J Dyn Diff Equat (2016) 28:1007–1030
DOI 10.1007/s10884-015-9457-7

Dirichlet Problem of a Delayed Reaction–Diffusion Equation on a Semi-infinite Interval

Taishan Yi[1] · **Xingfu Zou**[1,2]

Received: 9 October 2014 / Revised: 13 March 2015 / Published online: 23 May 2015

Abstract We consider a nonlocal delayed reaction–diffusion equation in a semi-infinite interval that describes mature population of a single species with two age stages (immature and mature) and a fixed maturation period living in a spatially semi-infinite environment. Homogeneous Dirichlet condition is imposed at the finite end, accounting for a scenario that boundary is hostile to the species. Due to the lack of compactness and symmetry of the spatial domain, the global dynamics of the equation turns out to be a very challenging problem. We first establish a priori estimate for nontrivial solutions after exploring the delicate asymptotic properties of the nonlocal delayed effect and the diffusion operator. Using the estimate, we are able to show the repellency of the trivial equilibrium and the existence of a positive heterogeneous steady state under the Dirichlet boundary condition. We then employ the dynamical system arguments to establish the global attractivity of the heterogeneous steady state. As a byproduct, we also obtain the existence and global attractivity of the heterogeneous steady state for the bistable evolution equation in the whole space.

Keywords Reaction-diffusion equation · Nonlocal · Delay · Dirichlet boundary condition · Half line domain

Mathematics Subject Classification 34D23 · 34G25 · 35K57 · 39A30

1 Introduction

Based on the interaction of intrinsic dynamics (birth and death) and the spatial diffusion in a structured population (Metz and Diekmann [19]), various spatially nonlocal and temporally

Dedicated to Professor John Mallet-Paret on the occasion of his 60th birthday.

✉ Xingfu Zou
xzou@uwo.ca

1 School of Mathematics and Statistics, Central South University, Changsha 410083, Hunan, People's Republic of China

2 Department of Applied Mathematics, University of Western Ontario, London, ON N6A 5B7, Canada

delayed reaction–diffusion equation models have been derived. Such models can well explain the population dynamics of species with age and spatial structures. For example, So et al. [21] derived the following equation,

$$\frac{\partial u(t,x)}{\partial t} = D_m \frac{\partial^2 u(t,x)}{\partial x^2} - d_m u(t,x) + \varepsilon \int_{\mathbb{R}} \Gamma_\alpha(x-y) b(u(t-\tau, y))\, \mathrm{dy}, \quad \mathrm{t} \geq 0, \quad \mathrm{x} \in \mathbb{R}. \tag{1.1}$$

for a species consisting immature and mature stages and living in the one dimensional whole space $\mathbb{R}$. Here $u(t,x)$ is the mature population of a species at time t and location x, D_m and d_m are the diffusion rate and death rate of the mature population, τ is the maturation time for the species, the other two indirect parameters ε and α are defined by $\varepsilon = exp(-\int_0^\tau d_I(a)da)$ and $\alpha = \int_0^\tau D_I(a)da$ where $D_I(a),\ d_I(a),\ a \in [0,\tau]$ are the age dependent diffusion rate and death rate of the immature population of the species, $b(u)$ is a birth function and the kernel $\Gamma_\alpha(u)$ parameterized by α is given by

$$\Gamma_\alpha(u) = \frac{1}{\sqrt{4\pi\alpha}} e^{-u^2/4\alpha}. \tag{1.2}$$

Obviously, α measures the mobility of the immature individuals, ε measures the probability that a new born can survive the immature period, and $f_\alpha(x-y)$ accounts for the probability that an individual born at location y at time $t-\tau$ will be at location x at time t. Since its derivation in [21], this equation has been extensively and intensively studied, leading to many interesting results on the existence and other qualitative properties of traveling wave fronts. See, for example, [1,4–6,8–10,13,15,18,21,23,25–27,33,34,37] and the references therein; in particular, the recent survey Gourley–Wu [10] offers a very nice review.

When considering a species that lives in a bounded domain Ω, similar models have also been derived/proposed in Liang et al. [14] and Xu and Zhao [29], in which the model equations are also in the form of (1.1) except that the integrals are in a bounded domain; moreover, depending on the boundary condition associated to the differential equation, the kernel function $\Gamma_\alpha(u)$ will take different forms. The global dynamics of the semiflow generated by such models subject to either the Dirichlet or the Neumann boundary condition have also been intensively and successfully studied (see, for example, [2,7,12,20,22,28,30–32,35,36,38]).

In the real world, there are also species whose individuals live in a semi-infinite domain which is neither bounded nor the whole space. For example, animals living in a big land that has the shore of an ocean or a lake at one side of the land provides such a scenario. For simplicity, we will consider $\mathbb{R}_+ = [0,\infty)$ in the one-dimensional space. For such a scenario, by the same approach used in [21] but imposing homogeneous Dirichlet boundary condition at $x=0$, we can obtain the following non-local reaction diffusion equation for the mature population:

$$\frac{\partial w}{\partial t} = D_m \frac{\partial^2 w}{\partial x^2} - d_m w + \varepsilon \int_0^\infty b(w(t-\tau, y))\, [\Gamma_\alpha(x-y) - \Gamma_\alpha(x+y)]\, \mathrm{dy}, \quad \mathrm{t} \geq 0, \quad \mathrm{x} \in \mathbb{R}_+, \tag{1.3}$$

where all parameters and functions remain the same as for (1.1). We give the detailed derivation in the Appendix.

Since (1.3) is derived under the Dirichlet condition at $x=0$, accordingly, we should also impose this condition for (1.3), together with an initial condition. In other words, we will consider the following homogeneous Dirichlet boundary value problem (DBVP) of nonlocal delayed reaction–diffusion equations in the semi-infinite domain $\mathbb{R}_+$:

$$\begin{cases} \frac{\partial w}{\partial t} &= D_m \frac{\partial^2 w}{\partial x^2} - d_m w + \varepsilon \int_0^\infty b(w(t-\tau, y))[\Gamma_\alpha(x-y) - \Gamma_\alpha(x+y)]\mathrm{d}y, \\ w(t,0) = 0, \quad t > 0, \\ w(t,x) = \varphi(t,x), \quad (t,x) \in [-\tau, 0] \times \mathbb{R}_+, \end{cases} \tag{1.4}$$

where $\varphi : [-\tau, 0] \times \mathbb{R}_+ \to \mathbb{R}_+$ is a bounded and continuous function with $\varphi(t, 0) = 0$ for all $t \in [-\tau, 0]$.

For convenience, by rescaling

$$\begin{aligned} &\frac{t}{\tau} \to t, \quad \frac{x}{\sqrt{\tau D_m}} \to x, \quad \tau d_m \to \mu, \quad w\left(\tau t, \sqrt{\tau D_m} x\right) \to u(t,x), \\ &\sqrt{\tau D_m}\Gamma_\alpha\left(\sqrt{\tau D_m} x\right) \to k(x), \quad \frac{\varepsilon}{d_m} b(\cdot) \to f(\cdot), \end{aligned} \tag{1.5}$$

we transform system (1.4) to the following the DBVP:

$$\begin{cases} \frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2} - \mu u + \mu \int_0^\infty f(u(t-1, y))[k(x-y) - k(x+y)]\mathrm{d}y, \quad \mathrm{t} > 0, \\ u(t,0) = 0, \quad t \geq 0, \\ u(t,x) = \varphi(t,x), \quad (t,x) \in [-1, 0] \times \mathbb{R}_+, \end{cases} \tag{1.6}$$

where $\varphi : [-1, 0] \times \mathbb{R}_+ \to \mathbb{R}_+$ is a bounded and continuous function with $\varphi(t, 0) = 0$ for all $t \in [-1, 0]$.

Since the spatial domain $\mathbb{R}_+$ is neither bounded nor the whole space, and the kernel $k(x-y) - k(x-y)$ is not symmetric, the dynamics of (1.6) turns out to be a mathematically challenging problem. In the rest of this paper, we will tackle this problem. In Sect. 2, we first show that there exists a uniformly bounded set which is positively invariant for (1.6). Due to the non-compactness and asymmetry of the semi-infinite interval, when there is a nontrivial equilibrium, it is quite difficult to show the repellency of the trivial equilibrium to any nonnegative solution. To overcome this difficulty, in Sect. 3, through describing the delicate asymptotic properties of the nonlocal delayed effect and the diffusion operator, we establish *a priori* estimate for nontrivial solutions and the repellency of the trivial equilibrium. Then we employ standard dynamical system theoretical arguments to obtain the global attractivity of the nontrivial equilibrium. In Sect. 4, we apply our general results to two particular models: one is the nonlocal diffusive Nicholson's blowfly equation and the other is the nonlocal diffusive Mackey–Glass equation, in the semi-infinite interval $\mathbb{R}_+$.

2 Preliminary Results

We first introduce some notations. Denote by $BUC(\mathbb{R}_+, \mathbb{R})$ the set of all bounded and uniformly continuous functions from $\mathbb{R}_+$ to $\mathbb{R}$. Let $X = \{\phi \in BUC(\mathbb{R}_+, \mathbb{R}) : \phi(0) = 0\}$ be equipped with the usual supremum norm $\|\cdot\|_X$, making X a Banach space. Let $X_+ = \{\phi \in X : \phi(x) \geq 0 \text{ for all } x \in \mathbb{R}_+\}$ and $X_+^\circ = \{\phi \in X : \phi(x) > 0 \text{ for all } x \in (0, \infty)\}$. Let $C = C([-1, 0], X)$ be the Banach space of continuous functions from $[-1, 0]$ into X with the supremum norm $||\cdot||_C$, and let $C_+ = C([-1, 0], X_+)$ and $C_+^\circ = C([-1, 0], X_+^\circ)$. It follows that X_+, C_+ is a closed cone in X, C, respectively. For convenience, we shall also treat an element $\varphi \in C$ as a function from $[-1, 0] \times \mathbb{R}_+$ into $\mathbb{R}$.

For any $\xi, \eta \in X$, we write $\xi \geq_X \eta$ if $\xi - \eta \in X_+$, $\xi >_X \eta$ if $\xi \geq \eta$ and $\xi \neq \eta$, $\xi \gg_X \eta$ if $\xi - \eta \in X_+^\circ$. Similarly, for any $\xi, \psi \in C$, we write $\varphi \geq_C \psi$ if $\varphi - \psi \in C_+$, $\varphi >_C \psi$ if $\varphi \geq_C \psi$ and $\varphi \neq \psi$, $\varphi \gg_C \psi$ if $\varphi - \psi \in C_+^\circ$. For simplicity of notations, when there is no confusion about the spaces, we just write $\geq, >, \gg$ and $||\cdot||$ for $\geq_*, >_*, \gg_*$ and $||\cdot||_*$, respectively, where $*$ stands for X or C.

For a real interval I, let $I+[-1,0]=\{t+\theta : t\in I \text{ and } \theta\in[-1,0]\}$. For $u:(I+[-1,0])\times\mathbb{R}_+\to\mathbb{R}$ and $t\in I$, we define $u_t(\cdot,\cdot)$ by $u_t(\theta,x)=u(t+\theta,x)$ for all $\theta\in[-1,0]$ and $x\in\mathbb{R}_+$.

We will consider the mild solution of system (1.6) with the initial value $\varphi\in C_+$, which solves the following integral equation with the given initial function,

$$\begin{cases} u(t,\cdot) = S(t)[\varphi(0,\cdot)]+\int_0^t S(t-s)[F(u_s)]\mathrm{d}s, & t\geq 0,\\ u_0 \quad = \varphi\in C_+, \end{cases} \tag{2.1}$$

where $F:C_+\to X_+$ is defined by

$$F(\varphi)(x)=\mu\int_0^\infty f(\varphi(-1,y))[k(x-y)-k(x+y)]\mathrm{d}y \quad \text{for } x\in\mathbb{R}_+ \text{ and } \varphi\in C_+,$$

and $S(t)$ is the semigroup generated by the linear system

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}-\mu u, & t>0,\\ u(t,0) = 0, & t\geq 0,\\ u(0,x)=\phi(x), & x\in\mathbb{R}_+, \end{cases} \tag{2.2}$$

that is, for $(x,\phi)\in\mathbb{R}_+\times X$,

$$\begin{cases} S(0)[\phi](x)=\phi(x)\\ S(t)[\phi](x)=\dfrac{\exp(-\mu t)}{\sqrt{4\pi t}}\displaystyle\int_0^\infty \phi(y)\left[\exp\left(-\frac{(x-y)^2}{4t}\right)-\exp\left(-\frac{(x+y)^2}{4t}\right)\right]\mathrm{d}y, \quad t>0. \end{cases} \tag{2.3}$$

For details of the above formula (2.3), see Eq. (10.5.39) and its derivation in [11].

Replacing $\mathbb{R}_+$ in (2.2) by $\mathbb{R}$ leads to another semigroup which is already well-known. More precisely, let $Z=BUC(\mathbb{R},\mathbb{R})$ be the set of all bounded and uniformly continuous functions from $\mathbb{R}$ to $\mathbb{R}$ equipped with the usual supremum norm $\|\cdot\|_Z$ and $U(t):Z\to Z$ be defined by

$$\begin{cases} U(0)[\phi](x)=\phi(x)\\ U(t)[\phi](x)=\dfrac{\exp(-\mu t)}{\sqrt{4\pi t}}\displaystyle\int_0^\infty \phi(y)\exp\left(-\frac{(x-y)^2}{4t}\right)\mathrm{d}y \quad \text{for all } t\in(0,\infty). \end{cases} \tag{2.4}$$

for $(x,\phi)\in\mathbb{R}\times Z$. Note that $U(t)$ $(t\geq 0)$ is an analytic and strongly continuous semigroup on Z generated by the Z-realization Δ_Z of $\Delta=\frac{\partial^2}{\partial x^2}$ (see, e.g., Daners and Medina [3]).

By some computations, we can easily establish the following results.

Lemma 2.1 *Then the following statements are true.*

(i) $S(t)[\phi](x)=e^{-\mu t}U(t)[\tilde{\phi}](x)$ *for all* $\phi\in X$ *and* $t\in\mathbb{R}_+$ *and* $x\in\mathbb{R}_+$, *where* $\tilde{\phi}$ *represents the odd extension of* ϕ.
(ii) $S(t)$ *is an analytic and strongly continuous semigroup on* X.
(iii) $S(t)[X_+]\subseteq X_+$ *for* $t\in\mathbb{R}_+$ *and* $S(t)[X_+\setminus\{\mathbf{0}\}]\subseteq X_+^\circ$ *for* $t>0$.
(iv) $\left.\frac{\partial S(t)[\phi](x)}{\partial x}\right|_{x=0}>0$ *for any* $(t,\varphi)\in(0,\infty)\times(C_+\setminus\{0\})$.
(v) *For all* $t\in(0,\infty)$ *and* $(x,\phi)\in\mathbb{R}_+\times X$, *there hold*

$$||S(t)[\phi]||_X\leq e^{-\mu t}||\phi||_X, \quad \left|\frac{\partial S(t)[\phi](x)}{\partial t}\right|\leq\frac{(1+\mu t)\exp(-\mu t)||\phi||_X}{t},$$

$$\left|\frac{\partial S(t)[\phi](x)}{\partial x}\right|\leq\frac{\exp(-\mu t)||\phi||_X}{\sqrt{\pi t}}, \quad \left|\frac{\partial^2 S(t)[\phi](x)}{\partial x^2}\right|\leq\frac{\exp(-\mu t)||\phi||_X}{t}.$$

(vi) *For any* $t_1, t_2 \in (0, \infty)$, $x_1, x_2 \in \mathbb{R}_+$ *and* $\phi \in X$, *there hold*

$$|S(t_1)[\phi](x_1) - S(t_2)[\phi](x_2)| \le \frac{(1 + \mu \min\{t_1, t_2\}) \exp(-\mu \min\{t_1, t_2\})||\phi||_X}{\min\{t_1, t_2\}} |t_2 - t_1| + \frac{\exp(-\mu \min\{t_1, t_2\})||\phi||_X}{\sqrt{\pi \min\{t_1, t_2\}}} |x_2 - x_1|.$$

Proof Obviously, statements (i), (iii), (iv) and the first inequality in (v) follow directly from the explicit expression of $S(t)$.

(ii) follows from statement (i) and the fact that $U(t)$ $(t \ge 0)$ is an analytic and strongly continuous semigroup on Z.

(vi) follows from the differential mean value theorem and the second and third inequalities in (v).

It remains to prove the second, third and fourth inequalities in (v). Indeed, for any $x \in \mathbb{R}_+$, $\phi \in X$ and $t > 0$, by (2.3), we have

$$\begin{aligned}\left|\frac{\partial S(t)[\phi](x)}{\partial x}\right| &= \frac{\exp(-\mu t)}{\sqrt{4\pi t}} \left| \int_0^\infty \phi(y) \left[\frac{y - x}{2t} \exp\left(-\frac{(x-y)^2}{4t}\right) \right.\right. \\ &\quad \left.\left. + \frac{x + y}{2t} \exp\left(-\frac{(x+y)^2}{4t}\right)\right] \mathrm{d}y \right| \\ &\le \frac{\exp(-\mu t)||\phi||_X}{\sqrt{4\pi t}} \int_0^\infty \left[\left| \frac{y - x}{2t} \exp\left(-\frac{(x-y)^2}{4t}\right) \right| \right. \\ &\quad \left. + \left| \frac{x + y}{2t} \exp\left(-\frac{(x+y)^2}{4t}\right)\right| \right] \mathrm{d}y \\ &\le \frac{\exp(-\mu t)||\phi||_X}{\sqrt{\pi t}}.\end{aligned}$$

Further computation shows that

$$\begin{aligned}\left|\frac{\partial^2 S(t)[\phi](x)}{\partial x^2}\right| &= \frac{\exp(-\mu t)}{\sqrt{4\pi t}} \left| \int_0^\infty \phi(y) \left[-\frac{1}{2t} \exp\left(-\frac{(x-y)^2}{4t}\right) + \frac{1}{2t} \exp\left(-\frac{(x+y)^2}{4t}\right) \right.\right. \\ &\quad \left.\left. + \left(\frac{y - x}{2t}\right)^2 \exp\left(-\frac{(x-y)^2}{4t}\right) - \left(\frac{x + y}{2t}\right)^2 \exp\left(-\frac{(x+y)^2}{4t}\right)\right] \mathrm{dy} \right| \\ &\le \frac{\exp(-\mu t)||\phi||_X}{\sqrt{4\pi t}} \int_0^\infty \left[\frac{1}{2t} \exp\left(-\frac{(x-y)^2}{4t}\right) + \frac{1}{2t} \exp\left(-\frac{(x+y)^2}{4t}\right) \right. \\ &\quad \left. + \left(\frac{y - x}{2t}\right)^2 \exp\left(-\frac{(x-y)^2}{4t}\right) + \left(\frac{x + y}{2t}\right)^2 \exp\left(-\frac{(x+y)^2}{4t}\right)\right] \mathrm{dy} \\ &\le \frac{\exp(-\mu t)||\phi||_X}{t}.\end{aligned}$$

Finally, from (2.2), we may obtain that for any $x \in \mathbb{R}_+$, $\phi \in X$ and $t > 0$, we have

$$\begin{aligned}\left|\frac{\partial S(t)[\phi](x)}{\partial t}\right| &\le \left|\frac{\partial^2 S(t)[\phi](x)}{\partial x^2}\right| + \mu |S(t)[\phi](x)| \\ &\le \frac{(1 + \mu t) \exp(-\mu t)||\phi||_X}{t}.\end{aligned}$$

The proof is completed. □

For given $\varphi \in C_+$, the method of steps and the definition of F, it is easy to see (2.1) has a solution which exists for all $t \geq 0$. Denote by $u^\varphi(t,x)$ represent the unique solution of (2.1), that is, the mild solution of (1.6) in the sense of Martin and Smith [16,17]. Then it is clear that $(u^\varphi)_t \in C_+$ for all $t \geq 0$ and $\varphi \in C_+$. Thus, the solution maps of (2.1) induce a continuous semiflow in C_+. Since the semigroup $S(t)$ is analytic, we know that a mild solution of (2.1) is also a classical solution of (1.6) for all $t > 1$ when f is continuously differentiable (see, e.g., [16,17,24,26]). Therefore, as far as asymptotic behaviors of solutions to (1.6) are concerned, it is sufficient to only consider solutions of (2.1).

Due to the non-compactness of the spatial domain, it is generally difficult and inconvenient to describe the global asymptotic behaviors with respect to the supremum norm. To overcome this difficulty, we introduce another more suitable topology called the compact open topology, and they are induced by the norms $\|\cdot\|^X_{co}$, $\|\cdot\|^C_{co}$ on X and C respectively given by $||\phi||^X_{co} \triangleq \sum_{n\in\mathbb{N}} 2^{-n}\sup\{|\phi(x)| : x \in [0,n]\}$ for $\phi \in X$ and $||\varphi||^C_{co} = \sup\{||\varphi(\theta)||^X_{co} : \theta \in [-1,0]\}$ for $\varphi \in C$, respectively, where $\mathbb{N} = \{1,2,\ldots\}$. Again for simplicity of notation, when there is no confusion about the spaces involved, we just write $||\cdot||_{co}$ for both norms defined above. Moreover, we denote the normed vector spaces $(X, ||\cdot||_{co})$, $(C, ||\cdot||_{co})$ by X_{co}, C_{co}, respectively.

In what follows, we shall always assume the following for the nonlinearity f:

(H1) $f : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous in $\mathbb{R}_+$ with $f(0) = 0$ and continuously differentiable in some right neighborhood of 0; moreover, there exists a sequence $\{B_n\}$ in $(0,\infty)$ such that $\lim\limits_{n\to\infty} B_n = \infty$ and $f([0, B_n]) \subseteq [0, B_n]$.

The following result establishes some basic properties for the solution semiflow of (2.1).

Proposition 2.1 *Let (H1) hold. Then for any positive integer n the following results are true:*

(i) $u^\varphi(t,\cdot) \in X^\circ_+$ *and* $\frac{\partial u^\varphi}{\partial x}(t,0) > 0$ *for any* $(t,\varphi) \in (0,\infty) \times (C_+ \setminus \{0\})$.
(ii) $(u^\varphi)_t \in C_{B_n}$ *for any* $(t,\varphi) \in \mathbb{R}_+ \times C_{B_n}$, *where* $C_{B_n} \triangleq \{\varphi \in C : \varphi(\theta,x) \in [0,B_n]$ *for all* $(\theta,x) \in [-1,0] \times \mathbb{R}_+\}$.
(iii) $(u^{(\cdot)})_t$ *is a continuous semiflow on* C_{B_n} *with respect to the compact open topology induced by the norm* $||\cdot||_{co}$.
(iv) *For any* $r, t^* > 0$, *there is* $M = M_{r,t^*} > 0$ *such that* $|u^\varphi(t,x) - u^\varphi(t,z)| \leq M|x - z|$, *where* $(t,x,z,\varphi) \in [t^*,\infty) \times \mathbb{R}_+ \times \mathbb{R}_+ \times C_r$.
(v) $\Phi(t, C_{B_n})(0,\cdot)$ *is precompact in* X_{co} *for* $t > 0$ *and* $\Phi(t, C_{B_n})$ *is precompact in* C_{co} *for* $t > 1$.

Proof (i) Let $(t,x,\varphi) \in (0,\infty) \times (0,\infty) \times (C_+ \setminus \{0\})$. From (2.1), we easily see that $u^\varphi(t,0) = 0$ and $u^\varphi(t,x) \geq S(t)[\varphi](x)$. The latter, combined with statements (iii) and (iv) of Lemma 2.1, implies that $\frac{\partial u^\varphi}{\partial x}(t,0) > 0$ and $u^\varphi(t,x) > 0$.

(ii) Suppose $\varphi \in C_{B_n}$. Let

$$v(t,x) = \begin{cases} u^\varphi(t,x) + B_n, & (t,x) \in [-1,\infty) \times [0,\infty) \\ -u^\varphi(t,-x) + B_n, & (t,x) \in [-1,\infty) \times (-\infty,0). \end{cases} \tag{2.5}$$

Then by (2.1) and Lemma 2.1-(i), v satisfies the following integral equation,

$$\begin{cases} v(t,x) = e^{-\mu t}U(t)[v(0,\cdot)](x) + \mu\int_0^t e^{-\mu(t-s)}U(t-s)[G(v_s)](x)\mathrm{d}s, & t \geq 0, \\ v_0 \quad = \psi \triangleq \tilde{\varphi} \in BUC([-1,0] \times \mathbb{R}, \mathbb{R}), \end{cases} \tag{2.6}$$

where $G : BUC([-1,0] \times \mathbb{R}, \mathbb{R}) \to X_+$ is defined by

$$G(\psi)(x) = \mu\int_{\mathbb{R}} g(\psi(-1,y))k(x-y)\mathrm{d}y \quad \text{for } x \in \mathbb{R}_+ \text{ and } \psi \in BUC([-1,0] \times \mathbb{R}, \mathbb{R})$$

with $g : \mathbb{R} \to \mathbb{R}$ given by

$$g(u) = \begin{cases} f(B_n) + B_n, & u \in (B_n, \infty), \\ f(u) + B_n, & u \in [0, B_n], \\ -f(-u) + B_n, & u \in [-B_n, 0], \\ -f(B_n) + B_n, & u \in (-\infty, -B_n), \end{cases} \tag{2.7}$$

Note that $g(\mathbb{R}_+) \subseteq [0, 2B_n]$. In the following, let $v^\psi(t, x)$ denote the solution of (2.6). Then, by the statements (i) and (ii) of Proposition 2.6 in [33], we obtain that $(v^\psi)(t, x) \in [0, 2B_n]$ for all $(t, x, \psi) \in \mathbb{R}_+ \times \mathbb{R} \times BUC([-1, 0] \times \mathbb{R}, [0, 2B_n])$. This, together with statement (i), implies (ii).

(iii) It is obvious that $(u^{(\cdot)})_t$ is a semigroup on C_{B_n}. Theorem 2.8-(i) in [33] together with the above discussions, shows that $(u^{(\cdot)})_t$ is continuous with respect to the compact open topology, proving (iii).

(iv) Without loss of generality, we may assume that $r = B_n$ for some $n \in \mathbb{N}$. Thus by the statement (ii), we have $0 \le (u^\varphi)_t \le r$ for all $(t, \varphi) \in \mathbb{R}_+ \times C_r$. Let $M_{r,t^*} = \frac{re^{-\mu t^*}}{\sqrt{\pi t^*}} + r\sqrt{\mu}$. Then for any $(t, x, z, \varphi) \in [t^*, \infty) \times \mathbb{R}_+ \times \mathbb{R}_+ \times C_r$, it follows from (2.1) and Lemma 2.1-(vi) that

$$\begin{aligned} \left|u^\varphi(t, x) - u^\varphi(t, z)\right| &\le |S(t)[\varphi(0, \cdot)](x) - S(t)[\varphi(0, \cdot)](z)| \\ &\quad + \int_0^t \left|S(t-s)\left[F\left(u_s^\varphi\right)\right](x) - S(t-s)\left[F\left(u_s^\varphi\right)\right](z)\right| \mathrm{d}s \\ &\le \frac{re^{-\mu t}}{\sqrt{\pi t}}|x - z| + \int_0^t \frac{\mu r e^{-\mu(t-s)}}{\sqrt{\pi(t-s)}}|x - z| \mathrm{d}s \\ &\le \left[\frac{re^{-\mu t}}{\sqrt{\pi t}} + \frac{\mu r}{\sqrt{\pi}} \int_0^\infty \frac{e^{-\mu s}}{\sqrt{s}} \mathrm{d}s\right] |x - z| \\ &\le \left[\frac{re^{-\mu t^*}}{\sqrt{\pi t^*}} + r\sqrt{\mu}\right] |x - z| \\ &= M_{r,t^*}|x - z|. \end{aligned}$$

(v) By the Arzèla–Ascoli theorem and the statements (ii) and (iv), it suffices to prove the following claim:

Claim *For any $\varepsilon > 0$ and $t^* > 0$, there is $\delta = \delta(\varepsilon, t^*) \in (0, 1)$ such that if $t^* \le t_1 \le t_2 < t_1 + \delta$, then $|u^\varphi(t_1, x) - u^\varphi(t_2, x)| < \varepsilon$ for all $(x, \varphi) \in \mathbb{R}_+ \times C_{B_n}$.*

Indeed, it follows from Lemma 2.1-(vi) and (2.1) that for any $\varepsilon > 0$ and $t^* > 0$, taking $M = [\frac{1}{t^*} + 1 + 5\mu]B_n$ and $\delta = \min\{\frac{\varepsilon^2}{4M^2}, \frac{t^*}{4}, 1\}$, we know that if $t^* \le t_1 \le t_2 < t_1 + \delta$ and $(x, \varphi) \in \mathbb{R}_+ \times C_{B_n}$, then

$$\begin{aligned} \left|u^\varphi(t_1, x) - u^\varphi(t_2, x)\right| &\le |S(t_1)[\varphi(0, \cdot)](x) - S(t_2)[\varphi(0, \cdot)](x)| \\ &\quad + \left|\int_0^{t_1} S(t_1 - s)\left[F\left(u_s^\varphi\right)\right](x)\mathrm{d}s - \int_0^{t_2} S(t_2 - s)\left[F\left(u_s^\varphi\right)\right](x)\mathrm{d}s\right| \\ &\le \frac{(1 + \mu t_1)B_n}{t_1}|t_2 - t_1| + \left|\int_{t_1}^{t_2} S(t_2 - s)\left[F\left(u_s^\varphi\right)\right](x)\mathrm{d}s\right| \\ &\quad + \left|\int_{t_1-\sqrt{\delta}}^{t_1} \left[S(t_1 - s)\left[F\left(u_s^\varphi\right)\right](x) - S(t_2 - s)\left[F\left(u_s^\varphi\right)\right](x)\right]\mathrm{d}s\right| \end{aligned}$$

$$
\begin{aligned}
&+\left|\int_0^{t_1-\sqrt{\delta}}\left[S(t_1-s)\left[F\left(u_s^{\varphi}\right)\right](x)-S(t_2-s)\left[F\left(u_s^{\varphi}\right)\right](x)\right]\mathrm{d}s\right|\\
\le\ &\frac{(1+\mu t^*)}{t^*}B_n\delta+\mu B_n\delta+2\mu B_n\sqrt{\delta}\\
&+\int_0^{t_1-\sqrt{\delta}}\mu B_n\delta\frac{1+\mu(t_1-s)}{t_1-s}e^{-\mu(t_1-s)}\mathrm{d}s\\
\le\ &\frac{(1+\mu t^*)}{t^*}B_n\delta+\mu B_n\delta+2\mu B_n\sqrt{\delta}+B_n\delta\frac{1+\mu\sqrt{\delta}}{\sqrt{\delta}}\\
\le\ &\left[\frac{(1+\mu t^*)}{t^*}B_n+\mu B_n+2\mu B_n+B_n(1+\mu)\right]\sqrt{\delta}\\
=\ &\left[\frac{1}{t^*}+1+5\mu\right]B_n\sqrt{\delta}<\varepsilon.
\end{aligned}
$$

Therefore, the Claim holds, and the proof is completed. □

Remark 2.1 Note that in the above discussion in this section, the initial functions are assumed to be bounded and *uniformly* continuous with respect to the spatial variable x. We point out that the same asymptotic behaviors of the integral equation (2.1) hold when the initial function φ is a bounded and continuous function from $[-1,0]\times\mathbb{R}_+$ to $\mathbb{R}_+$ with $\varphi|_{[-1,0]\times\{0\}}=0$. Indeed, by the step argument and the definition of F, it is easy to see that the solution $(u^{\varphi})_t$ of the integral equation (2.1) is also defined for all $t\in\mathbb{R}_+$; and by using some arguments similar to Remark 2.9 in [33], we may also obtain that $(u^{\varphi})_t\in C_+$ for all $t\in[2,\infty)$.

Denote by Φ the solution semiflow of (2.1), that is, $\Phi:\mathbb{R}_+\times C_+\to C_+$ is defined by $\Phi(t,\varphi)=(u^{\varphi})_t$ for all $(t,\varphi)\in\mathbb{R}_+\times C_+$, where the topology of C_+ is induced by the norm $||\cdot||_{co}$. Sometimes, we also write $\Phi(t,\varphi;f)$ for $\Phi(t,\varphi)$ to emphasize dependence on the nonlinearity f, if there is a need.

In what follows, we always assume that the tacit topology of C_+ is induced by the norm $||\cdot||_{co}$.

Definition 2.1 An element $\varphi\in C_+$ is called an equilibrium of Φ if $\Phi(t,\varphi)=\varphi$ for all $t\in\mathbb{R}_+$. A subset $\mathcal{A}$ of C_+ is said to be positively invariant under Φ if $\Phi(t,\varphi)\in\mathcal{A}$ for all $\varphi\in\mathcal{A}$ and $t\in\mathbb{R}_+$.

We write $O(\varphi)=\{\Phi(t,\varphi):t\in\mathbb{R}_+\}$ for the positive semi-orbit through the point φ. The ω-limit set of $O(\varphi)$ is defined by $\omega(\varphi)=\bigcap_{t\in\mathbb{R}_+}\overline{O(\Phi(t,\varphi))}$, where $\overline{O(\Phi(t,\varphi))}$ represents the closure of $O(\Phi(t,\varphi))$ with respect to the compact open topology.

Definition 2.2 Let $\mathbf{u}^*$ be an equilibrium and $\mathcal{A}$ be a positively invariant set of the semiflow Φ. We say that $\mathbf{u}^*$ is globally attractive in $\mathcal{A}$ if $\omega(\varphi)=\{\mathbf{u}^*\}$ for all $\varphi\in\mathcal{A}$.

Definition 2.3 We say that $\mathbf{0}$ is globally attractive in C_+ with respect to the usual supremum norm if $\lim\limits_{t\to\infty}||(u^{\varphi})_t||_C=0$ for all $\varphi\in C_+$.

Definition 2.4 Let $\mathbf{u}^*$ be an equilibrium. We say that $\mathbf{u}^*$ is globally attractive in $C_+\setminus\{\mathbf{0}\}$ with respect to the compact open topology if $\omega(\varphi)=\{u^*\}$, that is, $\lim_{t\to\infty}||(u^{\varphi})_t-\mathbf{u}^*||_{co}=0$ for all $\varphi\in C_+\setminus\{\mathbf{0}\}$.

In the sequel, we shall omit the term "with respect to the compact open topology" in Definition 2.4.

It is not difficult to establish the following result by using Theorem 2.15 in [33]. For the sake of completeness, the proof is provided.

Theorem 2.1 *If $f(x) < x$ for all $x > 0$, then $\mathbf{0}$ is a globally attractive equilibrium of* (2.1) *in C_+ with respect to the usual supremum norm.*

Proof Suppose that $\varphi \in C_+$. Take $\check{\varphi} \in C([-1,0] \times \mathbb{R}, \mathbb{R})$ such that $\check{\varphi}|_{[-1,0]\times[0,\infty)} = \varphi$ and $\check{\varphi}|_{[-1,0]\times(-\infty,0)} \equiv 0$. Theorem 2.17 in [33] shows $\lim_{t\to\infty} \sup\{(v^{\check{\varphi}}(t+\theta,x)| : (\theta,x) \in [-1,0] \times \mathbb{R}\} = 0$, where $(v^{\check{\varphi}})_t$ represents the solution of (2.6) with g replaced by the odd extension of f. Again, by Proposition 2.1-(i) together with (2.1) and (2.6), we have $0 \le (u^\varphi)_t \le (v^{\check{\varphi}})_t|_{[-1,0]\times\mathbb{R}_+}$. Therefore, $\lim_{t\to\infty} ||(u^\varphi)_t||_C = 0$ and the proof is completed. □

Clearly, assumption of Theorem 2.1 implies that $f'(0) \le 1$. If $f'(0) > 1$, then $\mathbf{0}$ is not a locally attractive equilibrium. In the next section, we tackle the global dynamics of (2.1) when $f'(0) > 1$.

3 Global Dynamics

In this section, we always assume that $f'(0) > 1$, In this case, to overcome the difficulty in showing that the trivial equilibrium repels nontrivial solutions caused by the lack of compactness of the spatial domain and by the non-symmetry of the spatial domain as well as the Dirichlet boundary conditions, we will first establish *a priori* estimate for nontrivial solutions. This will be done by looking at delicate asymptotic properties of the nonlocal delayed effect and the diffusion operator. The estimate then will enable us to show the existence and global attractivity of the nontrivial equilibrium by employing standard dynamical system theoretical arguments.

Let

$$l(x) = \frac{\sqrt{\mu}}{2} \exp\left(-\sqrt{\mu x^2}\right) \quad \text{and} \quad l(t,x) = \frac{\mu e^{-\mu t}}{\sqrt{4\pi t}} \exp\left(-\frac{x^2}{4t}\right) \quad \text{for} \quad t \in (0,\infty), \quad x \in \mathbb{R}.$$

Using these two functions, we define linear operators $K, K(0,\cdot), K(t,\cdot), K_t : X \to X$ by

$$\begin{aligned}
K[\zeta](x) &= \int_{\mathbb{R}_+^2} \zeta(y)[k(z-y)-k(z+y)][l(x-z)-l(x+z)]\mathrm{d}y\mathrm{d}z,\\
K(0,\zeta)(x) &= 0,\\
K(t,\zeta)(x) &= \int_{\mathbb{R}_+^2} \zeta(y)[k(z-y)-k(z+y)]\int_0^t [l(s,x-z)-l(s,x+z)]\mathrm{d}s\mathrm{d}y\mathrm{d}z,\\
K_t[\zeta] &= K - K(t,\zeta),
\end{aligned}$$

for all $\zeta \in X$, $t \in (0,\infty)$ and $x \in \mathbb{R}_+$. Note that

$$K_t[\zeta](x) = \int_{\mathbb{R}_+^2} \zeta(y)[k(z-y)-k(z+y)]\int_t^\infty [l(s,x-z)-l(s,x+z)]\mathrm{d}s\mathrm{d}y\mathrm{d}z$$

for all $t \in \mathbb{R}_+$ since $\int_{\mathbb{R}_+} l(s,x)\mathrm{d}s = l(x)$ for all $x \in \mathbb{R}$ due to Lemma 2.1(vi) in [33]. It is easy to verify that these operators are order preserving.

We point out that these operators can be naturally extended by the same formulas from the space $X$to the linear space of all measurable and bounded functions from $\mathbb{R}_+$ to $\mathbb{R}$ into itself, and the extended operators are also order preserving.

For given $T > 0$, define the function $h^T : \mathbb{R}_+ \to \mathbb{R}_+$ by $h^T(x) = 1$ for all $x \in [T, 2T]$ and $h^T(x) = 0$ for all $x \in \mathbb{R}_+ \setminus [T, 2T]$. Let $A^T = \{\phi \in X_+ : \phi(x) \geq h^T(x) \text{ for all } x \in \mathbb{R}_+\}$.

Lemma 3.1 *For any $n \in \mathbb{N}$ and $\delta \in (0, \frac{1}{2})$, there exists $T_{n,\delta} > 0$ such that $K^n[h^T] \geq (\frac{1}{2} - \delta)h^T$ for all $T \geq T_{n,\delta}$, and hence $K^n[A^T] \subseteq (\frac{1}{2} - \delta)A^T$ for all $T \geq T_{n,\delta}$, where K^n represents the nth-composition of K.*

Proof Fix $n \in \mathbb{N}$ and $\delta \in (0, \frac{1}{2})$. It suffices to prove that there exists $T_{n,\delta} > 0$ such that $K^n[h^T] \geq (\frac{1}{2} - \delta)h^T$ for all $T \geq T_{n,\delta}$, due to the monotonicity of K.

Define $g_n : \mathbb{R}^{2n} \to \mathbb{R}$ by $g_n(\mathbf{y}) = \prod_{i=1}^n (k(-y_i)l(-z_i))$ for all $\mathbf{y} = (y_1, z_1, y_2, z_2, \ldots, y_n, z_n) \in \mathbb{R}^{2n}$. It follows from Fubini's Theorem and the linear transformations of variables that for any $T > 0$ and $x \in \mathbb{R}_+$, we have

$$K^n[h^T](x) = \int_{\mathbb{R}^{2n}} \widetilde{h^T}\left(x + \sum_{i=1}^n (y_i + z_i)\right) g_n(\mathbf{y})\mathrm{d}\mathbf{y},$$

where $\widetilde{h^T}$ represents the odd extension of h^T. This together with the monotonicity and even property of k, l, shows that for any $T > 0$ and $x \in [T, 2T]$, we have

$$\begin{aligned} K^n[h^T](x) &= \int_{x+\sum_{i=1}^n (y_i+z_i)\in[T,2T]} g_n(\mathbf{y})\mathrm{d}\mathbf{y} - \int_{x+\sum_{i=1}^n (y_i+z_i)\in[-2T,-T]} g_n(\mathbf{y})\mathrm{d}\mathbf{y} \\ &\geq \int_{\sum_{i=1}^n (y_i+z_i)\in\left[0,\frac{T}{2}\right]} g_n(\mathbf{y})\mathrm{d}\mathbf{y} - \int_{\sum_{i=1}^n (y_i+z_i)\leq -T} g_n(\mathbf{y})\mathrm{d}\mathbf{y} \\ &= \int_{\sum_{i=1}^n (y_i+z_i)\in\left[0,\frac{T}{2}\right]} g_n(\mathbf{y})\mathrm{d}\mathbf{y} - \int_{\sum_{i=1}^n (y_i+z_i)\geq T} g_n(\mathbf{y})\mathrm{d}\mathbf{y}. \end{aligned}$$

By the definition of g_n and the fact that $\int_{\mathbb{R}} k(y)\mathrm{d}y = \int_{\mathbb{R}} l(y)\mathrm{d}y = 1$, we easily obtain

$$\lim_{T\to\infty} \int_{\sum_{i=1}^n (y_i+z_i)\in\left[0,\frac{T}{2}\right]} g_n(\mathbf{y})\mathrm{d}\mathbf{y} = \frac{1}{2}$$

and

$$\lim_{T\to\infty} \int_{\sum_{i=1}^n (y_i+z_i)\geq T} g_n(\mathbf{y})\mathrm{d}\mathbf{y} = 0.$$

Therefore, there exists $T_{n,\delta} > 0$ such that for all $T \geq T_{n,\delta}$,

$$\int_{\sum_{i=1}^n (y_i+z_i)\in\left[0,\frac{T}{2}\right]} g_n(\mathbf{y})\mathrm{d}\mathbf{y} - \int_{\sum_{i=1}^n (y_i+z_i)\geq T} g_n(\mathbf{y})\mathrm{d}\mathbf{y} \geq \frac{1}{2} - \delta.$$

So, for any $T \geq T_{n,\delta}$ and $x \in [T, 2T]$, we have $K^n\big[h^T\big](x) \geq \frac{1}{2} - \delta$, that is, $K^n\big[h^T\big] \geq \big(\frac{1}{2} - \delta\big)h^T$ and hence the proof is complete. □

To continue our discussions, we give some links together the nonlocal reaction and the diffusion, which shall be very useful to prove *a priori* estimate for nontrivial solutions for (2.1).

Lemma 3.2 *For any $n \in \mathbb{N}$ and $\delta \in \big(0, \frac{1}{2}\big)$, there exist $T_{n,\delta} > 0$ and $s_{n,\delta} > 0$ such that $K^n\big[h^T\big] \geq \big(\frac{1}{2} - \delta\big)h^T$ and $(K(s,\cdot))^n\big[h^T\big] \geq \big(\frac{1}{2} - \delta\big)h^T$ for all $T \geq T_{n,\delta}$ and $s \geq s_{n,\delta}$, where K^n and $(K(s,\cdot))^n$ represent the nth-composition of K and $K(s,\cdot)$, respectively.*

Proof Fix $n \in \mathbb{N}$ and $\delta \in \left(0, \frac{1}{2}\right)$. By Lemma 3.1, there exists $T_{n,\delta} > 0$ such that $K^n\left[h^T\right] \geq \frac{1-\delta}{2}h^T$ for all $T \geq T_{n,\delta}$.

Let $s_{n,\delta} = \frac{1}{\mu}\ln\left(\frac{2^{1+n}}{\delta}\right)$. Now fix $T \geq T_{n,\delta}$ and $s \geq s_{n,\delta}$. By the definitions of $K, K_s, K(s,\cdot)$, we have $K_s \circ K(s,\cdot) = K(s,\cdot) \circ K_s$, $K_s[1] \leq e^{-\mu s}$, $(K_s)^j[h^T] \leq 1$ and $(K(s,\cdot))^j\left[h^T\right] \leq 1$ for all $j \in \mathbb{N}$. It follows that

$$
\begin{aligned}
K^n\left[h^T\right] &= \sum_{j=0}^{n} C_n^j (K_s)^j (K(s,\cdot))^{n-j}\left[h^T\right] \\
&= (K(s,\cdot))^n\left[h^T\right] + K_s\left[\sum_{j=1}^{n} C_n^j (K_s)^{j-1}(K(s,\cdot))^{n-j}\left[h^T\right]\right] \\
&\leq (K(s,\cdot))^n\left[h^T\right] + (2^n - 1)K_s[1] \\
&\leq (K(s,\cdot))^n\left[h^T\right] + 2^n e^{-\mu s} \\
&\leq (K(s,\cdot))^n\left[h^T\right] + \frac{\delta}{2}.
\end{aligned}
$$

This, combined with the fact that $K^n\left[h^T\right] \geq \frac{1-\delta}{2}h^T$, implies that $(K(s,\cdot))^n\left[h^T\right] \geq \left(\frac{1}{2} - \delta\right)h^T$, completing the proof. □

The following result gives *a priori* estimate for nontrivial solutions for (2.1), which plays a key role in the proof of the repellency of the trivial equilibrium for any nonnegative solution and global attractivity of nontrivial equilibrium of (2.1).

Proposition 3.1 *Suppose that $f'(0) > 1$ and $B \in \{B_n : n \in \mathbb{N}\}$. Then there exist $\varepsilon_0 > 0$, $T_0 > 0$ and $T^* > 0$ such that for all $\varepsilon \in (0, \varepsilon_0]$, $T \in [T_0, \infty)$, and a solution $u : [-1, \infty) \times \mathbb{R}_+ \to [0, B]$ of (2.1) with $u(t,\cdot) \geq \varepsilon h^T$ for all $t \in [-1, T^*]$, we have $u(t,\cdot) \geq \varepsilon h^T$ for all $t \in [-1, \infty)$ and $u(t,\cdot) \gg \varepsilon h^T$ for all $t \in (T^*, \infty)$.*

Proof Obviously, there exist $n \geq 2$ and $\beta \in (1, f'(0))$ such that $\beta^n > 4$.

By the choice of β, one can easily see that there exists a $\varepsilon_1 \in (0, \min\{1, B\})$ such that $f(u) \geq \beta u$ for all $u \in [0, \varepsilon_1]$ and $f(u) \geq \beta\varepsilon_1$ for all $u \in [\varepsilon_1, B]$.

By applying Lemma 3.2 with $\delta = \frac{1}{4}$, we know that there exists $s_n > 0$ such that $K^n\left(h^T\right) \geq \frac{1}{4}h^T$ and $(K(s,\cdot))^n\left(h^T\right) \geq \frac{1}{4}h^T$ for all $T \geq s_n$ and $s \geq s_n$.

Let $\varepsilon_0 = \frac{\varepsilon_1}{\beta^{n+1}}$, $T_0 = s_n$ and $T^* = ns_n + n - 1$. Suppose that $\varepsilon \in (0, \varepsilon_0]$, $T \in [T_0, \infty)$, $u : [-1, \infty) \times \mathbb{R}_+ \to [0, B]$ is a solution of (2.1) such that $u(t,\cdot) \geq \varepsilon h^T$ for all $t \in [-1, T^*]$. Let $\varphi = u_0$. Then $u(t,x) = u^\varphi(t,x) = \Phi(t+1,\varphi)(-1,x)$ for all $(t,x) \in [-1,\infty) \times \mathbb{R}_+$. Due to the choices of ε and β, one can easily obtain $\beta^j(K(t,\cdot))^j\left[\varepsilon h^T\right] < \varepsilon_1$ and $f\left(\beta^j(K(t,\cdot))^j\left[\varepsilon h^T\right]\right) \geq \beta^{j+1}(K(t,\cdot))^j\left[\varepsilon h^T\right]$ for all $t \geq 0$ and $j = 0, 1, \ldots, n$.

Let $n^* = \sup\{j \in \{0,1,2,\cdots,n-1\} : u(t,\cdot) \geq \varepsilon\beta^j(K(s_n,\cdot))^j[h^T]$ for all $t \in [js_n + j - 1, T^*]\}$. We claim $n^* = n - 1$; otherwise, $n^* \in [0, n-2]$ and $u(t,\cdot) \geq \varepsilon\beta^{n^*}(K(s_n,\cdot))^{n^*}\left[h^T\right]$ for all $t \in [n^*s_n + n^* - 1, T^*]$. These, combined with (2.1) and Fubini's theorem implies that for any $t \in [(1+n^*)s_n + n^*, T^* + 1]$,

$$
\begin{aligned}
u(t,\cdot) &= u(s_n + (t - s_n), \cdot) \\
&= \Phi(s_n, \Phi(t - s_n, \varphi))(0,\cdot) \\
&= S(s_n)[u(t - s_n, \cdot)] + \int_0^{s_n} S(s_n - s)[F(u_{s+t-s_n})]\mathrm{d}s
\end{aligned}
$$

$$\begin{aligned}
&\geq \int_0^{s_n} S(s_n - s)[f(u(s+t-s_n-1,\cdot))]\mathrm{d}s \\
&\geq K(s_n, f(u(s+t-s_n-1,\cdot))) \\
&\geq K\big(s_n, \varepsilon\beta^{n^*+1}(K(s_n,\cdot))^{n^*}\big[h^T\big]\big) \\
&\geq \varepsilon\beta^{n^*+1}(K(s_n,\cdot))^{n^*+1}\big[h^T\big],
\end{aligned}$$

which yields a contradiction. This, together with the previous discussions with $n^* = n-1$, means that $u(t,\cdot) \geq \varepsilon\beta^n(K(s_n,\cdot))^n[h^T]$ for all $t \in [ns_n + n - 1, T^* + 1]$. In particular, $u(t,\cdot) \geq \varepsilon\beta^n(K(s_n,\cdot))^n[h^T]) \geq \varepsilon\frac{\beta^n}{4}h^T$ for all $t \in [T^*, T^*+1]$, and thus $u(t,\cdot) \gg \varepsilon h^T$ for all $t \in [T^*, T^*+1]$. Hence, the results easily follow from the semigroup property of $\Phi|_{\mathbb{R}_+\times C_B}$. □

The following shows that the positive limit set of a positive solution of the system is far away from zero at the infinity.

Theorem 3.1 *If $\varphi \in C_+ \setminus \{\mathbf{0}\}$, then there exist $\varepsilon_\varphi > 0$ and $T_\varphi > 0$ such that $\omega(\varphi) \geq \varepsilon_\varphi h^T$ for all $T \geq T_\varphi$. In other words, $\omega(\varphi) \geq \varepsilon_\varphi h^{T_\varphi,\infty}$, where $h^{T_\varphi,\infty}(x) = 1$ for all $x \in [T_\varphi, \infty)$ and $h^{T_\varphi,\infty}(x) = 0$ for all $x \in [0, T_\varphi)$.*

Proof By Proposition 2.1-(i) and (ii), we may assume that $u^\varphi(t,x) \in (0, B]$ for all $(t,x) \in [-1,\infty) \times (0,\infty)$ for some $B \in \{B_n : n \in \mathbb{N}\}$. Choose T_0, T^*, and ε_0 as in Proposition 3.1. Let $T_\varphi = T_0$, $\varepsilon_1 = \inf\{u(t,x) : (t,x) \in [-1,T^*] \times [T_0, 2T_0]\}$ and $\varepsilon_\varphi = \min\{\varepsilon_0, \varepsilon_1\}$. Then $\varepsilon_1 > 0$ and $\varepsilon_\varphi > 0$. By Proposition 3.1 and the choices of T_0, T^* and ε_0, we get $u^\varphi(t,\cdot) \geq \varepsilon_\varphi h^{T_\varphi}$ for all $t \geq -1$. This, combined with the definition of $\omega(\varphi)$, implies $\xi \geq \varepsilon_\varphi h^{T_\varphi}$ for all $\xi \in \omega(\varphi)$.

For any $\xi \in \omega(\varphi)$, let $a_\xi = \sup\{a \geq T_\varphi : \xi(\theta,x) \geq \varepsilon_\varphi$ for all $(t,x) \in [-1,0]\times[T_\varphi, 2a]\}$ and $T = \inf\{a_\xi : \xi \in \omega(\varphi)\}$. Then $a_\xi \geq T_\varphi$ for all $\xi \in \omega(\varphi)$, and thus $T \geq T_\varphi$.

We claim that $T = \infty$. By way of contradiction, suppose that $T < \infty$. Take $\xi^* \in \omega(\varphi)$. Then, the invariance of $\omega(\varphi)$ implies that $u^{\xi^*}(t,\cdot) \geq \varepsilon_\varphi h^T$ for all $t \in [-1, T^*]$. Again, by Proposition 3.1 and the choices of T_0, T^* and ε_0, we have $u^{\xi^*}(t,\cdot) \gg \varepsilon_\varphi h^T$ for all $t \in (T^*, \infty)$. In particular, there exists $\tilde{T} \in (T, 2T)$ such that $u^{\xi^*}(t,\cdot) \gg \varepsilon_\varphi h^{\tilde{T}}$ for all $t \in [1+T^*, 2+2T^*]$. On the other hand, by the definition of $\omega(\varphi)$, there exists a sequence $\{s_n\}_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} ||(u^\varphi)_{s_n} - \xi^*||_{co} = 0$. It follows that $\lim_{n\to\infty}(\sup\{|u^\varphi(s_n+t,x) - u^{\xi^*}(t,x)| : (t,x) \in [1+T^*, 2+2T^*] \times [\tilde{T}, 2\tilde{T}]\}) = 0$. Thus there exists $n^* > 1$ such that $u^\varphi(s_{n^*}+t,\cdot) \geq \varepsilon_\varphi h^{\tilde{T}}$ for all $t \in [1+T^*, 2+2T^*]$. It follows from Proposition 3.1 that $u^\varphi(s_{n^*}+t,\cdot) \geq \varepsilon_\varphi h^{\tilde{T}}$ for all $t \in [1+T^*, \infty)$. This and the definition of $\omega(\varphi)$ produce $\xi \geq \varepsilon_\varphi h^{\tilde{T}}$ for all $\xi \in \omega(\varphi)$. Since $2T > \tilde{T} > T$, we have $a_\xi \geq \tilde{T} > T$ for all $\xi \in \omega(\varphi)$. Then $T = \inf\{a_\xi : \xi \in \omega(\varphi)\} \geq \tilde{T} > T$, a contradiction. This proves the claim, that is, $T = \infty$. Hence, this claim and the choice of T imply that $\omega(\varphi) \geq \varepsilon_\varphi h^{T_\varphi,\infty}$ to complete the proof. □

The next lemma establishes a relation between the topologies defined by the local supremum norm and the local $C^1(\mathbb{R}_+)$ norm in $\{\varphi(0,\cdot) : \varphi \in \omega(\psi)\}$ for any given $\psi \in X_+$.

Lemma 3.3 *Let $\psi \in C_+$, and let φ and $\{\varphi^n\}_{n=1}^\infty$ be in $\omega(\psi)$. If $a \in (0,\infty)$ and $\lim_{n\to\infty} ||\varphi^n(0,\cdot) - \varphi(0,\cdot)||_{C([0,a],\mathbb{R})} = 0$, then $\lim_{n\to\infty} ||\varphi^n(0,\cdot) - \varphi(0,\cdot)||_{C^1([0,a],\mathbb{R})} = 0$. Thus, the local supremum norm and the local $C^1(\mathbb{R}_+)$ norm define the same topology on $\{\varphi(0,\cdot) : \varphi \in \omega(\psi)\}$.*

Proof Let $\psi \in C_+$. We *claim* that there exists $M = M_\psi > 0$ such that $||\varphi(0,\cdot)||_{C^1(\mathbb{R}_+)} \le M$ for all $\varphi \in \omega(\psi)$. Here $||g||_{C^1(\mathbb{R}_+)}$ is the sum of the supremum norms g, g'. Note that $\psi \le B$ and thus $\omega(\psi) \le B$ for some $B \in \{B_n : n \in \mathbb{N}\}$ due to Proposition 2.1. Now, it follows from (2.1) and Lemma 2.1-(v) that for any $x \in \mathbb{R}_+$ and $\varphi \in \omega(\psi)$, we have

$$\begin{aligned}\left|\frac{\partial u^\varphi(2,x)}{\partial x}\right| &= \left|\frac{\partial S(2)[\varphi(0,\cdot)](x)}{\partial x} + \frac{\partial \int_0^2 S(2-s)[F(u_s^\varphi)](x)\mathrm{d}s}{\partial x}\right| \\ &\le \frac{\exp(-2\mu)||\varphi(0,\cdot)||_X}{\sqrt{2\pi}} + \int_0^2 \frac{\exp(-\mu(2-s))\left||F\left(u_s^\varphi\right)\right||_X}{\sqrt{\pi(2-s)}}\mathrm{d}s \\ &\le \frac{B\exp(-2\mu)}{\sqrt{2\pi}} + \mu B\int_0^2 \frac{\exp(-\mu(2-s))}{\sqrt{\pi(2-s)}}\mathrm{d}s \\ &\le \frac{B}{\sqrt{2\pi}} + \sqrt{\mu}B < \left(1+\sqrt{\mu}\right)B.\end{aligned}$$

Taking $M_\psi = \left(2+\sqrt{\mu}\right)B$, the invariance of $\omega(\psi)$ leads to the above claim.

Now we also *claim* that $\{\frac{\partial\varphi(0,\cdot)}{\partial x} : \varphi \in \omega(\psi)\}$ is an equi-continuous functions family in $C(\mathbb{R}_+,\mathbb{R})$. Indeed, for any $\varepsilon > 0$, by choosing $t^* = 2 - \min\{(\frac{\varepsilon}{20\mu B})^2, \frac{1}{2}\}$ and $\delta = \min\{\frac{\varepsilon}{3B}, \frac{\varepsilon}{6B\mu\ln\frac{2}{2-t^*}}\}$, we may obtain that if $\varphi \in \omega(\psi)$ and $x_1, x_2 \in \mathbb{R}_+$ with $|x_1 - x_2| < \delta$, then by (2.1) and Lemma 2.1-(v), we have

$$\begin{aligned}\left|\frac{\partial u^\varphi(2,x_1)}{\partial x} - \frac{\partial u^\varphi(2,x_2)}{\partial x}\right| &\le \left|\frac{\partial S(2)[\varphi(0,\cdot)]}{\partial x}(x_1) - \frac{\partial S(2)[\varphi(0,\cdot)]}{\partial x}(x_2)\right| \\ &\quad + \left|\int_0^2 \frac{\partial S(2-s)\left[F\left(u_s^\varphi\right)\right]}{\partial x}(x_1)\mathrm{d}s\right. \\ &\quad \left. - \int_0^2 \frac{\partial S(2-s)\left[F\left(u_s^\varphi\right)\right]}{\partial x}(x_2)\mathrm{d}s\right| \\ &\le \frac{\exp(-2\mu)||\varphi(0,\cdot)||_X}{2}|x_1 - x_2| \\ &\quad + \int_0^{t^*}\left|\frac{\partial S(2-s)\left[F\left(u_s^\varphi\right)\right]}{\partial x}(x_1) - \frac{\partial S(2-s)\left[F\left(u_s^\varphi\right)\right]}{\partial x}(x_2)\right|\mathrm{d}s \\ &\quad + \int_{t^*}^2\left|\frac{\partial S(2-s)\left[F\left(u_s^\varphi\right)\right]}{\partial x}(x_1)\right| + \left|\frac{\partial S(2-s)\left[F\left(u_s^\varphi\right)\right]}{\partial x}(x_2)\right|\mathrm{d}s \\ &\le \frac{\exp(-2\mu)||\varphi(0,\cdot)||_X}{2}|x_1 - x_2| \\ &\quad + \int_0^{t^*}\frac{\exp(-\mu(2-s))\left\|F\left(u_s^\varphi\right)\right\|_X}{2-s}|x_1 - x_2|\mathrm{d}s \\ &\quad +2\int_{t^*}^2\frac{\exp(-\mu(2-s))\left\|F\left(u_s^\varphi\right)\right\|_X}{\sqrt{\pi(2-s)}}\mathrm{d}s \\ &\le \left[\frac{B}{2} + B\mu\ln\frac{2}{2-t^*}\right]|x_1 - x_2| + 4B\mu\sqrt{\frac{2-t^*}{\pi}} \\ &< \varepsilon,\end{aligned}$$

which together with the invariance of $\omega(\psi)$ implies the claim.

By the above claims, the Arzèla-Ascoli theorem , we know that $\{\varphi(0,\cdot) : \varphi \in \omega(\psi)\}$ is pre-compact under the topology induced by the $C^1_{loc}(\mathbb{R}_+)$-norm.

Now, let $a \in (0,\infty)$ and let the sequence $\{\varphi^n\}_{n=1}^{\infty}$ and φ be in $\omega(\psi)$ and satisfy $\lim\limits_{n\to\infty} ||\varphi^n(0,\cdot) - \varphi(0,\cdot)||_{C([0,a],\mathbb{R})} = 0$. We need to prove prove $\lim\limits_{n\to\infty} ||\varphi^n(0,\cdot) - \varphi(0,\cdot)||_{C^1([0,a],\mathbb{R})} = 0$. Otherwise, by the compactness of $\{\varphi(0,\cdot)|_{[0,a]} : \varphi \in \omega(\psi)\}$ in $C^1([0,a],\mathbb{R})$, there exist a $\phi \in C^1([0,a],\mathbb{R})$ and a subsequence $\{\varphi^{n_k}\}_{k=1}^{\infty}$ such that $\phi(0,\cdot) \neq \varphi(0,\cdot)|_{[0,a]}$ and $\lim\limits_{n\to\infty} ||\varphi^{n_k}(0,\cdot) - \phi(0,\cdot)||_{C^1([0,a],\mathbb{R})} = 0$. But, this, combining with the fact that $\lim\limits_{n\to\infty} ||\varphi^n(0,\cdot) - \varphi(0,\cdot)||_{C([0,a],\mathbb{R})} = 0$, we have $\phi(0,\cdot) = \varphi(0,\cdot)|_{[0,a]}$, a contradiction. Therefore, $\lim\limits_{n\to\infty} ||\varphi^n(0,\cdot) - \varphi(0,\cdot)||_{C^1([0,a],\mathbb{R})} = 0$ provided that $\lim\limits_{n\to\infty} ||\varphi^n(0,\cdot) - \varphi(0,\cdot)||_{C([0,a],\mathbb{R})} = 0$. In other words, the local supremum norm and the C^1_{loc}-norm define the same topology on $\{\varphi(0,\cdot) : \varphi \in \omega(\psi)\}$. □

According to Proposition 2.1-(iv-v) and the fact that $\Phi(t,\varphi)(\theta,x) = u^{\varphi}(t+\theta,x)$ for all $(t,\theta,x) \in [0,\infty) \times [-1,0] \times \mathbb{R}_+$, we easily obtain the following result,

Lemma 3.4 *Let $M > 0$, $t > 0$, $\tilde{\tau} \in (0,1]$ and $D \subseteq C_+$ with $D \le M$. Assume that $D|_{[-\tilde{\tau},0]\times\mathbb{R}_+}$ is pre-compact in $BUC_{co}([-\tilde{\tau},0] \times \mathbb{R}_+,\mathbb{R})$, where $D|_{[-\tilde{\tau},0]\times\mathbb{R}_+} \triangleq \{\varphi|_{[-\tilde{\tau},0]\times\mathbb{R}_+} : \varphi \in D\}$ and $BUC_{co}([-\tilde{\tau},0]\times\mathbb{R}_+,\mathbb{R})$ is the set of all bounded and uniformly continuous functions from $[-\tilde{\tau},0] \times \mathbb{R}_+$ to $\mathbb{R}$ equipped with the usual compact and open topology. Then $\Phi(t,D)|_{[-\min\{\frac{t}{2}+\tilde{\tau},1\},0]\times\mathbb{R}_+}$ is pre-compact in $BUC_{co}([-\min\{\frac{t}{2}+\tilde{\tau},1\},0]\times \mathbb{R}_+,\mathbb{R})$.*

Proposition 3.2 *Let (H1) hold. Then (2.1) has a positive steady state, located in X_+°.*

Proof Take $M \in \{B_n : n \in \mathbb{N}\}$. Then there is $\epsilon_M \in (0,M)$ such that $f'(x) > 0$ for all $x \in [0,\epsilon_M]$ and $f(\epsilon_M) = \min f([\epsilon_M,M])$. Define $\tilde{f} : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$\tilde{f}(u) = \begin{cases} f(u), & u \notin [0,\epsilon_M), \\ f(\epsilon_M), & u \in [\epsilon_M,\infty). \end{cases}$$

Then $\Phi(t,\psi;\tilde{f}) \le \Phi(t,\varphi;\tilde{f}) \le \Phi(t,\varphi)$ for all $(t,\varphi) \in \mathbb{R}_+ \times C_M$ and $\psi \in C_M$ with $\psi \le \varphi$. Choose $\varphi^* \in C_+ \setminus \{0\}$ with $\varphi^* \le M$ and let $\varepsilon_{\varphi^*}, T_{\varphi^*}$ defined as in Theorem 3.1 with f replaced by $\tilde{f}$. Then $\omega(\varphi^*;\tilde{f}) \ge \varepsilon_{\varphi^*} h^{T_{\varphi^*},\infty}$, and hence $\omega(\varphi^*;\tilde{f}) \subseteq X_+^{\circ}$. Let $D = \{\varphi \in C_+ : \omega(\varphi^*;\tilde{f}) \le \varphi \le M\}$. Clearly, D is a closed convex subset in C_{co} such that $\varepsilon_{\varphi^*} h^{T_{\varphi^*},\infty} \le D \subseteq C_+^{\circ}$ and $\Phi(t,D) \subseteq D$ for all $t \ge 0$. By applying Lemma 3.3 with f replaced by $\tilde{f}$, there exists $\gamma > 1$ such that $|\frac{\partial\varphi(0,x)}{\partial x}| \le \gamma$ for all $(x,\varphi) \in [0,\frac{M}{\gamma}] \times \omega(\varphi^*;\tilde{f})$. This, combined with the invariance of $\omega(\varphi^*;\tilde{f})$, implies that $\zeta \ge \omega(\varphi^*;\tilde{f})$ and $\zeta \le M$, where $\zeta(\cdot,x) = \gamma x$ for all $x \in [0,\frac{M}{\gamma}]$ and $\zeta(\cdot,x) = M$ for all $x \in (\frac{M}{\gamma},\infty)$. Thus $\zeta \in D$ and $D \neq \emptyset$.

Now suppose that $T \in I \triangleq \{\frac{1}{2^i} : i = 1,2,\cdots\}$. We claim that there exist a compact convex subset K_T in C_{co} and $\psi_T \in K_T$ such that $\Phi(T,K_T) \subseteq K_T$ and $\Phi(T,\psi_T) = \psi_T$. Indeed, by Lemma 3.4 and the fact that $\Phi(t,\varphi)(\theta,x) = u^{\varphi}(t+\theta,x)$ for all $(t,\theta,x) \in [0,\infty) \times [-1,0] \times \mathbb{R}_+$, we know that $\Phi(T,D)|_{[-\min\{1,\frac{T}{2}\},0]\times\mathbb{R}_+}$ is pre-compact with the compact and open topology in $BUC([-\min\{1,\frac{T}{2}\},0] \times \mathbb{R}_+,\mathbb{R})$. Let $g(K) \triangleq \overline{co}(\Phi(T,K))$ for any $K \subseteq D$. Then $g(D)|_{[-\min\{1,\frac{T}{2}\},0]\times\mathbb{R}_+}$ is compact with the compact and open topology in $BUC([-\min\{1,\frac{T}{2}\},0] \times \mathbb{R}_+,\mathbb{R})$. By applying Lemma 3.4 repeatedly, we may get that $g^k(D)|_{[-\min\{1,\frac{kT}{2}\},0]\times\mathbb{R}_+}$ is compact with the compact and open topology in $BUC([-\min\{1,\frac{kT}{2}\},0] \times \mathbb{R}_+,\mathbb{R})$. Choose a positive integer

k_0 such that $k_0 > \frac{2}{T}$. Then $K_T \triangleq g^{k_0}(D)$ is a compact convex subset in C_{co}. Let $J = \{k \in \mathbb{N} \cup \{0\} : k \leq k_0$ and $\Phi(T, g^k(D)) \subseteq g^k(D)\}$ and $k^* = \sup J$. Then $0 \in J$ and $k^* \in [0, k_0]$. We now prove $k^* = k_0$; otherwise $k^* < k_0$. Since $g^{k^*}(D)$ is a closed convex subset of C_{co}, by the definitions of g and k^* we have $g^{k^*+1}(D)) \subseteq g^{k^*}(D)$. This and the definition of g imply $\Phi(T, g^{k^*+1}(D)) \subseteq \Phi(T, g^{k^*}(D)) \subseteq g^{k^*+1}(D)$, which yields a contradiction with the definition of k^*. So, K_T is a compact convex subset in C_{co} such that $\Phi(T, K_T) \subseteq K_T$. Thus, by the Schauder fixed point theorem there is $\psi_T \in K_T$ such that $\Phi(T, \psi_T) = \psi_T$.

According to Proposition 2.1-(v) and the fact that $\{\psi_T : T \in I\} \subseteq \Phi(k, D)$ for any positive integer k, we know that $\{\psi_T : T \in I\}$ is pre-compact in C_{co}, and thus there exist $\psi \in D$ and a sequence $\{T_k\}$ in I such that $\lim_{T_k \to 0} \psi_{T_k} = \psi$. For any $t \in (0, \infty)$, there exist $r_k \in [0, T_k)$ and nonnegative integer N_k such that $t = N_k T_k + r_k$. Obviously, $\lim_{k\to\infty} r_k = 0$. Hence, for all $t \geq 0$, we have $\Phi(t, \psi) = \lim_{k\to\infty} \Phi(t, \psi_{T_k}) = \lim_{k\to\infty} \Phi(r_k, \psi_{T_k}) = \psi$, which implies that ψ is a positive steady state, located in X_+° of (2.1), completing the proof. □

In what follows, we denote by u_+ the positive steady state of (2.1) obtained in Proposition 3.2, and let $u_+^* = ||u_+||$. Now, we introduce some more assumptions on the nonlinearity f, which shall enable us to obtain the asymptotic behavior of positive steady state at infinity.

(A1) f^2 has a unique positive fixed point u^*.

(A2) There is a $u^* > 0$ such that $f(u^*) = u^*$, and $|f(b) - f(u^*)| \leq |b - u^*|$ for all $b \geq 0$; and the equality $|f(b) - f(u^*)| = |b - u^*|$ holds for some $b \geq 0$ if and only if either $b = 0$ or $b = u^*$.

(A3) f is continuously differentiable on $\mathbb{R}_+$ and has a unique critical point u^c and a unique fixed point u^* in $(0, \infty)$ such that either $u^c \geq u^*$ or ($u^c < u^*$ and $f(f(u)) > u$ for all $u \in [u^c, u^*)$).

By the proof of Theorems 3.11 and 3.12 in [33], we easily obtain the following results.

Lemma 3.5 *If (A2) or (A3) hold, then (A1) holds.*

Proposition 3.3 *If one of (A1)–(A3) holds, then* $\liminf_{x\to\infty} u(x) = \limsup_{x\to\infty} u(x) = u^*$ *and* $u_+^* < max\{f(x) : x \in [0, u_+^*]\}$.

Proof Let $\overline{u} = \limsup_{x\to\infty} u(x)$, $\underline{u} = \liminf_{x\to\infty} u(x)$ and $I = [\underline{u}, \overline{u}]$. Then by $u \geq \varepsilon h^{T,\infty}$ for some ε and $T \in (0, \infty)$ due to Theorem 3.1, we have $\underline{u} > 0$ and thus $I \subseteq (0, \infty)$.

Define $\underline{\phi} = \liminf_{x\to\infty} \phi(x)$, $\overline{\phi} = \limsup_{x\to\infty} \phi(x)$ and $P[\phi](x) = \int_0^\infty \phi(y)[p(x-y) - p(x+y)]\mathrm{d}y$ for all $x \in \mathbb{R}_+$ and $\phi \in X$, where $p : \mathbb{R} \to \mathbb{R}_+$ is a continuous and even function on $\mathbb{R}$ such that $\int_{\mathbb{R}} p(y) = 1$ and p is decreasing on $\mathbb{R}_+$. We claim that $\underline{\phi} \leq \underline{P[\phi]} \leq \overline{P[\phi]} \leq \overline{\phi}$ for all $\phi \in X_+$. Clearly, $\underline{\phi} \leq \overline{\phi}$ and $P[\phi](x) = \int_{-x}^\infty \phi(x+y)p(y)\mathrm{d}y - \int_{\mathrm{x}}^\infty \phi(\mathrm{y}-\mathrm{x})\mathrm{p}(\mathrm{y})\mathrm{d}\mathrm{y}$ for all $(x, \phi) \in \mathbb{R}_+ \times X_+$. Thus, for any $\phi \in X_+$, we have

$$\begin{aligned}\overline{P[\phi]} &= \limsup_{x\to\infty} P[\phi](x)\\ &\leq \limsup_{x\to\infty} \int_{-x}^\infty \phi(x+y)p(y)\mathrm{d}y - \liminf_{\mathrm{x}\to\infty} \int_{\mathrm{x}}^\infty \phi(\mathrm{y}-\mathrm{x})\mathrm{p}(\mathrm{y})\mathrm{d}\mathrm{y}\end{aligned}$$

$$
\begin{aligned}
&= \limsup_{x\to\infty} \int_{-x}^{\infty} \phi(x+y)p(y)\mathrm{d}y \\
&\leq \overline{\phi}.
\end{aligned}
$$

Similarly, $\underline{P[\phi]} \geq \underline{\phi}$ for all $\phi \in X_+$.

Since u_+ be the positive steady state of (2.1), we obtain that $u_+ = S(t)[u_+] + \int_0^t S(t-s)[F(u_+)]\mathrm{d}s$ for any $t \in \mathbb{R}_+$. This, together with the above claim, implies that

$$
\begin{aligned}
\underline{u_+} &\geq \underline{S(t)[u_+]} + \int_0^t \underline{S(t-s)[F(u_+)]}\mathrm{d}s \\
&\geq e^{-\mu t}\underline{u_+} + \int_0^t e^{-\mu(t-s)}\underline{F(u_+)}\mathrm{d}s \\
&\geq e^{-\mu t}\underline{u_+} + \mu\int_0^t e^{-\mu(t-s)}\underline{f(u_+)}\mathrm{d}s \\
&= e^{-\mu t}\underline{u_+} + \left(1-e^{-\mu t}\right)\underline{f(u_+)}.
\end{aligned}
$$

Thus, $\underline{u_+} \geq \underline{f(u_+)}$, and a similar argument yields $\overline{u_+} \leq \overline{f(u_+)}$. Consequently, $I \subseteq f(I)$ and hence the assumption implies $I = \{u^*\}$. In other word, $\liminf_{x\to\infty} u(x) = \limsup_{x\to\infty} u(x) = u^*$.

Since u_+ be the positive steady state of (2.1), we obtain that $u_+ = S(t)[u_+] + \int_0^t S(t-s)[F(u_+)]\mathrm{d}s$ for any $t \in \mathbb{R}_+$, which together with Lemma 2.1 and the represention of F, gives $u_+^* < e^{-\mu t}u_+^* + (1-e^{-\mu t})\max f([0,u_+^*])$ and thus $u_+^* < \max f([0,u_+^*])$. The proof is completed. □

In the sequel, we define $M^{u_+} : C_+ \times \mathbb{R}_+ \to \mathbb{R}$ by

$$
M^{u_+}(\varphi, x) = \begin{cases} \frac{\varphi(0,x)}{u_+(x)}, & x \in (0,\infty) \\ \frac{\partial\varphi(0,x)}{\partial x} / \frac{\partial u_+(x)}{\partial x}, & x = 0. \end{cases}
$$

Lemma 3.6 *Let $\psi \in C_+ \setminus \{0\}$. Then $M^{u_+}|_{\omega(\psi)\times\mathbb{R}_+}$ is a continuous positive function.*

Proof Suppose that $\psi \in C_+\setminus\{0\}$. Theorem 3.1 and Proposition 2.1 show that $\{u_+\}\cup\omega(\psi) \subseteq C_+^\circ$ and $\frac{\partial\varphi(0,x)}{\partial x}|_{x=0} > 0$ for all $\varphi \in \{u_+\}\cup\omega(\psi)$. These together with the definition of M^{u_+}, imply that $M^{u_+}(\omega(\psi)\times\mathbb{R}_+) \subseteq (0,\infty)$. Again, the definition of M^{u_+} gives that $M^{u_+}|_{\omega(\psi)\times(0,\infty)}$ and $M^{u_+}|_{\omega(\psi)\times\{0\}}$ are continuous functions.

It suffices to prove that for given $\varphi^* \in \omega(\psi)$ and given sequence $(\varphi_n, x_n) \in \omega(\psi) \times (0,\infty)$, if $\lim_{n\to\infty} ||\varphi_n - \varphi^*|| = \lim_{n\to\infty} |x_n| = 0$, then $\lim_{n\to\infty} M^{u_+}(\varphi_n, x_n) = M^{u_+}(\varphi^*, 0)$. Indeed, by Lemma 3.3, we have $\lim_{n\to\infty} \sup_{x\in[0,1]} |\frac{\partial\varphi_n(0,x)}{\partial x} - \frac{\partial\varphi^*(0,x)}{\partial x}| = 0$. It follows that

$$
\begin{aligned}
\lim_{n\to\infty} M^{u_+}(\varphi_n, x_n) &= \lim_{n\to\infty} \frac{\int_0^{x_n} \frac{\partial\varphi_n(0,x)}{\partial x}\mathrm{dx}}{\int_0^{x_n} \frac{\partial u_+(0,x)}{\partial x}\mathrm{dx}} \\
&= \lim_{n\to\infty} \frac{\int_0^{x_n} \frac{\partial\varphi^*(0,x)}{\partial x}\mathrm{dx}}{\int_0^{x_n} \frac{\partial u_+(0,x)}{\partial x}\mathrm{dx}} \\
&= M^{u_+}(\varphi^*, 0).
\end{aligned}
$$

Consequently, $M^{u_+}|_{\omega(\psi)\times\mathbb{R}_+}$ is a continuous and positive function. This completes the proof. □

To address the attractiveness of u_+, we further need the following conditions on the nonlinear function f:

(H2) $f(u) < f'(0)u$ for all $u \in (0, \infty)$.
(H3) For any closed interval $[a, b] \neq \{1\}$ with $0 < a \leq b < \infty$, either (i) $G((0, u_+^*] \times [a, b]) \subset (a, \infty)$ or (ii) $G((0, u_+^*] \times [a, b]) \subset (0, b)$, where $G : (0, u_+^*] \times (0, \infty) \to (0, \infty)$ is defined by $G(k, u) = \frac{f(ku)}{f(k)}$.

Theorem 3.2 *Assume that* (*H1*)–(*H3*) *hold. Then* (2.1) *has a unique positive steady state* u_+ *which attracts all solutions of* (2.1) *with the initial value* $\psi \in C_+ \setminus \{0\}$.

Proof The existence of u_+ is already established in Proposition 3.2, and the uniqueness will be a consequence of the global attractiveness of u_+ in $C_+ \setminus \{0\}$. So, we only need to show that u_+ attracts all solutions of (2.1) with the initial value $\psi \in C_+ \setminus \{0\}$.

We claim that if $l > 0$ and lu_+ satisfies (2.1), then $l = 1$. Otherwise $l \neq 1$. By (H3) with $[a, b] = \{l\} \neq \{1\}$, we know that either $f(lk) > lf(k)$ for all $k \in (0, u_+^*]$ or $f(lk) < lf(k)$ for all $k \in (0, u_+^*]$. This, combined with the monotonicity of $K(t, \cdot)$, yields either $K(t, f(lu_+)) > K(t, lf(u_+)))$ for all $t \in (0, \infty)$ or $K(t, f(lu_+)) < K(t, lf(u_+))$ for all $t \in (0, \infty)$. Since u_+ and lu_+ satisfy (2.1), we know that for any $t \in \mathbb{R}_+$, $u_+ = S(t)[u_+] + \int_0^t S(t-s)[F(u_+)]ds$ and $lu_+ = S(t)[lu_+] + \int_0^t S(t-s)[F(lu_+)]ds$. Therefore, $\int_0^t S(t-s)[lF(u_+)]ds = \int_0^t \mathrm{S(t-s)[F(lu_+)]ds}$, and thus, $K(t, lf(u_+)) = K(t, f(lu_+))$ for all $t \in \mathbb{R}_+$, a contradiction.

Suppose $\psi \in C_+ \setminus \{0\}$. By Lemma 3.6 we know that M^{u_+} is continuous and positive in $(\varphi, x) \in \omega(\psi) \times \mathbb{R}_+$.

Let $a^* = \inf\{M^{u_+}(\varphi, x) : (\varphi, x) \in \omega(\psi) \times \mathbb{R}_+\}$ and $b^* = \sup\{M^{u_+}(\varphi, x) : (\varphi, x) \in \omega(\psi) \times \mathbb{R}_+\}$. By the choices of a^*, b^*, Proposition 2.1 and Theorem 3.1 give $0 < a^* \leq M^{u_+}(\varphi, x) \leq b^* < \infty$ for all $(\varphi, x) \in \omega(\psi) \times \mathbb{R}_+$. To prove this theorem, it suffices to prove that $a^* = b^* = 1$. Otherwise, $a^* \neq 1$ or $b^* \neq 1$. If $a^* = b^* \neq 1$, then the invariance of $\omega(\psi)$ implies that $a^* u_+$ is also a positive steady state of (2.1). By the above claim, $b^* = a^* = 1$, a contradiction. If $a^* \neq b^*$, then $0 < a^* < b^*$, and by the assumption (H3) with $[a, b] = [a^*, b^*]$, we know that either (I) $f(ku) > a^* f(k)$ for all $(k, u) \in (0, u_+^*] \times [a^*, b^*]$ or (II) $f(ku) < b^* f(k)$ for all $(k, u) \in (0, u_+^*] \times [a^*, b^*]$.

For (I), there exists $\varepsilon > 0$ such that $f(ku) > a^* f(k) + \varepsilon$ for all $(k, u) \in [\frac{u_+^*}{2}, u_+^*] \times [a^*, b^*]$. Proposition 3.3 shows that there is $T_1 > 0$ such that $u_+(x) \geq \frac{u_+^*}{2}$ for all $x \geq T_1$. This, together with $\omega(\psi) \geq a^* u_+$ and the invariance of $\omega(\psi)$, implies that $b^* u_+^* \geq \varphi(-1, x) \geq \frac{a^* u_+^*}{2}$ for all $(x, \varphi) \in [T_1, \infty) \times \omega(\psi)$ and thus $f(\varphi(-1, \cdot)) - a^* f(u_+) \geq \varepsilon h^{T_1, \infty}$ for all $\varphi \in \omega(\psi)$. Applying Lemma 3.2 with $n = 1$ and $\delta = \frac{1}{4}$, we obtain that there is $T_2 > 0$ such that $K(s, h^T) \geq \frac{1}{4} h^T$ for all $s, T \geq T_2$. It follows from (2.1) and the invariance of $\omega(\psi)$ that we know that for all $\varphi \in \omega(\psi)$ and $t, T \geq T_3 \triangleq \max\{T_1, T_2\}$,

$$
\begin{aligned}
u^\varphi(t, \cdot) - a^* u_+ &= S(t)\big[u^\varphi(0, \cdot)\big] + \int_0^t S(t-s)\big[F\big(u_s^\varphi\big)\big]ds - \mathrm{a^* u_+} \\
&= S(t)\big[u^\varphi(0, \cdot) - a^* u_+\big] + \int_0^t S(t-s)\big[F\big(u_s^\varphi\big) - a^* F(u_+)\big]ds \\
&\geq K\big(t, \varepsilon h^T\big) \\
&= \varepsilon K\big(t, h^T\big) \\
&\geq \frac{\varepsilon}{4} h^T.
\end{aligned}
$$

Hence the invariance of $\omega(\psi)$ forces that $\varphi(0,\cdot) - a^*u_+ \geq \frac{\varepsilon}{4}h^T$ for all $(T,\varphi) \in [T_3,\infty) \times \omega(\psi)$, which gives $M^{u_+}(\varphi, x) \geq a^* + \frac{\varepsilon}{4}$ for all $(x,\varphi) \in [T_3,\infty) \times \omega(\psi)$. From (2.1), we have, for all $\varphi \in \omega(\psi)$ and $t \geq 1$,

$$\begin{aligned} u^{\varphi}(t,\cdot) - a^*u_+ &= S(t)\big[u^{\varphi}(0,\cdot)\big] + \int_0^t S(t-s)\big[F\big(u_s^{\varphi}\big)\big]\mathrm{ds} - \mathrm{a^*u_+} \\ &= S(t)\big[u^{\varphi}(0,\cdot) - a^*u_+\big] + \int_0^t S(t-s)\big[F\big(u_s^{\varphi}\big) - a^*F(u_+)\big]\mathrm{ds} \\ &\geq S(t)\big[u^{\varphi}(0,\cdot) - a^*u_+\big], \end{aligned}$$

and thus $u^{\varphi}(t,\cdot) - a^*u_+ \in X_+^{\circ}$ and $\frac{\partial u^{\varphi}(t,x) - a^*u_+(x)}{\partial x}|_{x=0} > 0$. Again the invariance of $\omega(\psi)$ forces that $\varphi(0,\cdot) - a^*u_+ \in X_+^{\circ}$ and $\frac{\partial \varphi(0,x) - a^*u_+(x)}{\partial x}|_{x=0} > 0$ for all $\varphi \in \omega(\psi)$. This combined with Lemma 3.6, gives that there is $\delta > 0$ such that $M^{u_+}(\varphi,x) \geq a^* + \delta$ for all $(x,\varphi) \in [0,T_3] \times \omega(\psi)$. So, the definition of M^{u_+} implies that $M^{u_+}(\varphi,x) \geq a^* + \min\{\frac{\varepsilon}{4},\delta\} > a^*$ for all $(x,\varphi) \in \mathbb{R}_+ \times \omega(\psi)$. This yields a contradiction to the choices of a^*.

For (II), we are similarly led to a contradiction. Consequently we see that $a^* = b^* = 1$ and hence $\omega(\psi) = \{u_+\}$. This completes the proof. □

Note that verifying (H3) becomes the key for applying these theorems. The following lemma shall be very useful for verifying (H3), which is closely related to the requirement for the map f to generate the dynamics of global convergence to a positive fixed point of f, and such a result is crucial in [33,36]. Generally, one cannot obtain u_+^* ($u_+^* = ||u_+||_X$) explicitly, therefore it is practically useful to introduce some sufficient conditions, among which is the following.

(H4) $\liminf\limits_{k\to 0+} G(k,u;f) \equiv u$ and $\frac{\partial G(k,u;f)}{\partial k}(1-u) > 0$ in $(0,f^*] \times ((0,\infty)\setminus\{1\})$. Here, $f^* \triangleq \max\{f(x) : x \in [0,u^*]\}$ and $G(k,u;f) = \frac{f(ku)}{f(k)}$.

By Lemma 3.1 in [38] and Lemma 3.5, we can present some useful and sufficient conditions for (H3).

Lemma 3.7 *If $G(f^*,\cdot;f)$ satisfies one of (A1)–(A3) and $G(\cdot,\cdot;f)|_{(0,f^*]\times(0,\infty)}$ satisfies (H4), then (H3) holds.*

As a direct corollary of Theorem 3.2 and Lemma 3.7, we have the following,

Theorem 3.3 *Suppose that (H1), (H2) and (H4) hold. If $G(f^*,\cdot;f)$ satisfies one of (A1), (A2) and (A3), then u_+ is a globally attractive positive steady state of (2.1) in $C_+\setminus\{\mathbf{0}\}$.*

As byproduct, we also obtain the existence and global attractivity of the heterogeneous steady state for the bistable evolution equation on $\mathbb{R}$. Let $Y \triangleq \{\psi \in BUC([-1,0]\times\mathbb{R},\mathbb{R}) : \psi(\cdot,x) = -\psi(\cdot,-x) \geq 0$ for all $x \in \mathbb{R}_+\}$. For given $\psi \in Y$, we have seen in Sect. 2 that (2.6) with the initial value $v_0 = \psi$, with g replaced by the odd extension of f has a unique solution $v^{\psi}(t,x)$ which exist on $\mathbb{R}_+$. Since the semigroup $U(t)$ is analytic, the solution maps of (2.6) induce continuous semiflows in Y.

Theorem 3.4 *Suppose that (H1)–(H3) hold. Let $\widetilde{u_+}$ represent the odd extension of u_+. Then $\widetilde{u_+}$ is a globally attractive equilibrium of (2.6) in Y with respect to the compact open topology*
.

Proof By the definitions of $S(t)$, $U(t)$, F, G, we easily see that $v^{\psi}(t,x) = u^{\psi|_{[-1,0]\times\mathbb{R}_+}}(t,x)$ for all $(t,x,\psi) \in [-1,0]\times\mathbb{R}_+\times Y$. Thus Theorem 3.2 gives theorem. This completes the proof. □

We conclude this section by pointing out that we shall consider Eq. (2.6) on $\mathbb{R}^n$ in a forthcoming paper. In particular, we shall obtain the existence, multiply, shape and attractivity of heterogeneous steady state for the spatially higher-dimensional case.

4 Examples

In this section, we illustrate the results of Theorems 2.1 and 3.3 by considering two concrete examples, that is, the non-local diffusive Nicholson's blowflies equation and the non-local diffusive Mackey-Glass equation.

Example 4.1 Consider the following diffusive Mackey and Glass equation

$$\begin{cases} \frac{\partial u}{\partial t} & = d\frac{\partial^2 u}{\partial x^2} - \delta u + \int_0^\infty \frac{pu(t-\tau,y)}{1+(u(t-\tau,y))^n}[\Gamma_\alpha(x-y) - \Gamma_\alpha(x+y)]\mathrm{d}y, \\ w(t,0) & = 0, \quad t > 0, \\ |w(t,\infty)| & < \infty, \quad t \in \mathbb{R}_+, \\ w(t,x) & = \varphi(t,x), \quad (t,x) \in [-r,0] \times \mathbb{R}_+, \end{cases} \tag{4.1}$$

where d, δ, p and n are all positive constants.

For this equation, the following lemma verifies the conditions needed for Theorems 2.1 and 3.3.

Lemma 4.1 *Let $f(u) = \frac{p}{\delta}\frac{u}{1+u^n}$ for all $u \geq 0$. Then the following statements are true:*

(i) *If $p \leq \delta$, then $f(u) < u$ for all $u > 0$.*
(ii) *If $p > \delta$, then the assumptions (H1) and (H4) hold.*
(iii) *If $p > \delta$ and $n \leq 2$, then the assumption (A2) holds.*
(iv) *If $p > \delta$ and $2 < n \leq \max\{2\frac{p^n(n-1)^{n-1}+n^n\delta^n}{p^n(n-1)^{n-1}}, \frac{p}{p-\delta}\}$, then the assumption (A3) holds.*

Proof (i)-(ii) are obvious. We only need to prove (iii) and (iv). Let $g(u) = G(f^*, u)$ and $h(u) = (1 + (f^*)^n)\frac{u}{1+u^n}$ for all $u \in \mathbb{R}_+$, where $f^* = \max\{f(u) : u \in [0, u^*]\}$. Then $g(u) = (1 + (f^*)^n)\frac{u}{1+(f^*u)^n}$ for all $u \in \mathbb{R}_+$. We easily check that g satisfies (A2) (or (A3)) if and only if h satisfies (A2) (or (A3)). On the other hand, by Lemma 4.2 in [33], h satisfies (A2) provided that $n \leq 2$. Thus, (iii) holds.

Next we shall finish the proof of (iv). Let $u_c = \frac{1}{(n-1)^{\frac{1}{n}}}$ and $u^* = (\frac{p}{\delta} - 1)^{\frac{1}{n}}$. Note that u_c, f^* are unique critical point and unique fixed point of h in $(0, \infty)$.

If $2 < n \leq \frac{p}{p-\delta}$, then $u_c \geq u^*$ and $f^* = u^* \leq u_c$. Thus, (A3) holds. If $\max\{2, \frac{p}{p-\delta}\} < n \leq 2\frac{p^n(n-1)^{n-1}+n^n\delta^n}{p^n(n-1)^{n-1}}$, then $u_c \geq u^*$ and $f^* = f(u_c) = \frac{p(n-1)^{\frac{n-1}{n}}}{\delta n} > u^* > u_c$. By Lemma 4.4 in [33] and the fact that $2 < n \leq 2\frac{1+(f^*)^n}{(f^*)^n} = 2\frac{p^n(n-1)^{n-1}+n^n\delta^n}{p^n(n-1)^{n-1}}$, we may obtain that h satisfies (A3). So, statement (iv) holds. This completes the proof. □

By applying Theorems 2.1 and 3.3, we then obtain the following results for (4.1).

Theorem 4.1 *The following statements hold for* (4.1).

(i) *If $p \leq \delta$, then the trivial steady state $u = 0$ attracts all solutions of* (4.1) *with the initial value in C_+;*

(ii) *If $p > \delta$, then* (4.1) *has a positive steady state u_+. Moreover, if*

$$n \leq \max\left\{\frac{p}{p-\delta}, 2\frac{p^n(n-1)^{n-1}+n^n\delta^n}{p^n(n-1)^{n-1}}\right\},$$

then the positive steady state u_+ attracts all solutions of (4.1) *with the initial functions in $C_+ \setminus \{0\}$.*

Example 4.2 Consider the diffusive Nicholson's blowflies equation with the nonlocal response,

$$\begin{cases} \frac{\partial u}{\partial t} & = d\frac{\partial^2 u}{\partial x^2} - \delta u + p\int_0^\infty u(t-\tau, y)e^{-u(t-\tau,y)}[\Gamma_\alpha(x-y) - \Gamma_\alpha(x+y)]\mathrm{d}y, \\ u(t,0) & = 0, \quad t > 0, \\ |u(t,\infty)| & < \infty, \quad t \in \mathbb{R}_+, \\ u(t,x) & = \varphi(t,x), \quad (t,x) \in [-r,0]\times\mathbb{R}_+, \end{cases} \tag{4.2}$$

where $d,\ p,\ \tau$ and δ are positive constants.

Let $f(u) = \frac{p}{\delta}ue^{-u}$ for all $u \geq 0$, we easily see that $f'(0) = \frac{p}{\delta}$ and f has a positive fixed point given by $u^* = \ln\frac{p}{\delta}$ if and only if $p > \delta$. The next lemma further summarizes the properties of f required by Theorems 2.1 and 3.3.

Lemma 4.2 *For the above f, the following statements hold:*

(i) *If $\delta \geq p$, then $f(u) < u$ for all $u > 0$.*
(ii) *If $p > \delta$, then f satisfies the assumptions (H1) and (H4).*
(iii) *If $\frac{p}{\delta} \in (1, 2e]$, then the assumption (A2) holds.*

Proof (i)–(ii) are obvious. To complete the proof of the statement (iii), we define $g(u) = G(f^*, u)$ and $h(u) = e^{f^*}ue^{-u}$ for all $u \in \mathbb{R}_+$, where $f^* = \max\{f(u) : u \in [0, u^*]\}$. Then $g(u) = e^{f^*}ue^{-f^*u}$ for all $u \in \mathbb{R}_+$. We easily check that g satisfies (A2) if and only if h satisfies (A2). If $\frac{p}{\delta} \leq e$, then $f^* = \ln\frac{p}{\delta}$ and $1 < e^{f^*} \leq e$. If $e < \frac{p}{\delta} \leq 2e$, then $f^* = \frac{p}{e\delta}$ and thus $1 < e^{f^*} \leq e^2$. These combined with Lemma 2.3 in [35], h satisfies (A2) provided that $1 < \frac{p}{\delta} \leq 2e$. Hence, statement (iii) holds. This completes the proof. □

Applying Theorems 3.1 and 3.3, Lemma 4.2, we then obtain the following results for the non-local problem (4.2).

Theorem 4.2 *The following statements are true:*

(i) *If $p \leq \delta$, then the trivial steady state $u = 0$ of* (4.2) *attracts all solutions of* (4.2) *with the initial value in C_+;*
(ii) *If $p > \delta$,* (4.2) *has a positive steady state u_+. Moreover, if $p < 2e\delta$, then the positive steady state u_+ attracts all solutions of* (4.2) *with the initial functions in $C_+ \setminus \{0\}$.*

As we mentioned in the introduction, Wu-Zhao [28] and Xu-Zhao [29] also obtained some results about the global dynamics of the Dirichlet problem for some nonlocal equations of type (4.2) on a typical bounded domain. The main tool in [28] and [29] is the theory of monotone dynamical systems, and hence, monotonicity (under some non-standard ordering) and sublinearity on the nonlocal nonlinear terms played a crucial role. In contrast, we assume an alternative condition (H3) rather than the sublinearity, and our approach is less dependent on ordering, and hence our main results in this paper are less demanding on monotonicity. Under the non-monotone condition, So and Yang [22] also obtained the global attractiveness of the positive steady state u_+ by dividing the bounded spatial domain according to some information from the positive steady state and these sub-domains were treated separately, both in L^2 norm and supremum norm. By modifying some of the arguments in Yi and Zou [35] and [36], more precisely, by combining a dynamical system argument with the maximum principle as well as some subtle inequalities, Yi et.al [32,38] also established the threshold dynamics of the Dirichlet problem and thus re-confirmed the existing results for the diffusion Nicholson's blowflies equation in [22].

Acknowledgments Research was supported partially by the National Natural Science Foundation of P. R. China (Grant No. 11171098) and the Hunan Provincial NSF (Grant No. 11JJ1001), and by NSERC of Canada.

Appendix: Derivation of Eq. (1.3)

Consider a species living in the semi-infinite interval $\mathbb{R}_+ = [0, \infty)$. Let $W(t, a, x)$ be the population density (w.r.t. age a) of the species at time $t \geq 0$, age $a \geq 0$ and location $x \geq 0$, and $D(a)$ and $d(a)$ be the diffusion rate and death rate, respectively, at age a. Then by Metz-Diekmann [19], $W(t, a, x)$ satisfies the following evolution equation

$$\frac{\partial W}{\partial t} + \frac{\partial W}{\partial a} = D(a)\frac{\partial^2 W}{\partial x^2} - d(a)W, \quad t > 0, \quad a > 0, \quad x \in (0, \infty). \tag{5.1}$$

In this paper, we consider the homogeneous Dirichlet boundary condition at $x = 0$

$$W(t, a, 0) = 0, \quad (t, a) \in \mathbb{R}^2_+, \tag{5.2}$$

which accounts for the scenario that the location $x = 0$ is hostile to the species. Assume that the species has a fixed maturation time $\tau > 0$. Then, the total mature population is given by

$$w(t, x) = \int_\tau^\infty W(t, a, x)\mathrm{da}. \tag{5.3}$$

By the biological meaning of $W(t, a, x)$, one should have the following conditions with respect to the age variable a:

$$\begin{cases} W(t, \infty, x) = 0, \\ W(t, 0, x) \;\; = b(w(t, x)) \end{cases} \quad (t, x) \in \mathbb{R}^2_+, \tag{5.4}$$

where b is the birth function.

Differentiating (5.3), making use of (5.4), yields

$$\frac{\partial w}{\partial t} = W(t, \tau, x) + D_m\frac{\partial^2 w}{\partial x^2} - d_m w, \quad t > 0, \quad x \in (0, \infty) \tag{5.5}$$

Here, for simplicity of presentation, we have assumed that the diffusion and death rates for the mature population are age independent, that is, $D(a) = D_m$ and $d(a) = d_m$ for $a \in [\tau, \infty)$. We need to determine $W(t, \tau, x)$ in terms of $w(t, x)$. To this end, for any $s \in \mathbb{R}_+$, let $V^s(t, x) = W(t, t - s, x)$ for $(t, x) \in [s, s + \tau] \times \mathbb{R}_+$. Then by (5.1), (5.2) and (5.4), we have

$$\begin{cases} \frac{\partial V^s}{\partial t}(t, x) = D(t - s)\frac{\partial^2 V^s}{\partial x^2}(t, x) - d(t - s)V^s(t, x), \quad (t, x) \in (s, s + \tau] \times (0, \infty), \\ V^s(t, 0) \;\; = 0, \quad t \in [s, s + \tau], \\ V^s(s, x) = b(w(s, x)), \quad x \in \mathbb{R}_+. \end{cases} \tag{5.6}$$

Applying the Fourier sine transform (see, e.g., [11, pp. 471–473]) to (5.6), we have

$$\begin{cases} \frac{d\hat{V}^s(t,\xi)}{dt} = -[D(t-s)\xi^2 + d(t-s)]\hat{V}^s(t,\xi), & (t,\xi) \in (s, s+\tau] \times (0,\infty), \\ \hat{V}^s(s,\xi) = c(s,\xi), \quad \xi \in \mathbb{R}_+, \end{cases} \tag{5.7}$$

where for all $(t,\xi) \in (s, s+\tau] \times [0,\infty)$,

$$\hat{V}^s(t,\xi) = \frac{2}{\pi}\int_0^\infty V^s(t,x)sin(\xi x)\mathrm{dx}$$

and

$$c(s,\xi) = \frac{2}{\pi}\int_0^\infty b(w(s,x))sin(\xi x)\mathrm{dx}.$$

Solving (5.7) leads to

$$\hat{V}^s(t,\xi) = c(s,\xi)e^{-\int_s^t[D(t-s)\xi^2+d(t-s)]\mathrm{dt}}..$$

Hence,

$$\hat{V}^{t-\tau}(t,\xi) = \varepsilon \cdot c(t-\tau,\xi)\cdot e^{-\alpha\xi^2}. \tag{5.8}$$

where $\varepsilon = \exp(-\int_0^\tau d(s)\mathrm{ds})$ and $\alpha = \int_0^\tau D(s)\mathrm{ds}$. Taking the inverse Fourier sine transform in (5.8) gives

$$V^{t-\tau}(t,x) = \int_0^\infty \varepsilon \cdot c(t-\tau,\xi)\cdot e^{-\alpha\xi^2}\sin(x\xi)\mathrm{d}\xi.$$

In order to obtain a concrete formula for $V^{t-\tau}(t,x)$, one only needs to follow (almost repeat) the steps on P477 in [11] for deriving the formula Eq. (10.5.39) (involving an odd extension of $b(w(s,x))$ from $[0,\infty)$ to the whole space $(-\infty,\infty)$, formulas of Fourier transforms of Gaussian functions, and the Convolution Theorem), and this will lead to

$$\begin{aligned} V^{t-\tau}(t,x) &= \frac{\varepsilon}{\sqrt{4\pi\alpha}}\int_0^\infty b(w(t-\tau,y))\left[\exp\left(-\frac{(x-y)^2}{4\alpha}\right) - \exp\left(-\frac{(x+y)^2}{4\alpha}\right)\right]\mathrm{dy} \\ &= \varepsilon\int_0^\infty b(w(t-\tau,y)[\Gamma_\alpha(x-y) - \Gamma_\alpha(x+y)]\mathrm{dy}. \end{aligned} \tag{5.9}$$

Note that $W(t,\tau,x) = V^{t-\tau}(t,x)$. Plugging (5.9) into (5.5) gives the PDE in (1.3), where we have used $D_I(a)$ and $d_I(a)$ to denote the diffusion and death rates of the immature respectively, that is, $D_I = D|_{[0,\tau]}$ and $d_I = d|_{[0,\tau]}$.

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