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Journal of Computational and Applied Mathematics 167 (2004) 73–90
JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS
www.elsevier.com/locate/cam

Hopf bifurcation in bidirectional associative memory neural networks with delays: analysis and computation☆

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Received 4 October 2002; received in revised form 28 July 2003

Abstract

In addition to giving some delay-independent conditions for global stability of the bidirectional associative memory neural networks with delays, we perform analysis of local stability and Hopf bifurcation. We also work out an algorithm for determining the direction and stability of the bifurcated periodic solutions.

Keywords: Bidirectional associative memory; Delay; Hopf bifurcation; Liapunov functional; Neural networks; Stability

1. Introduction

In this paper, we deal with the delayed bidirectional associative memory (BAM) neural networks described by the following system:

$$\begin{aligned} \dot{x}_i(t) &= -x_i(t) + \sum_{j=1}^{n} a_{ij} f_j(y_j(t-\tau_{ij})) + I_i, \\ \dot{y}_i(t) &= -y_i(t) + \sum_{j=1}^{n} b_{ij} g_j(x_j(t-r_{ij})) + J_i, \end{aligned} \tag{1.1}$$

where a_{ij}, b_{ij}, $i,j \in N(1,n) := \{1,2,\ldots,n\}$ are the connection weights through the neurons in two layers: I-layer and J-layer. On I-layer, the neurons whose states denoted by $x_i(t)$ receive the inputs I_i

☆ Research partially supported by NSERC of Canada and by a Petro-Canada Young Innovator Award.
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0377-0427/$ - see front matter
doi:10.1016/j.cam.2003.09.047

and the inputs outputted by those neurons in J-layer via activation functions (output–input functions) f_i, while on J-layer, the neurons whose associated states denoted by $y_i(t)$ receive the inputs J_i and the inputs outputted from those neurons in I-layer via activation functions (output–input functions) g_i. And $\tau_{ij}, r_{ij}, i, j \in N(1,n)$ are the associated delays due to the finite transmission speed among neurons in different layers.

When there is no delay present, (1.1) reduces to a system of ordinary differential equations which was investigated by Kosko [15–17]. Although system (1.1) can be mathematically regarded as a Hopfield-type neural network, which was extensively investigated recently (See, for example, [1,3,6,8,10,11,13,18–20,24,25,27]), with dimension $2n$, it really produces many nice properties due to the special structure of connection weights and has practical applications in storing paired patterns or memories and the ability to search the desired patterns via both directions: forward and backward directions. See [9,15–17,21] for details about the applications on learning and associative memories.

When delays are incorporated into the network models, stability analysis becomes much more difficult. For the bifurcation analysis to some special neural networks, we refer to [2,23,26,28]. Among them, either only a *single* delay is considered [28], or very *special connection structure (ring structure)* is required [2], or only networks of *two* neurons are studied [23,26]. To the best of our knowledge, no Hopf bifurcation analysis for BAM neural networks with $n > 2$ has been performed in the literature. The reason for this is that the characteristic equation for a general large size network with multiple delays is a transcendental polynomial of higher degree with multiple exponential terms, which is extremely hard to analyze. Our aim in this paper is to study stability and the Hopf bifurcation of (1.1), which has a block (or layer) structure, another special connection structure other than ring. In performing the analysis of local stability and Hopf bifurcation, we need to assume that the delays in each layer are identical. It is this assumption, that allows us to make use of the Schur complement theory for block matrices in analyzing the characteristic equations.

The rest of this paper is organized as follows. In Section 2, we give some delay-independent criteria for global stability of (1.1). In Section 3, we study the local stability and the Hopf bifurcation, and work out an computable algorithm for determining the direction and stability of the Hopf bifurcation. We give an example in Section 4 to demonstrate our analytical results with some numeric simulations confirming the results.

2. Global stability

In this section, we give some results on existence and global stability of an equilibrium of (1.1). First, using the Brourwer fixed point theorem [5], we can establish the existence as below.

Lemma 2.1. *Suppose the activation functions f_i, g_i, $i \in N(1,n)$ are continuous and bounded, then (1.1) has at least one equilibrium.*

Without loss of generality, in what follows, we assume that $I_i = J_i = 0$ and $f_i(0) = g_i(0)$ for $i \in N(1,n)$ (otherwise, one just needs to perform a translation to (1.1)), that is, consider the system

$$\dot{x}_i(t) = -x_i(t) + \sum_{j=1}^{n} a_{ij} f_j(y_j(t-\tau_1)),$$
$$\dot{y}_i(t) = -y_i(t) + \sum_{j=1}^{n} b_{ij} g_j(x_j(t-\tau_2)). \tag{2.1}$$

The following global stability results are independent of the delays.

Theorem 2.1. *Suppose there exist some* $p_i > 0$, $q_i > 0$, $i \in N(1,n)$ *such that*

$$\mathrm{Lip}(g_i) \sum_{j=1}^{n} |b_{ji}| q_j < p_i,$$
$$\mathrm{Lip}(f_i) \sum_{j=1}^{n} |a_{ji}| p_j < q_i. \tag{2.2}$$

Then the zero solution of (2.1) *is globally asymptotically stable* (*exponentially*).

Proof. The proof is similar to one in [25] by using the Liapunov functional

$$V(t) = \sum_{i=1}^{n} p_i \left(|x_i(t)| + \sum_{j=1}^{n} |a_{ij}| \mathrm{Lip}(f_j) \int_{t-\tau_{ij}}^{t} |y_j(s)|\,\mathrm{d}s \right)$$
$$+ \sum_{i=1}^{n} q_i \left(|y_i(t)| + \sum_{j=1}^{n} |b_{ij}| \mathrm{Lip}(g_j) \int_{t-r_{ij}}^{t} |x_j(s)|\,\mathrm{d}s \right). \quad \square$$

Theorem 2.2. *Suppose there are some real positive numbers* p_i, q_i, ξ, η_i, $i \in N(1,n)$ *such that*

$$p_i \sum_{j=1}^{n} |a_{ij}| \xi_j + \frac{\mathrm{Lip}^2(g_i)}{\eta_i} \sum_{j=1}^{n} |b_{ji}| q_j < 2p_i, \tag{2.3}$$

$$q_i \sum_{j=1}^{n} |b_{ij}| \eta_j + \frac{\mathrm{Lip}^2(f_i)}{\xi_i} \sum_{j=1}^{n} |a_{ji}| p_j < 2q_i. \tag{2.4}$$

Then the zero solution of system (2.1) *is globally asymptotically stable.*

Proof. The proof can be completed by using a Liapunov functional defined by

$$V(t)=\sum_{i=1}^{n} p_i\left(x_i^2(t)+\sum_{j=1}^{n}\frac{|a_{ij}|}{\xi_j}\int_{t-\tau_{ij}}^{t} f_j^2(y_j(s))\,\mathrm{d}s\right)$$
$$+\sum_{i=1}^{n} q_i\left(y_i^2(t)+\sum_{j=1}^{n}\frac{|b_{ij}|}{\eta_j}\int_{t-r_{ij}}^{t} g_j^2(x_j(s))\,\mathrm{d}s\right). \quad \square$$

Note that we can obtain various sufficient conditions to guarantee the global stability of (2.1) by various choices of the parameters p_i, q_i, ξ, η_i for $i\in N(1,n)$. For example, by taking $p_i=q_i=1$ for $i\in N(1,n)$ in Theorem 2.1, one obtains the following corollary which is one of the main results in [21].

Corollary 2.1. *Suppose*

$$\mathrm{Lip}(g_i)\sum_{j=1}^{n}|b_{ji}|<1 \tag{2.5}$$

and

$$\mathrm{Lip}(f_i)\sum_{j=1}^{n}|a_{ji}|<1. \tag{2.6}$$

Then system (2.1) *is globally* (*exponentially*) *asymptotically stable.*

3. Local stability and Hopf bifurcation

This section is the focus of this paper, in which we perform analysis for local stability and bifurcation. Such an analysis will show how delays will destroy the stability of the network. To this end, we assume in the rest of the this paper that the activation functions f_i and g_i are differentiable. As explained in the introduction, we only consider a special case of (2.1): $\tau_{ij}=\tau_1$ and $r_{ij}=\tau_2$ for all $i,j\in N(1,n)$. Then the linearization of (2.1) at the fixed point 0 is

$$\begin{aligned}\dot{x}_i(t)&=-x_i(t)+\sum_{j=1}^{n}\alpha_{ij}y_j(t-\tau_1),\\ \dot{y}_i(t)&=-y_i(t)+\sum_{j=1}^{n}\beta_{ij}x_j(t-\tau_2),\end{aligned} \tag{3.1}$$

where $\alpha_{ij}=a_{ij}f_j'(0)$, $\beta_{ij}=b_{ij}g_j'(0)$ for $i\in N(1,n)$. Denote the $n\times n$ identity matrix by E_n, $A=(\alpha_{ij})$, $B=(\beta_{ij})$ and $\tau=\tau_1+\tau_2$. Let

$$W=\begin{pmatrix}(z+1)E_n & -\mathrm{e}^{-z\tau_1}A\\ -\mathrm{e}^{-z\tau_2}B & (z+1)E_n\end{pmatrix}.$$

Then the associated characteristic equation is

$$\det W = 0. \tag{3.2}$$

If

$$d_0 := \det\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} \neq 0,$$

then $z = -1$ is not a root of (3.2), which implies that $(z+1)E_n$ is nonsingular. It follows from Theorem 1.23 of [7] that

$$\det W = \det((z+1)E_n)\det[W/(z+1)E_n],$$

where $[W/(z+1)E_n]$ is the Schur complement of the block $(z+1)E_n$ in W (To see the definition of Schur complement, we refer to [7]). Therefore, (3.2) is equivalent to

$$\det[(z+1)^2E_n - e^{-z\tau}BA] = 0. \tag{3.3}$$

It is easy to see that z is a solution of (3.3) if and only if there is $\lambda \in \sigma(BA)$ such that

$$(z+1)^2 - \lambda e^{-z\tau} = 0. \tag{3.4}$$

Hence, if λ_i, $i \in N(1,n)$ are eigenvalues of BA, then (3.3) is equivalent to n scalar equations

$$(z+1)^2 - \lambda_i e^{-z\tau} = 0, \quad i \in N(1,n). \tag{3.5}$$

Analyzing the distribution of roots of (3.5), we have

Theorem 3.1. *Let λ_i, $i \in N(1,n)$ be eigenvalues of BA and $d_0 \neq 0$. Then the following statements hold*:

(I) *The zero solution of system* (2.1) *is asymptotically stable when $\tau = 0$ if and only if*

$$|\mathrm{Re}(\sqrt{\lambda_i})| < 1, \; i \in N(1,n). \tag{3.6}$$

(II) *The zero solution of system* (2.1) *is asymptotically stable for all nonnegative τ if*

$$|\lambda_i| < 1, \quad i \in N(1,n). \tag{3.7}$$

Proof. Note that the zero solution is asymptotically stable if and only if all roots of (3.3) have negative real parts. *Case* (I): $\tau = 0$, then (3.5) reads for each $i \in N(1,n)$

$$(z+1)^2 - \lambda_i = 0, \tag{3.8}$$

which shows that

$$z = -1 \pm \sqrt{\lambda_i}.$$

It is easily seen that for any root z of (3.3), $\mathrm{Re}(z) < 0$ if and only if (3.6) holds.

Case (II): Letting $\lambda_i = a + ib$ and $z = u + iv$ and substituting them to (3.5), we have

$$(1 + u + iv)^2 = (a + ib)e^{-(u+iv)\tau},$$

which gives

$$(1+u)^2 - v^2 = \mathrm{e}^{-u\tau}[a\cos(v\tau) + b\sin(v\tau)],$$

$$2(1+u)v = \mathrm{e}^{-u\tau}[b\cos(v\tau) - a\sin(v\tau)].$$

Taking square on the both sides of the above two equations and summing them up, we get

$$[(1+u)^2 + v^2]^2 = [\mathrm{e}^{-u\tau}|\lambda_i|]^2$$

or

$$(1+u)^2 + v^2 = \mathrm{e}^{-u\tau}|\lambda_i|. \tag{3.9}$$

Hence, if $u > 0$, the left-hand side of (3.9) will be greater than 1. However, the right-hand side $\mathrm{e}^{-u\tau}|\lambda_i| < 1$ due to $u > 0$ and (3.7). This shows if (3.7) holds, then (3.3) does not admit a root with positive real part.

Thus the proof is complete. □

From the above theorem, we see that for positive τ, if (3.7) does not hold, then the stability of the zero solution of (2.1) may be destroyed. To check this point, in what follows we assume that

$$\max_{i\in N(1,n)} |\mathrm{Re}\sqrt{\lambda_i}| < 1 < \max_{i\in N(1,n)} |\lambda_i|. \tag{3.10}$$

Here and in what follows, we will restrict our attention to the case BA is a nonzero matrix having only real and purely imaginary eigenvalues. Since BA is a real matrix, its imaginary eigenvalues must appear in pairs, we may assume that

$$\sigma(BA) = \{\alpha_1, \alpha_2, \ldots, \alpha_p, \pm\mathrm{i}\delta_1, \pm\mathrm{i}\delta_2, \ldots, \pm\mathrm{i}\delta_q\}$$

with

$$p + 2q = n$$

and

$$\alpha_1 \leqslant \alpha_2 \leqslant \cdots < \alpha_s < 0 \leqslant \alpha_{s+1} \leqslant \cdots \leqslant \alpha_p$$

and

$$0 \leqslant \delta_q \leqslant \delta_{q-1} \leqslant \cdots \leqslant \delta_1.$$

It follows from (3.10) that

$$\alpha_p < 1 \tag{3.11}$$

and

$$\delta_1 < 2, \tag{3.12}$$

and

$$\alpha_1 < -1 \tag{3.13}$$

or

$$\delta_1 > 1. \tag{3.14}$$

Lemma 3.1. *Suppose that* (3.14) *holds. Then for each* $\delta_j > 1$, *define*

$$\tau(\delta_j) = \frac{1}{\omega_j} \arcsin \frac{1-\omega_j^2}{\delta_j}, \tag{3.15}$$

where $\omega_j = \sqrt{\delta_j - 1}$, *we have*:

(a) *At* $\tau(\delta_j)$, (3.3) *with* $\lambda = \mathrm{i}\delta_j$ *has a pair of purely imaginary simple roots* $\pm \mathrm{i}\omega_j$ *and all other roots have negative real parts.*
(b) $\tau \in [0, \tau(\delta_j))$, *all roots of* (3.3) *with* $\lambda = \delta_j$ *have negative real parts.*
(c) $$\mathrm{Re}\, \frac{\mathrm{d}z(\tau)}{\mathrm{d}\tau}\bigg|_{\tau=\tau(\delta_j)} > 0.$$
(d) $\tau(\delta_j) > \tau(\delta_1)$ *if* $\delta_1 > \delta_j > 1$.

Proof. Suppose $z = \mathrm{i}\omega$ is a root of (3.3) with $\lambda = \mathrm{i}\delta_j$. Then we may get

$$1 - \omega^2 = \delta_j \sin \omega\tau, \quad 2\omega = \delta_j \cos \omega\tau,$$

which shows that

$$\omega = \sqrt{\delta_j - 1}$$

and

$$\tau = \frac{1}{\omega}\left(\arcsin \frac{1-\omega^2}{\delta_j} + 2k\pi\right),$$

where $0 < \arcsin(1-\omega^2)/\delta_j < \pi$ and k is an integer. Clearly $\tau(\delta_j)$ is the least such positive τ. Hence at $\tau(\delta_j)$, (3.3) with $\lambda = \mathrm{i}\delta_j$ has a pair of purely imaginary roots $\pm \mathrm{i}\sqrt{\delta_j - 1}$. Let $H_{\delta_j}(z,\tau) = (z+1)^2 - \mathrm{i}\delta_j \mathrm{e}^{-z\tau}$. Then $\partial H_{\delta_j}/\partial z = 2(1+z) + \mathrm{i}\tau\delta_j \mathrm{e}^{-z\tau}$. Note that $H_{\delta_j} = 0$ and $\partial H_{\delta_j}/\partial z = 0$ give $\tau = -2/(z+1)$, which implies z is real. This shows that the multiple zeros of $H_{\delta_j}(z,\tau)$ have to be real, and hence $\mathrm{i}\sqrt{\delta_j - 1}$ is a simple purely imaginary root of (3.3) with $\lambda = \mathrm{i}\delta_j$.

Next we show that (3.3) with $\lambda = \mathrm{i}\delta_j$ has no root with positive real part. Suppose, on the contrary, that $z = u + \mathrm{i}v$ with $u > 0$ is a root of (3.3) with $\lambda = \mathrm{i}\delta_j$. Since the roots of (3.3) continuously depend on the parameter τ. Using Lemma 2.1 of [4], there exists $\hat{\tau} \in (0, \tau(\delta_j))$ such that (3.3) with $\lambda = \mathrm{i}\delta_j$ has a purely imaginary root at $\tau = \hat{\tau}$, which contradicts with the fact that $\tau(\delta_j)$ is the smallest such τ.

Thirdly, we will show that (c) of this lemma is true. Differentiating both sides of (3.4) with respect to τ leads to

$$\frac{\mathrm{d}z(\tau)}{\mathrm{d}\tau}\bigg|_{\tau=\tau(\delta_j)} = \frac{-\mathrm{i}\delta_j \mathrm{e}^{-z\tau} z}{\tau + 2(z+1)}\bigg|_{\tau=\tau(\delta_j)}.$$

A straightforward calculation yields

$$\mathrm{Re}\, \frac{\mathrm{d}z(\tau)}{\mathrm{d}\tau}\bigg|_{\tau=\tau(\delta_j)} = \frac{(\delta_j - 1)[2 + \delta_j + 2\tau(\delta_j)]}{(\tau+2)^2 + 4(\delta_j + 1)} > 0.$$

Let $\tau(x)$ be defined by

$$\tau(x) = \frac{1}{\sqrt{x-1}} \arcsin \frac{2-x}{x} \quad \text{for } x > 1.$$

Then (d) of this lemma follows from the fact that $\tau(x)$ is decreasing for $x > 1$. Thus the proof is complete. □

Similarly, we have

Lemma 3.2. *Suppose that* (3.13) *holds. Then for each* $\alpha_k < -1$, *define*

$$\tau(\alpha_k) = \frac{1}{\omega_k} \arcsin \frac{2\omega_k}{-\alpha_k}, \tag{3.16}$$

where $\omega_k = \sqrt{|\alpha_k| - 1}$, *we have*

(i) *At* $\tau(\alpha_k)$, (3.3) *with* $\lambda = \alpha_k$ *has a pair of purely imaginary simple roots* $\pm \mathrm{i}\omega_k$ *and all other roots have negative real parts.*

(ii) $\tau \in [0, \tau(\alpha_k))$, *all roots of* (3.3) *with* $\lambda = \alpha_k$ *have negative real parts.*

(iii) $$\operatorname{Re} \left.\frac{\mathrm{d}z(\tau)}{\mathrm{d}\tau}\right|_{\tau=\tau(\alpha_k)} > 0.$$

(iv) $\tau(\alpha_k) > \tau(\alpha_1)$ *if* $\alpha_1 < \alpha_k < -1$.

Proof. (i)–(iii) can be obtained from [26, Lemma 5]. (iv) follows from the fact that the function

$$\tau(x) = \frac{1}{\sqrt{x-1}} \arcsin \frac{2\sqrt{x-1}}{x} \quad \text{for } x > 1$$

is decreasing. □

If both (3.13) and (3.14) are satisfied, then we may define

$$\tau^* = \min\{\tau(\delta_1), \tau(\alpha_1)\}, \tag{3.17}$$

which is the least value of τ destabilizing the trivial solution of (2.1). Let $x = x^*(\delta_1)$ be the unique solution of the equation

$$\frac{1}{\sqrt{x-1}} \arcsin \frac{2\sqrt{x-1}}{x} = \tau(\delta_1). \tag{3.18}$$

Then we have

$$\tau^* = \begin{cases} \tau(\delta_1) & \text{if } |\alpha_1| < x^*(\delta_1), \\ \tau(\alpha_1) & \text{if } |\alpha_1| \geqslant x^*(\delta_1). \end{cases}$$

In order to use the general Hopf bifurcation theory for functional differential equations (see, [12]), we assume that

$$|\alpha_1| \neq x^*(\delta_1). \tag{3.19}$$

By the above lemmas, we immediately have the following result on local stability and bifurcation for system (2.1).

Theorem 3.2. *Suppose* (3.11)–(3.14) *and* (3.19) *are satisfied, then we have*:

(i) *If* $\tau \in [0, \tau^*)$, *then the zero solution of* (2.1) *is asymptotically stable*;
(ii) *If* $\tau > \tau^*$, *then the zero solution of* (2.1) *is unstable*;
(iii) *The Hopf bifurcation occurs at* $\tau = \tau^*$. *That is, system* (2.1) *has a branch of periodic solutions bifurcating from the zero solution near* $\tau = \tau^*$.

In the above, we have shown that the Hopf bifurcation occurs at some value $\tau^* = \tau_1^* + \tau_2^*$ for the BAM (2.1). Next, by using the normal form method and the center manifold theory [12], we will work out an algorithm for determining the direction, stability and the period of the bifurcated periodic solutions. Usually, the direction and stability of the Hopf bifurcation can be computed by the general algorithm developed in [12] (see also [14]). But in the practical application, it is not an easy job for high-dimensional cases. We will give a specific algorithm for a special case. More precisely, we will consider the BAM neural networks with two neurons in each layer, that is $n = 2$ in (2.1). Moreover, noting that most frequently used activation functions, such as $\tanh(x)$, satisfy

$$\tanh'(0) \neq 0, \ \tanh''(0) = 0 \quad \text{and} \quad \tanh'''(0) \neq 0,$$

we will assume that the activation functions in (2.1) satisfy

(P) for $i \in N(1, n)$ $f_i'(0) \neq 0$, $f_i''(0) = 0$, $f_i'''(0) \neq 0$ and $g_i'(0) \neq 0$, $g_i''(0) = 0$, $g_i'''(0) \neq 0$.

Since when $n = 2$, the matrix BA is a 2×2 matrix, based on the previous analysis, we have two cases to consider: case (1) both of the eigenvalues of BA are real; case (2) the eigenvalues of BA are a pair of purely imaginary numbers. In what follows, we will deal with case (1). In this case, the two real eigenvalues of BA α_1 and α_2 satisfy

$$\alpha_1 < -1 < \alpha_2 < 1,$$

and $\tau^* = \tau(\alpha_1)$. It is seen from the conclusions of Lemma 3.2 and Theorem 3.2 that all roots of (3.2) other than $\pm i\omega_0$ with $\omega_0 = \sqrt{|\alpha_1| - 1}$ have negative real parts, and the root of (3.2)

$$\lambda(\tau) := \alpha(\tau) + i\omega(\tau)$$

satisfying $\alpha(\tau^*) = 0$, $\omega(\tau^*) = \omega_0$ admits

$$\alpha'(\tau^*) := \frac{d\alpha(\tau^*)}{d\tau} > 0$$

and

$$\omega'(\tau^*) = \frac{d\omega(\tau^*)}{d\tau} = \frac{-\omega_0(2 + \tau^* + \tau^*\omega_0)}{(2 + \tau^*)^2 + (\tau^*\omega_0)^2}.$$

Follow the idea in [26], we let $\tau^* = \tau_1^* + \tau_2^*$ with $\tau_1^* < \tau_2^*$ and $\tau = \tau^* + \mu = (\tau_1^* + \mu) + \tau_2^*$, where $|\mu| \leqslant \tau_2^* - \tau_1^*$. Then $\mu = 0$ is the Hopf bifurcation value for system (2.1). Choosing the phase space as

$$C = C([-\tau_2^*, 0], C^4),$$

where we use C^4 instead of R^4 for the convenience in the later computation. Now system (2.1) can be rewritten as

$$\begin{aligned}
\dot{x}_1(t) &= -x_1(t) + \alpha_{11} y_1(t-\tau_1^*-\mu) + \alpha_{12} y_2(t-\tau_1^*-\mu) \\
&\quad + \alpha_{11}^* y_1^3(t-\tau_1^*-\mu) + \alpha_{12}^* y_2^3(t-\tau_1^*-\mu) + \mathrm{O}(y_1^4, y_2^4), \\
\dot{x}_2(t) &= -x_2(t) + \alpha_{21} y_1(t-\tau_1^*-\mu) + \alpha_{22} y_2(t-\tau_1^*-\mu) \\
&\quad + \alpha_{21}^* y_1^3(t-\tau_1^*-\mu) + \alpha_{22}^* y_2^3(t-\tau_1^*-\mu) + \mathrm{O}(y_1^4, y_2^4), \\
\dot{y}_1(t) &= -y_1(t) + \beta_{11} x_1(t-\tau_2^*) + \beta_{12} x_2(t-\tau_2^*) \\
&\quad + \beta_{11}^* x_1^3(t-\tau_2^*) + \beta_{12}^* x_2^3(t-\tau_2^*) + \mathrm{O}(x_1^4, x_2^4), \\
\dot{y}_2(t) &= -y_2(t) + \beta_{21} x_1(t-\tau_2^*) + \beta_{22} x_2(t-\tau_2^*) \\
&\quad + \beta_{21}^* x_1^3(t-\tau_2^*) + \beta_{22}^* x_2^3(t-\tau_2^*) + \mathrm{O}(x_1^4, x_2^4),
\end{aligned} \tag{3.20}$$

where $\alpha_{ij} = a_{ij} f_j'(0)$, $\alpha_{ij}^* = \frac{1}{6} a_{ij} f_j'''(0)$ and $\beta_{ij} = b_{ij} g_i'(0)$, $\beta_{ij}^* = \frac{1}{6} b_{ij} g_j'''(0)$ for $i, j \in N(1,2)$.

Let $U = (x_1(t), x_2(t), y_1(t), y_2(t))^{\mathrm{T}}$,

$$B_1 = \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad B_2 = \begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix}$$

and

$$F(\mu, \phi) = \begin{pmatrix} \alpha_{11}^* \phi_3^3(-\tau_1^*-\mu) + \alpha_{12}^* \phi_4^3(-\tau_1^*-\mu) + \mathrm{O}(\phi_3^4, \phi_4^4) \\ \alpha_{21}^* \phi_3^3(-\tau_1^*-\mu) + \alpha_{22}^* \phi_4^3(-\tau_1^*-\mu) + \mathrm{O}(\phi_3^4, \phi_4^4) \\ \beta_{11}^* \phi_1^3(-\tau_2^*) + \beta_{12}^* \phi_2^3(-\tau_2^*) + \mathrm{O}(\phi_1^4, \phi_2^4) \\ \beta_{21}^* \phi_1^3(-\tau_2^*) + \beta_{22}^* \phi_2^3(-\tau_2^*) + \mathrm{O}(\phi_1^4, \phi_2^4) \end{pmatrix}$$

for $\phi = (\phi_1, \phi_2, \phi_3, \phi_4)^{\mathrm{T}} \in C$. Then system (3.20) can be rewritten as

$$\dot{U}_t = -U_t + B_1 U_{t-\tau_1^*-\mu} + B_2 U_{t-\tau_2^*} + F(\mu, U_t), \tag{3.21}$$

where $U_t(\theta) = U(t+\theta)$ for $\theta \in [-\tau_2^*, 0]$. Let

$$\eta(\theta, \mu) = \begin{cases} -\mathrm{Id}, & \theta = 0, \\ B_1 \delta(\theta + \tau_1^* + \mu), & \theta \in [-\tau_1^* - \mu, 0), \\ -B_2 \delta(\theta + \tau_2^*), & \theta \in [-\tau_2^*, -\tau_1^* - \mu), \end{cases} \tag{3.22}$$

where Id is the identical matrix and δ is the usual Dirac function. For $\phi \in C$, define

$$A(\mu)\phi(\theta) = \begin{cases} \dot{\phi}, & \theta \in [-\tau_2^*, 0), \\ \int_{-\tau_2^*}^{0} \mathrm{d}\eta(s, \mu)\phi(s), & \theta = 0 \end{cases} \tag{3.23}$$

and

$$R(\mu)\phi = \begin{cases} 0, & \theta \in [-\tau_2^*, 0), \\ F(\mu, \phi), & \theta = 0. \end{cases} \tag{3.24}$$

Then (3.20) can be further rewritten as

$$\dot{U}_t = A(\mu)U_t + R(\mu)U_t. \tag{3.25}$$

Let $C^* = C^1([0, \tau_2^*], C^4)$, we define, for $\psi \in C^*$, the adjoint operator $A^*(0)$ of $A(0)$ by

$$A^*(0)\psi(s) = \begin{cases} -\dot{\psi}, & s \in (0, \tau_2^*], \\ \displaystyle\int_{-\tau_2^*}^{0} \mathrm{d}\eta^{\mathrm{T}}(\xi, 0)\psi(\xi), & s = 0, \end{cases} \tag{3.26}$$

where η^{T} is the transpose of η. For $\phi \in C([-\tau_2^*, 0], C^4)$ and $\psi \in C([0, \tau_2^*], C^4)$, we define the bilinear form

$$\langle \psi, \phi \rangle = \bar{\psi}(0) \cdot \phi(0) - \int_{\theta=-\tau_2^*}^{0} \int_{\xi=0}^{\theta} \bar{\psi}^{\mathrm{T}}(\xi - \theta)\,\mathrm{d}\eta(\theta, 0)\phi(\xi)\,\mathrm{d}\xi, \tag{3.27}$$

where $a \cdot b = \sum_{i=1}^{n} a_i b_i$ for $a = (a_1, a_2, \ldots, a_n)^{\mathrm{T}}$ and $b = (b_1, b_2, \ldots, b_n)^{\mathrm{T}}$. As usual, we have

$$\langle \psi, A(0)\phi \rangle = \langle A^*(0)\psi, \phi \rangle.$$

It is easily seen that $\lambda(0) = \mathrm{i}\omega_0$ is the eigenvalue of $A(0)$, then $-\mathrm{i}\omega_0$ is that of $A^*(0)$. Denote their corresponding eigenfunctions by $q(\theta)$ and $q^*(s)$, respectively, namely,

$$A(0)q(\theta) = \mathrm{i}\omega_0 q(\theta) \quad \text{and} \quad A^*(0)q^*(s) = -\mathrm{i}\omega_0 q^*(s).$$

We compute here with

$$q(\theta) = q(0)\mathrm{e}^{\mathrm{i}\omega_0\theta} \quad \text{and} \quad q^*(s) = q^*(0)\mathrm{e}^{\mathrm{i}\omega_0 s}, \tag{3.28}$$

where

$$q(0) = (q_1(0), q_2(0), q_3(0), q_4(0))^{\mathrm{T}}$$

and

$$\begin{aligned} q^*(0) &= (q_1^*(0), q_2^*(0), q_3^*(0), q_4^*(0))^{\mathrm{T}} \\ &= D(\psi_1(0), \psi_2(0), \psi_3(0), \psi_4(0))^{\mathrm{T}} \end{aligned}$$

with

$$q_1(0) = 1, \quad q_2(0) = \frac{\alpha_1\alpha_{21} + \beta_{21}\det A}{\alpha_1\alpha_{11} - \beta_{22}\det A},$$

$$q_3(0) = \frac{\mathrm{e}^{-\mathrm{i}\omega_0\tau_2^*}}{1 + \mathrm{i}\omega_0} P_3, \quad q_4(0) = \frac{\mathrm{e}^{-\mathrm{i}\omega_0\tau_2^*}}{1 + \mathrm{i}\omega_0} P_4$$

and

$$\psi_1(0) = 1, \quad \psi_2(0) = \frac{\alpha_1\beta_{12} + \alpha_{12}\det B}{\alpha_1\beta_{11} - \alpha_{22}\det A},$$

$$\psi_3(0) = \frac{e^{i\omega_0\tau_1^*}}{1 - i\omega_0} Q_3, \quad \psi_4(0) = \frac{e^{i\omega_0\tau_1^*}}{1 - i\omega_0} Q_4,$$

$$P_3 := \beta_{11} + \beta_{12}q_2(0), \quad P_4 := \beta_{21} + \beta_{22}q_2(0),$$

$$Q_3 := \alpha_{11} + \alpha_{21}\psi_2(0), \quad Q_4 := \alpha_{12} + \alpha_{22}\psi_2(0)$$

and

$$D = \left\{ 1 + q_2(0)\psi_2(0) + \frac{1}{\alpha_1}(P_3Q_3 + P_4Q_4) + \frac{1}{\alpha_1}(1 - i\omega_0) \right.$$
$$[\tau_1^*(P_3(\alpha_{11} + \alpha_{21}\psi_2(0)) + P_4(\alpha_{12} + \alpha_{22}\psi_2(0))$$
$$\left. +\tau_2^*(\beta_{11}Q_3 + \beta_{21}Q_4 + q_2(0)(\beta_{12}Q_3 + \beta_{22}Q_4)] \right\}^{-1}.$$

Then q and q^* satisfy

$$\langle q^*, q\rangle = 1 \quad \text{and} \quad \langle q^*, \bar{q}\rangle = 0.$$

Let U_t be a solution of (3.25), we define

$$z(t) = \langle q^*, U_t\rangle, \ w(t,\theta) = w(z,\bar{z},\theta) = U_t(\theta) - 2\,\mathrm{Re}\{z(t)q(\theta)\},$$

then on the center manifold for (3.25) for $\mu = 0$,

$$w(z,\bar{z},\theta) = w_{20}(\theta)\frac{z^2}{2} + w_{11}(\theta)z\bar{z} + w_{02}(\theta)\frac{\bar{z}^2}{2} + \cdots.$$

Therefore, at $\mu = 0$, (3.25) can be reduced to an ordinary differential equation

$$\dot{z}(t) = \langle q^*, A(0)U_t + RU_t\rangle = i\omega_0 z(t) + \bar{q}^*(0)\cdot F_0, \tag{3.29}$$

where

$$F_0 = F(0, w(z,\bar{z},\theta) + 2\,\mathrm{Re}\{z(t)q(\theta)\}).$$

We may rewrite (3.29) as

$$\dot{z}(t) = \langle q^*, A(0)U_t + RU_t\rangle = i\omega_0 z(t) + g(z,\bar{z}), \tag{3.30}$$

where

$$g(z,\bar{z}) = \bar{q}^*(0)\cdot F_0 \tag{3.31}$$

$$= g_{20}\frac{z^2}{2} + g_{11}z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + g_{21}\frac{z\bar{z}}{2} + \cdots. \tag{3.32}$$

Then from [12,14], we know, in order to study the stability and direction of the Hopf bifurcation, it is crucial to compute these coefficients g_{20}, g_{11}, g_{02} and g_{21}.

Note that

$$y_1(t-\tau_1^*) = w_3(t,-\tau_1^*) + z(t)q_3(-\tau_1^*) + \bar{z}(t)\bar{q}_3(-\tau_1^*),$$
$$y_2(t-\tau_1^*) = w_4(t,-\tau_1^*) + z(t)q_4(-\tau_1^*) + \bar{z}(t)\bar{q}_4(-\tau_1^*),$$

and

$$x_1(t-\tau_2^*) = w_1(t,-\tau_2^*) + z(t)q_1(-\tau_1^*) + \bar{z}(t)\bar{q}_1(-\tau_1^*),$$
$$x_2(t-\tau_2^*) = w_2(t,-\tau_2^*) + z(t)q_2(-\tau_1^*) + \bar{z}(t)\bar{q}_2(-\tau_1^*),$$

where

$$w_i(t,-\tau_j^*) = w_{20}^{(i)}(-\tau_j^*)\frac{z^2}{2} + w_{11}^{(i)}(-\tau_j^*)z\bar{z} + w_{02}^{(i)}(-\tau_j^*)\frac{\bar{z}^2}{2} + \cdots$$

for $i \in N(1,4)$ and $j = 1,2$, we then have

$$\begin{aligned}
g(z,\bar{z}) &= \bar{q}^*(0)\cdot F_0 \\
&= \bar{q}_1^*(0)(\alpha_{11}^* y_1^3(t-\tau_1^*) + \alpha_{12}^* y_2^3(t-\tau_1^*)) \\
&\quad + \bar{q}_2^*(0)(\alpha_{21}^* y_1^3(t-\tau_1^*) + \alpha_{22}^* y_2^3(t-\tau_1^*)) \\
&\quad + \bar{q}_3^*(0)(\beta_{11}^* x_1^3(t-\tau_2^*) + \beta_{12}^* x_2^3(t-\tau_2^*)) \\
&\quad + \bar{q}_4^*(0)(\beta_{21}^* x_1^3(t-\tau_2^*) + \beta_{22}^* x_2^3(t-\tau_2^*)) \\
&\quad + \mathrm{O}(u^4).
\end{aligned}$$

Expanding it and comparing the coefficients with (3.32), we have

$$g_{20} = g_{11} = g_{02} = 0$$

and

$$\begin{aligned}
g_{21} &= \bar{q}_1^*(0)(\alpha_{11}^*|q_3(-\tau_1^*)|^2 q_3(-\tau_1^*) + \alpha_{12}^*|q_4(-\tau_1^*)|^2 q_4(-\tau_1^*)) \\
&\quad + \bar{q}_2^*(0)(\alpha_{21}^*|q_3(-\tau_1^*)|^2 q_3(-\tau_1^*) + \alpha_{22}^*|q_4(-\tau_1^*)|^2 q_4(-\tau_1^*)) \\
&\quad + \bar{q}_3^*(0)(\beta_{11}^*|q_1(-\tau_2^*)|^2 q_1(-\tau_1^*) + \beta_{12}^*|q_2(-\tau_2^*)|^2 q_2(-\tau_1^*)) \\
&\quad + \bar{q}_4^*(0)(\beta_{21}^*|q_1(-\tau_2^*)|^2 q_1(-\tau_1^*) + \beta_{22}^*|q_2(-\tau_2^*)|^2 q_2(-\tau_1^*)).
\end{aligned}$$

Now we define

$$C_1(0) = \frac{1}{2}\,g_{21}$$

and

$$\mu_2 = -\mathrm{Re}\,C_1(0),\ \ \beta_2 = 2\,\mathrm{Re}\,C_1(0),\ \ T_2 = \frac{-1}{\omega_0}(\mathrm{Im}\,C_1(0) + \mu_2\omega'(\tau^*)).$$

The above analysis, the general theory on Hopf bifurcation [12] and the fact that $\mu_2\beta_2 < 0$ immediately give

Theorem 3.3. *Under the assumptions given in this subsection, the direction and stability of Hopf bifurcation of* (3.25) *and thus that of* (2.1) *can be determined by the sign of* μ_2. *Indeed, if* $\mu_2 > 0(< 0)$, *then the Hopf bifurcation of* (2.1) *at* $\tau = \tau^*$ *is supercritical* (*subcritical*) *and the periodic solution of* (2.1) *bifurcating from Hopf bifurcation value* $\tau = \tau^*$ *is asymptotically orbitally stable* (*unstable*). *Moreover,* T_2 *gives the* $O(\varepsilon^2)$ *term in the period*

$$P = \frac{2\pi}{\omega_0}(1 + T_2\varepsilon + O(\varepsilon^4))$$

for the periodic solutions bifurcating near τ^*.

4. An example

Consider the following BAM neural network with two neurons on each layer

$$\begin{aligned}
\dot{x}_1(t) &= -x_1(t) + a_{11}\tanh(y_1(t-\tau_1)) + a_{12}\tanh(y_2(t-\tau_1)),\\
\dot{x}_2(t) &= -x_2(t) + a_{21}\tanh(y_1(t-\tau_1)) + a_{22}\tanh(y_2(t-\tau_1)),\\
\dot{y}_1(t) &= -y_1(t) + b_{11}\tanh(x_1(t-\tau_2)) + b_{12}\tanh(x_2(t-\tau_2)),\\
\dot{y}_2(t) &= -y_2(t) + b_{21}\tanh(x_1(t-\tau_2)) + b_{22}\tanh(x_2(t-\tau_2)).
\end{aligned} \tag{4.1}$$

Corollary 4.1. *Assume that*

$$|a_{11}|p_1 + |a_{21}|p_2 < q_1,\ |a_{12}|p_1 + |a_{22}|p_2 < q_2 \tag{4.2}$$

and

$$|b_{11}|q_1 + |b_{21}|q_2 < p_1,\ |b_{12}|q_1 + |b_{22}|q_2 < p_2 \tag{4.3}$$

hold for some positive $p_i, q_i, i = 1, 2$. *Then the zero solution of* (4.1) *is globally asymptotically stable* (*exponentially*) *for any choice of* τ_1 *and* τ_2.

Remark 4.1. Let $p_i = q_i = 1,\ i = 1, 2$ in (4.2) and (4.3). Then Corollary 4.1 reproduces the main theorem in [21].

If we take

$$a_{11} = 1,\ a_{12} = -1,\ a_{21} = -1,\ a_{22} = 1.2 \tag{4.4}$$

and

$$b_{11} = 0.8,\ b_{12} = 1,\ b_{21} = 1,\ b_{22} = -2 \tag{4.5}$$

and $\tau_1^* = 0.1$. Then we have

$$\tau^* = 0.6568,\ \alpha_1 = -3.7391 < -1,\ \alpha_2 = 0.1391 < 1$$

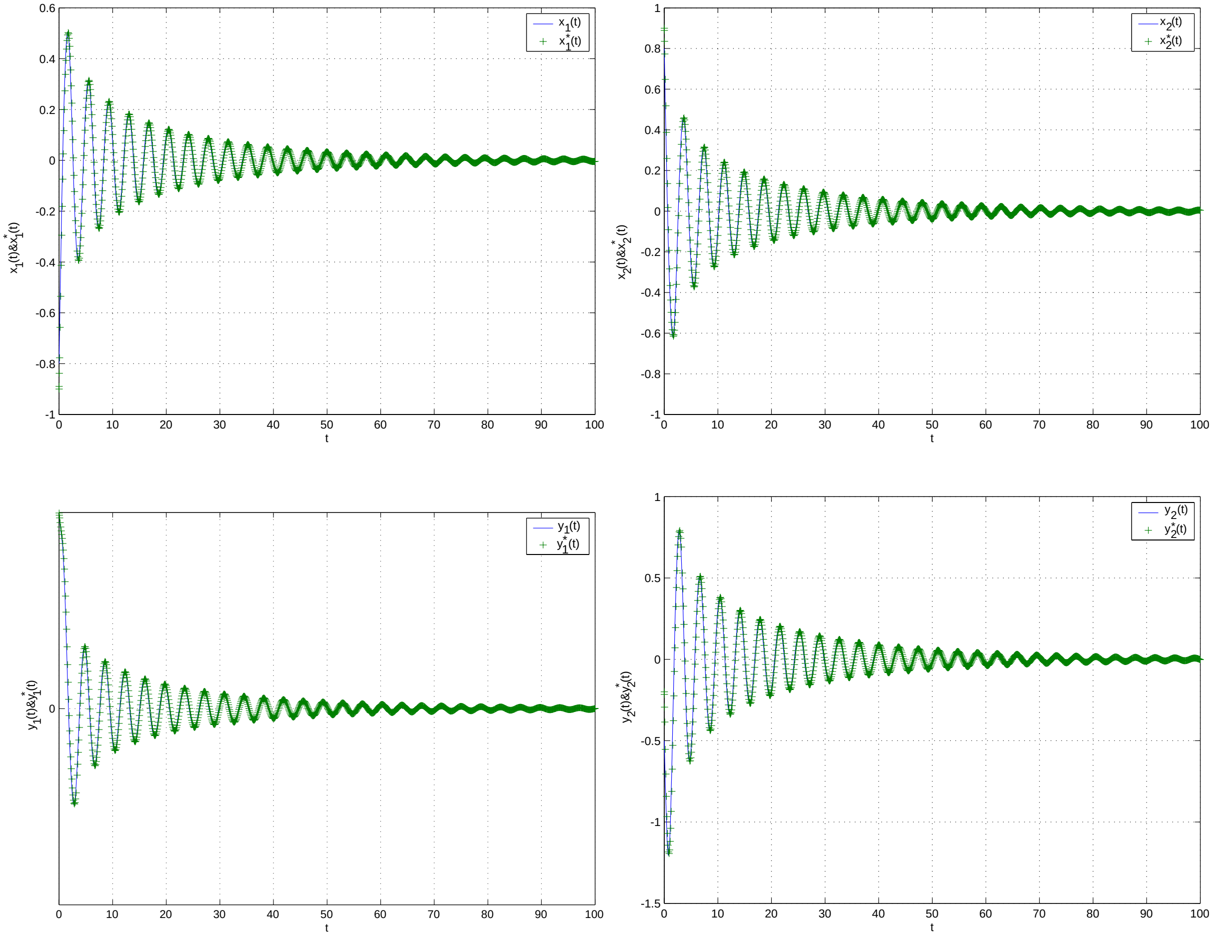

Fig. 1. $\tau_1 = 0.1, \tau_2 = 0.5$ and thus $\tau_1 + \tau_2 < \tau^*$.

and

$$\omega_0 = 1.6550,\ \ g_{21} = -4.4504 - 2.5948i,\ \ C_1(0) = -2.2252 - 1.2974i$$

and

$$\mu_2 = 2.2252,\ \ \beta_2 = -4.4504,\ \ T_2 = 1.9872.$$

This means the zero solution of system (4.1) with (4.4) and (4.5) is asymptotically stable if $\tau_1 + \tau_2 < \tau^* = 0.6568$, and the Hopf bifurcation occurs at $\tau_1 + \tau_2 = 0.6568$. Furthermore, the Hopf bifurcation is supercritical and the bifurcating periodic solutions are asymptotically orbitally stable. Moreover, the period of the bifurcation periodic solutions can be estimated by

$$T = \frac{2\pi}{\omega_0}(1 + T_2\varepsilon^2) = 3.7965 + 7.5443\varepsilon^2$$

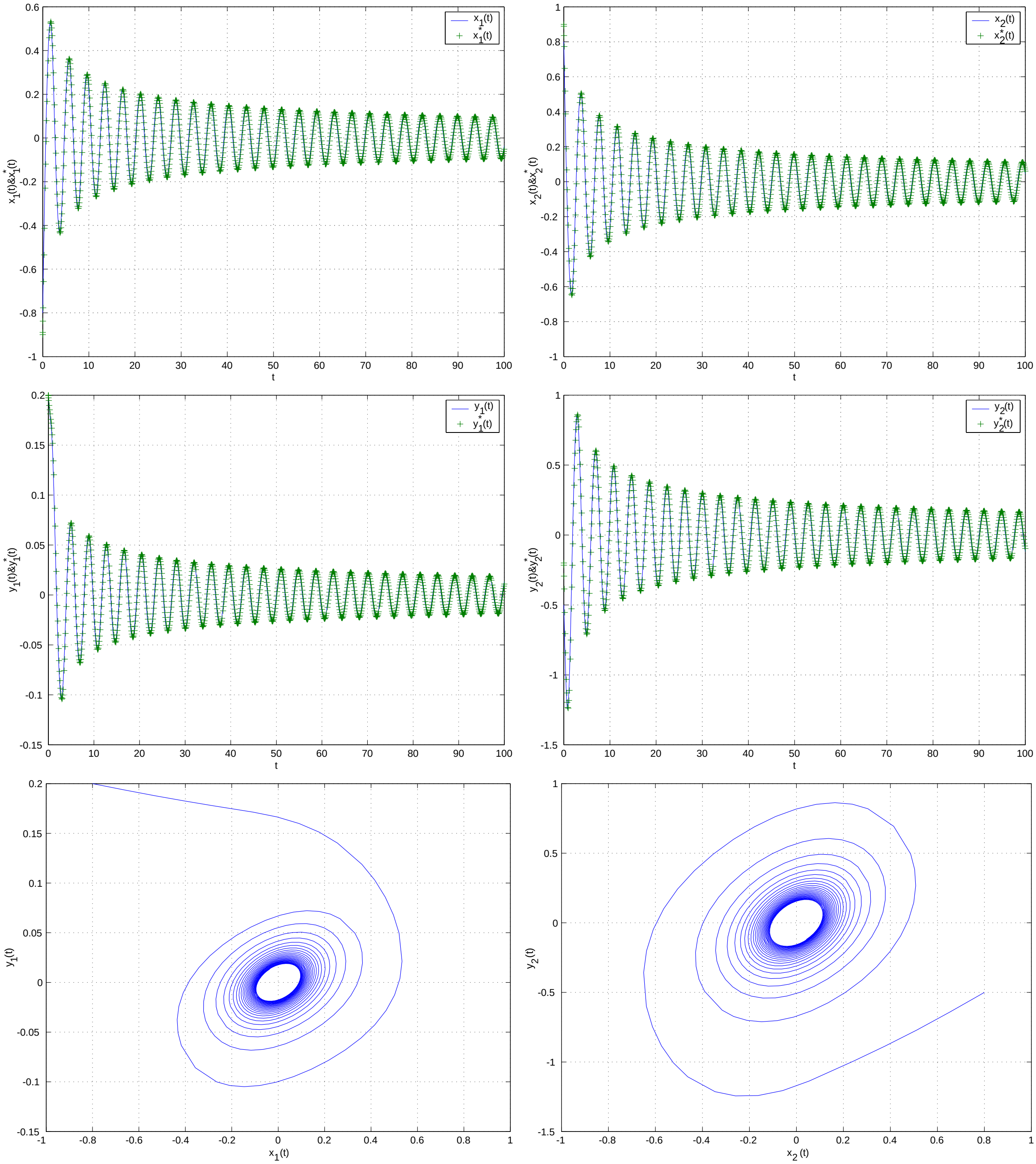

Fig. 2. $\tau_1 + \tau_2 = 0.1 + 0.5569 > \tau^*$.

with $\varepsilon = (|\mu|/\mu_2)^{1/2}$. The numerical simulations, which are performed by the DDEs solver developed by Shampine and Thompson [22], are given in Figs. 1 and 2.

Acknowledgements

The authors would like to thank the referee for his/her valuable comments which have led to an improvement of the presentation of this paper.

References

[1] J. Bélair, Stability in a model of a delayed neural network, J. Dyn. Differential Equations 5 (1993) 607–623.
[2] S.A. Campbell, S. Ruan, J. Wei, Qualitative analysis of a neural network model with multiple time delays, Internat. J. Bifur. Chaos Appl. Sci. Eng. 9 (8) (1999) 1585–1595.
[3] Y. Cao, Q. Wu, A note on stability of analog neural networks with time delays, IEEE Trans. Neural Networks 7 (1996) 1533–1535.
[4] K. Cooke, P. van den Driessche, On zeros of some transcendental equation, Funkcial. Ekvac. 29 (1985) 77–90.
[5] K. Deimling, Nonlinear Functional Analysis, Springer, New York, 1985.
[6] C. Feng, R. Plamondon, On the stability analysis of delayed neural networks systems, Neural Networks 14 (2000) 1181–1188.
[7] M. Fielder, Special Matrices and their Applications in Numerical Mathematics, Martinus Nijhoff Publishers, Boston, 1986.
[8] M. Forti, On global asymptotic stability of a class of nonlinear systems arising in neural network theory, J. Differential Equations 113 (1994) 246–264.
[9] K. Gopalsamy, X. He, Delay-independent stability in bidirectional associative memory networks, IEEE Trans. Neural Networks 5 (1994) 998–1002.
[10] K. Gopalsamy, X. He, Stability in asymmetric Hopfield nets with transmission delays, Phys. D 76 (1994) 344–358.
[11] Z. Guan, G. Chen, Y. Qin, On equilibria, stability, and instability of Hopfield neural networks, IEEE Trans. Neural Networks 2 (2000) 534–540.
[12] B.D. Hassard, N.D. Kazavinoff, Y.H. Wan, Theory and Applications of the Hopf Bifurcation, Cambridge University Press, Cambridge, MA, 1981.
[13] J. Hopfield, Neurons with graded response have collective computational properties like those of two-stage neurons, Proc. Natl. Acad. Sci. USA 81 (1984) 3088–3092.
[14] N.D. Kazarinoff, Y.H. Wan, P. van den Driessche, Hopf bifurcation and stability of periodic solutions of differential-difference and integro-differential equations, J. Inst. Math. Appl. 21 (1978) 461–477.
[15] B. Kosko, Adaptive bidirectional associative memories, Appl. Opt. 26 (1987) 4947–4960.
[16] B. Kosko, Bidirectional associative memories, IEEE Trans. Systems Man Cybernet. 18 (1988) 49–60.
[17] B. Kosko, Unsupervised learning in noise, IEEE Trans. Neural Networks 1 (1990) 1–12.
[18] H. Lu, On stability of nonlinear continuous-time neural networks with delays, Neural Networks 13 (2000) 1135–1143.
[19] C. Marcus, R. Westervelt, Stability of analog neural networks with delay, Phys. Rev. A 39 (1989) 347–359.
[20] K. Matsuoka, Stability conditions for nonlinear continuous neural networks with asymmetric connection weights, Neural Networks 5 (1992) 495–500.
[21] S. Mohamad, Globaly exponential stability in continuous-time and discrete-time delayed bidirectional neural networks, Physica D 159 (2001) 233–251.
[22] L.F. Shampine, S. Thompson, Solving DDEs in MATLAB, Appl. Numer. Math. 37 (2001) 441–458.
[23] L.P. Shayer, S.A. Campbell, Stability, bifurcation, and multistability in a system of two coupled neurons with multiple time delays, SIAM J. Appl. Math. 61 (2) (2000) 673–700.
[24] D. Tank, J. Hopfield, Simple neural optimization networks: an A/D converter, signal decision circuit and a linear programming circuit, IEEE Trans. Circuits Systems 33 (1986) 533–541.

[25] P. van den Driessche, X. Zou, Global attractivity in delayed Hopfield neural network models, SIAM J. Appl. Math. 58 (1998) 1878–1890.
[26] J. Wei, S. Ruan, Stability and bifurcation in a neural network model with two delays, Phys. D 130 (1999) 255–272.
[27] J. Wu, Symmetric functional-differential equations and neural networks with memory, Trans. Amer. Math. Soc. 350 (1999) 4799–4838.
[28] J. Wu, X. Zou, Patterns of sustained oscillations in neural networks with delayed iteractions, Appl. Math. Comput. 73 (1995) 55–75.