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Dynamics in numerics: on two different finite difference schemes for ODEs[☆]

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Abstract

We compare two finite difference schemes for Kolmogorov type of ordinary differential equations: Euler's scheme (a derivative approximation scheme) and an integral approximation (IA) scheme, from the view point of dynamical systems. Among the topics we investigate are equilibria and their stability, periodic orbits and their stability, and topological chaos of these two resulting nonlinear discrete dynamical systems.
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1. Introduction

Consider the scalar differential equation of the Kolmogorov type

$$\dot{x} = xf(x), \tag{1.1}$$

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where f is C^1 mapping from $\mathbb{R}^+$ into $\mathbb{R}$ with

$$f'(x) < 0 \quad \forall x \in \mathbb{R}^+, \quad f(\bar{x}) = 0 \quad \text{for some } \bar{x} > 0. \tag{1.2}$$

Almost all ODE models for the population growth of single species have the form of (1.1), and from biological point of view, condition (1.2) is typical and standard.

When seeking numerical solutions of (1.1) on the unbounded interval $[0, \infty)$, Euler's finite difference scheme is frequently used, leading to the following difference equation:

$$x_{n+1} = x_n + \mu x_n f(x_n). \tag{1.3}$$

Here, $x_n = x(n\mu)$ with μ being the uniform mesh. It is known that Euler's method is based on the approximation of the derivative $x'(t) \approx [x(t + \mu) - x(t)]/\mu$ for small μ. On the other hand, one can rewrite (1.1) as

$$x(t + \mu) = x(t)\mathrm{e}^{\int_t^{t+\mu} f(x(s))\,\mathrm{d}s}. \tag{1.4}$$

Taking $t = n\mu$ and using the approximation of the integral $\int_t^{t+\mu} f(x(s))\,\mathrm{d}s \approx f(x(t))\mu$ results in an alternative finite difference scheme:

$$x_{n+1} = x_n \mathrm{e}^{\mu f(x_n)}. \tag{1.5}$$

We call (1.3) the Euler's scheme and (1.5) the integral approximation scheme or simply the IA scheme for (1.1). We see that the same ODE can lead to different nonlinear discrete dynamical systems.

Taking the prototype $f(x) = r(1 - x/K)$ in (1.1) gives the well-known logistic equation $\dot{x} = rx(1 - x/K)$, the dynamics of which is quite simple: all positive solutions converge to the positive equilibrium $x = K$. However, the corresponding Euler's scheme is of the form

$$x_{n+1} = x_n(a - bx_n), \tag{1.6}$$

which could demonstrate very complicated long-term behaviour (see, e.g., [9, pp. 41–47]). This simple example warns that one has to be careful when choosing a numeric scheme for a given ordinary differential equation.

An immediate observation is that the interval $[0, \infty)$ is invariant under (1.5), but is not under (1.3). On the other hand, in numerical analysis, it is known that approximations of integrals is *generally* more effective than approximations of derivatives. This makes one wonder if the IA scheme is any better than the Euler's scheme in terms of the long-term behaviour of the solution sequences of (1.3) and (1.5), comparing with the solution of (1.1). Thus, it is interesting and worthwhile to compare these two schemes from the view point of dynamical systems, and this is the right purpose of this paper. Among the topics of investigation are equilibria and their stability (local and global), periodic doubling bifurcations and their stability, and topological chaos, for the two nonlinear discrete dynamical systems (1.3) and (1.5) resulted from the same differential equation (1.1). The topic of chaos is especially worth exploring, since although there is no chaos in one or two-dimensional ordinary differential systems (ODEs), for discrete dynamical systems, chaos can occur even in one-dimensional systems. Indeed, it has been shown that the Euler's scheme sometimes exhibits a very complicated dynamical behaviour [11]. For the chaotic behaviour of Euler's finite difference scheme for high-dimensional ODEs, see [7].

2. Local and global stability

For convenience, let

$$F_\mu(x) = x(1 + \mu f(x)), \quad G_\mu(x) = x\mathrm{e}^{\mu f(x)}. \tag{2.1}$$

We will discuss the dynamics of iterates of F and G, respectively.

Under the assumption (1.2), F and G have two fixed points $x = 0$ and $\bar{x}$. A direct calculation shows that

$$F'_\mu(0) = 1 + \mu f(0) > 1, \quad G'_\mu(0) = \mathrm{e}^{\mu f(0)} > 1 \tag{2.2}$$

and

$$F'_\mu(\bar{x}) = 1 + \mu\bar{x} f'(\bar{x}), \quad G'_\mu(\bar{x}) = 1 + \mu\bar{x} f'(\bar{x}). \tag{2.3}$$

Thus, we immediately have the following local stability result.

Theorem 2.1. (i) *Under the assumption* (1.2), *the fixed point* 0 *is a repeller of* F_μ *and* G_μ *for all* $\mu > 0$; (ii) *The positive fixed point* $\bar{x}$ *is locally stable for both* F_μ *and* G_μ *if* $0 < \mu < \mu_0$ *and unstable if* $\mu > \mu_0$, *where*

$$\mu_0 = -\frac{2}{\bar{x} f'(\bar{x})}. \tag{2.4}$$

In general, global stability is more demanding, and it is usually in respect to some invariant set of F_μ and G_μ. A typical condition to ensure an invariant set for maps is the so called unimodal property. The following lemma addresses this property for F_μ and G_μ.

Lemma 2.2. *In addition to* (1.2), *assume that there exists a constant* $\delta_0 > 0$ *and* $x_0 > 0$ *such that*

$$f'(x) \leqslant -\frac{1}{\delta_0} \quad \forall x > x_0 \quad \textit{and} \quad f''(x) \leqslant 0 \quad \textit{in} \ [0, \infty). \tag{2.5}$$

Then for any $\mu > 0$, F_μ *and* G_μ *are unimodal on* $\mathbb{R}^+$. *That is, for any* $\mu > 0$, *there exist* $x_{m1}(\mu) > 0$ *and* $x_{m2}(\mu) > 0$ *such that* $F_\mu(x)$ *and* $G_\mu(x)$ *increase on* $[0, x_{m1}(\mu)]$ *and* $[0, x_{m2}(\mu)]$, *respectively, and decrease on* $[x_{m1}(\mu), +\infty)$ *and* $[x_{m2}(\mu), +\infty)$, *respectively*.

Proof. We first compute the derivatives of F_μ and $G_\mu(x)$ as below

$$F'_\mu(x) = 1 + \mu f(x) + \mu x f'(x), \tag{2.6}$$

$$G'_\mu(x) = \mathrm{e}^{\mu f(x)}[1 + \mu x f'(x)]. \tag{2.7}$$

By (1.2), (2.5) and (2.6), we know that there exists $x_{m1}(\mu) > 0$ such that $F'_\mu(x) > 0$ for $x < x_{m1}(\mu)$, and $F'_\mu(x) < 0$ for $x > x_{m1}(\mu)$, giving the unimodal property of $F'_\mu(x)$ on $\mathbb{R}^+$. Similarly, from (1.2), (2.5) and (2.7), it follows that there exists $x_{m2}(\mu) > 0$ such that $G'_\mu(x) > 0$ for $x < x_{m2}(\mu)$, and $G'_\mu(x) < 0$ for $x > x_{m2}(\mu)$, implying that $G_\mu(x)$ is also unimodal on $\mathbb{R}^+$. This completes the proof. □

Denote by $M_1(\mu)$ and $M_2(\mu)$ the maximum values of $F_\mu(x)$ and $G_\mu(x)$ on $\mathbb{R}^+$, respectively. By the proof of Lemma 2.2, we know

$$M_1(\mu) = F_\mu(x_{m1}(\mu)), \quad M_2(\mu) = G_\mu(x_{m2}(\mu)), \tag{2.8}$$

where the maximum points $x_{m1}(\mu)$ and $x_{m2}(\mu)$ satisfy, respectively, the following equations:

$$f(x_{m1}(\mu)) + x_{m1}(\mu) f'(x_{m1}(\mu)) = -\frac{1}{\mu}, \quad x_{m2}(\mu) f'(x_{m2}(\mu)) = -\frac{1}{\mu}. \tag{2.9}$$

Let

$$h_1(x) \triangleq f(x) + xf'(x), \quad h_2(x) \triangleq xf'(x).$$

Then, by (2.5), $h_i(x)$, $i = 1, 2$, are decreasing functions. Hence, as functions of μ, $x_{m1}(\mu)$ and $x_{m2}(\mu)$ are decreasing. Therefore, we have

Lemma 2.3. *Under the assumptions of Lemma* 2.2, *let* $\mu_1 = -\frac{1}{\bar{x} f'(\bar{x})}$, *then*

$$\bar{x} \leqslant x_{mi}(\mu), \quad i = 1, 2 \tag{2.10}$$

if and only if $0 < \mu \leqslant \mu_1$.

Proof. The conclusion follows from the above observation on $h_i(x)$, $i = 1, 2$, and the fact that

$$h_i(\bar{x}) = \bar{x} f'(\bar{x}) = -\frac{1}{\mu_1}, \quad i = 1, 2. \qquad \square$$

For any $\mu > 0$, let $x_F(\mu)$ be the unique positive solution of the equation

$$1 + \mu f(x) = 0.$$

Lemma 2.4. *Under the assumptions of Lemma* 2.2, *we have*

(i) *for* $i = 1, 2$, $M_i(\mu)$ *is monotonically increasing and continuously differentiable on* $\mu \in [\mu_1, +\infty)$, *and*

$$M_i(\mu_1) = \bar{x}, \quad \lim_{\mu \to +\infty} M_i(\mu) = +\infty;$$

(ii) $x_F(\mu)$ *is monotonically decreasing and continuously differentiable on* $\mu > 0$, *and*

$$x_F(0) = +\infty, \quad \lim_{\mu \to +\infty} x_F(\mu) = \bar{x}.$$

Proof. Let $M_i'(\mu)$ denote the derivative of $M_i(\mu)$ with respect to μ, $i = 1, 2$. It follows from Lemma 2.3 that for $\mu > \mu_1$, $x_{mi}(\mu) < \bar{x}$. Hence, for such $\mu > \mu_1$

$$M_1'(\mu) = x_{m1}(\mu) f(x_{m1}(\mu)) > 0, \quad M_2'(\mu) = x_{m2}(\mu) f(x_{m2}(\mu)) \mathrm{e}^{\mu f(x_{m2}(\mu))} > 0. \tag{2.11}$$

So (i) follows. The proof of (ii) is similar. $\square$

By Lemma 2.4, there exists a unique $\mu_2 > \mu_1$ such that

$$M_1(\mu_2) = x_F(\mu_2). \tag{2.12}$$

In terms of μ_1 and μ_2, we have the following results on invariant sets.

Lemma 2.5. *Under the assumptions of Lemma* 2.2, *we have the following*:

(i) *For* $0 < \mu < \mu_1$, F_μ *has an invariant interval* $[0, x_F(\mu)]$. *For* $\mu_1 < \mu < \mu_2$, F_μ *has two invariant intervals* $[0, x_F(\mu)]$ *and* $I_1 \triangleq [F_\mu(M_1(\mu)), M_1(\mu)]$. *Furthermore,* I_1 *is an absorbing interval in the sense that for any* $x \in (0, x_F(\mu)]$, *there exists an integer* $k > 0$ *such that* $F_\mu^k(x) \in I_1$.

(ii) *For all* $\mu > 0$, $\mathbb{R}^+$ *is always invariant under* G_μ, *while for* $\mu > \mu_1$, $I_2 \triangleq [G_\mu(M_2(\mu)), M_2(\mu)]$ *is also invariant under* G_μ *and* I_2 *is an absorbing interval in the sense that for any* $x > 0$, *there exists an integer* $k > 0$ *such that* $G_\mu^k(x) \in I_2$.

From the above lemmas, we immediately obtain the following results on the global stability of the maps F_μ and G_μ.

Theorem 2.6. *Under the assumptions of Lemma* 2.2, *if* $0 < \mu \leqslant \mu_1$, *then*

(i) F_μ *is global asymptotically stable on* $(0, x_F(\mu)]$, *that is, for any* $x \in (0, x_F(\mu)]$, $F_\mu^n(x)$ *converges to* $\bar{x}$ *as* $n \to \infty$, *and the convergence is eventually monotone*.

(ii) G_μ *is global asymptotically stable on* $\mathbb{R}^+$, *that is, for any* $x > 0$, $G_\mu^n(x)$ *converges to* $\bar{x}$ *as* $n \to \infty$, *and the convergence is eventually monotone*.

Note that in the above theorem, while we can only establish the convergence of $F_\mu^n(x)$ for $x \in (0, x_F(\mu)]$, the convergence of $G_\mu^n(x)$ is for all $x \in \mathbb{R}^+$, showing an advantage of the IA scheme.

Next, we consider $\mu > \mu_1$. Obviously, $\bar{x} \leqslant M_1(\mu) < M_2(\mu)$ in this case. Define

$$g_i(\mu) \triangleq \mu M_i(\mu) f'(M_i(\mu)) \quad \text{for} \ \ \mu \in [\mu_1, +\infty), \qquad i = 1, 2. \tag{2.13}$$

Then it is easily verified that

$$g_i(\mu_1) = -1, \qquad \lim_{\mu \to \infty} g_i(\mu) = -\infty,$$

$$g_i(\mu_0) < \mu_0 \bar{x} f'(\bar{x}) = -2, \qquad g_1(\mu) > g_2(\mu),$$

$$g_i'(\mu) = [M_i(\mu) + \mu M_i'(\mu)] f'(M_i(\mu)) + \mu M_i(\mu) f''(M_i(\mu)) M_i'(\mu) < 0.$$

Thus, there exist constants μ_F and μ_G with $\mu_1 < \mu_G \leqslant \mu_F < \mu_0$ such that

$$g_1(\mu_F) = -2, \qquad g_2(\mu_G) = -2. \tag{2.14}$$

Let

$$\mu_F^0 = \min\{\mu_F, \mu_2\}, \tag{2.15}$$

where μ_2 is given by (2.12).

Theorem 2.7. *Under the assumptions of Lemma* 2.2 *and assuming* $f \in C^2$, *the following conclusions hold*:

(i) *If* $\mu_1 < \mu < \mu_F^0$, *then for any* $x \in (0, x_F(\mu))$, $F_\mu^n(x)$ *also converges to* $\bar{x}$ *as* $n \to \infty$, *but the convergence is eventually oscillatory around* $\bar{x}$.
(ii) *If* $\mu_1 < \mu < \mu_G$, *then for any* $x > 0$, $G_\mu^n(x)$ *also converges to* $\bar{x}$ *as* $n \to \infty$ *but the convergence is eventually oscillatory around* $\bar{x}$.

Here μ_F^0 *and* μ_G *are defined as in* (2.14) *and* (2.15).

Proof. When $\mu_1 < \mu < \mu_2$, from Lemma 2.5, I_1 and I_2 are absorbing interval of F_μ and G_μ, respectively. It suffices to prove the theorem for $x \in I_1$ and I_2, respectively.

For (i), fix $x \in I_1 = [F_\mu(M_1(\mu)), M_1(\mu)]$. Taking the Liapunov function

$$V(x_n) = (x_n - \bar{x})^2,$$

where $x_n = F_\mu^n(x)$, we have

$$\begin{aligned} V(x_{n+1}) - V(x_n) &= (x_{n+1} - \bar{x})^2 - (x_n - \bar{x})^2 \\ &= (x_{n+1} - x_n)(x_{n+1} + x_n - 2\bar{x}) \\ &= \mu x_n f(x_n)(\mu x_n (f(x_n) - f(\bar{x})) + 2(x_n - \bar{x})) \\ &= \mu x_n f(x_n)(x_n - \bar{x})(\mu x_n f'(\xi_n) + 2), \end{aligned}$$

where ξ_n is in between $\bar{x}$ and x_n. Since

$$\mu x_n f(x_n)(x_n - \bar{x}) \leqslant 0,$$

we see that

$$V(x_{n+1}) - V(x_n) \leqslant 0,$$

provided that

$$\mu x_n f'(\xi_n) + 2 > 0.$$

Since $x_n \in I_1$ by the invariant of I_1, a sufficient condition for the last equation to hold is

$$g_1(\mu) \triangleq \mu M_1(\mu) f'(M_1(\mu)) > -2.$$

Thus, if $\mu_1 < \mu < \mu_F^0$, then

$$V(x_{n+1}) \leqslant V(x_n).$$

That is, $\{V(x_n)\}$ is a nonincreasing and bounded sequence, thus it converges. Let

$$\lim_{n \to \infty} V(x_n) = A.$$

We now claim that $A = 0$. Indeed, if $A > 0$, then there exists a integer N such that for any $n > N$, we have

$$|x_n - \bar{x}| > \sqrt{A/2}.$$

Since $x_n \in I_1$, from the above inequalities and (2.5), for $n > N$, we have

$$\begin{aligned} &V(x_{n+1}) - V(x_n) \\ &\quad = \mu x_n f(x_n)(x_n - \bar{x})(\mu x_n f'(\xi_n) + 2) \\ &\quad \leqslant \mu x_n \max\{-f(\bar{x} - \sqrt{A/2}),\ f(\bar{x} + \sqrt{A/2})\}\sqrt{A/2}(\mu x_n f'(\xi_n) + 2) \\ &\quad \leqslant \mu F_\mu(M_1(\mu)) \max\{-f(\bar{x} - \sqrt{A/2}),\ f(\bar{x} + \sqrt{A/2})\}\sqrt{A/2}(\mu M_1(\mu) f'(M_1(\mu)) + 2). \end{aligned}$$

Letting $n \to \infty$, we have

$$0 < \mu F_\mu(M_1(\mu)) \max\{-f(\bar{x} - \sqrt{A/2}),\ f(\bar{x} + \sqrt{A/2})\}\sqrt{A/2}(\mu M_1(\mu) f'(M_1(\mu)) + 2) < 0$$

for $\mu_1 < \mu < \mu_F^0$. A contradiction. This shows that $A = 0$, and so $x_n \to \bar{x}$ as $n \to \infty$.

On the other hand, $\mu > \mu_1$ guarantees that there exists a neighbourhood $(\bar{x} - \varepsilon_0, \bar{x} + \varepsilon_0)$ of $\bar{x}$ on which $F_\mu(x)$ is decreasing, and thus, $F_\mu(x) > \bar{x}$ if $x \in (\bar{x} - \varepsilon_0, \bar{x})$ and $F_\mu(x) < \bar{x}$ if $x \in (\bar{x}, \bar{x} + \varepsilon_0)$. Therefore, the convergence of x_n to $\bar{x}$ is in an eventually oscillatory way $n \to \infty$.

For (ii), we take the Liapunov function

$$V(x_n) = (\ln x_n - \ln \bar{x})^2.$$

Then

$$\begin{aligned} V(x_{n+1}) - V(x_n) &= (\ln x_{n+1} - \ln x_n)(\ln x_{n+1} + \ln x_n - 2\ln \bar{x}) \\ &= \mu f(x_n)[2(\ln x_n - \ln \bar{x}) + \mu(f(e^{\ln x_n}) - f(e^{\ln \bar{x}}))] \\ &= \mu f(x_n)[2(\ln x_n - \ln \bar{x}) + \mu e^{\zeta_n} f(e^{\zeta_n})(\ln x_n - \ln \bar{x})] \\ &= \mu f(x_n)(\ln x_n - \ln \bar{x})[2 + \mu \xi_n f'(\xi_n)], \end{aligned}$$

since $f(e^{\ln \bar{x}}) = f(\bar{x}) = 0$, where ζ_n is in between $\ln \bar{x}$ and $\ln x_n$ and so $\xi_n = e^{\zeta_n}$ is in between $\bar{x}$ and x_n.

The rest of the proof is in a way similar to the proof of (i). □

This theorem identifies the ranges of mesh μ within which, the Euler's scheme and the IA scheme each converges to the positive fixed point $\bar{x}$, in eventually oscillatory way. Since $\mu_G \leqslant \mu_F \leqslant \mu_F^0$, the Euler's scheme is better than IA scheme if the global stability of $\bar{x}$ is concerned. However, the basin for the former is only a finite interval $(0, x_\mu]$ while the basin for the later is the infinite interval $\mathbb{R}^+$.

3. Period doubling bifurcation and negative Schwarzian derivative

In Section 2, we have seen that the range of the mesh μ for local stability of the positive fixed point $\bar{x}$ is the same for both F_μ and G_μ (the range for global stability may be different though). In this section, we will see that for a class of f, there is essentially difference between F_μ and G_μ on the period doubling bifurcation, as well on the sign of Schwarzian derivative which plays an important role in one-dimensional discrete dynamical systems.

Firstly, let us discuss period doubling bifurcation as the parameter μ exceed the critical value μ_0, which is given by (2.4). To this end, we need the following lemma from [10].

Lemma 3.1 (*Robinson [10, p. 246, Theorem 3.1]*). *Assume that $g : \mathbb{R}^2 \to \mathbb{R}$ is a C^r function jointly in both variable with $r \geqslant 3$, and that g satisfies the following conditions*:

(i) *The point x_0 is a fixed point for $s = s_0$: $g(x_0, s_0) = x_0$.*
(ii) $\frac{\partial g}{\partial x}(x_0, s_0) = -1$.
(iii)

$$\alpha \triangleq \left[\frac{\partial^2 g}{\partial s \partial x} + \frac{1}{2}\left(\frac{\partial g}{\partial s}\right)\left(\frac{\partial^2 g}{\partial x^2}\right)\right]_{(x_0, s_0)} < 0.$$

(iv)

$$\beta \triangleq \left(\frac{1}{6}\frac{\partial^3 g}{\partial x^3}(x_0, s_0)\right) + \left(\frac{1}{2}\frac{\partial^2 g}{\partial x^2}(x_0, s_0)\right)^2 \neq 0.$$

Then, there is a period doubling bifurcation at (x_0, s_0). More specifically, there is a periodic orbit with period 2 when s is in a small right neighbourhood of s_0. The stability type of the period 2 orbit depends on the sign of β: the period 2 orbit is attracting if $\beta > 0$; and it is repelling if $\beta < 0$.

For F_μ and G_μ, we can establish the following.

Theorem 3.2. *Let $f(x)$ be C^3 such that $f(\bar{x}) = 0$, $f'(\bar{x}) < 0$ for some $\bar{x} > 0$ and $\mu_0 = -\frac{2}{\bar{x} f'(\bar{x})}$. Then*

$$F'_{\mu_0}(\bar{x}) = -1, \quad G'_{\mu_0}(\bar{x}) = -1, \tag{3.1}$$

$$\alpha_F = \alpha_G = \bar{x} f'(\bar{x}) < 0, \tag{3.2}$$

$$\beta_F = -\frac{1}{\alpha_F}\left(f''(\bar{x}) + \frac{1}{3}\bar{x} f'''(\bar{x})\right) + \frac{1}{\alpha_F^2}(2f'(\bar{x}) + \bar{x} f''(\bar{x}))^2, \tag{3.3}$$

$$\beta_G = \beta_F + \mu_0 f''(\bar{x}) - \frac{10}{3\bar{x}^2}. \tag{3.4}$$

Here α_F, α_G, β_F and β_G, are defined as in Lemma 3.1 but are, instead of g, for F and G respectively.

Proof. (3.1) is a direct result of (2.6)–(2.7) and the definition of μ_0. It is also easily seen

$$\left.\frac{\partial F_\mu}{\partial \mu}\right|_{(\bar{x}, \mu_0)} = \left.\frac{\partial G_\mu}{\partial \mu}\right|_{(\bar{x}, \mu_0)} = \bar{x} f(\bar{x}) = 0.$$

and

$$\left.\frac{\partial^2 F_\mu(x)}{\partial \mu \partial x}\right|_{(\bar{x}, \mu_0)} = \bar{x} f'(\bar{x}) = \left.\frac{\partial^2 G_\mu(x)}{\partial \mu \partial x}\right|_{(\bar{x}, \mu_0)}.$$

So, we have $\alpha_F = \alpha_G = \bar{x} f'(\bar{x}) < 0$. A direct calculation shows that

$$F'''_\mu(x)|_{(\bar{x}, \mu_0)} = 3\mu_0 f''(\bar{x}) + \mu_0 \bar{x} f'''(\bar{x}),$$

$$G'''_\mu(x)|_{(\bar{x}, \mu_0)} = 3\mu_0 f''(\bar{x}) + \mu_0 \bar{x} f'''(\bar{x}) + 3\mu_0^2 (f'(\bar{x}))^2 + 3\mu_0^2 \bar{x} f'(\bar{x}) f''(\bar{x}) + \bar{x}(\mu_0 f'(\bar{x}))^3.$$

A routine check combined with (2.6)–(2.7) can show (3.3) and (3.4). □

Lemma 3.1 and Theorem 3.2 show that, F_μ and G_μ have period doubling bifurcation at the same parameter value $\mu=\mu_0$, and the stability of the respective periodic 2 orbits of F_μ and G_μ are determined by the signs of β_F and β_G which are given by the convenient formulas (3.3) and (3.4). From these formulas, we see that if $f''(x)<0$ then $\beta_G<\beta_F$, and thus the Euler scheme is better than IA scheme. But in general we cannot make comparison about the stability of the bifurcated periodic 2 solutions of F_μ and G_μ, as is numerically illustrated in the following example.

Example 3.1. Consider

$$f(x)=(1-x)+a(1-x)^2+b(1-x)^3, \tag{3.5}$$

where a and b are nonnegative positive constants. Then we have $\bar{x}=1$ and

$$f'(1)=-1, \quad f''(1)=2a, \quad f'''(1)=-6b,$$

$$\mu_0=2, \quad \alpha_F=\alpha_G=-1.$$

By (3.2) and (3.3), we obtain

$$\beta_F=2(a-b)+4(a-1)^2,$$

$$\beta_G=\beta_F+4a-\tfrac{10}{3}.$$

Consider the following cases for the parameters a and b:

(i) $a=0$ and $b=1$. In this case, $\beta_F=2>0$ and $\beta_G=2-\frac{10}{3}=-\frac{4}{3}<0$.
(ii) $a=1$ and $b=2$. In this case, $\beta_F=-2<0$ and $\beta_G=\frac{-4}{3}<0$.
(iii) $a=b=2$. In this case, $\beta_F=4>0$ and $\beta_G=12-\frac{10}{3}>0$.
(iv) $a=1$ and $b=1.1$. In this case, $\beta_F=-0.2<0$ and $\beta_G=\frac{7}{15}>0$.

The negative Schwarzian derivative condition is a much more subtle property and it provides a powerful tool in one-dimensional dynamics. There are many theorems which are proved only for maps with negative Schwarzian derivatives (see, e.g., [8]). We quote below some results for C^1-unimodal maps with negative Schwarzian derivative, which will be used later.

Assume that g is C^3. The *Schwarzian derivative* of g at x, denote by $Sg(x)$ is defined

$$Sg(x)=\frac{g'''(x)}{g'(x)}-\frac{2}{3}\left(\frac{g''(x)}{g'(x)}\right)^2. \tag{3.6}$$

Definition 3.1. We call g a S-unimodal map if

(S_1) g is C^1-unimodal. That is, $g:[a,b]\to[a,b]$ is continuously differentiable and there exists $a<c<b$ such that $g'(x)>0$ if $a<x<c$ and $g'(x)<0$ if $c<x<b$.
(S_2) g is third-order continuously differentiable.
(S_3) $Sg(x)<0$ for all $x\in[a,b]$ (here we allow the value $-\infty$ for $Sg(x)$ at $x=c$).
(S_4) $[g(b),b]$ is invariant under g.
(S_5) $g''(c)<0$.

The following two lemmas are from [4].

Lemma 3.3 (*Collet and Eckmann [4, p. 105, Lemma II.4.6]*). *If g is a polynomial of degree large than or equal to* 2 *and all zeros of* g' *are real then* $Sg < 0$.

Lemma 3.4 (*Collet and Eckmann [4, p. 95, Corollary II.4.2; p. 119, Proposition II.5.7]*). *Assume that g is S-unimodal, then*

(i) *g has at most one stable periodic orbit, plus possibly a stable fixed point in the interval* $[a, g(b))$.
(ii) *If g has a stable period orbit, then*

$$\mathscr{L}(E_g) = 0, \tag{3.7}$$

where $E_g = \{x \in [a, b] \mid g^n(x)$ *does not tend to the stable periodic orbit of g*$\}$, *and* $\mathscr{L}$ *is Lebesgue measure on* $\mathbb{R}$.

Roughly speaking, Lemma 3.4(ii) implies that for a S-unimodal map, a local stable periodic orbit is also almost globally stable.

Concerning the Schwarzian derivatives of F_μ and G_μ, an observation is that for general f the corresponding F_μ and G_μ may have different signs for their Schwarzian derivatives. For example, let

$$f(x) = s - x$$

with $s > 0$, then

$$F_\mu(x) = x(1 + \mu(s - x))$$

is a quadratic map, which, by Lemma 3.3, has negative Schwarzian derivative for all $\mu > 0$ and $s > 0$. On the other hand, for this f the corresponding G_μ is given by

$$G_\mu = x\mathrm{e}^{\mu(s-x)}$$

and a direct computation shows that its Schwarzian derivative is

$$SG_\mu(x) = \frac{\mu^2(3 - \mu x)}{1 - \mu x} - \frac{2}{3}\left(\frac{\mu(2 - \mu x)}{1 - \mu x}\right)^2 = \frac{\mu^2}{3(1 - \mu x)^2}((\mu x - 2)^2 - 3). \tag{3.8}$$

Obviously, $SG_\mu(0) = \mu^2/3$, and thus $SG_\mu(x)$ is also positive in the right neighbourhood of $x = 0$.

To show the feasibility of the results established above, in the rest of this section, we consider the following particular ODE.

$$\dot{x} = x(1 - x^2). \tag{3.9}$$

For this equation, the corresponding Euler's difference scheme is

$$x_{n+1} = x_n(1 + \mu(1 - x_n^2)) \triangleq F_\mu(x_n) \tag{3.10}$$

and the IA difference scheme becomes

$$x_{n+1} = x_n \mathrm{e}^{\mu(1-x_n^2)} \triangleq G_\mu(x_n). \tag{3.11}$$

For (3.10), since

$$F'_\mu(x) = 1 + \mu - 3\mu x^2,$$

has two real zeros, we have

$$SF_\mu < 0 \quad \forall x \geqslant 0,$$

by Lemma 3.3. On the other hand, the positive zero of $F_\mu(x) = 0$ is

$$x_F(\mu) = \sqrt{\frac{1+\mu}{\mu}}, \tag{3.12}$$

the maximum point is

$$x_m(\mu) = \sqrt{\frac{1+\mu}{3\mu}} \tag{3.13}$$

and the corresponding maximum value of F_μ is

$$M_1(\mu) = \frac{2}{3}(\mu+1)\sqrt{\frac{1+\mu}{3\mu}}. \tag{3.14}$$

Calculation shows

$$\mu_0 = 1, \quad \mu_1 = \frac{1}{2}, \quad \mu_F = \frac{3}{4^{1/3}} - 1 \approx 0.89, \quad \mu_2 = \frac{3\sqrt{3}}{2} - 1 \approx 1.59.$$

By these values and the results above, we can summarize the dynamics of (3.10) as below.

(A1) If $0 < \mu \leqslant \mu_1 = -\frac{1}{\bar{x} f'(\bar{x})} = \frac{1}{2}$, then $1 \leqslant x_m(\mu)$ and $F_\mu^n(x)$ tends, in an eventually monotone way, to the fixed point $\bar{x} = 1$ as $n \to \infty$ for any $x \in (0, x_F(\mu))$, by Theorem 2.6.

(A2) If $\mu_1 = 0.5 < \mu < \mu_F^o = \min\{\mu_F, \mu_2\} \approx 0.89$, for any $(0, x_F(\mu), F_\mu^n(x))$ converges, in an eventually oscillatory way, to $\bar{x} = 1$, by Theorem 2.7.

(A3) Finally, by Theorem 3.2, F_μ has a period doubling bifurcation at $(\bar{x}, \mu_0) = (1, 1)$ and, according to (3.2)

$$\beta_F = 8 > 0.$$

It follows from Lemma 3.1 that the period 2 orbit bifurcated from $\bar{x} = 1$ is stable. On the other hand, it is easy to check that $F_\mu(x)$ is S-unimodal on $[0, x_F(\mu)]$. By Lemma 3.4, there exists a $\mu_F^1 > 1$ such that $F_\mu^n(x)$ tends to the stable period 2 orbit as $n \to \infty$ for almost all $x \in [0, x_F(\mu)]$ if $1 < \mu < \mu_F^1$.

We remark that there is a gap for μ between $\mu_F^o = 0.89$ and the critical value $\mu = 1$ for periodic bifurcation. For general f, one may not be able to address the dynamics of $F_\mu(x)$ when μ is in this range. However for this particular f, one can. Indeed, for all $\mu \in (0.5, 1)$, we have $-1 < F'_\mu(1) < 0$. By

$$F^2_\mu(x_m(\mu)) = F_\mu(M_1(\mu)) = \frac{2}{3}(\mu+1)x_m(\mu)\left(1+\mu\left(1-\frac{4}{9}\,\frac{(\mu+1)^3}{3\mu}\right)\right)$$
$$= \frac{2}{81}x_m(\mu)(\mu+1)(23-4\mu^3-12\mu^2+15\mu),$$

we see that

$$F^2_\mu(x_m(\mu)) > x_m(\mu), \tag{3.15}$$

provided that

$$h(\mu) \triangleq \tfrac{2}{81}(\mu+1)(23-4\mu^3-12\mu^2+15\mu) > 1. \tag{3.16}$$

Since

$$h''(\mu) = -\tfrac{12}{81}(8\mu^2+16\mu-1) < 0$$

for $\mu \in [0.5, 1]$, and $h(0.5) = 1$, $h(1) = \frac{88}{81} > 1$, we do have $h(\mu) > 1$ for all $\mu \in (0.5, 1)$, and thus (3.15) holds. Therefore, if $0.5 < \mu < 1$, $F^n_\mu(x)$ tends, in an eventually oscillatory way, to the fixed point $\bar{x} = 1$ for all $x \in (0, x_F(\mu))$.

For (3.11), in a similar way we can summarize the corresponding results as below.

(B1) If $0 < \mu \leqslant \frac{1}{2} = \mu_1$, then $G^n_\mu(x)$ tends in an eventually monotone way, to $\bar{x} = 1$ as $n \to \infty$ for any $x > 0$.

(B2) If $\frac{1}{2} < \mu < 1 = \mu_0$, then $G^n_\mu(x)$ tends, in an eventually oscillatory way, to $\bar{x} = 1$ as $n \to \infty$ for any $x > 0$.

(B3) $G_\mu(x)$ has a period doubling bifurcation at $(\bar{x}, \mu_0) = (1, 1)$ and by (3.4)

$$\beta_G = \beta_F + \mu_0 f''(\bar{x}) - \frac{10}{3\bar{x}^2} = 8 - 2 - \frac{10}{3} > 0.$$

Hence the corresponding period 2 orbit is stable. In order to apply Lemma 3.4, we examine the sign of the Schwarzian derivative of $G_\mu(x)$. From (3.6), G_μ has negative Schwarzian derivative iff

$$3G'_\mu(x)G'''_\mu(x) - 2(G''_\mu(x))^2 < 0.$$

A direct computation shows that

$$3G'_\mu(x)G'''_\mu(x) - 2(G''_\mu(x))^2 = 2\mu e^{2\mu(1-x^2)}(8\mu^3x^5 - 36\mu^2x^4 + 18\mu x^2 - 9)$$
$$= 2\mu e^{2\mu(1-x^2)}(8y^3 - 36y^2 + 18y - 9)$$
$$\triangleq h(y),$$

where $y = \mu x^2$. Thus if $0 \leqslant y \leqslant 4$, that is, if

$$0 \leqslant x \leqslant \frac{2}{\sqrt{\mu}}, \tag{3.17}$$

then $SG_\mu(x) < 0$. Therefore, again from Lemma 3.4, there exists a $\mu_G^0 > 1$ such that when $1 < \mu < \mu_G^0$, $G_\mu^n(x)$ tends to the stable period 2 orbit as $n \to \infty$, for almost all $x \in (0, \frac{2}{\sqrt{\mu}}]$. Note that the interval $(0, \frac{2}{\sqrt{\mu}}]$ attracts all $x > 0$ after one iteration, we conclude that, indeed for almost all $x > 0$, $G_\mu^n(x)$ converges to the stable period 2 orbit as $n \to \infty$.

4. Topological chaos

In this section, we will discuss the chaotic behaviour of F_μ and G_μ for μ large enough. To this end, we need some definitions from one-dimensional discrete dynamical systems.

Let I be a compact interval in $\mathbb{R}$ and g be a continuous map from I into itself. The definitions of fixed points and periodic points are in the usual sense. To study the nature of orbits which are not periodic, we define a "scrambled" set. A set $S \subset I$ is called a *scrambled set* if it possesses the following two properties:

(i) If $x, y \in S$ with $x \neq y$, then

$$\limsup_{n\to\infty} |g^n(x) - g^n(y)| > 0, \quad \liminf_{n\to\infty} |g^n(x) - g^n(y)| = 0.$$

(ii) If $x \in S$ and y is any periodic point of g,

$$\limsup_{n\to\infty} |g^n(x) - g^n(y)| > 0.$$

Thus, orbits starting from points in a scrambled set are not even asymptotically periodic. Moreover, for any pair of initial points in the scrambled set, the orbits move apart and return close to each other infinitely often.

As we known, there are many different notions for describing the dynamical complexity of a dynamical system, such as Li–Yorke's chaos [6], Devaney's chaos [5], positive Liapunov exponent, etc., each reflecting its own background. Another mathematical concept that may make the notion of chaos more precise is the "topological entropy" which is a kind of quantitative measurement of chaos. Topological entropy was first introduced in 1960s by Adler et al. [1] for a compact dynamical system. Later in 1970s, Bowen [3] gave a new but equivalent definition for a uniformly continuous map on a (not necessarily compact) metric space. In this section, we will say that a map g exhibits *topological chaos* if g has positive topological entropy. For details on topological chaos and other notions of chaos, we refer, for example, to [2,12], where the following equivalent statements can be found.

Lemma 4.1. *Let g be a continuous map from I into itself. Then the following condition are equivalent.*

(i) *g has a periodic point whose period is not a power of* 2.
(ii) *g exhibits topological chaos*.

Furthermore, each of the above conditions implies that g has an uncountable scrambled set $S \subset I$.

From the above preparation, we see that in order to apply the above results on chaos to explore the topological chaos of F_μ and G_μ, the differentiability of f is not needed. However, in order to be consistent and for convenience, we keep this condition and adopt all the assumptions and notations from the previous sections.

By (2.14), we know that

$$\mu_2 \triangleq \sup\{\mu \geqslant 0 \mid M_1(\mu) - x_F(\mu) \leqslant 0\} \tag{4.1}$$

and Lemma 2.5 shows that the interval $[0, x_F(\mu)]$ is invariant under F_μ when $0 < \mu < \mu_2$. The next theorem describe the chaotic dynamics of F_μ when $\mu > \mu_2$.

Theorem 4.2. *Assume that f satisfies* (1.2). *If* $\mu > \mu_2$, *then the interval* $[0, x_F(\mu)]$ *is no longer invariant under* F_μ. *Indeed, in this case,* F_μ *has a periodic orbit of period three and hence has topological chaos.*

Proof. Since $M_1(\mu) > x_F(\mu)$ when $\mu > \mu_2$, there exists $x^* \in [0, x_1(\mu)]$ such that

$$0 = F_\mu^3(x^*) < x^* < F_\mu(x^*) < F_\mu^2(x^*), \tag{4.2}$$

which implies F_μ has periodic orbit of period three. This completes the proof. □

In the sense of Sarkovskii's ordering, one knows that "periodic three" is the strongest chaos. Chaos of F_μ in a little bit weaker sense also exists for $\mu < \mu_2$. To see this, let

$$\mu_3 \triangleq \inf\{0 < \mu < \mu_2 \mid F_\mu^2(M_1(\mu)) < \bar{x}\}. \tag{4.3}$$

Then, we have the following theorem.

Theorem 4.3. *Assume that f satisfies* (1.2). *Then, for* $\mu_3 < \mu \leqslant \mu_2$, F_μ *has a periodic orbit of period* 6 *and hence it also exhibits topological chaos.*

Proof. For $\mu_3 < \mu \leqslant \mu_2$, we have

$$F_\mu^2(M_1(\mu)) < \bar{x}.$$

There are two cases: either

$$F_\mu(M_1(\mu)) < F_\mu^2(M_1(\mu)) < x_{m1}(\mu) < \bar{x}, \tag{4.4}$$

or

$$F_\mu(M_1(\mu)) < x_{m1}(\mu) < F_\mu^2(M_1(\mu)) < \bar{x}. \tag{4.5}$$

In the first case, F_μ has a periodic orbit of period three. Thus it has also period 6 orbit by Sarkovskii's theorem. In the later case, F_μ^2 has a periodic orbit of period three and so F_μ has a period 6 orbit. □

To study the chaos of the map G_μ, define

$$\mu_4 \triangleq \inf\{0 < \mu \mid G_\mu^2(M_2(\mu)) < \bar{x}\}. \tag{4.6}$$

Similar to Theorem 4.3, we have

Theorem 4.4. *Under the assumption of Theorem* 4.2, *if* $\mu > \mu_4$, *then* G_μ *has a periodic orbit of period* 6 *and hence it exhibits topological chaos.*

Remark 4.1. We note that the constants μ_3 and μ_4 are in general not optimal in the sense that F_μ and G_μ may have topological chaos for some $\mu < \mu_3$ and $\mu < \mu_4$, respectively.

Finally, as an example, we study chaotic dynamics of the difference schemes (3.10) and (3.11) resulted from the ordinary differential equation (3.9). We have seen in Section 3 that $\mu_2 = (3\sqrt{3}/2) - 1 \approx 1.5981$, and F_μ is global stable for $(0, x_F(\mu)]$ when $0 < \mu < 1$ where $x_F(\mu)$ is given by (3.12), and has a period doubling bifurcation at $\mu = 1$. Numerical computation shows that $\mu_3 \approx 1.360$, where μ_3 is defined by (4.3). Thus, if $1.36 < \mu < 1.5981$, F_μ has an invariant interval $[0, x_F(\mu_2)]$ and has periodic orbit of period 6 in this interval, and thus, exhibits topological chaos. F_μ experiences period doubling bifurcation when μ increases in (1, 1.360).

For the difference scheme (3.11), from Section 3, the dynamics of G_μ is the same as that of F_μ when $0 < \mu < 1$, and G_μ has a period doubling bifurcation also at $\mu = 1$. For any $\mu > 0$, the maximum point (critical point) of G_μ is

$$x_{2m}(\mu) = \sqrt{\frac{1}{2\mu}}.$$

Solving the inequality

$$G_\mu^2(M_2(\mu)) = G_\mu^3(x_{2m}(\mu)) < \bar{x} = 1,$$

numerically, we obtain

$$\mu > \approx 1.417.$$

That is, $\mu_4 \approx 1.417$. Hence, by Theorem 4.4, if $\mu > 1.417$, then G_μ has a periodic orbit of period 6 and so it exhibits topological chaos. Also, there is period doubling bifurcations when μ increases in (1, 1.417) for G_μ.

5. Conclusion

We have discussed the stability of the common equilibrium, the periodic doubling bifurcations and their stability, and chaos in the Euler scheme (1.3) and the IA scheme (1.4) by developing some general formulas. We have shown the feasibility of these formulas by applying them to some particular forms of $f(x)$. By comparing these formulas and some numerical examples, we have found that (i) as long as the global stability of the equilibrium is concerned, the Euler scheme is better than the IA scheme in terms of the step size parameter μ; (ii) as long as the stability of the periodic 2 solutions are concerned, the Euler scheme is also better than the IA scheme in the same sense provided that $f''(x) < 0$; and in the case $f''(x) < 0$ is not satisfied, example shows that there is no general comparison for the two schemes; (iii) for the occurrence of chaos in terms of μ, we are unable to obtain a general comparison result and the conclusion seems to depend on particular forms of $f(x)$.

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