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Abstract We consider a very general class of delayed reaction–diffusion equations in which the reaction term can be non-monotone as well as spatially non-local. By employing comparison technique and a dynamical system approach, we study the global asymptotic behavior of solutions to the equation subject to the homogeneous Dirichlet condition. Established are threshold results and global attractiveness of the trivial steady state, as well as the existence, uniqueness and global attractiveness of a positive steady state solution to the problem. As illustrations, we apply our main results to the local delayed diffusive Mackey–Glass equation and the nonlocal delayed diffusive Nicholson blowfly equation, leading to some very sharp results for these two particular models.

Keywords Dirichlet condition · Global asymptotic behavior · Nonlocal reaction–diffusion equation · Threshold dynamics

Mathematics Subject Classification (2000) 34G25 · 35K57

1 Introduction

When considering the population growth of a species with age structure and diffusion in a spatial domain Ω, one faces a reaction–diffusion equation of the form

$$\frac{\partial u(x,t)}{\partial t} = d\Delta u(t,x) + \int_{\Omega} \Gamma(D_i, \tau, x, y) b(u(t-\tau, y))\, dy - \delta u(t,x), \quad (t,x) \in (0,\infty)\times\Omega. \tag{1.1}$$

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Here $u(t, x)$ is the mature population of the species at time t and location x, d and D_i are the diffusion rates of the mature and immature individuals, τ is the average mature time for the species, δ and δ_i are the death rates of mature and immature, $b(u)$ is the birth function. The kernel function $\Gamma(D_i, \tau, x, y)$ accounts for the probability that an individual born at location y can survive the immature period $[0, \tau]$ and has moved to location x when becoming mature (τ time units after birth). Therefore, the temporally delayed and spatially nonlocal term given by the integral reflects the combined effect of the survivability and mobility of the immature individuals. It is related to the semi-group $S(t)$ generated by solutions of the equation $v_t(t, x) = D_i \Delta v(t, x) - \delta_i v(t, x)$ under respective boundary conditions on $\partial\Omega$ in the sense that $(S(\tau)\phi)(x) = \int_\Omega \Gamma(D_i, \tau, x, y)\phi(y)dy$. In general, an explicit form of $\Gamma(D_i, \tau, x, y)$ can not be obtained except for some simple cases. For example, when $\Omega = \mathbb{R}$, (a, b) or $(a, b) \times (c, d)$, applying the method of separation of variables to the above PDE, one may obtain explicit forms of $\Gamma(D_i, \tau, x, y)$ for these simple cases. See [10,17] or the survey [5] for details of derivation of such a model. In particular, for a species whose immature individuals do not move ($D_i = 0$), the kernel $\Gamma(0, \tau, x, y)$ is given by the Dirac delta function centered at x multiplied by the survival probability, reducing (1.1) to the following spatially local equation

$$\frac{\partial u(x,t)}{\partial t} = d\Delta u(t,x) + e^{-\delta_i \tau} b(u(t-\tau, x) - \delta u(t,x), \ \ (t,x) \in (0,\infty) \times \Omega. \quad (1.2)$$

When the spatial domain Ω is unbounded, existence of traveling wave solutions of (1.1) or (1.2) is one of the main concerns, and has been investigated by many authors under some particular forms of the kernel function $\Gamma(D_i, \tau, x, y)$ (see, e,g., [4,13,17,19,23,31] and the references therein). When the spatial domain Ω is bounded, various boundary conditions can be posed for (1.1) or (1.2), among which are the homogeneous Neumann boundary value condition or zero flux condition (NBVC) and the homogeneous Dirichlet boundary value condition (DBVC) with the former accounting for an isolated domain and the latter explaining a scenario that the boundary is hostile for the species.

In the case of bounded Ω and under NBVC, Yi and Zou [27,28] developed a dynamical system approach which has allowed the authors to obtain very sharp conditions for delay independent global stability of steady states of the spatially local (1.2). When such a sharp condition is violated for the Ricker's birth function (leading to the so-called diffusive Nicholson's blowfly equation), Yang and So [25] found that the stability of the positive steady state of (1.2) can be destroyed by large delay, giving rise to stable periodic solutions through Hopf bifurcation. By applying a fluctuation method, Zhao [30] has recently established the global attractiveness of the positive steady state of (1.1). Using the idea of exponential ordering, Wu and Zhao [22] also obtained some threshold results for (1.1) in terms of the principal eigenvalue of a non-local eigenvalue problem.

In the case of bounded Ω but under DBVC, the dynamics of (1.1) or (1.2) is usually much more difficult to determine. This is mainly due to the fact that a non-trivial positive steady state is not a constant and it has to possess some spatial pattern which is unknown. Thus the uniqueness/multiplicity and stability of a positive steady state all become hard problems. For the local version (1.2) with the Ricker's birth function, So and Yang [18] obtained some results, by energy method, on the existence and stability of a positive steady of (1.2). By modifying some of the arguments in Yi and Zou [27], Yi et al. [26] described the threshold dynamics of (1.2) subject to DBVC which include some results in [18] as special cases. When (1.1) possesses *monotonicity*, Xu and Zhao [24] obtained some results on the uniqueness and global attractiveness of a positive steady state of (1.1) under DBVC by appealing to the theory of monotone dynamical systems. Also along the line of monotone dynamical systems, Wu and Zhao [22] established some results on threshold between extinction and persistence for (1.1) under DBVC, as well as existence and attractiveness of a positive steady state in terms of the principal eigenvalue of a non-local eigenvalue problem. In both [24] and [22], the monotonicity (in some sense) and the sublinearity played crucial roles. However, in the non-monotone case, as remarked in [30], uniqueness and stability of a positive steady state for (1.1) or (1.2) subject to DBVC still largely remain open and challenging. Most recently, Guo et al. [6] made another attempt, in which the authors focused on the existence and uniqueness of positive steady state of (1.1) under DBVC. Developed in [6] was a technique that combines the method of sup-sub solutions and an estimation of the integral kernel, which enabled the authors to obtain some sufficient conditions for the non-existence, existence and uniqueness of a positive steady state.

In this paper, we are interested in the global dynamics of (1.1) in a bounded domain with DBVC. Motivated by [27,28], we will develop a dynamical system approach to study the global asymptotic behavior for (1.1) subject to DBVC. It turns out that this approach offers a unified treatment for the global dynamics of a class of very general delayed reaction–diffusion equations, including spatially local as well as nonlocal cases. Our results not only re-confirm the existing results in [18,26] for local cases, but also establish uniqueness and global attractiveness of a positive steady state (the open problem raised in [30]) under the condition that population can persist, without assuming monotonicity. In Section 2, we present and prove our main results for a more general setting (2.1) [than (1.1)]. We first transform (2.1) to its associated integral equation with the given initial function; then we obtain some basic information about the solution semiflow of the associated integral equation [mild solution of (2.1)]. By employing the comparison technique, the Krein–Rutman theorem in [9] and the spectral mapping theorem for semigroups in [2], we derive conditions that assure the global attractiveness of the trivial steady state, the existence and global attractiveness of a unique positive steady state of (2.1) with the nonlinearity *not necessarily being monotone*. In Sect. 3, we apply the main results obtained in Sect. 2 to the local delayed diffusive Mackey–Glass equation and to the nonlocal delayed diffusion Nicholson blowfly equation with DBVC, leading to some very sharp results for these two particular models. Our very different approach to study the global dynamics of the Dirichlet problem provides a new and unified treatment applicable for both local or non-local equations.

2 Main Results

Let $\mathbb{R}$ and $\mathbb{R}_+$ be the sets of all reals and nonnegative reals, respectively, and let Ω be a bounded domain in $\mathbb{R}^m$ with smooth boundary $\partial\Omega$, Δ be the Laplacian operator and $\frac{\partial}{\partial\nu}$ be the derivative in the outward normal direction of $\partial\Omega$. Let $C^0 = \{\phi \in C(\overline{\Omega}, \mathbb{R}) : \phi|_{\partial\Omega} = 0\}$ and $X = \{\varphi \in C([-\tau, 0] \times \overline{\Omega}, \mathbb{R}) : \varphi|_{[-\tau,0]\times\partial\Omega} = 0\}$ be equipped with the usual supremum norm $\|\cdot\|$. Also, let $C_+ = \{\phi \in C^0 : \phi|_{\overline{\Omega}} \geq 0\}$ and $X_+ = \{\varphi \in X : \varphi|_{[-\tau,0]\times\overline{\Omega}} \geq 0\}$. For any $a \in \mathbb{R}$, we also use a to denote the constant function in the corresponding function space taking constant value a, when no confusion arises.

For any $\xi, \eta \in C^0$, we write $\xi \geq_C \eta$ if $\xi - \eta \in X_+$, $\xi >_X \eta$ if $\xi \geq_C \eta$ and $\xi \neq \eta$. Similarly, for any $\varphi, \psi \in X$, we write $\varphi \geq_X \psi$ if $\varphi - \psi \in X_+$, $\varphi >_X \psi$ if $\varphi \geq_X \psi$ and $\varphi \neq \psi$. For simplicity of notations, we shall write $\geq, >$, respectively for $\geq_*, >_*$, where $*$ stands for one of X and C.

For convenience, we embed (1.1) into the following more general form

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = d\Delta u(t,x) - \mu u(t,x) + \mu K(f(u(t-\tau,\cdot)))(x), & (t,x) \in (0,\infty)\times\Omega, \\ u(\theta,x) = 0, \quad (\theta,x) \in [-\tau,0]\times\partial\Omega, \\ u(\theta,x) = \varphi(\theta,x), \quad (\theta,x)\in[-\tau,0]\times\overline{\Omega}, \end{cases} \tag{2.1}$$

where μ is a positive constant, Δ is the Laplacian operator on Ω, $\varphi \in X_+$. The nonlinear function f is a continuously differentiable function on $[0,\infty)$ with $f(0) = 0$ and $f((0,\infty)) \subseteq (0,\infty)$. The operator K is linear from C^0 into C^0 such that $K(C_+ \setminus \{0\}) \subseteq C_+ \setminus \{0\}$ and

$$||K|| \triangleq \sup\left\{\frac{||K(\phi)||}{||\phi||} : \phi \in C_+ \setminus \{0\}\right\} = 1.$$

This condition is motivated by the form of K given by in (1.1), that is $K(\phi)(x) = \int_\Omega \Gamma(D_i,\tau,x,y)\phi(y)\,dy$.

For an interval $I \subseteq \mathbb{R}$, let $I + [-\tau,0] = \{t+\theta : t \in I \text{and} \theta \in [-\tau,0]\}$. For $u : (I+[-\tau,0])\times\overline{\Omega} \to \mathbb{R}$ and $t \in I$, we write $u_t(\cdot,\cdot)$ for the element of X defined by $u_t(\theta,x) = u(t+\theta,x)$ for $-\tau \le \theta \le 0$ and $x \in \overline{\Omega}$.

Let $T(\cdot)$ be the semigroup on C^0 generated by the operator $d\Delta - \mu Id$ under the Dirichlet boundary condition. It is well-known that $T(\cdot)$ is an analytic, compact and positive semigroup on C^0 (see [15,16,21]). Denote by A_T the generator of the strongly continuous semigroup $T(\cdot)$ and let $D(A_T) \subseteq C^0$ be the domain of A_T. Define $F : X_+ \to C^0$ by

$$F(\varphi) = \mu K(f(\varphi(-\tau,\cdot))).$$

Associate to (2.1) with the following integral equation with the given initial condition,

$$\begin{cases} u(t,\cdot) = T(t)\varphi(0,\cdot) + \int_0^t T(t-s)F(u_s)\mathrm{d}s, \quad t \ge 0, \\ u_0 = \varphi \in X_+. \end{cases} \tag{2.2}$$

A solution of (2.2) is called a mild solution of (2.1) in the sense of Martin and Smith [11,12].

By the step argument and the definition of F, it is easy to see that, for any $\varphi \in X_+$, (2.2) has a unique solution which exists for all $t \ge 0$. Denote this solution by u^φ. Moreover, $(u^\varphi)_t \in X_+$ for all $t \in \mathbb{R}_+$. Since the semigroup $T(t)$ is analytic, by Corollary 2.2.5 [21], we know that a mild solution of (2.1) is also a classical solution of (2.1) for $t > \tau$ (see also [3,11,12,20]).

Define the map $U : \mathbb{R}_+ \times X_+ \to X_+$ by $U(t,\varphi) = (u^\varphi)_t$ for all $(t,\varphi) \in X$. The following result is a direct consequence of the abstract results in [3,20,21].

Lemma 2.1 *The map U is a semiflow defined on X_+ and satisfies the following properties:*

(i) For a given $\varphi \in X_+$, $U(t,\varphi)(0,\cdot)$ is a classical solution of (2.1) for $t > \tau$;
(ii) At any given $t > \tau$, $U(t,\cdot) : X_+ \to X_+$ is completely continuous. More precisely, if $B \subset X_+$ is bounded, then $U(t,\cdot)B$ is precompact for $t > \tau$.

As a simple application of Corollary 7.2.3 and 7.2.4 in [16], and Theorem 2.5.2 in [14], we easily obtain the following lemma.

Lemma 2.2 *Suppose $\phi \in C_+ \setminus \{0\}$. Then*

(i) $T(t)(\phi)(x) > 0$ for all $(t,x) \in (0,\infty)\times\Omega$;
(ii) $\frac{\partial[T(t)(\phi)(x)]}{\partial\nu} < 0$ for all $(t,x) \in (0,\infty)\times\partial\Omega$;
(iii) $||T(t)(\phi)|| \le ||\phi||e^{-\mu t}$ for all $t \in (0,\infty)$;

(iv) there exists a constant C_1 *such that* $||A_T[T(t)(\phi)]|| \le \frac{C_1}{t}||\phi||$ *for all* $t \in (0, \infty)$.

In what follows, we shall always assume the following for the nonlinearity f:

(H1) There exists a sequence $\{B_n\}_{n\ge 1}$ in $(0, \infty)$ such that $\lim_{n\to\infty} B_n = \infty$ and $f([0, B_n]) \subseteq [0, B_n])$.

The following lemma establishes some further properties of the semiflow $U(t, \varphi)$.

Lemma 2.3 *Suppose* $\varphi \in X_+ \setminus \{0\}$. *Then*

(i) $U(t, \varphi)(0, x) > 0$ *for all* $(t, x) \in (\tau, \infty) \times \Omega$;
(ii) $\frac{\partial U(t,\varphi)(0,x)}{\partial \nu} < 0$ *for all* $(t, x) \in (\tau, \infty) \times \partial\Omega$;
(iii) if $||\varphi|| \le B_n$ *for some positive integer n, then* $||U(t, \varphi)|| \le B_n$ *for all* $t \in [0, \infty)$.

Proof We first prove the statement (i). Since $\varphi(\theta^*, \cdot) > 0$ for some $\theta^* \in (-\tau, 0)$, we know by Lemma 2.2(i) that $T(-\theta^*)(K(f(\varphi(\theta^*, \cdot))))(x) > 0$ for all $x \in \Omega$. It follows from (2.2) that $u^\varphi(\tau, x) \ge \int_0^\tau T(\tau - s)K(f(u^\varphi(s - \tau, \cdot)))(x)\mathrm{d}s > 0$ for all $x \in \Omega$. Thus, Lemma 2.2(i), combined with the semigroup properties of U and (2.2), implies that $U(t + \tau, \varphi)(0, x) = U(t, U(\tau, \varphi))(0, x) \ge T(t)(U(\tau, \varphi)(0, \cdot))(x) = T(t)[u^\varphi(\tau, \cdot)](x) > 0$ for all $(t, x) \in [0, \infty) \times \Omega$, proving (i).

We next prove statement (ii). From (i), the semigroup properties of U and Lemma 2.1(i), Lemma 2.2(ii) and (2.2), it follows that

$$\frac{\partial[U(t, \varphi)(0, x)]}{\partial \nu} = \frac{\partial u^\varphi(t, x)}{\partial \nu} \le \frac{\partial T(t - \tau)(u^\varphi(\tau, \cdot))(x)}{\partial \nu} < 0,$$

where $(t, x) \in (\tau, \infty) \times \partial\Omega$, that is, statement (ii) holds.

Finally, we prove the statement (iii). By the assumption (H1), $f([0, B_n]) \subseteq [0, B_n]$. It follows from (2.2) and Lemma 2.2(iii) that for any $(t, x) \in [0, \tau] \times \overline{\Omega}$,

$$\begin{aligned} u^\varphi(t, x) &= T(t)(\varphi(0, \cdot))(x) + \int_0^t T(t - s)(K(f(u^\varphi(s - \tau, \cdot))))(x)\mathrm{d}s, \\ &\le ||\varphi||e^{-\mu t} + \mu \int_0^t B_n e^{-\mu(t-s)}\mathrm{d}s, \\ &\le B_n, \end{aligned}$$

which, combined with the semigroup properties of U, implies (iii). □

For $\varphi \in X_+$, we define $O(\varphi) = \{U(t, \varphi) : t \ge 0\}$ and $\omega(\varphi) = \bigcap_{t\ge 0} \overline{O(U(t, \varphi))}$. By Lemma 2.1(ii) and Lemma 2.3(iii), we know that $\overline{O(\varphi)}$ is compact and hence $\omega(\varphi)$ is nonempty, compact, connected and invariant. According to the invariant property of $\omega(\varphi)$, for $\psi \in \omega(\varphi)$, there is a global classical solution $u : \mathbb{R} \times \overline{\Omega} \to \mathbb{R}$ of (2.1) with $u_0 = \psi$ and $u_t \in \omega(\varphi)$ for all $t \in \mathbb{R}$ (see Hale [7]). The following theorem addresses the global attractiveness of the trivial steady state.

Theorem 2.1 *If* $f(u) < u$ *for all* $u > 0$, *then the trivial steady state of (2.1) attracts all solutions of (2.1) with the initial value* $\psi \in X_+$.

Proof Suppose that $\psi \in X_+$. Take $\varphi^* \in \omega(\psi)$ such that $||\varphi^*|| = \sup\{||\varphi|| : \varphi \in \omega(\psi)\}$. We claim that $||\varphi^*|| = 0$. Otherwise, $||\varphi^*|| > 0$. By the invariance of $\omega(\psi)$, there exists a global classical solution $u(t,x) : \mathbb{R} \times \overline{\Omega} \to [0,\infty)$ of (2.1) such that $u_{2\tau} = \varphi^*$. By employing Lemma 2.2(iii) and (2.2), we obtain that for all $(t,x) \in (0,\infty) \times \overline{\Omega}$,

$$\begin{aligned} u(t,x) &= T(t)(u(0,\cdot))(x) + \mu \int_0^t T(t-s)(K(f(u(s-\tau,\cdot))))(x)\mathrm{d}s, \\ &\leq ||\varphi^*||e^{-\mu t} + \mu \int_0^t ||f(u(s-\tau,\cdot))||e^{-\mu(t-s)}\mathrm{d}s, \\ &< ||\varphi^*||e^{-\mu t} + \mu \int_0^t ||\varphi^*||e^{-\mu(t-s)}\mathrm{d}s, \\ &= ||\varphi^*||, \end{aligned}$$

which yields a contradiction. This completes the proof. □

To proceed further, we introduce the following subset of C_+:

$$Y_+ = \{\phi \in C_+ : \phi \in C^1(\overline{\Omega}, \mathbb{R}), \phi(x) > 0 \quad \text{for all} \quad x \in \Omega \text{ and} \frac{\partial \phi}{\partial \nu}(x) < 0 \quad \text{for all} \quad x \in \partial\Omega\}.$$

Lemma 2.4 *Let I be a closed interval in $\mathbb{R}$ and let $\phi^* \in Y_+$. Assume that $u \in C(I \times \overline{\Omega}, [0,\infty))$ is continuously differentiable with respect to $x \in \overline{\Omega}$, and $u(t,\cdot) \in Y_+$ for all $t \in I$. Let $M : I \times \overline{\Omega} \to \mathbb{R}$ be defined by*

$$M(t,x) = \begin{cases} \frac{u(t,x)}{\phi^*(x)}, & x \in \Omega \quad \text{and} \quad t \in I, \\ \frac{\partial u(t,x)}{\partial \nu} / \frac{\partial \phi^*(x)}{\partial \nu}, & x \in \partial\Omega \quad \text{and} \quad t \in I. \end{cases}$$

Then M is a continuous function and $M(I \times \overline{\Omega}) \subseteq (0,\infty)$.

Proof From the definition of M, we can see easily that $M|_{I\times\Omega}$ and $M|_{I\times\partial\Omega}$ are continuous functions. It suffices to prove that for a given $(t^*, x^*) \in I \times \partial\Omega$ and a given sequence $(t_n, x_n) \in I \times \Omega$, if $\lim_{n\to\infty}(t_n, x_n) = (t^*, x^*)$ then $\lim_{n\to\infty} M(t_n, x_n) = M(t^*, x^*)$. Indeed, in view of the smoothness of $\partial\Omega$ and Lemma 4.2 in [8], we know that, for sufficient large n, there exist $x_n^0 \in \partial\Omega$ and $s_n > 0$ such that $s_n = dist(x_n, \partial\Omega)$, $x_n = x_n^0 - s_n \nu_{x_n^0}$ and $x_n^0 - \eta s_n \nu_{x_n^0} \in \overline{\Omega}$ for all $\eta \in [0,1]$, where $\nu_{x_n^0}$ is the unit outward normal vector at x_n^0. Thus

$$\begin{aligned} \lim_{n\to\infty} M(t_n, x_n) &= \lim_{n\to\infty} \frac{u(t_n, x_n)}{\phi^*(x_n)}, \\ &= \lim_{n\to\infty} \frac{\int_0^1 \frac{\partial u(t_n, x_n^0 - \eta s_n \nu_{x_n^0})}{\partial \eta}\mathrm{d}\eta}{\int_0^1 \frac{\partial \phi^*(x_n^0 - \eta s_n \nu_{x_n^0})}{\partial \eta}\mathrm{d}\eta}, \\ &= \lim_{n\to\infty} \frac{\int_0^1 \nabla_x u(t_n, x_n^0 - \eta s_n n_{x_n^0})\nu_{x_n^0}\mathrm{d}\eta}{\int_0^1 \nabla_x \phi^*(x_n^0 - \eta s_n n_{x_n^0})\nu_{x_n^0}\mathrm{d}\eta}, \\ &= M(t^*, x^*). \end{aligned}$$

Consequently, M is a continuous function. This completes the proof. □

The next lemma establishes a relation between the topologies defined by the supremum norm and the $C^1(\overline{\Omega})$ norm in $\{\varphi(0,\cdot) : \varphi \in \omega(\psi)\}$ for any given $\psi \in X_+$.

Lemma 2.5 *Let $\psi \in X_+$, and φ and $\{\varphi^n\}_{n=1}^{\infty}$ be in $\omega(\psi)$. If $\lim_{n\to\infty} ||\varphi^n(0,\cdot) - \varphi(0,\cdot)|| = 0$, then $\lim_{n\to\infty} ||\varphi^n(0,\cdot) - \varphi(0,\cdot)||_{C^1(\overline{\Omega})} = 0$. Thus, the supremum norm and the $C^1(\overline{\Omega})$ norm define the same topology on $\{\varphi(0,\cdot) : \varphi \in \omega(\psi)\}$.*

Proof Suppose $\psi \in X_+$. We *claim* that there exists $M = M_\psi > 0$ such that $||\varphi(0,\cdot)||_{C^{1,1/2}(\overline{\Omega})} \le M$ for all $\varphi \in \omega(\psi)$. Here $||g||_{C^{1,1/2}(\overline{\Omega})}$ is the sum of $||g||_{C^1(\overline{\Omega})}$ and the Hölder seminorm (with exponent $1/2$) of all first order partial derivatives of g.

Let $\gamma = \sup\{||\varphi|| : \varphi \in \omega(\psi)\}$. By employing the proof of Theorem 4.3.1 in [14], combined with the compactness of $\omega(\psi)$ and the smoothness of f, we know that there exist constants $C_2 > 0$ and $\delta > 0$ such that $|f(\varphi(0,x)) - f(\varphi(-h,x))| \le C_2 h^\delta$ for all $(\varphi, x, h) \in \omega(\psi) \times \overline{\Omega} \times [0,\tau]$. It follows from (2.2), Lemma 2.2(iv), the invariance of $\omega(\psi)$ and the semigroup properties of U that for any $\varphi \in \omega(\psi)$, there holds

$$\begin{aligned}
\left|\left|A_T u^\varphi(2\tau,\cdot)\right|\right| &= \left|\left|A_T\left[T(\tau)(u^\varphi(\tau,\cdot)) + \mu\int_0^\tau T(\tau-s)(K(f(u^\varphi(s,\cdot))))\mathrm{d}s\right]\right|\right|,\\
&\le \left|\left|A_T[T(\tau)(u^\varphi(\tau,\cdot))]\right|\right| + \left|\left|\mu A_T\left[\int_0^\tau T(\tau-s)(K(f(u^\varphi(s,\cdot))))\mathrm{d}s\right]\right|\right|,\\
&\le \frac{\gamma C_1}{\tau} + \left|\left|\mu A_T\left[\int_0^\tau T(\tau-s)(K(f(u^\varphi(\tau,\cdot))))\mathrm{d}s\right]\right|\right|,\\
&\quad + \left|\left|\mu A_T\left[\int_0^\tau T(\tau-s)(K(f(u^\varphi(s,\cdot)) - f(u^\varphi(\tau,\cdot))))\mathrm{d}s\right]\right|\right|,\\
&\le \frac{\gamma C_1}{\tau} + \mu\left|\left|T(\tau)[K(f(u^\varphi(\tau,\cdot)))] - K(f(u^\varphi(\tau,\cdot)))\right|\right|,\\
&\quad +\mu\int_0^\tau \frac{C_1C_2\left|\left|K\right|\right|}{\tau-s}(\tau-s)^\delta\mathrm{d}s,\\
&\le C_3 \triangleq \frac{\gamma C_1}{\tau} + \mu^\varphi(1+e^{-\mu\tau})\sup f([0,\gamma]) + \mu\int_0^\tau \frac{C_1C_2}{\tau-s}(\tau-s)^\delta\mathrm{d}s.
\end{aligned}$$

On the other hand, by Theorem 5.4(II) in [1] and Theorem 7.3.1 in [14], we can easily see that there exist $C_4 > 0$ and $C_5 > 0$ such that $||u^\varphi(2\tau,\cdot)||_{C^{1,1/2}(\overline{\Omega})} \le C_4 + C_5||A_T u^\varphi(2\tau,\cdot)||$ for all $\varphi \in \omega(\psi)$. Thus, $||u^\varphi(2\tau,\cdot)||_{C^{1,1/2}(\overline{\Omega})} \le C_4 + C_3C_5$ for all $\varphi \in \omega(\psi)$. This and the invariance of $\omega(\psi)$ show that

$$||\varphi(0,\cdot)||_{C^{1,1/2}(\overline{\Omega})} \le M \triangleq C_4 + C_3C_5 \ \text{ for all } \ \varphi \in \omega(\psi),$$

confirming that *claim* holds.

By the above claim and the Arzèla–Ascoli theorem, we know that $\{\varphi(0,\cdot) : \varphi \in \omega(\psi)\}$ is pre-compact under the topology induced by $C^1(\overline{\Omega})$-norm. Now, let the sequence $\{\varphi^n\}_{n=1}^\infty$ and φ be in $\omega(\psi)$ and satisfy $\lim_{n\to\infty} ||\varphi^n(0,\cdot) - \varphi(0,\cdot)|| = 0$. We need to prove prove

$\lim_{n\to\infty} ||\varphi^n(0,\cdot)-\varphi(0,\cdot)||_{C^1(\overline{\Omega})} = 0$. Otherwise, by the compactness of $\{\varphi(0,\cdot) : \varphi \in \omega(\psi)\}$ in $C^1(\overline{\Omega})$, there exist a $\bar{\varphi} \in C^1(\overline{\Omega})$ and a subsequence $\{\varphi^{n_k}\}_{k=1}^{\infty}$ such that $\bar{\varphi}(0,\cdot) \neq \varphi(0,\cdot)$ and $\lim_{n\to\infty} ||\varphi^{n_k}(0,\cdot)-\bar{\varphi}(0,\cdot)||_{C^1(\overline{\Omega})} = 0$. But, this, combining with the fact that $\lim_{n\to\infty} ||\varphi^n(0,\cdot)-\varphi(0,\cdot)|| = 0$, we have $\bar{\varphi}(0,\cdot) = \varphi(0,\cdot)$, a contradiction. It is easy to see that $\lim_{n\to\infty} ||\varphi^n(0,\cdot)-\varphi(0,\cdot)|| = 0$ whenever $\lim_{n\to\infty} ||\varphi^n(0,\cdot)-\varphi(0,\cdot)||_{C^1(\overline{\Omega})} = 0$. Therefore, the supremum norm and the C^1-norm define the same topology on $\{\varphi(0,\cdot) : \varphi \in \omega(\psi)\}$. □

In the sequel, for a given $\xi \in Y_+$, we define $M^{\xi} : C([-\tau,0],Y_+) \times \overline{\Omega} \to \mathbb{R}$ by

$$M^{\xi}(\varphi,x) = \begin{cases} \frac{\varphi(0,x)}{\xi(x)}, & x \in \Omega, \\ \frac{\partial\varphi(0,x)}{\partial\nu} / \frac{\partial\xi(x)}{\partial\nu}, & x \in \partial\Omega. \end{cases}$$

Lemma 2.6 *Let $\psi \in X_+$ and $\xi \in Y_+$. Then $M^{\xi}|_{\omega(\psi)\times\overline{\Omega}}$ is a continuous positive function.*

Proof By the definition of M^{ξ}, Lemmas 2.4 and 2.5, we easily see that $M^{\xi}(\omega(\psi)\times\overline{\Omega}) \subseteq (0,\infty)$, and $M^{\xi}|_{\omega(\psi)\times\Omega}$, $M^{\xi}|_{\omega(\psi)\times\partial\Omega}$ are continuous functions.

It suffices to prove that for given $(\varphi^*,x^*) \in \omega(\psi)\times\partial\Omega$ and given sequence $(\varphi_n,x_n) \in \omega(\psi)\times\Omega$, if $\lim_{n\to\infty} ||\varphi_n-\varphi^*|| = \lim_{n\to\infty} ||x_n-x^*|| = 0$, then $\lim_{n\to\infty} M^{\xi}(\varphi_n,x_n) = M^{\xi}(\varphi^*,x^*)$. Indeed, if $\varphi^* = 0$, then by Lemma 2.5 and $\lim_{n\to\infty} ||\varphi_n|| = 0$, we have $\lim_{n\to\infty} ||\varphi_n(0,\cdot)||_{C^1(\overline{\Omega})} = 0$. Thus by the definition of M^{ξ}, we easily see that $\lim_{n\to\infty} M^{\xi}(\varphi_n,x_n) = 0 = M^{\xi}(\varphi^*,x^*)$. Next, Suppose that $\varphi^* > 0$. Without loss of generality, we may assume that $\varphi_n \in X_+\setminus\{0\}$. By the proof of Lemma 2.4, for sufficient large n, there exist $x_n^0 \in \partial\Omega$ and $s_n > 0$ such that $s_n = dist(x_n,\partial\Omega)$, $x_n = x_n^0 - s_n\nu_{x_n^0}$ and $x_n^0 - \eta s_n\nu_{x_n^0} \in \overline{\Omega}$ for all $\eta \in [0,1]$, where $\nu_{x_n^0}$ is the unit outward normal vector at x_n^0. It follows from $\varphi_n \in \omega(\psi)\cap Y_+$, Lemmas 2.4 and 2.5 that

$$\begin{aligned} \lim_{n\to\infty} M^{\xi}(\varphi_n,x_n) &= \lim_{n\to\infty} \frac{\varphi_n(0,x_n)}{\xi(x_n)}, \\ &= \lim_{n\to\infty} \left[\frac{\varphi^*(0,x_n)}{\xi(x_n)} \cdot \frac{\varphi_n(0,x_n)}{\varphi^*(0,x_n)}\right], \\ &= \lim_{n\to\infty} \left[\frac{\varphi^*(0,x_n)}{\xi(x_n)}\right] \cdot \lim_{n\to\infty} \left[\frac{\varphi_n(0,x_n)}{\varphi^*(0,x_n)}\right], \\ &= M^{\xi}(\varphi^*,x^*) \lim_{n\to\infty} \frac{\int_0^1 \frac{\partial\varphi_n(0,x_n^0-\eta s_n\nu_{x_n^0})}{\partial\eta} d\eta}{\int_0^1 \frac{\partial\varphi^*(0,x_n^0-\eta s_n\nu_{x_n^0})}{\partial\eta} d\eta}, \\ &= M^{\xi}(\varphi^*,x^*). \end{aligned}$$

Consequently, $M^{\xi}|_{\omega(\psi)\times\overline{\Omega}}$ is a continuous positive function. This completes the proof. □

To continue to investigate the dynamics of (2.1) or its abstract form (2.2), we now formulate another basic condition on the nonlinearity f:

(H2) $f(u) < f'(0)u$ for all $u \in (0,\infty)$.

Let $A = d\Delta - \mu Id + \mu f'(0)K$ and denote by $s(A)$ the spectral bound of the linear operator A. By the Krein–Rutman theorem and the spectral mapping theorems for semigroups, we know that $s(A)$ is a real and simple eigenvalue of A and there exists a unique $\xi \in Y_+$ such that $A\xi = s(A)\xi$ and $||\xi|| = 1$.

Theorem 2.2 *Assume that (H1) and (H2) hold. If $s(A) \le 0$, then the trivial equilibrium* 0 *of (2.1) attracts all solutions of (2.1) with the initial value $\psi \in X_+$.*

Proof In view of $s(A) \le 0$, there exists $\mu^* \le \mu$ such that $s(d\Delta - \mu^* Id + \mu f'(0)K) = 0$ which implies that there exists $\phi^* \in Y_+$ such that $d\Delta\phi^* - \mu^*\phi^* + \mu f'(0)K\phi^* = 0$. Thus, for all $t \in [0, \infty)$, we have

$$\phi^* = e^{-(\mu^*-\mu)t}T(t)(\phi^*) + \mu \int_0^t e^{-(\mu^*-\mu)(t-s)}T(t-s)(K(f'(0)\phi^*))\mathrm{d}s,$$
$$\ge T(t)(\phi^*) + \mu \int_0^t T(t-s)(K(f'(0)\phi^*))\mathrm{d}s.$$

Suppose that $\psi \in X_+$. By Lemma 2.6, we know that M^{ϕ^*} is continuous in $(\varphi, x) \in \omega(\psi) \times \overline{\Omega}$. Thus, there exists $\varphi^* \in \omega(\psi)$ such that $||M^{\phi^*}(\varphi^*, \cdot)|| = \sup\{||M^{\phi^*}(\varphi, \cdot)|| : \varphi \in \omega(\psi)\} \triangleq b^*$.

We now prove $\varphi^* = 0$. Otherwise, $\varphi^* \in X_+ \setminus \{0\}$ and thus $b^* > 0$. By the invariance of $\omega(\psi)$, there exists a global classical solution $u(t, x) : \mathbb{R} \times \overline{\Omega} \to [0, \infty)$ of (2.1) such that $u_{2\tau} = \varphi^*$. Obviously, by the choice of b^*, we have $b^*\phi^* - u(t, \cdot) \in C_+$ for all $t \in \mathbb{R}$.

We claim that $b^*\phi^*(x^*) - u(t^*, x^*) > 0$ for some $(t^*, x^*) \in [-\tau, \tau] \times \Omega$; otherwise, $u(t, x) = b^*\phi^*(x)$ for all $(t, x) \in [-\tau, \tau] \times \overline{\Omega}$. It follows from the semigroup properties of U that $u(t, x) = b^*\phi^*(x)$ for all $(t, x) \in [-\tau, \infty) \times \overline{\Omega}$. From (2.2) and (H2), we have for all $t \in (0, \infty)$,

$$b^*\phi^* = T(t)(b^*\phi^*) + \mu \int_0^t T(t-s)[K(f(b^*\phi^*))]\mathrm{d}s,$$
$$< T(t)(b^*\phi^*) + \mu \int_0^t T(t-s)[K(f'(0)b^*\phi^*)]\mathrm{d}s.$$

Thus, $\phi^* < T(t)(\phi^*) + \mu \int_0^t T(t-s)[K(f'(0)\phi^*)]\mathrm{d}s$ for all $t \in (0, \infty)$, a contradiction.

Let $v(t, x) = b^*\phi^*(x) - u(t + t^*, x)$ for all $(t, x) \in \mathbb{R} \times \overline{\Omega}$. By the semigroup properties of U and (2.2), we know that for all $(t, x) \in [0, \infty) \times \Omega$,

$$u(t + t^*, x) = T(t)(u(t^*, \cdot))(x) + \mu \int_0^t T(t-s)(K(f(u(s + t^* - \tau, \cdot))))(x)\mathrm{d}s.$$

This implies that for all $(t, x) \in [0, \infty) \times \Omega$, we have

$$\begin{aligned}
v(t, x) &= b^*\phi^*(x) - u(t + t^*, x), \\
&\ge T(t)(b^*\phi^* - u(t^*, \cdot))(x) \\
&\quad + \mu \int_0^t T(t-s)[b^*K(f'(0)\phi^*) - K(f(u(s + t^* - \tau, \cdot)))](x)\mathrm{d}s, \\
&\ge T(t)(b^*\phi^* - u(t^*, \cdot))(x) \\
&\quad + \mu \int_0^t T(t-s)[b^*K(f'(0)\phi^*) - K(f'(0)u(s + t^* - \tau, \cdot))](x)\mathrm{d}s,
\end{aligned}$$

$$= T(t)(v(0,\cdot))(x) + \mu \int_0^t T(t-s)[K(f'(0)v(s-\tau,\cdot))](x)\mathrm{d}s,$$
$$\geq T(t)(v(0,\cdot))(x),$$

which, together with Lemma 2.2(i–ii) and $v(0,\cdot) \in X_+ \setminus \{0\}$, implies that $v(t,x)|_\Omega > 0$ and $\frac{\partial v(t,x)}{\partial \nu}|_{\partial\Omega} < 0$ for all $t \in (0,\infty)$. Thus, $M^{\phi^*}(v_t, x) > 0$ for all $(t,x) \in (0,\infty) \times \overline{\Omega}$, in particular, $M^{\phi^*}(v_{2\tau-t^*}, x) > 0$ for all $x \in \overline{\Omega}$. Thus, the definitions of v and M^{ϕ^*} imply that for all $x \in \overline{\Omega}$,

$$M^{\phi^*}(\varphi^*, x) = M^{\phi^*}(u_{2\tau}, x) = b^* - M^{\phi^*}(v_{2\tau-t^*}, x) < b^*,$$

This, combined with Lemma 2.6, yields a contradiction to the choice of b^*. So, $\varphi^* = 0$ and hence $\omega(\psi) = \{0\}$. This completes the proof. □

When $s(A) > 0$, we will see in the next proposition that (2.1) is persistent, implying that the trivial steady state becomes unstable.

Proposition 2.1 *Assume that (H1) and (H2) hold. If $s(A) > 0$, then there exists $\xi \in Y_+$ with $||\xi|| = 1$ such that for any $M \in \{B_n : n \geq 1\}$, there exist $\epsilon_M > 0$ such that $\{\varphi \in X_+ : \epsilon_M \xi \leq \varphi \leq M\}$ is a positively invariant set of U. Moreover, $\omega(\varphi) \subseteq \epsilon_M \xi + X_+$ for any $\varphi \in X_+ \setminus \{0\}$ with $\varphi \leq M$.*

Proof In view of $s(A) > 0$, there exists $c^* \in (0, f'(0))$ such that $s(d\Delta - \mu Id + \mu c^* K) = 0$. Hence there exists $\xi \in Y_+$ such that $||\xi|| = 1$ and $d\Delta\xi - \mu\xi + \mu c^* K(\xi) = 0$.

Let $M \in \{B_n : n \geq 1\}$. Then $f([0,M]) \subseteq [0,M]$ from (H1). Thus, by (H2) there is $\epsilon_M > 0$ such that $f(x) > c^* x$ for all $x \in (0, \epsilon_M]$ and $f(x) > c^* \epsilon_M$ for all $x \in [\epsilon_M, M]$. Let $H(\varphi) = \mu c^* K(\min\{\varphi(-\tau,\cdot), \epsilon_M\})$ for all $\varphi \in X_+$. Then H in nondecreasing on X_+, and thus H is quasimonotone in the sense of [11,21].

Suppose that $\varphi \in X_+ \setminus \{0\}$ with $\epsilon_M \xi \leq \varphi \leq M$. Lemma 2.3(iii) implies that $U(t,\varphi) \leq M$ for all $t \in [0,\infty)$. Let $u(t,x) = u^\varphi(t,x)$ for all $(t,x) \in [-\tau,\infty) \times \overline{\Omega}$. It follows from (2.2) that, for any $t \geq 0$,

$$u(t,\cdot) = T(t)\varphi(0,\cdot) + \int_0^t T(t-s)F(u_s)ds,$$
$$\geq T(t)\varphi(0,\cdot) + \int_0^t T(t-s)H(u_s)ds.$$

By letting $v(t,x) = \epsilon_M \xi(x)$ for all $(t,x) \in [-\tau,\infty) \times \overline{\Omega}$, we have $v(t,\cdot) = T(t)v(0,\cdot) + \int_0^t T(t-s)H(v_s)ds$. Thus by $\varphi \geq \epsilon_M \xi$ and Corollary 8.1.11 in [21], we have $u(t,x) \geq v(t,x) = \epsilon_M \xi(x)$ for all $(t,x) \in [0,\infty) \times \overline{\Omega}$, which yields the first conclusion.

Next, suppose that $\varphi \in X_+ \setminus \{0\}$ with $\varphi \leq M$. Let $u(t,x) = u^\varphi(t,x)$ for all $(t,x) \in [-\tau,\infty) \times \overline{\Omega}$. Without loss of generality, we may assume that u is a classical solution and $u(t,\cdot) \in Y_+$ for all $t \in [-\tau,\infty)$. By Lemma 2.4, there exists $\epsilon_0 \in (0, \epsilon_M]$ such that $u(t,\cdot) - \epsilon_0 \xi \in C_+$ for all $t \in [-\tau, 0]$. By the proof of the first conclusion, we easily see that $u(t,\cdot) - \epsilon_0 \xi \in C_+$ for all $t \in [-\tau,\infty)$. Thus, $\omega(\varphi) \subseteq \epsilon_0 \xi + X_+$. Let $\epsilon^* = \sup\{\epsilon \geq \epsilon_0 : \psi \geq \epsilon\xi$ for all $\psi \in \omega(\varphi)\}$. Obviously, $\omega(\varphi) \subseteq \epsilon^* \xi + X_+$. It suffice to show $\epsilon^* \geq \epsilon_M$; otherwise, $\epsilon^* < \epsilon_M$. By Lemma 2.6 and the compactness of $\omega(\varphi)$, there exists $\psi \in \omega(\varphi)$ such that $\inf\{M^\xi(\psi,x) : x \in \overline{\Omega}\} = \epsilon^*$. By the invariance of $\omega(\varphi)$, there exists a global classical solution $v(t,x) : \mathbb{R} \times \overline{\Omega} \to [0,\infty)$ of (2.1) such that $v_{2\tau} = \psi$. Let $w(t,x) = v(t,x) - \epsilon^* \xi(x)$ for all $(t,x) \in \mathbb{R} \times \overline{\Omega}$. Clearly, $v_t \geq \epsilon^* \xi$ and $w_t \geq 0$ for all $t \in \mathbb{R}$.

We claim that $w(t^*, x^*) > 0$ for some $(t^*, x^*) \in [-\tau, \tau] \times \Omega$; otherwise, $v(t, x) = \epsilon^* \xi^*(x)$ for all $(t, x) \in [-\tau, \tau] \times \overline{\Omega}$. It follows from the semigroup properties of U that $u(t, x) = \epsilon^* \xi(x)$ for all $(t, x) \in [-\tau, \infty) \times \overline{\Omega}$. Thus, $\epsilon^* \xi$ is a steady state of (2.1), that is, $d\Delta(\epsilon^*\xi) - \mu\epsilon^*\xi + \mu K(f(\epsilon^*\xi)) = 0$. It follows from the choices ξ, ε_M that $c^*\epsilon^* K(\xi) = K(f(\epsilon^*\xi)) > K(c^*\epsilon^*\xi)$, a contradiction.

By the above claim, without loss of generality we may assume that $w(0, \cdot) > 0$. It follows from (2.2) that, for any $t \geq 0$,

$$
\begin{aligned}
w(t, \cdot) &= T(t)(w(0, \cdot)) + \int_0^t T(t-s)F(v_s)ds - \int_0^t T(t-s)H(\epsilon^*\xi)ds, \\
&= T(t)(w(0, \cdot)) + \int_0^t T(t-s)[H(v_s) - H(\epsilon^*\xi)]ds, \\
&\geq T(t)(w(0, \cdot)).
\end{aligned}
$$

Combining this with Lemma 2.2(i–ii) and $w(0, \cdot) > 0$, we obtain that $w(t, x)|_\Omega > 0$ and $\frac{\partial w(t, \cdot)}{\partial \nu}|_{\partial\Omega} < 0$ for all $t \in (0, \infty)$. Thus, $M^\xi(w_t, x) > 0$ for all $(t, x) \in (0, \infty) \times \overline{\Omega}$, in particular, $M^\xi(w_{2\tau}, x) > 0$ for all $x \in \overline{\Omega}$. Thus the definitions of w and M^ξ imply that for all $x \in \overline{\Omega}$, we have

$$
M^\xi(\psi, x) = M^\xi(v_{2\tau}, x) = \epsilon^* + M^\xi(w_{2\tau}, x) > \epsilon^*,
$$

This, combined with Lemma 2.6, yields a contradiction to the choice of ψ. This completes the proof. □

According to Lemma 2.1(ii) and the fact that $U(t, \varphi)(\theta, x) = u^\varphi(t + \theta, x)$ for all $(t, \theta, x) \in [0, \infty) \times [-\tau, 0] \times \overline{\Omega}$, we easily obtain the following result,

Lemma 2.7 *Let $M > 0$, $t > 0$, $\tilde{\tau} \in (0, \tau]$ and $D \subseteq X_+$ with $D \leq M$. Assume that $D|_{[-\tilde{\tau},0]\times\overline{\Omega}}$ is pre-compact in $C([-\tilde{\tau}, 0] \times \overline{\Omega}, \mathbb{R})$, where $D|_{[-\tilde{\tau},0]\times\overline{\Omega}} \triangleq \{\varphi|_{[-\tilde{\tau},0]\times\overline{\Omega}} : \varphi \in D\}$ and $C([-\tilde{\tau}, 0] \times \overline{\Omega}, \mathbb{R})$ is equipped with the usual supremum norm. Then $U(t, D)|_{[-\min\{\frac{t}{2}+\tilde{\tau},\tau\},0]\times\overline{\Omega}}$ is pre-compact in $C([-\min\{\frac{t}{2} + \tilde{\tau}, \tau\}, 0] \times \overline{\Omega}, \mathbb{R})$.*

Proposition 2.2 *Assume that (H1) and (H2) hold. If $s(A) > 0$, then the trivial steady state $u = 0$ is unstable and (2.1) has a positive steady state, located in Y_+.*

Proof The instability of $u = 0$ is implied by the persistence of (2.1) established in Proposition 2.1, thus we only need to prove the existence of a positive steady state.

Let $M = B_1$ and ξ, ϵ_M defined as in Proposition 2.1. Let $D = \{\varphi \in X_+ : \epsilon_M\xi \leq \varphi \leq M\}$. By Proposition 2.1, $U(t, D) \subseteq D$ for all $t \geq 0$.

Now suppose that $T \in I \triangleq \left\{\frac{1}{2^i} : i = 1, 2, \cdots\right\}$. We claim that there exists a compact convex subset K_T in D such that $U(T, K_T) \subseteq K_T$. Indeed, by Lemma 2.1(ii) and the fact that $U(t, \varphi)(\theta, x) = u^\varphi(t + \theta, x)$ for all $(t, \theta, x) \in [0, \infty) \times [-\tau, 0] \times \overline{\Omega}$, we know that $U(T, D)|_{[-\min\{\tau,\frac{T}{2}\},0]\times\overline{\Omega}}$ is pre-compact in $C([-\min\{\tau, \frac{T}{2}\}, 0] \times \overline{\Omega}, \mathbb{R})$. Let $g(K) \triangleq \overline{co}(U(T, K))$ for any $K \subseteq D$. Then $g(D)|_{[-\min\{\tau,\frac{T}{2}\},0]\times\overline{\Omega}}$ is pre-compact in $C([-\min\{\tau, \frac{T}{2}\}, 0] \times \overline{\Omega}, \mathbb{R})$. By applying Lemma 2.7 repeatedly, we may get that $g^k(D)|_{[-\min\{\tau,\frac{kT}{2}\},0]\times\overline{\Omega}}$ is pre-compact in $C([-\min\{\tau, \frac{kT}{2}\}, 0] \times \overline{\Omega}, \mathbb{R})$. Choose a positive integer k_0 such that $k_0 > \frac{2\tau}{T}$. Then $K_T \triangleq g^{k_0}(D)$ is a compact convex subset in D such that $U(T, K_T) \subseteq K_T$. By the Schauder fixed point theorem, there is $\psi_T \in K_T$ such that $U(T, \psi_T) = \psi_T$. According to Lemma 2.1(ii) and the fact that $\{\psi_T : T \in I\} \subseteq U(k, D)$ for any positive integer k, we know that $\{\psi_T : T \in I\}$ is pre-compact in X_+, and thus there exist $\psi \in D$ and a sequence $\{T_k\}$ in I such that $\lim_{T_k \to 0} \psi_{T_k} = \psi$. For any $t \in (0, \infty)$, there exist $r_k \in [0, T_k)$ and a nonnegative integer N_k such that $t = N_k T_k + r_k$. Obviously, $\lim_{k\to\infty} r_k = 0$. Hence, $U(t, \psi) = \lim_{k\to\infty} U(t, \psi_{T_k}) = \lim_{k\to\infty} U(r_k, \psi_{T_k}) = \psi$, which implies that ψ is a positive steady state, located in Y_+ of (2.1), completing the proof. □

In what follows, we denote by u_+ be the positive steady state of (2.1) obtained in Proposition 2.2, and let $u_+^* = ||u_+||$. To address the attractiveness of u^+, we further need the following condition on the nonlinear function f:

(H3) For any closed interval $[a, b] \neq \{1\}$ with $0 < a \leq b < \infty$, either (i) $G((0, u_+^*] \times [a, b]) \subset (a, \infty)$ or (ii) $G((0, u_+^*] \times [a, b]) \subset (0, b)$, where $G : (0, u_+^*] \times (0, \infty) \to (0, \infty)$ is defined by $G(k, u) = \frac{f(ku)}{f(k)}$.

Theorem 2.3 *Assume that (H1)–(H3) hold. If $s(A) > 0$, then (2.1) has a unique positive steady state u_+ which attracts all solutions of (2.1) with the initial value $\psi \in X_+ \setminus \{0\}$.*

Proof The existence of u_+ is already established in Proposition 2.2, and the uniqueness will be a consequence of the global attractiveness of u_+ in $X_+ \setminus \{0\}$. So, we only need to show that u_+ attracts all solutions of (2.1) with the initial value $\psi \in X_+ \setminus \{0\}$.

Suppose $\psi \in X_+ \setminus \{0\}$. By Lemma 2.6 we know that M^{u_+} is continuous in $(\varphi, x) \in \omega(\psi) \times \overline{\Omega}$.

Let $a^* = \inf\{M^{u_+}(\varphi, x) : (\varphi, x) \in \omega(\psi) \times \overline{\Omega}\}$ and $b^* = \sup\{||M^{u_+}(\varphi, \cdot)|| : \varphi \in \omega(\psi)\}$. By the choices of a^*, b^*, and Lemma 2.6, we have $0 < a^* \leq M^{u_+}(\varphi, x) \leq b^* < \infty$ for all $(\varphi, x) \in \omega(\psi) \times \overline{\Omega}$. To prove this theorem, it suffices to prove that $a^* = b^* = 1$. Otherwise, $a^* \neq 1$ or $b^* \neq 1$. We shall show that this is impossible by discussing the two possible cases.

Case 1: $a^* = b^* \neq 1$. In this case, $\varphi(\theta, \cdot) = a^* u_+$ for all $\theta \in [-\tau, 0]$ and $\varphi \in \omega(\psi)$. By (2.1) and the invariance of $\omega(\psi)$, we know that $a^* u_+$ is also a positive steady state of (2.1). Multiplying the steady state equation for u_+ by a^* and subtracting the the resulting equation from the steady state equation for $a^* u_+$ (noticing that K is linear), we then obtain $K(f(a^* u_+(\cdot))) = K(a^* f(u_+(\cdot)))$. On the other hand, by (H3) with $[a, b] = \{a^*\} \neq \{1\}$, we know that either $f(ka^*) > a^* f(k)$ for all $k \in (0, u_+^*]$ or $f(ka^*) < a^* f(k)$ for all $k \in (0, u_+^*]$. This, combined with the monotonicity of K, yields $K(f(a^* u_+(\cdot))) > K(a^* f(u_+(\cdot)))$ or $K(f(a^* u_+(\cdot))) < K(a^* f(u_+(\cdot)))$, a contradiction.

Case 2: $0 < a^* < b^*$. In this case, by the assumption (H3) with $[a, b] = [a^*, b^*]$, we know that either (I) $f(ku) > a^* f(k)$ for all $(k, u) \in (0, u_+^*] \times [a^*, b^*]$ or (II) $f(ku) < b^* f(k)$ for all $(k, u) \in (0, u_+^*] \times [a^*, b^*]$.

For (I), by Lemma 2.6, we have $a^* = M^{u_+}(\varphi^*, x^*)$ for some $(\varphi^*, x^*) \in \omega(\psi) \times \overline{\Omega}$. By the invariance of $\omega(\psi)$, there exists a global classical solution $u(t, x) : \mathbb{R} \times \overline{\Omega} \to [0, \infty)$ of (2.1) such that $u_{2\tau} = \varphi^*$. Obviously, by the choice of a^*, we have $u(t, \cdot) - a^* u_+ \in C_+$ for all $t \in \mathbb{R}$.

We now claim $u(t^*, x^*) - a^* u_+(x^*) > 0$ for some $(t^*, x^*) \in [-\tau, \tau] \times \Omega$; otherwise by the invariance of $\omega(\psi)$ and the semigroup properties of U, $u(t, x) = a^* u_+(x)$ for all $(t, x) \in [0, \infty) \times \Omega$, that is, $a^* u_+$ is a positive steady state of (2.1). Thus, $K(f(a^* u_+(\cdot))) = K(a^* f(u_+(\cdot)))$, yielding a contradiction to the monotonicity of K and the fact that $f(ku) > a^* f(k)$ for all $u \in [a^*, b^*]$ and $k \in (0, u_+^*]$. Hence, the claim holds.

Let $v(t,x) = u(t+t^*,x) - a^*u_+(x)$ for all $(t,x) \in [-\tau,\infty) \times \overline{\Omega}$. By the semigroup properties of U and (2.2), we know that for all $(t,x) \in [0,\infty) \times \Omega$,

$$u(t+t^*,x) = T(t)(u(t^*,\cdot))(x) + \mu \int_0^t T(t-s)(K(f(u(s+t^*-\tau,\cdot))))(x)\mathrm{d}s.$$

This, combined with the monotonicity of K and the fact that $f(ku) > a^*f(k)$ for all $(u,k) \in [a^*,b^*] \times (0,u_+^*]$ and $u_+(x) = T(t)(u_+)(x) + \mu\int_0^t T(t-s)(K(f(u_+)))(x)ds$, implies that for all $(t,x) \in [0,\infty) \times \Omega$,

$$\begin{aligned} v(t,x) &= T(t)(u(t^*,\cdot))(x) + \int_0^t T(t-s)F(u_{s+t^*})\mathrm{ds} - \mathrm{a}^*\mathrm{u}_+(\mathrm{x}), \\ &= T(t)(u(t^*,\cdot) - a^*u_+)(x) \\ &\quad + \mu\int_0^t T(t-s)(K[f(u(s+t^*-\tau,\cdot)) - a^*f(u_+)])(x)\mathrm{ds}, \\ &= T(t)(u(t^*,\cdot) - a^*u_+)(x) \\ &\quad + \mu\int_0^t T(t-s)(K[f(M^{u_+}(u(s+t^*-\tau,\cdot),\cdot)u_+) - a^*f(u_+)])(x)\mathrm{ds}, \\ &\geq T(t)(u(t^*,\cdot) - a^*u_+)(x), \\ &= T(t)(v(0,\cdot))(x). \end{aligned}$$

Combining this with Lemma 2.2(i–ii) and $v(0,\cdot) > 0$, we obtain that $v(t,x)|_\Omega > 0$ and $\frac{\partial v(t,\cdot)}{\partial \nu}|_{\partial\Omega} < 0$ for all $t \in (0,\infty)$. Thus, $M^{u_+}(v_t,x) > 0$ for all $(t,x) \in (0,\infty) \times \overline{\Omega}$, in particular, $M^{u_+}(v_{2\tau-t^*},x) > 0$ for all $x \in \overline{\Omega}$. Thus the definitions of v and M^{u_+} imply that

$$M^{u_+}(\varphi^*,x^*) = M^{u_+}(u_{2\tau},x^*) = a^* + M^{u_+}(v_{2\tau-t^*},x^*) > a^*.$$

This yields a contradiction to the choices of x^* and φ^*.

For (II), we are similarly led to a contradiction.

Summarizing the above Cases 1–2, we see that $a^* = b^* = 1$ and hence $\omega(\psi) = \{u_+\}$. This completes the proof. □

For some special forms of the linear operator K, by applying Theorem 2.2 and 2.3, we may obtain some more explicit results. We start with the *local* case, represented by $K = Id$, the identity operator. In this case, we now formulate the following condition to replace (H3):

(H4) There exists $u^* > 0$ such that $f(u^*) = u^*$. Moreover, for any closed interval $[a,b] \neq \{1\}$ with $0 < a \leq b < \infty$, either (i) $G((0,u^*] \times [a,b]) \subset (a,\infty)$ or (ii) $G((0,u^*] \times [a,b]) \subset (0,b)$, where $G : (0,u^*] \times (0,\infty) \to (0,\infty)$ defined by $G(k,u) = \frac{f(ku)}{f(k)}$.

Denote by λ_1 the *first eigenvalue* of the operator $-\Delta$ with the homogeneous Dirichlet boundary condition. We have following results for the local case.

Theorem 2.4 *Consider $K = Id$ and assume that (H1) and (H2) hold.*

(i) If $\mu f'(0) \leq d\lambda_1 + \mu$, then the trivial steady state $u = 0$ of (2.1) attracts all solutions of (2.1) with the initial value in X_+;

(ii) $\mu f'(0) > d\lambda_1 + \mu$, *then* $u = 0$ *becomes unstable and* (2.1) *is persistent and has a positive steady state* u_+. *Moreover, if (H4) holds, then* u_+ *attracts all solutions of* (2.1) *with initial function in* $X_+ \setminus \{0\}$.

Proof For $K = Id$, $A = d\Delta - \mu Id + \mu f'(0)Id$. Note that there is a unique $\xi_1 \in Y_+$ such that $\Delta\xi_1 = -\lambda_1\xi_1$ and $||\xi_1|| = 1$. Thus, $A\xi_1 = d\Delta\xi_1 - \mu\xi_1 + \mu f'(0)\xi_1 = (-d\lambda_1 - \mu + \mu f'(0))\xi_1$, meaning that ξ_1 is also a positive eigenfunction of A associated to the eigenvalue $-d\lambda_1 - \mu + \mu f'(0)$. By the uniqueness of the positive principal eigenfucntion in the Krein–Rutman Theorem and the spectral mapping theorem for semigroups, we conclude that $s(A) = -d\lambda_1 - \mu + \mu f'(0)$. Thus, $\mu f'(0) \le d\lambda_1 + \mu$ is equivalent to $s(A) \le 0$, and hence, the conclusion in (i) follows from Theorem 2.2.

For (ii), by Theorem 2.3, we only need to verify (H3) under (H4). According to the assumption (H4), for any $u \in (0,\infty) \setminus \{1\}$, by taking $a = b = u$ and $k = u^*$, we have $\frac{f(uu^*)}{u^*} > u$ or $\frac{f(uu^*)}{u^*} < u$, and hence $f(u) \ne u$ for all $u \in (0,\infty) \setminus \{u^*\}$, concluding that u^* is a unique positive fixed of f. This together with (H2) and the condition $f'(0) > 1 + d\lambda_1/\mu$, we have both $f(u) > u$ for all $u \in (0, u^*)$ and $f(u) < u$ for all $u \in (u^*, \infty)$. By the continuity of u_+, there exists $x^* \in \Omega$ such that $u_+(x^*) = u_+^* = \sup\{u_+(x) : x \in \Omega\}$. Thus, $\Delta u_+(x^*) \le 0$, which, combined with the fact that $d\Delta u_+(x) - \mu u_+(x) + \mu f(u_+(x)) = 0$, implies that $-\mu u_+(x^*) + \mu f(u_+(x^*)) \ge 0$ and hence $f(u_+(x^*)) \ge u_+(x^*)$. Hence, $u_+^* = u_+(x^*) \le u^*$. This and (H4) imply that (H3) holds. Therefore, the statement (ii) follows from Theorem 2.3. □

Next, we consider an integral form for K related to the non-local equation (1.1) in the Introduction, that is,

$$K(\phi)(x) = \int_\Omega \Gamma(x, y)\phi(y)\mathrm{d}y \quad \text{for } x \in \overline{\Omega}. \tag{2.3}$$

In this case, we need the following replacement for (H3):

(H5) There exists $u^* > 0$ such that $f(u^*) = u^*$. Moreover, for any closed interval $[a, b] \ne \{1\}$ with $0 < a \le b < \infty$, either (i) $G((0, f^*] \times [a, b]) \subset (a, \infty)$ or (ii) $G((0, f^*] \times [a, b]) \subset (0, b)$, where $f^* = \sup_{k\in[0,u^*]} f(k)$ and $G : (0, f^*] \times (0,\infty) \to (0,\infty)$ is defined by $G(k, u) = \frac{f(ku)}{f(k)}$.

Theorem 2.5 *Assume that* K *is given by* (2.3) *with* $\Gamma : \overline{\Omega} \times \overline{\Omega} \to [0,\infty)$ *being continuous satisfying* $||K|| = 1$. *Suppose that (H1) and (H2) hold.*

(i) If $s(A) \le 0$, *then the trivial steady state of* (2.1) *attracts all solutions of* (2.1) *with the initial value in* X_+;

(ii) If $s(A) > 0$, *then* $u =$ *becomes unstable, and* (2.1) *is persistent and has a positive steady state* u_+. *Moreover, if* f *satisfies the assumption (H5), then* u_+ *attracts all solutions of* (2.1) *with initial functions in* $X_+ \setminus \{0\}$.

Proof (i) is a direct consequent of Theorem 2.2. (ii) follows from Theorem 2.3 if we can verify (H3) under (H5). Indeed, by the definition of u_+^*, there exists $x^* \in \Omega$ such that $u_+(x^*) = u_+^* = \sup\{u_+(x) : x \in \Omega\}$. Thus, $\Delta u_+(x^*) \le 0$, which, combined with (2.1), implies that $u_+^* = u_+(x^*) \le ||K(f(u_+)|| \le ||f(u_+)||$. We claim that $u_+^* \le f^*$; otherwise, $u_+^* > f^* \ge f(u^*) = u^*$. This implies $||f(u_+)|| = \sup_{x\in\Omega} |f(u_+(x))| = \sup_{x\in\Omega} f(u_+(x)) \le \sup_{u\in[0,u_+^*]} f(u) = \max\{f^*, \sup_{u\in[u^*,u_+^*]} f(u)\} < u_+^*$ due to the fact $f(u) < u$ for all $u > u^*$, a contradiction. Thus (H3) is verified and hence, the statement of (ii) holds by Theorem 2.3. □

Remark 2.1 Similar results to those in Theorem 2.1–2.5 can be established if $d\Delta$ is replaced by a uniformly elliptic operator.

3 Examples

In this section, we illustrate the results of Theorems 2.4 and 2.5 by considering two concrete examples, one is the local diffusive Mackey and Glass equation and the other is the non-local diffusive Nicholson's blowflies equation.

Note that (H1) and (H2) are quite standard and minimal conditions and are easy to check. Thus, verifying (H4) or (H5) becomes the key for applying these two theorems. The following lemma shall be very useful for verifying (H4) and (H5), which is closely related to the requirement that the map $f(u)$ generates a dynamics of global convergence to a positive fixed point of f which is a condition crucial in [27–29].

Lemma 3.1 *For given $k^* > 0$, let $G : (0, k^*] \times (0, \infty) \to (0, \infty)$ be a continuously differentiable function. Assume that $G(0^+, u) = u$ for all $u > 0$ and $\frac{\partial G(k,u)}{\partial k} \cdot (1 - u) > 0$ for all $(k, u) \in (0, k^*] \times ((0, \infty) \setminus \{1\})$. If $\{u > 0 : G(k^*, G(k^*, u)) = u\}$ consists of single point, then for any closed interval $[a, b] \neq \{1\}$ with $0 < a \leq b < \infty$, either (i) $G((0, k^*] \times [a, b]) \subset (a, \infty)$ or (ii) $G((0, k^*] \times [a, b]) \subset (0, b)$.*

Proof Note that

$$\frac{\partial G(k,u)}{\partial k}(k,u) = \begin{cases} > 0, & (k,u) \in (0,k^*] \times (0,1), \\ < 0, & (k,u) \in (0,k^*] \times (1,\infty), \\ = 0, & (k,u) \in (0,k^*] \times \{1\}. \end{cases} \tag{3.1}$$

Obviously, $G(k, 1) = G(0^+, 1) = 1$ for all $k \in (0, k^*]$. Let $[a, b] \subseteq (0, \infty)$ be such $[a, b] \neq \{1\}$. We will complete the proof by distinguishing three cases.

Case 1: $a \leq b \leq 1$ and $[a, b] \neq \{1\}$. In this case, $G(k, 1) = 1 > a$ for all $k \in (0, k^*]$. Since $G(k, u)$ is strictly increasing in k when $u \in (0, 1)$, we know that for any $(k, u) \in (0, k^*] \times [a, 1)$, $G(k, u) > \lim_{k\to 0^+} G(k, u) = u \geq a$. Hence, $G(k, u) > a$ for all $(k, u) \in (0, k^*] \times [a, b]$.

Case 2: $1 \leq a \leq b$ and $[a, b] \neq \{1\}$. In this case, since $G(k, u)$ is strictly decreasing in k when $u \in (1, \infty)$, we know that $G(k, u) < \lim_{k\to 0^+} G(k, u) = u \leq b$ for any $u \in (1, b]$. Note that $b > 1$, and thus $G(k, 1) = 1 < b$ for all $k \in (0, k^*]$. So, $G(k, u) < b$ for all $(k, u) \in (0, k^*] \times [a, b]$.

Case 3: $a < 1 < b$. By way of contradiction, we assume there exist (k^a, u^a) and (k^b, u^b) in $(0, k^*] \times [a, b]$ such that $G(k^a, u^a) \leq a$ and $G(k^b, u^b) \geq b$. By the discussions in Cases 1-2, we easily see that $u^a > 1$ and $u^b < 1$. Thus, $G(k^*, u^a) \leq G(k^a, u^a) \leq a$ and $G(k^*, u^b) \geq G(k^b, u^b) \geq b$.

Let $h(u) = G(k^*, u)$ for all $u \in (0, \infty)$. It follows from (3.1) and $G(0^+, u) \equiv u$ that $h(u) > u$ for all $u \in (0, 1)$ and $h(u) < u$ for all $u \in (1, \infty)$. Thus, there exist two positive numbers ε, M such that $[\varepsilon, M] \supseteq [a, b]$ and $[a, b] \subseteq h([a, b]) \subseteq h([\varepsilon, M]) \subseteq [\varepsilon, M]$. Let $I = [\varepsilon, M]$ and $g = h_I$. According to the assumption that $\{u > 0 : G(k^*, G(k^*, u)) = u\}$ is a single point set, we easily see that g^2 has a unique positive fixed point 1 in I. By Proposition 2.1 in [28], we have $I_g = \{1\}$, where $I_g = \bigcap_{n\geq 1} \overline{g^n(I)}$. By $g([a, b]) \supseteq [a, b]$ and $I \supseteq [a, b]$, we get $I_g \supseteq [a, b] \neq \{1\}$, a contradiction.

Summarizing the above cases 1–3, we see that for any closed interval $[a, b] \neq \{1\}$ with $0 < a \le b < \infty$, either (i) $G((0, k^*] \times [a, b]) \subset (a, \infty)$ or (ii) $G((0, k^*] \times [a, b]) \subset (0, b)$. This completes the proof. □

Remark 3.1 When $u^* > 0$ is a fixed point of f and $G(k, u) = \frac{f(ku)}{f(k)}$, calculations show that $G(u^*, G(u^*, u)) = \frac{f^2(u^*u)}{u^*}$. Hence,

$$\{u > 0 : G(u^*, G(u^*, u)) = u\} = \{1\} \quad \text{if and only if} \quad \{u > 0 : f^2(u) = u\} = \{u^*\}. \tag{3.2}$$

Note that if (3.2) holds, then all positive orbits of the dynamical system generated by the map f converge to the positive fixed point u^*. Thus, Theorems 2.4 and 2.5 together with Lemma (3.1) and (3.2) reveal a relationship, to certain extent, between the global convergence of the map dynamic $\{f^n\}$ and global convergence of the corresponding reaction–diffusion equation (1.1) under DBVC.

Example 3.1 Consider the following diffusive Mackey and Glass equation

$$\begin{cases} \frac{\partial u}{\partial t}(t, x) = d\Delta u(t, x) - \delta u(t, x) + \frac{pu(t-\tau, x)}{1+(u(t-\tau, x))^n}, \\ u|_{\partial\Omega} = 0, \\ u(\theta, x) = \phi(\theta, x) \quad \text{for} \quad (\theta, x) \in [-\tau, 0] \times \bar{\Omega}, \end{cases} \tag{3.3}$$

where d, δ, p and n are all positive constants.

For this equation, the following lemma verifies the conditions

Lemma 3.2 *Let $f(u) = \frac{p}{\delta}\frac{u}{1+u^n}$ for all $u \ge 0$. Then the following statements are true:*

(i) if $p \le \delta$, then $f(u) < u$ for all $u > 0$;
(ii) if $p > \delta$, then the assumptions (H1) and (H2) hold;
(iii) if $p > \delta$ and $n \le \frac{2p}{p-\delta}$, then the assumption (H4) holds with $u^* = (\frac{p}{\delta} - 1)^{\frac{1}{n}}$.

Proof (i)–(ii) are obvious. We only need to prove (iii). Firstly, it is straightforward to verify that $u^* = (\frac{p}{\delta} - 1)^{\frac{1}{n}}$ satisfied $f(u^*) = u^*$, and it is positive when $p > \delta$. Let $k^* = u^*$ and let

$$G(k, u) = \frac{f(ku)}{f(k)} = \frac{u(1 + k^n)}{1 + k^n u^n} \quad \text{for all} \quad (k, u) \in (0, k^*] \times (0, \infty).$$

Then

$$\frac{\partial G}{\partial k}(k, u) = \frac{nuk^{n-1}(1 - u^n)}{(1 + k^n u^n)^2} \quad \text{for all} \quad (k, u) \in (0, k^*] \times (0, \infty).$$

From the above formulas of $G(k, u)$ and $\frac{\partial G}{\partial k}(k, u)$, we easily see that $G(0^+, u) \equiv u$ and $\frac{\partial G}{\partial k}(k, u)(1 - u) > 0$ when $u \neq 1$. According to the proof of Theorem 4.5, Remarks 4.6 and 4.7 in [28], we know that $\{u > 0 : f^2(u) = u\} = \{u^*\}$ if and only if $p > \delta$ and $n \le \max\{2, \frac{2p}{p-\delta}\} = \frac{2p}{p-\delta}$. By Lemma (3.1) and (3.2), we see that the assumption (H4) holds. This completes the proof. □

By applying Theorems 2.1, 2.4, and Lemma 3.2, we then obtain the following results for (3.3).

Theorem 3.1 *The following statements hold for (3.3).*

(i) If $p \le d\lambda_1 + \delta$, then the trivial steady state $u = 0$ of (3.3) attracts all solutions of (3.3) with the initial functions in X_+;

(ii) *If* $p > d\lambda_1 + \delta$, *then* $u = 0$ *becomes unstable, and (3.3) is persistent and has a positive steady state* u_+. *Moreover, if* $n \leq \frac{2p}{p-\delta}$, *the positive steady state* u_+ *attracts all solutions of (3.3) with the initial functions in* $X_+ \setminus \{0\}$.

Example 3.2 Consider the diffusive Nicholson's blowflies equation with the nonlocal response,

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = d\Delta u(t,x) - \delta u(t,x) + \varepsilon \int_\Omega \Gamma_1(D_i,\tau,x,y) pu(t-\tau,y)e^{-u(t-\tau,y)}\mathrm{d}y, \\ t > 0, x \in \Omega, \\ u(0,\cdot)|_{\partial\Omega} = 0, \end{cases} \tag{3.4}$$

where $d,\ q,\ \tau$ and δ are positive constants, $\varepsilon = e^{-\delta_i\tau}$ with δ_i being the death rate of immature individuals and D_i is the diffusion rate of the immature individuals. The kernel $\Gamma_1(D_i,\tau,x,y)$ is the Green function for the operator $D_i\Delta$ associated with homogeneous Dirichlet boundary condition.

Biologically ε is the probability that a new born individual can survive the maturation period $[0,\tau]$, and $\Gamma_1(D_i,\tau,x,y)$ is the probability that a surviving individual born at location y has moved to location x at maturation (i.e., after τ time units). Therefore, in the case when immature individuals do not move around ($D_i = 0$), $\Gamma_1(0,\tau,x,y)$ reduces to the Dirac delta function centered at x, and (3.4) reduces to the following local equation:

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = d\Delta u(t,x) - \delta u(t,x) + \varepsilon pu(t-\tau,x)e^{-u(t-\tau,x)} \quad t > 0, x \in \Omega, \\ u(0,\cdot)|_{\partial\Omega} = 0. \end{cases} \tag{3.5}$$

When Ω is one dimensional (an interval), $\Gamma_1(D_i,\tau,x,y)$ can be given by a sine series (see, e.g., [10]).

Let $L = L(D_i,t)$ be defined by $[L(D_i,t)(\phi)](x) = \int_\Omega \Gamma_1(D_i,t,x,y)\phi(y)\mathrm{d}y$ for all $(t,x,\phi) \in \mathbb{R}_+ \times \overline{\Omega} \times C^0$. That is, $L(D_i,t)$ is the semigroup on C^0 generated by the operator $D_i\Delta$ under the Dirichlet boundary condition. Thus, $||L(D_i,t)|| \leq 1$ and $L(D_i,t) = e^{\frac{\mu D_i}{d}t}T(\frac{D_i}{d}t)$ for all $t \geq 0$. Note that $L(0,\tau) = Id$. Comparing (3.4) and (2.1), we find that K is given by $[K(\phi)](x) = \frac{[L(D_i,\tau)(\phi)](x)}{||L(D_i,\tau)||}$ for all $(x,\phi) \in \overline{\Omega} \times C^0$, and $f(u) = \frac{p\varepsilon||L(D_i,\tau)||}{\delta}ue^{-u}$ for $u \in [0,\infty)$.

For the above function f, we easily see that $f'(0) = \frac{p\varepsilon||L(D_i,\tau)||}{\delta}$ and f has a positive fixed point given by $u^* = \ln\frac{p\varepsilon||L(D_i,\tau)||}{\delta}$ if and only if $\frac{p\varepsilon||L(D_i,\tau)||}{\delta} > 1$. The next lemma further summarizes some properties of f required by Theorem 2.5.

Lemma 3.3 *For the above* f, *the following statements hold:*

(i) *if* $\frac{p\varepsilon||L(D_i,\tau)||}{\delta} \leq 1$, *then* $f(u) < u$ *for all* $u > 0$;
(ii) *if* $\frac{p\varepsilon||L(D_i,\tau)||}{\delta} > 1$, *then* f *satisfies the assumptions (H1) and (H2);*
(iii) *if* $\frac{p\varepsilon||L(D_i,\tau)||}{\delta} \in (1,e^2]$, *then the assumption (H4) holds with* $u^* = \ln(\frac{p\varepsilon||L(D_i,\tau)||}{\delta})$; *in particular, if* $\frac{p\varepsilon}{\delta} \in (1,e^2]$, *then the assumption (H4) holds for* $D_i = 0$ *with* $u^* = \frac{1}{q}\ln\left(\frac{p\varepsilon}{\delta}\right)$;
(iv) *if* $\frac{p\varepsilon||L(D_i,\tau)||}{\delta} \in (1,e]$, *then the assumption (H5) holds with* $f^* = u^* = \ln(\frac{p\varepsilon||L(D_i,\tau)||}{\delta})$.
(v) *if* $\frac{p\varepsilon||L(D_i,\tau)||}{\delta} \in (e,2e]$, *then the assumption (H5) holds with* $u^* = \ln(\frac{p\varepsilon||L(D_i,\tau)||}{\delta})$ *and* $f^* = \frac{p\varepsilon||L(D_i,\tau)||}{\delta e}$.

Proof (i)–(ii) are obvious and (iv) follows from (iii). To complete the proof of the statements (iii) and (v), we suppose that $\frac{p\varepsilon||L(D_i,\tau)||}{\delta} > 1$. For any given $k^* \in$

$\left\{\ln\left(\frac{p\varepsilon||L(D_i,\tau)||}{\delta}\right), \frac{p\varepsilon||L(D_i,\tau)||}{\delta e}\right\}$, let

$$G(k,u)=\frac{f(ku)}{f(k)}=ue^{k(1-u)} \quad \text{for all} \quad (k,u)\in(0,k^*]\times(0,\infty).$$

Then

$$\frac{\partial G}{\partial k}(k,u)=u(1-u)e^{k(1-u)} \quad \text{for all} \quad (k,u)\in(0,k^*]\times(0,\infty).$$

Thus by the explicit expressions of G and $\frac{\partial G}{\partial k}$, to apply Lemma 3.1, it suffices to check that $\{u>0: G(k^*,G(k^*,u))=u\}=\{1\}$ which is equivalent to $\{u>0: h^2(u)=u\}=\{k^*\}$, where $h(u)=e^{k^*}ue^{-u}$ for all $u\in(0,\infty)$. According to the proof of Theorem 4.2 and Remark 4.3 in [28], we know that $\{u>0: h^2(u)=u\}=\{k^*\}$ if and only if $k^*\in(0,2]$.

For (iii), taking $k^*=\ln(\frac{p\varepsilon||L(D_i,\tau)||}{\delta})$, we have $k^*\in(0,2]$, and thus $\{u>0: G(k^*,G(k^*,u))=u\}=\{1\}$. By Lemma 3.1, the assumption (H4) holds with $u^*=\ln(\frac{p\varepsilon||L(\tau)||}{\delta})$. The conclusion for $D_i=0$ in (iii) follows from the facts that $L(0,\tau)=Id$ and $||L(0,\tau)||=1$.

For (v), taking $k^*=\frac{p\varepsilon||L(d_i,\tau)||}{\delta e}$, we have $k^*\in(1,2]\subseteq(0,2]$, and thus $\{u>0: G(k^*,G(k^*,u))=u\}=\{1\}$. By Lemma 3.1, the assumption (H5) holds with $u^*=\ln(\frac{p\varepsilon||L(D_i,\tau)||}{\delta})$ and $f^*=\frac{p\varepsilon||L(D_i,\tau)||}{\delta e}$. The proof is completed. □

Note that in such a non-local case, the linear operator A is given by $A=d\Delta-\delta Id+\delta f'(0)K=d\Delta-\delta Id+p\varepsilon L(D_i,\tau)$. When $D_i=0$ (local case), we have seen in the proof of Theorem 2.4 that $s(A)=-d\lambda_1-\delta+p\varepsilon$. For the true non-local case, that is, $D_i>0$, the following lemma gives a similar formula for $s(A)$.

Lemma 3.4 *For $A=d\Delta-\delta Id+p\varepsilon L(\tau)$, $s(A)=-d\lambda_1-\delta+p\varepsilon e^{-\lambda_1\alpha}$, where $\alpha=D_1\tau$.*

Proof By the definition of λ_1, there is a unique $\xi_1\in Y_+$ with $||\xi_1||=1$ such that $\Delta(\xi_1)=-\lambda_1\xi_1$. Note $L(D_i,t)$ is the semi-group generated by the linear problem

$$\begin{cases}\frac{\partial u}{\partial t}(t,x)=D_i\Delta u(t,x)\ t>0, \quad x\in\Omega,\\ u(0,\cdot)|_{\partial\Omega}=0.\end{cases} \tag{3.6}$$

That is,

$$[L(D_i,t)\phi](x)=\int_\Omega \Gamma_1(D_i,t,x,y)\phi(y)\,dy.$$

It is easy to see that $e^{-\lambda_1 D_i t}\xi_1(x)$ satisfies (3.6). This implies that

$$\int_\Omega \Gamma_1(D_i,t,x,y)\xi_1(y)\,dy=e^{-\lambda_1 D_i t}\xi_1(x),$$

and particularly,

$$[L(D_i,\tau)\xi_1](x)=\int_\Omega \Gamma_1(D_i,\tau,x,y)\xi_1(y)\,dy=e^{-\lambda_1 D_i\tau}\xi_1(x)=e^{-\lambda_1\alpha}\xi_1(x). \tag{3.7}$$

Therefore,

$$A(\xi_1)=[d\Delta-\delta Id+p\varepsilon L(D_i,\tau)]\xi_1=[-d\lambda_1-\delta+p\varepsilon e^{-\lambda_1\alpha}]\xi_1,$$

meaning that $-d\lambda_1-\delta+p\varepsilon e^{-\lambda_1\alpha}$ is a real eigenvalue of A corresponding to which, there is the unit positive eigenfunction ξ_1. By the uniqueness of the positive eigenfunction in Krein–Rutman theorem and the spectral mapping theorem for semigroups, we conclude that $s(A)=-d\lambda_1-\delta+p\varepsilon e^{-\lambda_1\alpha}$, completing the proof. □

Applying Theorems 2.1 and 2.5, Lemma 3.3(i)–(ii)–(iv)–(v) and Lemma 3.4, we then obtain the following results for the non-local problem (3.4).

Theorem 3.2 *If* $\frac{p\varepsilon||L(D_i,\tau)||}{\delta} \le 1$ *and* $\alpha > 0$, *then the trivial equilibrium* 0 *of (3.4) attracts all solutions of (3.4) with the initial value* $\psi \in X_+$. *If* $\frac{p\varepsilon||L(D_i,\tau)||}{\delta} \in (1, 2e]$, *then the following statements are true:*

(i) If $p\varepsilon e^{-\lambda_1\alpha} \le d\lambda_1 + \delta$, *then the trivial steady state* $u = 0$ *of (3.4) attracts all solutions of (3.4) with the initial value in* X_+;
(ii) if $p\varepsilon e^{-\lambda_1\alpha} > d\lambda_1 + \delta$, *then there exists a positive steady state* u_+ *of (3.4) which attracts all solutions of (3.4) with the initial value in* $X_+ \setminus \{0\}$.

Remark 3.2 As we mentioned in the introduction, Wu–Zhao [22] and Xu–Zhao [24] also obtained some results about the global dynamics of the Dirichlet problem for some nonlocal equations of type (3.4). The main tool in [22] and [24] is the theory of monotone dynamical systems, and hence, monotonicity (under some non-standard ordering) and sublinearity on the nonlocal nonlinear terms played a crucial role. In contrast, we assume an alternative condition (H3) [or (H4) or (H5)] rather than the sublinearity, and our approach is less dependent on ordering, and hence our results in Theorems 2.5 and 3.2 are less demanding on monotonicity, which can be clearly reflected by the condition $\frac{p\varepsilon||L(D_i,\tau)||}{\delta} \in (1, 2e]$. Indeed, we can demonstrate this by considering Theorem 3.2(ii). Assume that $\frac{p\varepsilon||L(D_i,\tau)||}{\delta} \in (1, 2e]$ and $p\varepsilon e^{-1-\lambda_1\alpha} > d\lambda_1 + \delta$, let u_+ be the unique positive steady state. Then, we can show that $u_+^* = \sup_{x\in\Omega} u_+(x) > 1$. Otherwise, assume that $u_+ \le 1$. Let $\xi_1 \in Y_+$ satisfy $\Delta(\xi_1) = -\lambda_1\xi_1$ with $||\xi_1|| = 1$. By a similar argument to that in the proof of Lemma 3.4, we see that $[d\Delta - \delta Id + p\varepsilon e^{-1}L(D_i,\tau)]\xi_1 = [-d\lambda_1 - \delta + p\varepsilon e^{-1-\lambda_1\alpha}]\xi_1$. Choose $\varepsilon_0 > 0$ sufficiently small that that $u_+ \ge \varepsilon_0\xi_1$. Then, straightforward verification shows that $v(t,x) = \varepsilon_0\xi_1(x)e^{(-d\lambda_1-\delta+p\varepsilon e^{-1-\lambda_1\alpha})t}$ satisfies the following equation,

$$\begin{cases} \frac{\partial v}{\partial t}(t,x) = d\Delta v(t,x) - \delta v(t,x) + \varepsilon p e^{-1}\int_\Omega \Gamma(D_i,\tau,x,y)v(t,y)\mathrm{d}y, & t>0, x\in\Omega, \\ v(0,\cdot)|_{\partial\Omega} = 0. \end{cases} \tag{3.8}$$

Note that (3.8) is cooperative and $ue^{-u} \ge ue^{-1}$ for all $u \in [0,1]$. By the comparison theorem (see, e.g., Corollary 5 in [11]), we know that $u(t,x) \triangleq u_+(x) \ge v(t,x)$ for all $(t,x) \in [0,\infty) \times \overline{\Omega}$. Now, in view of $p\varepsilon e^{-1-\lambda_1\alpha} > d\lambda_1 + \delta$, we have $\lim_{t\to\infty} ||v(t,\cdot)|| = \infty$, a contradiction to $u_+ \le 1$. So, $u_+^* > 1$, which implies the range of $u_+(x)$ contains a domain on which the non-local delayed reaction function is not monotone. Thus the main results in Wu–Zhao [22] and Xu–Zhao [24] do not apply in this case, but our results may be applied as is shown in Theorem 3.2.

Replacing Lemma 3.3(iv)–(v) by Lemma 3.3(iii), we obtain the following results for the local ($D_i = 0$) problem (3.5).

Corollary 3.1 *The following statements hold.*

(i) If $\varepsilon p \le d\lambda_1 + \delta$, *then the trivial steady state* $u = 0$ *of (3.5) attracts all solutions of (3.5) with the initial functions in* X_+;
(ii) If $\varepsilon p > d\lambda_1 + \delta$, *then* $u = 0$ *becomes unstable, and (3.5) is persistence and has a positive steady state* u_+. *Moreover, if* $\frac{p\varepsilon}{\delta} \in (1, e^2]$, *the* u_+ *attracts all solutions of (3.5) with the initial functions in* $X_+ \setminus \{0\}$.

Remark 3.3 Clearly, the range $(0, 2e]$ for $\frac{p\varepsilon||L(D_i,\tau)||}{\delta}$ in Theorem 3.2 for the true nonlocal case has been expanded to $(0, e^2]$ for $\frac{p\varepsilon||L(D_i,0)||}{\delta} = \frac{p\varepsilon}{\delta}$. We point out that in [25], the authors also proved that under $\varepsilon p \leq d\lambda_1 + \delta$, $u = 0$ is globally attractive in the sense of $L^2(\Omega))$ norm. Under the condition (ii) in Corollary 3.1, Yang and So [25] also obtained the global attractiveness of the positive steady state u_+ by dividing the spatial domain according to some information from the positive steady state and these sub-domains were treated separately, both in $L^2(\Omega))$ norm and $C(\Omega))$ norm. By modifying some of the arguments in Yi and Zou [27], more precisely, by combining a dynamical system argument with the maximum principle as well as some subtle inequalities, Yi et al. [26] also established the threshold dynamics of the Dirichlet problem (3.5) and thus re-confirmed the existing results for the diffusion Nicholson's blowflies equation in [25]. Our results cover all these previous results.

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