) \le r(\infty)$ *for* $i = 1, 2$, *and* $\phi_1(x) > 0$ *on a closed interval, then for any* $\varepsilon > 0$,

$$\lim_{t\to\infty}\left[\sup_{x\le t(c_1^*(\infty)-\varepsilon)} u_2^{(1)}(t,x,\phi)\right]=0;$$

(ii) *If* $0 \le \phi_2(x) \le r(\infty)$ *and* $\phi_2(x) \equiv 0$ *for all sufficiently large* x, *then for any* $\varepsilon > 0$,

$$\lim_{t\to\infty}\left[\sup_{x\ge t(c_2^*(\infty)+\varepsilon)} u_2^{(1)}(t,x,\phi)\right]=0.$$

(iii) *If* $0 \le \phi_2(x) \le r(\infty)$, *and* $\phi_2(x) > 0$ *on a closed interval, then for every* ε *with* $0 < \varepsilon < (c_2^*(\infty) - c_1^*(\infty))/2$,

$$\lim_{t\to\infty}\left[\sup_{t(c_1^*(\infty)+\varepsilon)\le x\le t(c_2^*(\infty)-\varepsilon)}\left|r(\infty)-u_2^{(1)}(t,x,\phi)\right|\right]=0.$$

Lemmas 2.2, 2.3, 2.5 and 2.6 motivates us to consider the following iteration scheme:

$$\begin{cases}\dfrac{\partial u_1^{(n)}}{\partial t}=d_1\dfrac{\partial^2 u_1^{(n)}}{\partial x^2}+u_1^{(n)}\left[r(x-ct)-u_1^{(n)}-a_1u_2^{(n-1)}\right], & t>0,\ x\in\mathbb{R},\\ \dfrac{\partial u_2^{(n)}}{\partial t}=d_2\dfrac{\partial^2 u_2^{(n)}}{\partial x^2}+u_2^{(n)}\left[r(x-ct)-u_2^{(n)}-a_2u_1^{(n)}\right], & t>0,\ x\in\mathbb{R},\\ u(0,x)=\phi(x),\quad x\in\mathbb{R},\\ n=1,2,\ldots.\end{cases}\tag{2.40}$$

With $\left(u_1^{(0)}, u_2^{(0)}\right)$ being the solution of the system (2.2), this iteration generates a sequence $\left\{\left(u_1^{(n)}, u_2^{(n)}\right)\right\}_{n=0}^{\infty}$ of functions. By Lemmas 2.2, 2.3, 2.5 and 2.6, this sequence obviously satisfies the following properties:

(a) the function sequence $\left\{u_1^{(n)}\right\}_{n=0}^{\infty}$ is non-increasing and the function sequence $\left\{u_2^{(n)}\right\}_{n=0}^{\infty}$ non-decreasing:

$$r(\infty)\ge u_1^{(0)}\ge u_1^{(1)}\ge\cdots\ge u_1^{(n)}\ge u_1^{(n+1)}\ge\cdots\ge 0$$

and

$$0 \le u_2^{(0)} \le u_2^{(1)} \le \cdots \le u_2^{(n)} \le u_2^{(n+1)} \le \cdots \le r(\infty);$$

(b) for any $\varepsilon > 0$,

$$\lim_{t\to\infty}\left[\sup_{x\le t(c-\varepsilon)} u_1^{(n)}(t,x,\phi)\right] = 0, \quad n = 1, 2, \ldots; \tag{2.41}$$

(c) if $\phi_i(x) \equiv 0$ for all sufficiently large x, then for any $\varepsilon > 0$,

$$\lim_{t\to\infty}\left[\sup_{x\ge t(c_i^*(\infty)+\varepsilon)} u_i^{(n)}(t,x,\phi)\right] = 0, \quad i = 1, 2, \quad n = 1, 2\ldots; \tag{2.42}$$

(d) if $\phi_1(x) > 0$ on a closed interval, then for every ε with $0 < \varepsilon < (c_1^*(\infty) - c)/2$,

$$\lim_{t\to\infty}\left[\sup_{t(c+\varepsilon)\le x\le t(c_1^*(\infty)-\varepsilon)} \left|r(\infty) - u_1^{(n)}(t,x,\phi)\right|\right] = 0 \tag{2.43}$$

and

$$\lim_{t\to\infty}\left[\sup_{x\le t(c_1^*(\infty)-\varepsilon)} u_2^{(n)}(t,x,\phi)\right] = 0, \quad n = 1, 2, \ldots; \tag{2.44}$$

(e) if $\phi_i(x) > 0$ on a closed interval for $i = 1, 2$, then for every ε with $0 < \varepsilon < (c_2^*(\infty) - c_1^*(\infty))/2$,

$$\lim_{t\to\infty}\left[\sup_{t(c_1^*(\infty)+\varepsilon)\le x\le t(c_2^*(\infty)-\varepsilon)} \left|r(\infty) - u_2^{(n)}(t,x,\phi)\right|\right] = 0, \quad n = 1, 2, \ldots. \tag{2.45}$$

By Property (a), the sequences $\left\{u_1^{(n)}\right\}_{n=0}^{\infty}$ and $\left\{u_2^{(n)}\right\}_{n=0}^{\infty}$ both converge pointwise, as $n \to \infty$, that is, there exist $u_1^*(t,x,\phi)$ and $u_2^*(t,x,\phi)$ such that for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$,

$$\lim_{n\to\infty} u_i^{(n)}(t,x,\phi) = u_i^*(t,x,\phi), \quad i = 1, 2. \tag{2.46}$$

Now we are in a position to prove our main result, Theorem 2.4.

Proof of Theorem 2.4 Denote by z the vector (t,x). For any given $T > 0$ and $M > 0$, let $\Lambda = [0,T] \times [-M,M]$. Now we show that (2.46) uniformly holds for all $z \in \Lambda$. Indeed, by (2.40), we can obtain that for any $n = 1, 2, \ldots$ and $p = 1, 2, \ldots,$

$$u_1^{(n)}(z,\phi) - u_1^{(n+p)}(z,\phi) = \int_0^t \int_{-\infty}^{+\infty} k_{11}(t-s, x-y) h_1(s,y,n,p)\,dyds, \quad \forall z \in \Lambda, \tag{2.47}$$

where

$$k_{11}(t,x) = \frac{1}{\sqrt{4\pi d_1 t}} e^{-\frac{x^2}{4d_1 t}} \tag{2.48}$$

and

$$\begin{aligned} h_1(s,y,n,p) = &\left[u_1^{(n+p)}(s,y) + u_1^{(n)}(s,y) + a_1 u_2^{(n-1)}(s,y) - r(y-cs)\right] \\ &\times \left[u_1^{(n+p)}(s,y) - u_1^{(n)}(s,y)\right] + a_1 u_1^{(n+p)}(s,y) \\ &\left[u_2^{(n+p-1)}(s,y) - u_2^{(n-1)}(s,y)\right]. \end{aligned} \tag{2.49}$$

Let $\widetilde{h}_1 = 2(2+a_1)r^2(\infty)$. Then $|h_1(s, y, n, p)| \leq \widetilde{h}_1$ for all $s \in \mathbb{R}_+$, $y \in \mathbb{R}$, $n = 1, 2, \ldots$ and $p = 1, 2, \ldots$

For any given $\varepsilon > 0$, there exists $L > 0$ such that

$$\int_{-L}^{L} \frac{1}{\sqrt{\pi}} e^{-x^2} dx \geq 1 - \frac{\varepsilon}{5T\widetilde{h}_1}.$$

Therefore, for any $s > 0$,

$$\int_{-L\sqrt{4d_1 s}}^{L\sqrt{4d_1 s}} \frac{1}{\sqrt{4\pi d_1 s}} e^{-\frac{x^2}{4d_1 s}} dx \geq 1 - \frac{\varepsilon}{5T\widetilde{h}_1}.$$

It is clear from (2.47) that

$$\begin{aligned}
&\left|u_1^{(n)}(z,\phi) - u_1^{(n+p)}(z,\phi)\right| \\
&= \left|\int_0^t \int_{-\infty}^{+\infty} k_{11}(s,y) h_1(t-s, x-y, n, p) dy ds\right| \\
&\leq \frac{\varepsilon}{5} + \left|\int_0^t \int_{-L\sqrt{4d_1 s}}^{L\sqrt{4d_1 s}} k_{11}(s,y) h_1(t-s, x-y, n, p) dy ds\right| \\
&= \frac{\varepsilon}{5} + \left|\int_0^t \int_{x-L\sqrt{4d_1(t-s)}}^{x+L\sqrt{4d_1(t-s)}} k_{11}(t-s, x-y) h_1(s, y, n, p) dy ds\right|, \quad \forall z \in \Lambda.
\end{aligned}$$

Let $\delta_1 = \frac{\varepsilon}{5\widetilde{h}_1}$. In the case of $t \leq \delta_1$, we obtain that

$$\begin{aligned}
&\left|u_1^{(n)}(z,\phi) - u_1^{(n+p)}(z,\phi)\right| \\
&\leq \frac{\varepsilon}{5} + \int_0^t \int_{x-L\sqrt{4d_1(t-s)}}^{x+L\sqrt{4d_1(t-s)}} k_{11}(t-s, x-y) \left|h_1(s, y, n, p)\right| dy ds \\
&\leq \frac{\varepsilon}{5} + \widetilde{h}_1 \int_0^{\delta_1} \int_{-\infty}^{+\infty} k_{11}(s,y) dy ds \\
&= \frac{2\varepsilon}{5}, \quad \forall z \in \Lambda.
\end{aligned}$$

In the case of $t > \delta_1$, we have

$$\begin{aligned}
&\left|\int_{t-\delta_1}^t \int_{x-L\sqrt{4d_1(t-s)}}^{x+L\sqrt{4d_1(t-s)}} k_{11}(t-s, x-y) h_1(s, y, n, p) dy ds\right| \\
&\leq \int_0^{\delta_1} \int_{-\infty}^{+\infty} k_{11}(s,y) \left|h_1(t-s, x-y, n, p)\right| dy ds \\
&\leq \widetilde{h}_1 \int_0^{\delta_1} \int_{-\infty}^{+\infty} k_{11}(s,y) dy ds \\
&= \frac{\varepsilon}{5}, \quad \forall z \in \Lambda.
\end{aligned}$$

Therefore, it follows that

$$\begin{aligned}
&\left|u_1^{(n)}(z,\phi)-u_1^{(n+p)}(z,\phi)\right|\\
&\quad\le \frac{\varepsilon}{5}+\left|\int_0^{t-\delta_1}\int_{x-L\sqrt{4d_1(t-s)}}^{x+L\sqrt{4d_1(t-s)}}k_{11}(t-s,x-y)h_1(s,y,n,p)dyds\right|\\
&\qquad+\left|\int_{t-\delta_1}^{t}\int_{x-L\sqrt{4d_1(t-s)}}^{x+L\sqrt{4d_1(t-s)}}k_{11}(t-s,x-y)h_1(s,y,n,p)dyds\right|\\
&\quad\le \frac{2\varepsilon}{5}+\left|\int_0^{t-\delta_1}\int_{x-L\sqrt{4d_1(t-s)}}^{x+L\sqrt{4d_1(t-s)}}k_{11}(t-s,x-y)h_1(s,y,n,p)dyds\right|,\quad \forall z\in\Lambda.
\end{aligned} \tag{2.50}$$

Let

$$\Lambda_1=[0,T]\times[-M-L\sqrt{4d_1T},M+L\sqrt{4d_1T}]$$

and

$$\Lambda_z=\left\{(s,y)|0\le s\le t-\delta_1,\, x-L\sqrt{4d_1(t-s)}\le y\le x+L\sqrt{4d_1(t-s)}\right\}.$$

Then $\Lambda_z\subset\Lambda_1$ is bounded. Thus, by Egorov's Theorem (see, e.g., Theorem 3.2.8 in [39]), for above $\varepsilon>0$ there exists a measurable subset Λ_ε of Λ_1 such that $m(\Lambda_1-\Lambda_\varepsilon)<\sqrt{4\pi d_1}(5\widetilde{h}_1)^{-\frac{3}{2}}\varepsilon^{\frac{3}{2}}$,

$$\lim_{n\to\infty}u_1^{(n)}(z,\phi)=u_1^*(z,\phi)\quad\text{and}\quad\lim_{n\to\infty}u_2^{(n)}(z,\phi)=u_2^*(z,\phi) \tag{2.51}$$

uniformly for all $z\in\Lambda_\varepsilon$, where $m(\Lambda_1-\Lambda_\varepsilon)$ is the measure of the set $\Lambda_1-\Lambda_\varepsilon$. So, for above $\varepsilon>0$ there exist $K_\varepsilon>0$ and $P_\varepsilon>0$ such that

$$\begin{cases}\left|u_1^{(n+p)}(z,\phi)-u_1^{(n)}(z,\phi)\right|<\dfrac{2\varepsilon}{5T(3+2a_1)r(\infty)},\\[2mm] \left|u_2^{(n+p-1)}(z,\phi)-u_2^{(n-1)}(z,\phi)\right|<\dfrac{2\varepsilon}{5T(3+2a_1)r(\infty)},\end{cases}\quad\text{for all } n>K_\varepsilon,\ p>P_\varepsilon,\ z\in\Lambda_\varepsilon. \tag{2.52}$$

It follows from (2.49) and (2.52) that

$$|h_1(s,y,n,p)|\le\frac{2\varepsilon}{5T}\quad\text{for all } n>K_\varepsilon,\ p>P_\varepsilon,\ (s,y)\in\Lambda_\varepsilon. \tag{2.53}$$

By (2.50) and (2.53), we obtain that

$$
\begin{aligned}
&\left|u_1^{(n)}(z,\phi)-u_1^{(n+p)}(z,\phi)\right| \\
&\leq \frac{2\varepsilon}{5}+\int\int_{\Lambda_\varepsilon\bigcap\Lambda_z} k_{11}(t-s,x-y)\,|h_1(s,y,n,p)|\,dyds \\
&\quad+\int\int_{(\Lambda_1-\Lambda_\varepsilon)\bigcap\Lambda_z} k_{11}(t-s,x-y)\,|h_1(s,y,n,p)|\,dyds \\
&\leq \frac{2\varepsilon}{5}+\frac{2\varepsilon}{5T}\int\int_{\Lambda_\varepsilon\bigcap\Lambda_z} k_{11}(t-s,x-y)dyds \\
&\quad+\widetilde{h}_1\int\int_{(\Lambda_1-\Lambda_\varepsilon)\bigcap\Lambda_z} k_{11}(t-s,x-y)dyds \qquad (2.54)\\
&\leq \frac{2\varepsilon}{5}+\frac{2\varepsilon}{5T}\int_0^t\int_{-\infty}^{+\infty} k_{11}(s,y)dyds+\widetilde{h}_1\int\int_{\Lambda_2^*}\frac{1}{\sqrt{\pi}}e^{-y^2}dyds \\
&\leq \frac{4\varepsilon}{5}+\frac{\widetilde{h}_1}{\sqrt{\pi}}m(\Lambda_2^*) \\
&\leq \frac{4\varepsilon}{5}+\frac{\widetilde{h}_1}{\sqrt{4d_1\pi\delta_1}}m(\Lambda_1-\Lambda_\varepsilon) \\
&\leq \varepsilon \quad \text{for all } n>K_\varepsilon,\ p>P_\varepsilon,\ z\in\Lambda,
\end{aligned}
$$

where $\Lambda_2^*=f((\Lambda_1-\Lambda_\varepsilon)\bigcap\Lambda_z)$, $m(\Lambda_2^*)$ is the measure of the set Λ_2^* and $f:(\Lambda_1-\Lambda_\varepsilon)\bigcap\Lambda_z\to\mathbb{R}^2$ is a bijective function defined by

$$f(s,y)=(s,\frac{y-x}{\sqrt{4d_1(t-s)}}),\quad \forall(s,y)\in(\Lambda_1-\Lambda_\varepsilon)\bigcap\Lambda_z.$$

Thus, the convergence in $\lim_{n\to\infty}u_1^{(n)}(z,\phi)=u_1^*(z,\phi)$ is uniform for all $z\in\Lambda$. Similarly, we obtain that the limit $\lim_{n\to\infty}u_2^{(n)}(z,\phi)=u_2^*(z,\phi)$ uniformly holds for all $z\in\Lambda$. Because T and M are arbitrary, we have actually shown that the function sequence $\left\{\left(u_1^{(n)},u_2^{(n)}\right)\right\}_{n=0}^{\infty}$ converges to (u_1^*,u_2^*) uniformly on each bounded subset of $\mathbb{R}_+\times\mathbb{R}$. Therefore, it follows from (2.40) that (u_1^*,u_2^*) is a solution to (2.1) with the initial function ϕ. By the uniqueness of solution to the initial value problem associated to (2.1), we have $(u_1^*,u_2^*)=(u_1(t,x,\phi),u_2(t,x,\phi))$.

What left is to pass the properties (b)–(e) for the sequence $(u_1^{(n)},u_2^{(n)})$ to its limit $(u_1^{(*)},u_2^{(*)})$ to obtain the conclusions in (i)–(iv) from (b)–(e) respectively.

We first show Theorem 2.4-(i) holds. By (2.41) in (b), for any $\varepsilon>0$ and for any $\delta>0$, there exists a positive constant T_δ such that

$$
\begin{aligned}
&\left|u_1^{(n)}(z,\phi)\right|<\frac{\delta}{2}\ \text{ for}\\
&z\in\Lambda_0=\{(t,x)|T_\delta\leq t\leq T_\delta+\tau,-M\leq x\leq t(c-\varepsilon)\},\ M>0,\ \tau>0. \qquad (2.55)
\end{aligned}
$$

For above $\delta>0$, there exists $K_{\delta,\Lambda_0}>0$ such that

$$\left|u_1^{(n)}(z,\phi)-u_1^*(z,\phi)\right|<\frac{\delta}{2}\quad\text{for all } z\in\Lambda_0,\ n>K_{\delta,\Lambda_0}. \qquad (2.56)$$

Thus, for above $\delta>0$ there holds

$$\left|u_1^*(z,\phi)\right|\leq\left|u_1^{(n)}(z,\phi)\right|+\left|u_1^{(n)}(z,\phi)-u_1^*(z,\phi)\right|<\delta\quad\text{for all } z\in\Lambda_0.$$

Because τ and M are arbitrary, we have actually shown that

$$|u_1^*(z,\phi)| < \delta \quad \text{for all } z \in \{(t,x)|t \geq T_\delta, x \leq t(c-\varepsilon)\}.$$

Since $\delta > 0$ is arbitrary, we conclude that

$$\lim_{t\to\infty}\left[\sup_{x\leq t(c-\varepsilon)} u_1(t,x,\phi)\right] = \lim_{t\to\infty}\left[\sup_{x\leq t(c-\varepsilon)} u_1^*(t,x,\phi)\right] = 0, \tag{2.57}$$

proving Theorem 2.4-(i). Applying the similar arguments to (2.42), (2.43), (2.44) and (2.45), we can obtain (ii), (iii) and (iv) of the theorem respectively. The proof is completed. □

3 Some Numeric Simulations

In this section, we present some numerical simulations to model (2.1) with

$$r(x) = \frac{1.7}{1+e^{-0.4x}} - 0.7 \tag{3.1}$$

and the initial data

$$\phi_1(x) = \begin{cases} 0.9\sin(x-16), & \text{if } 16 \leq x \leq 16+\pi, \\ 0, & \text{elsewhere} \end{cases} \tag{3.2}$$

and

$$\phi_2(x) = \begin{cases} 0.5\sin(x-11), & \text{if } 11 \leq x \leq 11+\pi, \\ 0, & \text{elsewhere.} \end{cases} \tag{3.3}$$

Firstly, we choose

$$d_1 = 0.81, \quad d_2 = 1, \quad a_1 = 0.2, \quad a_2 = 1.5. \tag{3.4}$$

Then we can calculate to obtain $c_1^*(\infty) = 1.8$, $c_2^*(\infty) = 2$.

Now, if $c = 2.1$, then, $c > c_2^*(\infty)$, a scenario that the environment is worsening very fast. Not surprisingly, the two species will eventually go to extinction, as the simulation results show in Fig. 1.

Next, we consider a case that worsening speed c is a little bit slower: $c = 1.83 \in (c_1^*(\infty), c_2^*(\infty))$, a scenario that the spreading capability of species 1 without competition is not enough to allow this species to survive the environment worsening speed, but the spreading capability of species 2 without competition enables it to survive the environment worsening speed. The numerical results, presented in Fig. 2, indicate that species 1 eventually becomes extinct in the habitat and the species 2 persists by spreading to the right with spread speed $c_2^*(\infty) = 2$.

We further consider an even smaller value of c, $c = 1.4$. Then, $c < c_1^*(\infty)$, a scenario of Theorem 2.4. The numeric simulations (see Fig. 3) confirm that the two competing species co-persist in a spreading pattern, and their respective asymptotical spreading speeds seem to be $c_1^*(\infty) = 1.8$ and $c_2^*(\infty) = 2$, respectively.

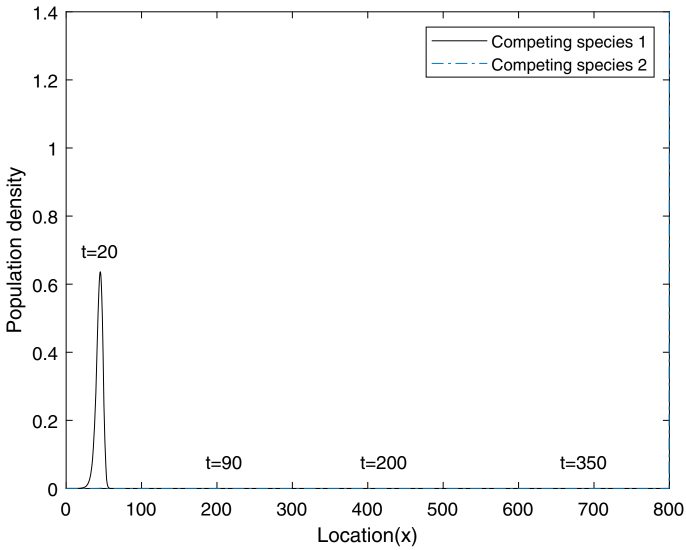

Fig. 1 Numerical simulations on (2.1) with (3.1)–(3.4): when the environment worsening rate is too large ($c = 2.1 > c_2^*(\infty)$), both species go to extinct in the habitat

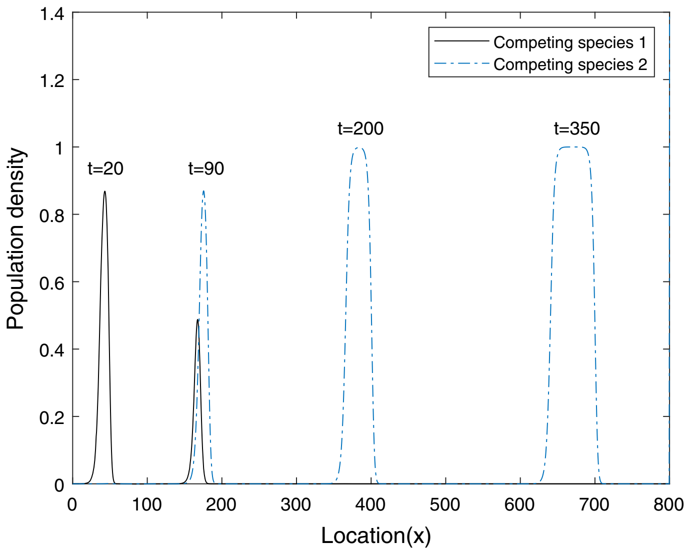

Fig. 2 Numerical simulations on (2.1) with (3.1)–(3.4): when the environment worsening rate is neutral in the sense that $c = 1.83 \in (c_1^*(\infty), c_2^*(\infty))$, species 1 becomes extinct in the habitat and species 2 persist by spreading to the right with the speed $c_2^*(\infty) = 2$

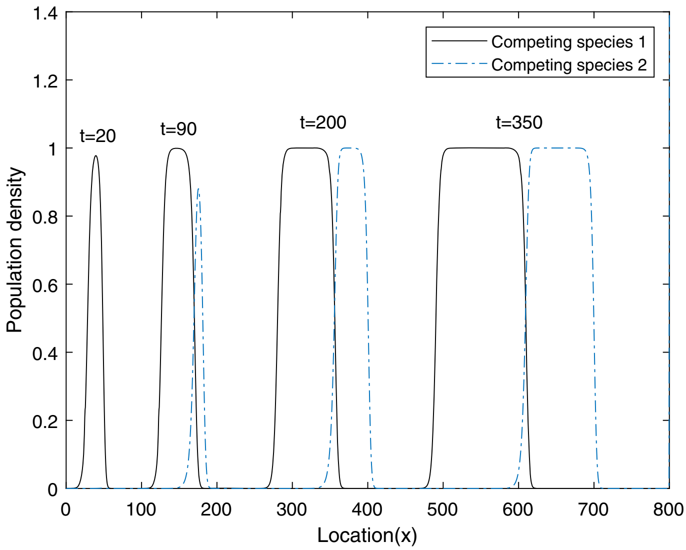

Fig. 3 Numerical simulations on (2.1) with (3.1)–(3.4): when the environment worsening rate is very small ($c = 1.4$) in the sense that $c < c_1^*(\infty)$, both species co-persist by spreading to the right respective speeds $c_1^*(\infty) = 1.8$ and $c_2^*(\infty) = 2$

4 Conclusion and Discussion

We have seen that the two combined parameters $c_i^*(\infty) = 2\sqrt{d_i r(\infty)}$, $i = 1, 2$ play a crucial role in determining whether the species i will go to extinction or can persist by spreading. Summarizing the results in this paper and those in [40,41], one can conclude that when the habitat is worsening with a constant speed c and the grow functions satisfying (A), competition strengths become less important and it is the the intrinsic spreading speeds $c_1^*(\infty)$ and $c_2^*(\infty)$ that mainly determine the extinction or persistence of the respective species. This is understandable because in the assumption (A), $r(-\infty)$ is assumed to be negative, implying that the habitat worsening is severe or extreme. A natural question then arises: what happens if $r(-\infty) \geq 0$, meaning that the habitat worsening is mild. In such a scenario of mild worsening for the habitat, we would expect that the competition strengths will play an important role because a species will be able to persist without spreading to the right, and hence. Under such a circumstance, there may be a trade-off between diffusion and local competition in terms of the species' fitness. We believe that there should be some more interesting yet challenging mathematical problems in studying the interplays of d_i, a_i and $r(\infty)$ and $r(-\infty)$ if the the condition $r(-\infty) < 0$ is replaced by $r(-\infty) \geq 0$. We leave this as an *open problem* for future exploration.

Appendix

In this appendix, we give the detailed proof of Lemmas 2.2 and 2.5.

Proof of Lemma 2.2 For any ϵ with $0 < \epsilon < \min\left\{\frac{1}{3}, r^{(0)}(\infty), \frac{c_{2,0}^*(\infty)-c_1^*(\infty)-\varepsilon^{(0)}}{3}\right\}$, let ℓ be a real number such that

$$c_{2,0}^*(\ell) = c_{2,0}^*(\infty) - \epsilon.$$

We choose $0 < \mu_1 < \mu_2 < \mu_{2,0}^*(\ell)$ such that $\psi_2(\mu_1) = c_1^*(\infty) + \varepsilon^{(0)} + \epsilon$ and $\psi_2(\mu_2) = c_{2,0}^*(\infty) - 2\epsilon$. By Lemma 2.1, for any $\mu \in [\mu_1, \mu_2]$ and sufficiently small $\beta > 0$ and $\gamma > 0$, $\frac{\beta}{\varphi(\mu;\sigma(\mu))}\varphi(\mu; x - \ell - \psi_2(\mu)t)$ with φ given by (2.7) is a continuous weak lower solution of (2.5).

Since $\phi_2(x) \geq 0$ and $\phi_2(x) \not\equiv 0$, $u_2^{(0)}(t, x, \phi) > 0$ for all $(t, x) \in (0, +\infty) \times \mathbb{R}$. Choose $T_0 < t_0 < T_0 + \frac{\sigma(\mu_1)}{\psi_2(\mu_1)}$, and choose sufficiently small $\beta > 0$ and $\gamma > 0$ such that

$$\frac{\pi/\gamma + \sigma(\mu_1) - \sigma(\mu_2)}{\psi_2(\mu_2) - \psi_2(\mu_1)} > T_0 \tag{5.1}$$

and

$$u_2^{(0)}(t_0, x, \phi) \geq \beta, \quad \forall x \in [\ell + \psi_2(\mu_1)T_0, \ell + 4\pi/\gamma + \psi_2(\mu_2)T_0]. \tag{5.2}$$

Define

$$w(T_0, x) = \begin{cases} \frac{\beta}{\varphi(\mu_1;\sigma(\mu_1))}\varphi(\mu_1; x - \ell - \psi_2(\mu_1)T_0) \\ \quad \text{if } \ell + \psi_2(\mu_1)T_0 \leq x \leq \ell + \sigma(\mu_1) + \psi_2(\mu_1)T_0; \\ \beta \quad \text{if } \ell + \sigma(\mu_1) + \psi_2(\mu_1)T_0 \leq x \leq \ell + 3\pi/\gamma + \sigma(\mu_2) + \psi_2(\mu_2)T_0; \\ \frac{\beta}{\varphi(\mu_2;\sigma(\mu_2))}\varphi(\mu_2; x - \ell - 3\pi/\gamma - \psi_2(\mu_2)T_0) \\ \quad \text{if } \ell + 3\pi/\gamma + \sigma(\mu_2) + \psi_2(\mu_2)T_0 \leq x \leq \ell + 4\pi/\gamma + \psi_2(\mu_2)T_0; \\ 0 \quad \text{elsewhere.} \end{cases}$$

It is easily seen that

$$w(T_0, x) \geq \frac{\beta}{\varphi(\mu_1; \sigma(\mu_1))}\varphi(\mu_1; x - \ell - \psi_2(\mu_1)T_0 - s), \quad \forall s \in [0, 2\pi/\gamma]$$

and

$$w(T_0, x) \geq \frac{\beta}{\varphi(\mu_2; \sigma(\mu_2))}\varphi(\mu_2; x - \ell - 3\pi/\gamma - \psi_2(\mu_2)T_0 + s), \quad \forall s \in [0, 2\pi/\gamma].$$

By (5.2) and Lemma 2.1, for any $t \geq t_0$, we have

$$\begin{aligned} &u_2^{(0)}(t, x, \phi) \\ &\quad \geq \frac{\beta}{\varphi(\mu_1; \sigma(\mu_1))}\varphi(\mu_1; x - \ell - \psi_2(\mu_1)(T_0 + t - t_0) - s), \quad \forall s \in [0, 2\pi/\gamma] \end{aligned} \tag{5.3}$$

and

$$\begin{aligned} u_2^{(0)}(t, x, \phi) \geq \frac{\beta}{\varphi(\mu_2; \sigma(\mu_2))}&\varphi(\mu_2; x - \ell - 3\pi/\gamma \\ &-\psi_2(\mu_2)(T_0 + t - t_0) + s), \quad \forall s \in [0, 2\pi/\gamma]. \end{aligned} \tag{5.4}$$

Inequalities (5.3) and (5.4) imply that for any $t \geq t_0$,

$$u_2^{(0)}(t,x,\phi) \geq \begin{cases} \frac{\beta}{\varphi(\mu_1;\sigma(\mu_1))}\varphi(\mu_1; x-\ell-\psi_2(\mu_1)(T_0+t-t_0)) \\ \quad \text{if } \ell+\psi_2(\mu_1)(T_0+t-t_0) \leq x \leq \ell+\sigma(\mu_1)+\psi_2(\mu_1)(T_0+t-t_0); \\ \beta \quad \text{if } \ell+\sigma(\mu_1)+\psi_2(\mu_1)(T_0+t-t_0) \leq x \leq \ell+2\pi/\gamma \\ \quad +\sigma(\mu_1)+\psi_2(\mu_1)(T_0+t-t_0); \\ 0 \quad \text{elsewhere} \end{cases} \tag{5.5}$$

and

$$u_2^{(0)}(t,x,\phi) \geq \begin{cases} \beta \quad \text{if } \ell+\pi/\gamma+\sigma(\mu_2)+\psi_2(\mu_2)(T_0+t-t_0) \leq x \leq \ell+3\pi/\gamma \\ \quad +\sigma(\mu_2)+\psi_2(\mu_2)(T_0+t-t_0); \\ \frac{\beta}{\varphi(\mu_2;\sigma(\mu_2))}\varphi(\mu_2; x-\ell-3\pi/\gamma-\psi_2(\mu_2)(T_0+t-t_0)) \\ \quad \text{if } \ell+3\pi/\gamma+\sigma(\mu_2)+\psi_2(\mu_2)(T_0+t-t_0) \leq x \leq \ell+4\pi/\gamma \\ \quad +\psi_2(\mu_2)(T_0+t-t_0); \\ 0 \quad \text{elsewhere.} \end{cases} \tag{5.6}$$

Let

$$h = \frac{\pi/\gamma+\sigma(\mu_1)-\sigma(\mu_2)}{\psi_2(\mu_2)-\psi_2(\mu_1)} - T_0.$$

Then (5.1) implies that $h > 0$. Since

$$\begin{aligned} &\ell+2\pi/\gamma+\sigma(\mu_1)+\psi_2(\mu_1)(T_0+t-t_0) \\ &\geq \ell+\pi/\gamma+\sigma(\mu_2)+\psi_2(\mu_2)(T_0+t-t_0), \quad \forall t \in [t_0, t_0+h], \end{aligned}$$

inequalities (5.5) and (5.6) imply that

$$u_2^{(0)}(t,x,\phi) \geq w(t-t_0^*, x), \quad \forall t \in [t_0, t_0+h], \tag{5.7}$$

where $t_0^* = t_0 - T_0$ and

$$w(t-t_0^*,x) = \begin{cases} \frac{\beta}{\varphi(\mu_1;\sigma(\mu_1))}\varphi(\mu_1; x-\ell-\psi_2(\mu_1)(t-t_0^*)) \\ \quad \text{if } \ell+\psi_2(\mu_1)(t-t_0^*) \leq x \leq \ell+\sigma(\mu_1)+\psi_2(\mu_1)(t-t_0^*); \\ \beta \quad \text{if } \ell+\sigma(\mu_1)+\psi_2(\mu_1)(t-t_0^*) \leq x \leq \ell+3\pi/\gamma+\sigma(\mu_2)+\psi_2(\mu_2)(t-t_0^*); \\ \frac{\beta}{\varphi(\mu_2;\sigma(\mu_2))}\varphi(\mu_2; x-\ell-3\pi/\gamma-\psi_2(\mu_2)(t-t_0^*)) \\ \quad \text{if } \ell+3\pi/\gamma+\sigma(\mu_2)+\psi_2(\mu_2)(t-t_0^*) \leq x \leq \ell+4\pi/\gamma+\psi_2(\mu_2)(t-t_0^*); \\ 0 \quad \text{elsewhere.} \end{cases} \tag{5.8}$$

We claim that (5.7) holds true for all $t \geq t_0$. Assume that (5.7) is valid for $t \in [t_0, t_0 + nh]$ for some positive integer n. Then for any $s \in [0, 2\pi/\gamma + (\psi_2(\mu_2) - \psi_2(\mu_1))(T_0 + nh)]$,

$$w(T_0 + nh, x) \geq \frac{\beta}{\varphi(\mu_1; \sigma(\mu_1))}\varphi(\mu_1; x - \ell - \psi_2(\mu_1)(T_0 + nh) - s)$$

and

$$w(T_0 + nh, x) \geq \frac{\beta}{\varphi(\mu_2; \sigma(\mu_2))}\varphi(\mu_2; x - \ell - 3\pi/\gamma - \psi_2(\mu_2)(T_0 + nh) + s).$$

Therefore, Lemma 2.1 implies that for any $t \geq t_0 + nh$ and $s \in [0, 2\pi/\gamma + (\psi_2(\mu_2) - \psi_2(\mu_1))(T_0 + nh)]$,

$$u_2^{(0)}(t, x, \phi) \geq \frac{\beta}{\varphi(\mu_1; \sigma(\mu_1))}\varphi(\mu_1; x - \ell - \psi_2(\mu_1)(T_0 + nh) - \psi_2(\mu_1)(t - (t_0 + nh)) - s) \tag{5.9}$$

and

$$\begin{aligned} u_2^{(0)}(t, x, \phi) \geq \frac{\beta}{\varphi(\mu_2; \sigma(\mu_2))}\varphi(\mu_2; x - \ell - 3\pi/\gamma \\ -\psi_2(\mu_2)(T_0 + nh) - \psi_2(\mu_2)(t - (t_0 + nh)) + s). \end{aligned} \tag{5.10}$$

Inequalities (5.9) and (5.10) imply that for any $t \geq t_0 + nh$,

$$u_2^{(0)}(t, x, \phi) \geq \begin{cases} \frac{\beta}{\varphi(\mu_1;\sigma(\mu_1))}\varphi(\mu_1; x - \ell - \psi_2(\mu_1)(T_0 + t - t_0)) \\ \quad \text{if } \ell + \psi_2(\mu_1)(T_0 + t - t_0) \leq x \leq \ell + \sigma(\mu_1) + \psi_2(\mu_1)(T_0 + t - t_0); \\ \beta \quad \text{if } \ell + \sigma(\mu_1) + \psi_2(\mu_1)(T_0 + t - t_0) \leq x \leq \ell + 2\pi/\gamma \\ \quad +\sigma(\mu_1) + \psi_2(\mu_1)(t - (t_0 + nh)) + \psi_2(\mu_2)(T_0 + nh); \\ 0 \quad \text{elsewhere} \end{cases} \tag{5.11}$$

and

$$u_2^{(0)}(t, x, \phi) \geq \begin{cases} \beta \quad \text{if } \ell + \pi/\gamma + \sigma(\mu_2) + \psi_2(\mu_2)(t - (t_0 + nh)) + \psi_2(\mu_1)(T_0 + nh) \\ \quad \leq x \leq \ell + 3\pi/\gamma + \sigma(\mu_2) + \psi_2(\mu_2)(T_0 + t - t_0); \\ \frac{\beta}{\varphi(\mu_2;\sigma(\mu_2))}\varphi(\mu_2; x - \ell - 3\pi/\gamma - \psi_2(\mu_2)(T_0 + t - t_0)) \\ \quad \text{if } \ell + 3\pi/\gamma + \sigma(\mu_2) + \psi_2(\mu_2)(T_0 + t - t_0) \leq x \leq \ell + 4\pi/\gamma \\ \quad +\psi_2(\mu_2)(T_0 + t - t_0); \\ 0 \quad \text{elsewhere.} \end{cases} \tag{5.12}$$

Since

$$\begin{aligned} &\ell + 2\pi/\gamma + \sigma(\mu_1) + \psi_2(\mu_1)(t - (t_0 + nh)) + \psi_2(\mu_2)(T_0 + nh) \\ &\geq \ell + \pi/\gamma + \sigma(\mu_2) + \psi_2(\mu_2)(t - (t_0 + nh)) + \psi_2(\mu_1)(T_0 + nh) \end{aligned}$$

for all $t \in [t_0 + nh, t_0 + (2n + 1)h + 2T_0]$, inequalities (5.11) and (5.12) imply that (5.7) holds true for all $t \in [t_0 + nh, t_0 + (n + 1)h]$. By induction, (5.7) is true for all $t \geq t_0$.

For the chosen $\epsilon > 0$ and $\beta > 0$, there exists $L > 0$ such that

$$\int_{-L}^{L} \frac{1}{\sqrt{\pi}} e^{-x^2} dx \geq 1 - \beta\epsilon.$$

Therefore, for any $s > 0$, we have

$$\int_{-L\sqrt{4d_2 s}}^{L\sqrt{4d_2 s}} \frac{1}{\sqrt{4\pi d_2 s}} e^{-\frac{x^2}{4d_2 s}} dx \geq 1 - \beta\epsilon.$$

Let $t_1 > t_0$ be sufficiently large. Then, for $t > t_1$, the solution $u_2^{(0)}(t, x, \phi)$ of (2.5) satisfies the integral equation

$$\begin{aligned} u_2^{(0)}(t, x, \phi) = &\int_{-\infty}^{+\infty} k_2(t - t_1, x - y) u_2^{(0)}(t_1, y, \phi) dy \\ &+ \int_{t_1}^{t} \int_{-\infty}^{+\infty} k_2(t - s, x - y) u_2^{(0)}(s, y, \phi) \left[\rho + R^{(0)}(s, y) - u_2^{(0)}(s, y, \phi)\right] dyds, \end{aligned} \tag{5.13}$$

where $\rho > 3r(\infty) - r(-\infty)$ is a real number and

$$k_2(t, x) = \frac{1}{\sqrt{4\pi d_2 t}} e^{-\rho t - \frac{x^2}{4d_2 t}}. \tag{5.14}$$

It follows from (5.7) and (5.13) that for any $t > t_1$,

$$\begin{aligned} u_2^{(0)}(t, x, \phi) \geq &\int_{-\infty}^{+\infty} k_2(t - t_1, x - y) w(t_1 - t_0^*, y) dy \\ &+ \int_{t_1}^{t} \int_{-\infty}^{+\infty} k_2(t - s, x - y) w(s - t_0^*, y) \left[\rho + R^{(0)}(s, y) - w(s - t_0^*, y)\right] dyds. \end{aligned} \tag{5.15}$$

For $t > t_1$ and x, y satisfying

$$\begin{aligned} &\ell + \sigma(\mu_1) + \psi_2(\mu_1)(t_1 - t_0^*) + L\sqrt{4d_2(t - t_1)} \\ &\quad \leq x \leq \ell + \sigma(\mu_2) + \psi_2(\mu_2)(t_1 - t_0^*) + 3\pi/\gamma - L\sqrt{4d_2(t - t_1)} \end{aligned} \tag{5.16}$$

and

$$-L\sqrt{4d_2(t - t_1)} \leq y \leq L\sqrt{4d_2(t - t_1)}, \tag{5.17}$$

we have that

$$\ell + \sigma(\mu_1) + \psi_2(\mu_1)(t_1 - t_0^*) \leq x - y \leq \ell + \sigma(\mu_2) + \psi_2(\mu_2)(t_1 - t_0^*) + 3\pi/\gamma. \tag{5.18}$$

It follows from (5.8) and (5.18) that

$$\begin{aligned}
&\int_{-\infty}^{+\infty} k_2(t-t_1, x-y) w(t_1 - t_0^*, y) dy \\
&= \int_{-\infty}^{+\infty} k_2(t-t_1, y) w(t_1 - t_0^*, x-y) dy \\
&\geq e^{-\rho(t-t_1)} \int_{-L\sqrt{4d_2(t-t_1)}}^{L\sqrt{4d_2(t-t_1)}} \frac{1}{\sqrt{4\pi d_2(t-t_1)}} e^{-\frac{y^2}{4d_2(t-t_1)}} w(t_1 - t_0^*, x-y) dy \\
&= \beta e^{-\rho(t-t_1)} \int_{-L\sqrt{4d_2(t-t_1)}}^{L\sqrt{4d_2(t-t_1)}} \frac{1}{\sqrt{4\pi d_2(t-t_1)}} e^{-\frac{y^2}{4d_2(t-t_1)}} dy \\
&\geq (1-\beta\epsilon)\beta e^{-\rho(t-t_1)}
\end{aligned} \tag{5.19}$$

for all x satisfying (5.16). For $t > t_1$ and x, y satisfying

$$\begin{aligned}
&\ell + \sigma(\mu_1) + \psi_2(\mu_1)(s - t_0^*) + L\sqrt{4d_2(t-s)} \\
&\quad \leq x \leq \ell + \sigma(\mu_2) + \psi_2(\mu_2)(s - t_0^*) + 3\pi/\gamma - L\sqrt{4d_2(t-s)}, \quad \forall s \in [t_1, t]
\end{aligned} \tag{5.20}$$

and

$$-L\sqrt{4d_2(t-s)} \leq y \leq L\sqrt{4d_2(t-s)}, \quad \forall s \in [t_1, t], \tag{5.21}$$

we have

$$\begin{aligned}
&\ell + \sigma(\mu_1) + \psi_2(\mu_1)(s - t_0^*) \\
&\quad \leq x - y \leq \ell + \sigma(\mu_2) + \psi_2(\mu_2)(s - t_0^*) + 3\pi/\gamma, \quad \forall s \in [t_1, t]
\end{aligned} \tag{5.22}$$

and

$$x - y - cs \geq \ell + \sigma(\mu_1) + \psi_2(\mu_1)(s - t_0^*) - cs > \ell + \sigma(\mu_1) + \psi_2(\mu_1) T_0 > \ell, \quad \forall s \in [t_1, t]. \tag{5.23}$$

Then it follows from (5.8), (5.22) and (5.23) that

$$\begin{aligned}
&\int_{t_1}^{t} \int_{-\infty}^{+\infty} k_2(t-s, x-y) w(s - t_0^*, y) \left[\rho + R^{(0)}(s, y) - w(s - t_0^*, y)\right] dy ds \\
&= \int_{t_1}^{t} \int_{-\infty}^{+\infty} k_2(t-s, y) w(s - t_0^*, x-y) \left[\rho + R^{(0)}(s, x-y) - w(s - t_0^*, x-y)\right] dy ds \\
&\geq \int_{t_1}^{t} e^{-\rho(t-s)} \int_{-L\sqrt{4d_2(t-s)}}^{L\sqrt{4d_2(t-s)}} \frac{1}{\sqrt{4\pi d_2(t-s)}} e^{-\frac{y^2}{4d_2(t-s)}} w(s - t_0^*, x-y) \\
&\quad \times \left[\rho + R^{(0)}(s, x-y) - w(s - t_0^*, x-y)\right] dy ds \\
&= \beta \int_{t_1}^{t} e^{-\rho(t-s)} \int_{-L\sqrt{4d_2(t-s)}}^{L\sqrt{4d_2(t-s)}} \frac{1}{\sqrt{4\pi d_2(t-s)}} e^{-\frac{y^2}{4d_2(t-s)}} \left[\rho + R^{(0)}(s, x-y) - \beta\right] dy ds \\
&\geq \beta(\rho + r^{(0)}(\infty) - \mu_{2,0}^*(\infty)\epsilon - \beta) \int_{t_1}^{t} e^{-\rho(t-s)} \int_{-L\sqrt{4d_2(t-s)}}^{L\sqrt{4d_2(t-s)}} \frac{1}{\sqrt{4\pi d_2(t-s)}} e^{-\frac{y^2}{4d_2(t-s)}} dy ds \\
&\geq \beta(1-\beta\epsilon)(\rho + r^{(0)}(\infty) - \mu_{2,0}^*(\infty)\epsilon - \beta) \int_{t_1}^{t} e^{-\rho(t-s)} ds
\end{aligned} \tag{5.24}$$

for all x satisfying (5.20). Here we have used the fact that for x satisfying (5.20) and y satisfying (5.21),

$$R^{(0)}(s, x-y) > r^{(0)}(\ell) > r^{(0)}(\infty) - \frac{\epsilon}{2d_2} c^*_{2,0}(\infty) = r^{(0)}(\infty) - \mu^*_{2,0}(\infty)\epsilon.$$

By (5.15), (5.19) and (5.24), we obtain that

$$u_2^{(0)}(t, x, \phi) \geq \hat{v}_2^{(1)}(t) \tag{5.25}$$

for all $t > t_1$ and x satisfying (5.16) and (5.20), where

$$\begin{aligned}\hat{v}_2^{(1)}(t) = \beta(1-\beta\epsilon)e^{-\rho(t-t_1)} + \beta(1-\beta\epsilon)(\rho + r^{(0)}(\infty) \\ -\mu^*_{2,0}(\infty)\epsilon - \beta)\int_{t_1}^t e^{-\rho(t-s)} ds.\end{aligned} \tag{5.26}$$

It then further follows from induction and (5.13) that

$$u_2^{(0)}(t, x, \phi) \geq \hat{v}_2^{(n)}(t) \tag{5.27}$$

for all $t > t_1$ and x satisfying (5.16) and

$$\begin{aligned}&\ell + \sigma(\mu_1) + \psi_2(\mu_1)(s - t_0^*) + nL\sqrt{4d_2(t-s)} \\ &\quad \leq x \leq \ell + \sigma(\mu_2) + \psi_2(\mu_2)(s - t_0^*) + 3\pi/\gamma - nL\sqrt{4d_2(t-s)}, \ \forall s \in [t_1, t],\end{aligned} \tag{5.28}$$

where

$$\begin{aligned}\hat{v}_2^{(n)}(t) =& \beta(1-\beta\epsilon)e^{-\rho(t-t_1)} + (1-\beta\epsilon)\int_{t_1}^t e^{-\rho(t-s)}\hat{v}_2^{(n-1)}(s) \\ &\times \left[\rho + r^{(0)}(\infty) - \mu^*_{2,0}(\infty)\epsilon - \hat{v}_2^{(n-1)}(s)\right] ds.\end{aligned} \tag{5.29}$$

Direct calculations and induction show that

$$\hat{v}_2^{(n)}(t) = \hat{a}_2^{(n)} + \hat{b}_2^{(n)}(t)e^{-\rho(t-t_1)}, \tag{5.30}$$

where

$$\hat{a}_2^{(n)} = \hat{a}_2^{(n-1)}(1-\beta\epsilon)(\rho + r^{(0)}(\infty) - \mu^*_{2,0}(\infty)\epsilon - \hat{a}_2^{(n-1)})/\rho, \tag{5.31}$$
$$\hat{a}_2^{(1)} = \beta(1-\beta\epsilon)(\rho + r^{(0)}(\infty) - \mu^*_{2,0}(\infty)\epsilon - \beta)/\rho \tag{5.32}$$

and $\hat{b}_2^{(n)}(t)$ is a sum of products of polynomials and exponential functions of the form $e^{-j\rho(t-t_1)}$ with j being a non-negative integer. Therefore,

$$\lim_{t\to\infty} \hat{v}_2^{(n)}(t) = \hat{a}_2^{(n)} \tag{5.33}$$

and $\hat{a}_2^{(n)} \leq r(\infty)$ for all $n \geq 1$. Let $\hat{a}_2^{(0)} = \beta$. Then for small ϵ and β, we have

$$\hat{a}_2^{(1)} - \hat{a}_2^{(0)} = (r^{(0)}(\infty) - \mu^*_{2,0}(\infty)\epsilon - \beta - \beta^2\rho\epsilon)/\rho > 0. \tag{5.34}$$

It follows from (5.34) and induction that

$$\begin{aligned}
&\hat{a}_2^{(n+1)} - \hat{a}_2^{(n)} \\
&\quad = \Big[\hat{a}_2^{(n)}(\rho + r^{(0)}(\infty) - \mu_{2,0}^*(\infty)\epsilon - \hat{a}_2^{(n)}) \\
&\qquad -\hat{a}_2^{(n-1)}(\rho + r^{(0)}(\infty) - \mu_{2,0}^*(\infty)\epsilon - \hat{a}_2^{(n-1)})\Big] \frac{1-\beta\epsilon}{\rho} \\
&\quad = (\hat{a}_2^{(n)} - \hat{a}_2^{(n-1)}) \Big[\rho + r^{(0)}(\infty) - \mu_{2,0}^*(\infty)\epsilon - \hat{a}_2^{(n-1)} - \hat{a}_2^{(n)}\Big] \frac{1-\beta\delta}{\rho} \\
&\quad > 0, \quad \forall n \geq 1.
\end{aligned} \tag{5.35}$$

Thus, $\left\{\hat{a}_2^{(n)}\right\}_{n=0}^{\infty}$ is increasing and $\beta < \hat{a}_2^{(n)} \leq r(\infty)$ for all $n \geq 1$. So, $\lim_{n\to\infty} \hat{a}_2^{(n)}$ exists. Let

$$\lim_{n\to\infty} \hat{a}_2^{(n)} = \hat{a}_2^*. \tag{5.36}$$

Then $\beta \leq \hat{a}_2^* \leq r(\infty)$ and by (5.31), we obtain that

$$\hat{a}_2^* = \hat{a}_2^*(1-\beta\epsilon)(\rho + r^{(0)}(\infty) - \mu_{2,0}^*(\infty)\epsilon - \hat{a}_2^*)/\rho. \tag{5.37}$$

Therefore, it follows from (5.37) that

$$\hat{a}_2^* = r^{(0)}(\infty) - \left(\mu_{2,0}^*(\infty) + \frac{\beta\rho}{1-\beta\epsilon}\right)\epsilon. \tag{5.38}$$

Thus, by (5.30), (5.33), (5.36) and (5.38), we obtain that there exist a positive integer N and $t_2 > t_1$ such that

$$\hat{v}_2^{(n)}(t) > r^{(0)}(\infty) - \left(1 + \mu_{2,0}^*(\infty) + \frac{\beta\rho}{1-\beta\epsilon}\right)\epsilon, \quad \forall t > t_2,\ n \geq N. \tag{5.39}$$

Clearly, if

$$\begin{aligned}
&\ell + \sigma(\mu_1) + \psi_2(\mu_1)(t - t_0^*) + NL\sqrt{4d_2(t-t_1)} \\
&\qquad \leq x \leq \ell + \sigma(\mu_2) + \psi_2(\mu_2)(t_1 - t_0^*) + 3\pi/\gamma - NL\sqrt{4d_2(t-t_1)},
\end{aligned} \tag{5.40}$$

then (5.16) holds and (5.28) with n replaced by N also holds. Choose $t_1 = ml + t_0^*$ and $t - t_1 = l$, where $m > 1$ and $l > 0$ are both sufficiently large. Then we can rewrite (5.40) as

$$\begin{aligned}
&\ell + \sigma(\mu_1) + \psi_2(\mu_1)l(m+1) + NL\sqrt{4d_2 l} \\
&\qquad \leq x \leq \ell + \sigma(\mu_2) + ml\psi_2(\mu_2) + 3\pi/\gamma - NL\sqrt{4d_2 l},
\end{aligned} \tag{5.41}$$

that is,

$$\begin{aligned}
&(t_0^* + l(m+1))\left[\psi_2(\mu_1) + \tfrac{\ell+\sigma(\mu_1)}{l(m+1)} + \tfrac{NL\sqrt{4d_2}}{(m+1)\sqrt{l}}\right]\tfrac{l(m+1)}{t_0^*+l(m+1)} \\
&\quad \leq x \leq (t_0^* + l(m+1))\left[\tfrac{m}{m+1}\psi_2(\mu_2) + \tfrac{\ell+\sigma(\mu_2)+3\pi/\gamma}{l(m+1)} - \tfrac{NL\sqrt{4d_2}}{(m+1)\sqrt{l}}\right]\tfrac{l(m+1)}{t_0^*+l(m+1)}.
\end{aligned} \tag{5.42}$$

Now for any given ε with $0 < \varepsilon < (c_{2,0}^*(\infty) - c_1^*(\infty) - \varepsilon^{(0)})/2$, choose ϵ sufficiently small such that $\epsilon < \varepsilon/3$. Then there exist l_0 and m_0 sufficiently large such that for any

$m > m_0,\ l > l_0$ and $t = t_0^* + l(m+1) > t_2$, we have

$$(t_0^* + l(m+1))\left[\psi_2(\mu_1) + \frac{\ell + \sigma(\mu_1)}{l(m+1)} + \frac{NL\sqrt{4d_2}}{(m+1)\sqrt{l}}\right]\frac{l(m+1)}{t_0^* + l(m+1)}$$
$$< t(\psi_2(\mu_1) + \epsilon) = t(c_1^*(\infty) + \varepsilon^{(0)} + \epsilon + \epsilon)$$
$$< t(c_1^*(\infty) + \varepsilon^{(0)} + \varepsilon)$$

and

$$(t_0^* + l(m+1))\left[\frac{m}{m+1}\psi_2(\mu_2) + \frac{\ell + \sigma(\mu_2) + 3\pi/\gamma}{l(m+1)} - \frac{NL\sqrt{4d_2}}{(m+1)\sqrt{l}}\right]\frac{l(m+1)}{t_0^* + l(m+1)}$$
$$> t(\psi_2(\mu_2) - \epsilon) = t(c_{2,0}^*(\infty) - 2\epsilon - \epsilon)$$
$$> t(c_{2,0}^*(\infty) - \varepsilon).$$

Let $t_3 = t_0^* + l_0(m_0+1)$. If $t > t_3$, then $t(c_1^*(\infty) + \varepsilon^{(0)} + \varepsilon) \le x \le t(c_{2,0}^*(\infty) - \varepsilon)$ implies that (5.40) holds. Thus, by (5.27) and (5.39), we obtain that

$$\lim_{t\to\infty}\left[\inf_{t(c_1^*(\infty)+\varepsilon^{(0)}+\varepsilon)\le x\le t(c_{2,0}^*(\infty)-\varepsilon)} u_2^{(0)}(t,x,\phi)\right]$$
$$\ge r^{(0)}(\infty) - \left(1 + \mu_{2,0}^*(\infty) + \frac{\beta\rho}{1-\beta\epsilon}\right)\epsilon. \tag{5.43}$$

Because ϵ can be arbitrarily small and (5.43), we have actually shown that

$$\lim_{t\to\infty}\left[\inf_{t(c_1^*(\infty)+\varepsilon^{(0)}+\varepsilon)\le x\le t(c_{2,0}^*(\infty)-\varepsilon)} u_2^{(0)}(t,x,\phi)\right] \ge r^{(0)}(\infty). \tag{5.44}$$

Since $\varepsilon^{(0)} > 0$ is arbitrary, we have actually shown that for every ε with $0 < \varepsilon < (c_2^*(\infty) - c_1^*(\infty))/2$,

$$\lim_{t\to\infty}\left[\inf_{t(c_1^*(\infty)+\varepsilon)\le x\le t(c_2^*(\infty)-\varepsilon)} u_2^{(0)}(t,x,\phi)\right] \ge r(\infty). \tag{5.45}$$

It follows from $u_2^{(0)}(t,x,\phi) \le r(\infty)$ for all $(t,x) \in [0,+\infty)\times\mathbb{R}$ that

$$\lim_{t\to\infty}\left[\sup_{t(c_1^*(\infty)+\varepsilon)\le x\le t(c_2^*(\infty)-\varepsilon)} \left|r(\infty) - u_2^{(0)}(t,x,\phi)\right|\right] = 0. \tag{5.46}$$

The proof of Lemma 2.2 is completed. □

Proof of Lemma 2.5 For any ϵ with $0 < \epsilon < \min\left\{\frac{1}{3}, r^{(1)}(\infty), \frac{c_{1,1}^*(\infty)-c-\varepsilon^{(1)}}{3}\right\}$, let ℓ be a real number such that

$$c_{1,1}^*(\ell) = c_{1,1}^*(\infty) - \epsilon.$$

We choose $0 < \mu_1 < \mu_2 < \mu_{1,1}^*(\ell)$ such that $\psi_1(\mu_1) = c + \varepsilon^{(1)} + \epsilon$ and $\psi_1(\mu_2) = c_{1,1}^*(\infty) - 2\epsilon$. By Lemma 2.4, for any $\mu \in [\mu_1, \mu_2]$ and sufficiently small $\beta > 0$ and $\gamma > 0$, $\frac{\beta}{\varphi(\mu;\sigma(\mu))}\varphi(\mu; x - \ell - \psi_1(\mu)t)$ with φ given by (2.7) is a continuous weak lower solution of (2.34).

Since $\phi_1(x) \geq 0$ and $\phi_1(x) \not\equiv 0$, $u_1^{(1)}(t, x, \phi) > 0$ for all $(t, x) \in (0, +\infty) \times \mathbb{R}$. Choose $T_1 < t_0 < T_1 + \frac{\sigma(\mu_1)}{\psi_1(\mu_1)}$, and choose sufficiently small $\beta > 0$ and $\gamma > 0$ such that

$$\frac{\pi/\gamma + \sigma(\mu_1) - \sigma(\mu_2)}{\psi_1(\mu_2) - \psi_1(\mu_1)} > T_1 \tag{5.47}$$

and

$$u_1^{(1)}(t_0, x, \phi) \geq \beta, \quad \forall x \in [\ell + \psi_1(\mu_1)T_1, \ell + 4\pi/\gamma + \psi_1(\mu_2)T_1]. \tag{5.48}$$

Define

$$w(T_1, x) = \begin{cases} \frac{\beta}{\varphi(\mu_1;\sigma(\mu_1))}\varphi(\mu_1; x - \ell - \psi_1(\mu_1)T_1) \\ \quad \text{if } \ell + \psi_1(\mu_1)T_1 \leq x \leq \ell + \sigma(\mu_1) + \psi_1(\mu_1)T_1; \\ \beta \quad \text{if } \ell + \sigma(\mu_1) + \psi_1(\mu_1)T_1 \leq x \leq \ell + 3\pi/\gamma + \sigma(\mu_2) + \psi_1(\mu_2)T_1; \\ \frac{\beta}{\varphi(\mu_2;\sigma(\mu_2))}\varphi(\mu_2; x - \ell - 3\pi/\gamma - \psi_1(\mu_2)T_1) \\ \quad \text{if } \ell + 3\pi/\gamma + \sigma(\mu_2) + \psi_1(\mu_2)T_1 \leq x \leq \ell + 4\pi/\gamma + \psi_1(\mu_2)T_1; \\ 0 \quad \text{elsewhere.} \end{cases}$$

It is easily seen that

$$w(T_1, x) \geq \frac{\beta}{\varphi(\mu_1; \sigma(\mu_1))}\varphi(\mu_1; x - \ell - \psi_1(\mu_1)T_1 - s), \quad \forall s \in [0, 2\pi/\gamma]$$

and

$$w(T_1, x) \geq \frac{\beta}{\varphi(\mu_2; \sigma(\mu_2))}\varphi(\mu_2; x - \ell - 3\pi/\gamma - \psi_1(\mu_2)T_1 + s), \quad \forall s \in [0, 2\pi/\gamma].$$

It follows from (5.48) and Lemma 2.4 that for any $t \geq t_0$,

$$\begin{aligned} &u_1^{(1)}(t, x, \phi) \\ &\quad \geq \frac{\beta}{\varphi(\mu_1; \sigma(\mu_1))}\varphi(\mu_1; x - \ell - \psi_1(\mu_1)(T_1 + t - t_0) - s), \quad \forall s \in [0, 2\pi/\gamma] \end{aligned} \tag{5.49}$$

and

$$\begin{aligned} u_1^{(1)}(t, x, \phi) \geq \frac{\beta}{\varphi(\mu_2; \sigma(\mu_2))}&\varphi(\mu_2; x - \ell - 3\pi/\gamma \\ &-\psi_1(\mu_2)(T_1 + t - t_0) + s), \quad \forall s \in [0, 2\pi/\gamma]. \end{aligned} \tag{5.50}$$

Inequalities (5.49) and (5.50) imply that for any $t \geq t_0$,

$$u_1^{(1)}(t,x,\phi) \geq \begin{cases} \frac{\beta}{\varphi(\mu_1;\sigma(\mu_1))}\varphi(\mu_1; x-\ell-\psi_1(\mu_1)(T_1+t-t_0)) \\ \quad \text{if } \ell+\psi_1(\mu_1)(T_1+t-t_0) \leq x \leq \ell+\sigma(\mu_1)+\psi_1(\mu_1)(T_1+t-t_0); \\ \beta \quad \text{if } \ell+\sigma(\mu_1)+\psi_1(\mu_1)(T_1+t-t_0) \leq x \leq \ell+2\pi/\gamma \\ \quad +\sigma(\mu_1)+\psi_1(\mu_1)(T_1+t-t_0); \\ 0 \quad \text{elsewhere} \end{cases} \tag{5.51}$$

and

$$u_1^{(1)}(t,x,\phi) \geq \begin{cases} \beta \quad \text{if } \ell+\pi/\gamma+\sigma(\mu_2)+\psi_1(\mu_2)(T_1+t-t_0) \leq x \leq \ell+3\pi/\gamma \\ \quad +\sigma(\mu_2)+\psi_1(\mu_2)(T_1+t-t_0); \\ \frac{\beta}{\varphi(\mu_2;\sigma(\mu_2))}\varphi(\mu_2; x-\ell-3\pi/\gamma-\psi_1(\mu_2)(T_1+t-t_0)) \\ \quad \text{if } \ell+3\pi/\gamma+\sigma(\mu_2)+\psi_1(\mu_2)(T_1+t-t_0) \leq x \leq \ell+4\pi/\gamma \\ \quad +\psi_1(\mu_2)(T_1+t-t_0); \\ 0 \quad \text{elsewhere.} \end{cases} \tag{5.52}$$

Let

$$h = \frac{\pi/\gamma+\sigma(\mu_1)-\sigma(\mu_2)}{\psi_1(\mu_2)-\psi_1(\mu_1)} - T_1.$$

Then (5.47) implies that $h > 0$. Since

$$\ell+2\pi/\gamma+\sigma(\mu_1)+\psi_1(\mu_1)(T_1+t-t_0) \geq \ell+\pi/\gamma+\sigma(\mu_2)+\psi_1(\mu_2)(T_1+t-t_0)$$

for all $t \in [t_0, t_0+h]$, inequalities (5.51) and (5.52) imply that

$$u_1^{(1)}(t,x,\phi) \geq w(t-t_0^*, x) \tag{5.53}$$

for all $t \in [t_0, t_0+h]$, where $t_0^* = t_0 - T_1$ and

$$w(t-t_0^*,x) = \begin{cases} \frac{\beta}{\varphi(\mu_1;\sigma(\mu_1))}\varphi(\mu_1; x-\ell-\psi_1(\mu_1)(t-t_0^*)) \\ \quad \text{if } \ell+\psi_1(\mu_1)(t-t_0^*) \leq x \leq \ell+\sigma(\mu_1)+\psi_1(\mu_1)(t-t_0^*); \\ \beta \quad \text{if } \ell+\sigma(\mu_1)+\psi_1(\mu_1)(t-t_0^*) \leq x \leq \ell+3\pi/\gamma+\sigma(\mu_2)+\psi_1(\mu_2)(t-t_0^*); \\ \frac{\beta}{\varphi(\mu_2;\sigma(\mu_2))}\varphi(\mu_2; x-\ell-3\pi/\gamma-\psi_1(\mu_2)(t-t_0^*)) \\ \quad \text{if } \ell+3\pi/\gamma+\sigma(\mu_2)+\psi_1(\mu_2)(t-t_0^*) \leq x \leq \ell+4\pi/\gamma+\psi_1(\mu_2)(t-t_0^*); \\ 0 \quad \text{elsewhere.} \end{cases} \tag{5.54}$$

We claim that (5.53) holds true for all $t \geq t_0$. Assume that (5.53) is valid for $t \in [t_0, t_0+nh]$ for some positive integer n. Then for any $s \in [0, 2\pi/\gamma + (\psi_1(\mu_2)-\psi_1(\mu_1))(T_1+nh)]$,

$$w(T_1+nh, x) \geq \frac{\beta}{\varphi(\mu_1;\sigma(\mu_1))}\varphi(\mu_1; x-\ell-\psi_1(\mu_1)(T_1+nh)-s)$$

and

$$w(T_1+nh,x)\geq \frac{\beta}{\varphi(\mu_2;\sigma(\mu_2))}\varphi(\mu_2;x-\ell-3\pi/\gamma-\psi_1(\mu_2)(T_1+nh)+s).$$

Therefore, Lemma 2.4 implies that

$$u_1^{(1)}(t,x,\phi)\geq \frac{\beta}{\varphi(\mu_1;\sigma(\mu_1))}\varphi(\mu_1;x-\ell-\psi_1(\mu_1)(T_1+nh)-\psi_1(\mu_1)(t-(t_0+nh))-s) \tag{5.55}$$

and

$$\begin{aligned} u_1^{(1)}(t,x,\phi)\geq{}& \frac{\beta}{\varphi(\mu_2;\sigma(\mu_2))}\varphi(\mu_2;x-\ell-3\pi/\gamma \\ &-\psi_1(\mu_2)(T_1+nh)-\psi_1(\mu_2)(t-(t_0+nh))+s) \end{aligned} \tag{5.56}$$

for all $t\geq t_0+nh$ and $s\in[0,2\pi/\gamma+(\psi_1(\mu_2)-\psi_1(\mu_1))(T_1+nh)]$. Inequalities (5.55) and (5.56) imply that for $t\geq t_0+nh$,

$$u_1^{(1)}(t,x,\phi)\geq \begin{cases} \frac{\beta}{\varphi(\mu_1;\sigma(\mu_1))}\varphi(\mu_1;x-\ell-\psi_1(\mu_1)(T_1+t-t_0)) \\ \qquad \text{if } \ell+\psi_1(\mu_1)(T_1+t-t_0)\leq x\leq \ell+\sigma(\mu_1)+\psi_1(\mu_1)(T_1+t-t_0); \\ \beta \quad \text{if } \ell+\sigma(\mu_1)+\psi_1(\mu_1)(T_1+t-t_0)\leq x\leq \ell+2\pi/\gamma \\ \qquad +\sigma(\mu_1)+\psi_1(\mu_1)(t-(t_0+nh))+\psi_1(\mu_2)(T_1+nh); \\ 0 \quad \text{elsewhere} \end{cases} \tag{5.57}$$

and

$$u_1^{(1)}(t,x,\phi)\geq \begin{cases} \beta \quad \text{if } \ell+\pi/\gamma+\sigma(\mu_2)+\psi_1(\mu_2)(t-(t_0+nh))+\psi_1(\mu_1)(T_1+nh) \\ \qquad \leq x\leq \ell+3\pi/\gamma+\sigma(\mu_2)+\psi_1(\mu_2)(T_1+t-t_0); \\ \frac{\beta}{\varphi(\mu_2;\sigma(\mu_2))}\varphi(\mu_2;x-\ell-3\pi/\gamma-\psi_1(\mu_2)(T_1+t-t_0)) \\ \qquad \text{if } \ell+3\pi/\gamma+\sigma(\mu_2)+\psi_1(\mu_2)(T_1+t-t_0)\leq x\leq \ell+4\pi/\gamma \\ \qquad +\psi_1(\mu_2)(T_1+t-t_0); \\ 0 \quad \text{elsewhere.} \end{cases} \tag{5.58}$$

Since

$$\begin{aligned} &\ell+2\pi/\gamma+\sigma(\mu_1)+\psi_1(\mu_1)(t-(t_0+nh))+\psi_1(\mu_2)(T_1+nh) \\ &\geq \ell+\pi/\gamma+\sigma(\mu_2)+\psi_1(\mu_2)(t-(t_0+nh))+\psi_1(\mu_1)(T_1+nh) \end{aligned}$$

for all $t\in[t_0+nh,t_0+(2n+1)h+2T_1]$, inequalities (5.57) and (5.58) imply that (5.53) holds true for all $t\in[t_0+nh,t_0+(n+1)h]$. By induction, (5.53) is true for all $t\geq t_0$.

For the chosen $\epsilon>0$ and $\beta>0$, there exists $L>0$ such that

$$\int_{-L}^{L}\frac{1}{\sqrt{\pi}}e^{-x^2}dx\geq 1-\beta\epsilon.$$

Therefore, for any $s > 0$, we have

$$\int_{-L\sqrt{4d_1 s}}^{L\sqrt{4d_1 s}} \frac{1}{\sqrt{4\pi d_1 s}} e^{-\frac{x^2}{4d_1 s}} dx \geq 1 - \beta\epsilon.$$

Let $t_1 > t_0$ be sufficiently large. Then, for $t > t_1$, the solution $u_1^{(1)}(t, x, \phi)$ of (2.34) satisfies the integral equation

$$\begin{aligned} u_1^{(1)}(t, x, \phi) = & \int_{-\infty}^{+\infty} k_1(t - t_1, x - y) u_1^{(1)}(t_1, y, \phi) dy \\ & + \int_{t_1}^{t} \int_{-\infty}^{+\infty} k_1(t - s, x - y) u_1^{(1)}(s, y, \phi) \\ & \left[\rho + R^{(1)}(s, y) - u_1^{(1)}(s, y, \phi)\right] dyds, \end{aligned} \tag{5.59}$$

where $\rho > 3r(\infty) - r(-\infty)$ is a real number and

$$k_1(t, x) = \frac{1}{\sqrt{4\pi d_1 t}} e^{-\rho t - \frac{x^2}{4d_1 t}}. \tag{5.60}$$

It follows from (5.53) and (5.59) that for any $t > t_1$,

$$\begin{aligned} u_1^{(1)}(t, x, \phi) \geq & \int_{-\infty}^{+\infty} k_1(t - t_1, x - y) w(t_1 - t_0^*, y) dy \\ & + \int_{t_1}^{t} \int_{-\infty}^{+\infty} k_1(t - s, x - y) w(s - t_0^*, y) \\ & \left[\rho + R^{(1)}(s, y) - w(s - t_0^*, y)\right] dyds. \end{aligned} \tag{5.61}$$

For $t > t_1$ and x, y satisfying

$$\begin{aligned} & \ell + \sigma(\mu_1) + \psi_1(\mu_1)(t_1 - t_0^*) + L\sqrt{4d_1(t - t_1)} \\ & \leq x \leq \ell + \sigma(\mu_2) + \psi_1(\mu_2)(t_1 - t_0^*) + 3\pi/\gamma - L\sqrt{4d_1(t - t_1)} \end{aligned} \tag{5.62}$$

and

$$-L\sqrt{4d_1(t - t_1)} \leq y \leq L\sqrt{4d_1(t - t_1)}, \tag{5.63}$$

we have that

$$\ell + \sigma(\mu_1) + \psi_1(\mu_1)(t_1 - t_0^*) \leq x - y \leq \ell + \sigma(\mu_2) + \psi_1(\mu_2)(t_1 - t_0^*) + 3\pi/\gamma. \tag{5.64}$$

It follows from (5.54) and (5.64) that

$$\begin{aligned} & \int_{-\infty}^{+\infty} k_1(t - t_1, x - y) w(t_1 - t_0^*, y) dy \\ & = \int_{-\infty}^{+\infty} k_1(t - t_1, y) w(t_1 - t_0^*, x - y) dy \\ & \geq e^{-\rho(t - t_1)} \int_{-L\sqrt{4d_1(t-t_1)}}^{L\sqrt{4d_1(t-t_1)}} \frac{1}{\sqrt{4\pi d_1(t - t_1)}} e^{-\frac{y^2}{4d_1(t-t_1)}} w(t_1 - t_0^*, x - y) dy \\ & = \beta e^{-\rho(t - t_1)} \int_{-L\sqrt{4d_1(t-t_1)}}^{L\sqrt{4d_1(t-t_1)}} \frac{1}{\sqrt{4\pi d_1(t - t_1)}} e^{-\frac{y^2}{4d_1(t-t_1)}} dy \\ & \geq (1 - \beta\epsilon)\beta e^{-\rho(t - t_1)} \end{aligned} \tag{5.65}$$

for all x satisfying (5.62). For $t > t_1$ and x, y satisfying

$$\begin{aligned}&\ell+\sigma(\mu_1)+\psi_1(\mu_1)(s-t_0^*)+L\sqrt{4d_1(t-s)}\\&\quad\le x\le \ell+\sigma(\mu_2)+\psi_1(\mu_2)(s-t_0^*)+3\pi/\gamma-L\sqrt{4d_1(t-s)},\quad \forall s\in[t_1,t]\end{aligned} \tag{5.66}$$

and

$$-L\sqrt{4d_1(t-s)}\le y\le L\sqrt{4d_1(t-s)},\quad \forall s\in[t_1,t], \tag{5.67}$$

we have

$$\begin{aligned}&\ell+\sigma(\mu_1)+\psi_1(\mu_1)(s-t_0^*)\\&\quad\le x-y\le \ell+\sigma(\mu_2)+\psi_1(\mu_2)(s-t_0^*)+3\pi/\gamma,\quad \forall s\in[t_1,t]\end{aligned} \tag{5.68}$$

and

$$\begin{aligned}&x-y-cs\ge \ell+\sigma(\mu_1)+\psi_1(\mu_1)(s-t_0^*)-cs>\ell\\&\quad+\sigma(\mu_1)+\psi_1(\mu_1)T_1>\ell,\quad \forall s\in[t_1,t].\end{aligned} \tag{5.69}$$

Then it follows from (5.54), (5.68) and (5.69) that

$$\begin{aligned}&\int_{t_1}^{t}\int_{-\infty}^{+\infty}k_1(t-s,x-y)w(s-t_0^*,y)\left[\rho+R^{(1)}(s,y)-w(s-t_0^*,y)\right]dyds\\
&=\int_{t_1}^{t}\int_{-\infty}^{+\infty}k_1(t-s,y)w(s-t_0^*,x-y)\left[\rho+R^{(1)}(s,x-y)-w(s-t_0^*,x-y)\right]dyds\\
&\ge\int_{t_1}^{t}e^{-\rho(t-s)}\int_{-L\sqrt{4d_1(t-s)}}^{L\sqrt{4d_1(t-s)}}\frac{1}{\sqrt{4\pi d_1(t-s)}}e^{-\frac{y^2}{4d_1(t-s)}}w(s-t_0^*,x-y)\\
&\quad\times\left[\rho+R^{(1)}(s,x-y)-w(s-t_0^*,x-y)\right]dyds\\
&=\beta\int_{t_1}^{t}e^{-\rho(t-s)}\int_{-L\sqrt{4d_1(t-s)}}^{L\sqrt{4d_1(t-s)}}\frac{1}{\sqrt{4\pi d_1(t-s)}}e^{-\frac{y^2}{4d_1(t-s)}}\left[\rho+R^{(1)}(s,x-y)-\beta\right]dyds\\
&\ge\beta(\rho+r^{(1)}(\infty)-\mu_{1,1}^*(\infty)\epsilon-\beta)\int_{t_1}^{t}e^{-\rho(t-s)}\int_{-L\sqrt{4d_1(t-s)}}^{L\sqrt{4d_1(t-s)}}\frac{1}{\sqrt{4\pi d_1(t-s)}}e^{-\frac{y^2}{4d_1(t-s)}}dyds\\
&\ge\beta(1-\beta\epsilon)(\rho+r^{(1)}(\infty)-\mu_{1,1}^*(\infty)\epsilon-\beta)\int_{t_1}^{t}e^{-\rho(t-s)}ds\end{aligned} \tag{5.70}$$

for all x satisfying (5.66). Here we have used the fact that for x satisfying (5.66) and y satisfying (5.67),

$$R^{(1)}(s,x-y)>r^{(1)}(\ell)>r^{(1)}(\infty)-\frac{\epsilon}{2d_1}c_{1,1}^*(\infty)=r^{(1)}(\infty)-\mu_{1,1}^*(\infty)\epsilon.$$

By (5.61), (5.65) and (5.70), we obtain that

$$u_1^{(1)}(t,x,\phi)\ge\hat{v}_1^{(1)}(t) \tag{5.71}$$

for all $t > t_1$ and x satisfying (5.62) and (5.66), where

$$\begin{aligned}\hat{v}_1^{(1)}(t)&=\beta(1-\beta\epsilon)e^{-\rho(t-t_1)}+\beta(1-\beta\epsilon)(\rho+r^{(1)}(\infty)\\&\quad-\mu_{1,1}^*(\infty)\epsilon-\beta)\int_{t_1}^{t}e^{-\rho(t-s)}ds.\end{aligned} \tag{5.72}$$

It then further follows from induction and (5.59) that

$$u_1^{(1)}(t, x, \phi) \geq \hat{v}_1^{(n)}(t) \tag{5.73}$$

for all $t > t_1$ and x satisfying (5.62) and

$$\begin{aligned} &\ell + \sigma(\mu_1) + \psi_1(\mu_1)(s - t_0^*) + nL\sqrt{4d_1(t-s)} \\ &\quad \leq x \leq \ell + \sigma(\mu_2) + \psi_1(\mu_2)(s - t_0^*) + 3\pi/\gamma - nL\sqrt{4d_1(t-s)}, \ \forall s \in [t_1, t], \end{aligned} \tag{5.74}$$

where

$$\begin{aligned} \hat{v}_1^{(n)}(t) =& \beta(1-\beta\epsilon)e^{-\rho(t-t_1)} + (1-\beta\epsilon)\int_{t_1}^{t} e^{-\rho(t-s)}\hat{v}_1^{(n-1)}(s) \\ & \times \left[\rho + r^{(1)}(\infty) - \mu_{1,1}^*(\infty)\epsilon - \hat{v}_1^{(n-1)}(s)\right] ds. \end{aligned} \tag{5.75}$$

Direct calculations and induction show that

$$\hat{v}_1^{(n)}(t) = \hat{a}_1^{(n)} + \hat{b}_1^{(n)}(t)e^{-\rho(t-t_1)}, \tag{5.76}$$

where

$$\hat{a}_1^{(n)} = \hat{a}_1^{(n-1)}(1-\beta\epsilon)(\rho + r^{(1)}(\infty) - \mu_{1,1}^*(\infty)\epsilon - \hat{a}_1^{(n-1)})/\rho, \tag{5.77}$$

$$\hat{a}_1^{(1)} = \beta(1-\beta\epsilon)(\rho + r^{(1)}(\infty) - \mu_{1,1}^*(\infty)\epsilon - \beta)/\rho \tag{5.78}$$

and $\hat{b}_1^{(n)}(t)$ is a sum of products of polynomials and exponential functions of the form $e^{-j\rho(t-t_1)}$ with j being a non-negative integer. Therefore,

$$\lim_{t\to\infty} \hat{v}_1^{(n)}(t) = \hat{a}_1^{(n)} \tag{5.79}$$

and $\hat{a}_1^{(n)} \leq r(\infty)$ for all $n \geq 1$. Let $\hat{a}_1^{(0)} = \beta$. Then for small ϵ and β, we have

$$\hat{a}_1^{(1)} - \hat{a}_1^{(0)} = (r^{(1)}(\infty) - \mu_{1,1}^*(\infty)\epsilon - \beta - \beta^2\rho\epsilon)/\rho > 0. \tag{5.80}$$

It follows from (5.80) and induction that

$$\begin{aligned} &\hat{a}_1^{(n+1)} - \hat{a}_1^{(n)} \\ &= \Big[\hat{a}_1^{(n)}(\rho + r^{(1)}(\infty) - \mu_{1,1}^*(\infty)\epsilon - \hat{a}_1^{(n)}) - \hat{a}_1^{(n-1)}(\rho + r^{(1)}(\infty) \\ &\quad -\mu_{1,1}^*(\infty)\epsilon - \hat{a}_1^{(n-1)})\Big]\frac{1-\beta\epsilon}{\rho} \\ &= (\hat{a}_1^{(n)} - \hat{a}_1^{(n-1)})\left[\rho + r^{(1)}(\infty) - \mu_{1,1}^*(\infty)\epsilon - \hat{a}_1^{(n-1)} - \hat{a}_1^{(n)}\right]\frac{1-\beta\delta}{\rho} \\ &> 0, \quad \forall n \geq 1. \end{aligned} \tag{5.81}$$

Thus, $\left\{\hat{a}_1^{(n)}\right\}_{n=0}^{\infty}$ is increasing and $\beta < \hat{a}_1^{(n)} \leq r(\infty)$ for $n \geq 1$. So, $\lim_{n\to\infty} \hat{a}_1^{(n)}$ exists. Let

$$\lim_{n\to\infty} \hat{a}_1^{(n)} = \hat{a}_1^*. \tag{5.82}$$

Then $\beta \leq \hat{a}_1^* \leq r(\infty)$ and by (5.77), we obtain that

$$\hat{a}_1^* = \hat{a}_1^*(1-\beta\epsilon)(\rho + r^{(1)}(\infty) - \mu_{1,1}^*(\infty)\epsilon - \hat{a}_1^*)/\rho. \tag{5.83}$$

Therefore, it follows from (5.83) that

$$\hat{a}_1^* = r^{(1)}(\infty) - \left(\mu_{1,1}^*(\infty) + \frac{\beta\rho}{1-\beta\epsilon}\right)\epsilon. \tag{5.84}$$

Thus, by (5.76), (5.79), (5.82) and (5.84), we obtain that there exist a positive integer N and $t_2 > t_1$ such that

$$\hat{v}_1^{(n)}(t) > r^{(1)}(\infty) - \left(1 + \mu_{1,1}^*(\infty) + \frac{\beta\rho}{1-\beta\epsilon}\right)\epsilon, \quad \forall t > t_2,\ n \geq N. \tag{5.85}$$

Clearly, if

$$\begin{aligned} &\ell + \sigma(\mu_1) + \psi_1(\mu_1)(t - t_0^*) + NL\sqrt{4d_1(t-t_1)} \\ &\quad \leq x \leq \ell + \sigma(\mu_2) + \psi_1(\mu_2)(t_1 - t_0^*) + 3\pi/\gamma - NL\sqrt{4d_1(t-t_1)}, \end{aligned} \tag{5.86}$$

then (5.62) holds and (5.74) with n replaced by N also holds. Choose $t_1 = ml + t_0^*$ and $t - t_1 = l$, where $m > 1$ and $l > 0$ are both sufficiently large. Then we can rewrite (5.86) as

$$\begin{aligned} &\ell + \sigma(\mu_1) + \psi_1(\mu_1)l(m+1) + NL\sqrt{4d_1 l} \\ &\quad \leq x \leq \ell + \sigma(\mu_2) + ml\psi_1(\mu_2) + 3\pi/\gamma - NL\sqrt{4d_1 l}, \end{aligned} \tag{5.87}$$

that is,

$$\begin{aligned} &(t_0^* + l(m+1))\left[\psi_1(\mu_1) + \tfrac{\ell+\sigma(\mu_1)}{l(m+1)} + \tfrac{NL\sqrt{4d_1}}{(m+1)\sqrt{l}}\right]\tfrac{l(m+1)}{t_0^*+l(m+1)} \\ &\leq x \leq (t_0^* + l(m+1))\left[\tfrac{m}{m+1}\psi_1(\mu_2) + \tfrac{\ell+\sigma(\mu_2)+3\pi/\gamma}{l(m+1)} - \tfrac{NL\sqrt{4d_1}}{(m+1)\sqrt{l}}\right]\tfrac{l(m+1)}{t_0^*+l(m+1)}. \end{aligned} \tag{5.88}$$

Now for any given ε with $0 < \varepsilon < (c_{1,1}^*(\infty) - c - \varepsilon^{(1)})/2$, choose ϵ sufficiently small such that $\epsilon < \varepsilon/3$. Then there exist l_0 and m_0 sufficiently large such that for $m > m_0,\ l > l_0$ and $t = t_0^* + l(m+1) > t_2$,

$$\begin{aligned} &(t_0^* + l(m+1))\left[\psi_1(\mu_1) + \frac{\ell+\sigma(\mu_1)}{l(m+1)} + \frac{NL\sqrt{4d_1}}{(m+1)\sqrt{l}}\right]\frac{l(m+1)}{t_0^*+l(m+1)} \\ &< t(\psi_1(\mu_1) + \epsilon) = t(c + \varepsilon^{(1)} + \epsilon + \epsilon) \\ &< t(c + \varepsilon^{(1)} + \varepsilon) \end{aligned}$$

and

$$\begin{aligned} &(t_0^* + l(m+1))\left[\frac{m}{m+1}\psi_1(\mu_2) + \frac{\ell+\sigma(\mu_2)+3\pi/\gamma}{l(m+1)} - \frac{NL\sqrt{4d_1}}{(m+1)\sqrt{l}}\right]\frac{l(m+1)}{t_0^*+l(m+1)} \\ &> t(\psi_1(\mu_2) - \epsilon) = t(c_{1,1}^*(\infty) - 2\epsilon - \epsilon) \\ &> t(c_{1,1}^*(\infty) - \varepsilon). \end{aligned}$$

Let $t_3 = t_0^* + l_0(m_0+1)$. If $t > t_3$, then $t(c + \varepsilon^{(1)} + \varepsilon) \leq x \leq t(c_{1,1}^*(\infty) - \varepsilon)$ implies that (5.86) holds. Thus, by (5.73) and (5.85), we obtain that

$$\begin{aligned} &\lim_{t\to\infty}\left[\inf_{t(c+\varepsilon^{(1)}+\varepsilon)\leq x\leq t(c_{1,1}^*(\infty)-\varepsilon)} u_1^{(1)}(t,x,\phi)\right] \\ &\quad \geq r^{(1)}(\infty) - \left(1 + \mu_{1,1}^*(\infty) + \frac{\beta\rho}{1-\beta\epsilon}\right)\epsilon. \end{aligned} \tag{5.89}$$

Because ϵ can be arbitrarily small and (5.89), we have actually shown that

$$\lim_{t\to\infty}\left[\inf_{t(c+\varepsilon^{(1)}+\varepsilon)\le x\le t(c^*_{1,1}(\infty)-\varepsilon)} u_1^{(1)}(t,x,\phi)\right]\ge r^{(1)}(\infty). \tag{5.90}$$

Since $\varepsilon^{(1)}>0$ is arbitrary, we have actually shown that for every ε with $0<\varepsilon<(c_1^*(\infty)-c)/2$,

$$\lim_{t\to\infty}\left[\inf_{t(c+\varepsilon)\le x\le t(c^*_{1}(\infty)-\varepsilon)} u_1^{(1)}(t,x,\phi)\right]\ge r(\infty). \tag{5.91}$$

It follows from $u_1^{(1)}(t,x,\phi)\le r(\infty)$ for all $(t,x)\in[0,+\infty)\times\mathbb{R}$ that

$$\lim_{t\to\infty}\left[\sup_{t(c+\varepsilon)\le x\le t(c^*_{1}(\infty)-\varepsilon)} \left|r(\infty)-u_1^{(1)}(t,x,\phi)\right|\right]=0. \tag{5.92}$$

The proof of Lemma 2.5 is completed. □

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