ELSEVIER
Journal of Computational and Applied Mathematics 146 (2002) 309–321
JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS
www.elsevier.com/locate/cam

Delay induced traveling wave fronts in reaction diffusion equations of KPP-Fisher type ☆

Xingfu Zou

Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, NF, Canada A1C 5S7

Received 2 August 2001; received in revised form 4 November 2001

Abstract

This paper explores the impact of delay on the existence of traveling wave fronts in reaction–diffusion equations of KPP–Fisher type. For two such equations, one being local and the other with nonlocal effect arising from the age structure of the population, we show that delay can induce slow traveling wave fronts.

Keywords: Traveling wave fronts; Delay; Reaction–diffusion equation; Iteration

1. Introduction

The most classic and the simplest case of nonlinear reaction–diffusion equation that was first shown to have traveling wave fronts is the so-called Fisher–KPP equation (Fisher [4] and Kolmogorov, Petrovskii and Piskunov [10])

$$\frac{\partial u}{\partial t} = ru(t,x)\left[1 - \frac{u(t,x)}{K}\right] + D\,\frac{\partial^2 u}{\partial x^2}, \tag{1.1}$$

where r and D are positive parameters. Eq. (1.1) was first suggested by Fisher [4] as a deterministic version of a stochastic model for the spatial spread of a favored gene in a population. It is also a natural extension of the logistic growth population ODE model.

☆ Research supported by Natural Sciences and Engineering Research Council of Canada.
E-mail address: xzou@math.mun.ca (X. Zou).

0377-0427/02/$ - see front matter
PII: S0377-0427(02)00363-1

Eq. (1.1) can be normalized into the simpler form

$$\frac{\partial u}{\partial t} = u(t,x)[1 - u(t,x)] + \frac{\partial^2 u}{\partial x^2} \tag{1.2}$$

by rescaling $t^* = rt$, $x^* = \sqrt{(r/D)}x$, $u* = u/K$ and omitting the asterisks for notational simplicity. Eq. (1.2) and its traveling wave solutions have been widely studied (see, e.g., [3,13] and references therein), not only because it has in itself wide applicability but also because it is the prototype equation which admits the traveling wave front solutions. It is also a convenient equation from which many techniques can be developed for analyzing single species models with spatial dispersal.

As far as traveling wave fronts are concerned, a reaction–diffusion equation with a *monostable* nonlinearity typically admits a family of wave speeds, while an equation with a *bistable* nonlinearity usually can only have an unique (if any) wave speed. Eq. (1.2) has an monostable reaction term $f(u) = u(1-u)$, and it has been shown that (1.2) can have traveling wave front with any speed $c \geqslant 2$, but does not allow wave front with speed $c < 2$. In other words, $c^* = 2$ serves as the minimal wave speed for (1.2) (see, e.g., [3,13]).

It has been widely argued and accepted (see, e.g., [7,11,18]) that for various reasons, time delay should be taken into consideration in modeling. If one wants to incorporate a single discrete time delay $\tau \geqslant 0$ into (1.2), as was done by many researchers to the corresponding logistic ODE model, one has the following two choices:

$$\frac{\partial u}{\partial t} = u(t,x)[1 - u(t-\tau,x)] + \frac{\partial^2 u}{\partial x^2} \tag{1.3}$$

and

$$\frac{\partial u}{\partial t} = u(t-\tau,x)[1 - u(t,x)] + \frac{\partial^2 u}{\partial x^2}. \tag{1.4}$$

The former comes from the corresponding ODE model (also called Hutchinson equation), and has been extensively studied in the literature (see [8,11,16], and references therein), and the latter was derived by Kobayashi [9] from a branching process.

In population biology, age structure cannot be ignored for some species. Recently, So et al. [15] derived a model for the growth of the matured population of a single species, taking into account the age structure, and the model turns out to be the following reaction-diffusion equation with both time delay and spatially nonlocal effect

$$\frac{\partial w}{\partial t} = D_{\rm m}\frac{\partial^2 w}{\partial x^2} - d_{\rm m} w + \delta \int_{-\infty}^{\infty} b(w(t-\tau, y)) f_\alpha(x-y)\,{\rm d}y. \tag{1.5}$$

Here,

$$\delta = {\rm e}^{-\int_0^\tau d_{\rm I}(a)\,{\rm d}a}, \quad \alpha = \int_0^\tau D_{\rm I}(a)\,{\rm d}a$$

and

$$f_\alpha(x) = \frac{1}{\sqrt{4\pi\alpha}}{\rm e}^{-x^2/4\alpha},$$

where $\tau \geqslant 0$ is the mature age, $D_{\rm m}$ and $d_{\rm m}$ are the diffusion and death rates of the mature, respectively, which are assumed to be constants, while $D_{\rm I}$ and $d_{\rm I}$ are the diffusion and death rates of the immature, respectively, and $b(\cdot)$ is the birth function. Here δ reflects the impact of the death rate of the immature and α represents the effect of the dispersal rate of the immature on the matured population. Notice that the death rate and diffusion rate of the immature enter (1.5) in a totally different way from that of the matured. When $\tau=0$ (hence $\alpha=0$ and $\delta=1$), which means neglecting the age structure, (1.5) with the birth function $b(w)=rw(1-\beta w)$ with $r>d_{\rm m}$ reduces to the KPP–Fisher equation (1.1) (see (3.11)). For a justification of such a birth function, one may recall that the widely used logistic nonlinearity $ru(1-u/K)$ in (1.1) is actually a result of lumping together such a birth term $bw(1-\beta u)$ and a death term $\mathrm{d}u$. Other functions with a single hump reflecting the peak of productivity may also be used, but in order to avoid hiding the main ideas behind complicated mathematical computations, we adopt this quadratic birth function in the rest of this paper. For the same birth function but without assuming $\alpha=0$ and $\delta=1$, (1.5) becomes

$$\frac{\partial w}{\partial t}=D_{\rm m}\frac{\partial^2 w}{\partial x^2}-d_{\rm m}w+\delta\int_{-\infty}^{\infty} rw(t-\tau,y)[1-\beta w(t-\tau,y)]f_{\alpha}(x-y)\,\mathrm{d}y. \qquad (1.6)$$

It is well known that delays can have very complicated impact on the dynamics of a system (see, e.g., [7,11,18]). For example, delays can cause the loss of stability, and can induce various oscillations and even chaos. In this paper, we are interested in the impact of delay on the existence of traveling wave fronts of reaction–diffusion equations. For simplicity, we only report results on the equations of KPP–Fisher types as mentioned above. By comparing the existence results for the aforementioned equations, we will show that delay can induce *slow* (e.g., in the sense of $c<2$ for (1.4)) traveling wave fronts. As far as the author knows, delay induced traveling wave fronts have not been reported elsewhere. Our approach will be the iteration technique recently developed in [19], which has also been successfully employed by So and Zou [17], So et al. [15], Gourley [6], and Ma [12].

It is worth pointing out that existence of traveling wave fronts of (1.3) has already been addressed in [19] by using the iteration method. Roughly speaking, it is shown in [19] that for any $c>2$, (1.3) still admits a wave front with speed c, provided that $\tau\geqslant 0$ is sufficiently small. But when τ increases, the numeric simulations in [1] and [6] shows that the monotonicity of the wave fronts will disappear, and oscillatory and even periodic waves may occur. Thus, as far as *monotone* wave fronts go, delay can *prevent* fronts as well.

Also note that traveling wave fronts of (1.4) may be obtained as well by using the general theory of Schaaf [14], which is the pioneering work in this topic. But Schaaf's theory is only for scalar reaction–diffusion equations with a single discrete delay and with nonlinear term that is monotone with respect to the delayed term and is spatially local, and thus cannot be applied to (1.6), but our method can. Another merit of our approach is that the iteration provides a scheme for approximation of the profile of the wave front.

2. Delay induced traveling wave fronts in (1.4)

By a traveling wave front of (1.4), we mean a solution of (1.4) of the form $u(t,x)=\phi(x+ct)$, where $c>0$, ϕ satisfies the following delay ordinary differential equation

$$c\phi'(t)=\phi''(t)+\phi(t-c\tau)[1-\phi(t)],\quad t\in R \qquad (2.1)$$

and is subject to the following asymptotic boundary condition

$$\lim_{t\to-\infty} \phi(t)=0, \qquad \lim_{t\to\infty} \phi(t)=1. \tag{2.2}$$

Note that the wave speed $c>0$ in (2.1) is unknown in advance and needs to be determined while solving (2.1)–(2.2) for ϕ. Thus indeed (2.1)–(2.2) is an eigenvalue problem.

The following result is well known (see e.g., [3,13]).

Theorem 2.1. *When* $\tau=0$, (2.1)–(2.2) *has a monotone solution if and only if* $c \geqslant 2$. *In other words, in the absence of delay,* $c=2$ *is the minimal wave speed for* (1.2).

The main result in this section is the following:

Theorem 2.2. *The following hold*:

(i) *For every* $c \geqslant 2$, (2.1)–(2.2) *always has a monotone solution*, **regardless** *of the value of* $\tau \geqslant 0$.
(ii) *For every* $c \in (0,2)$, (2.1)–(2.2) **also** *has a monotone solution if*

$$\tau \geqslant \left(\frac{2}{c}\right)^2 \ln\frac{2}{c}. \tag{2.3}$$

Remark 2.1. Comparing Theorems 2.1 and 2.2, we see the delay can induce arbitrarily slow ($c<2$) traveling wave fronts for (1.4).

The proof of Theorem 2.2 consists of several lemmas, which will be given in the remainder of this section.

Define the *profile set* for (2.1)–(2.2) by

$$\Gamma=\left\{\phi\in C(\mathbb{R};\mathbb{R}):\ \begin{array}{l}\text{(i) } \phi \text{ is nondecreasing in } \mathbb{R};\\ \text{(ii) } \lim\limits_{t\to-\infty}\phi(t)=0, \quad \lim\limits_{t\to\infty}\phi(t)=1.\end{array}\right\}.$$

A function $\phi\in C(\mathbb{R},\mathbb{R})$ is called an *upper* (resp. *lower*) *solution* of (2.1) if it is differentiable almost everywhere (a.e.) and satisfies

$$c\phi' \geqslant \phi''(t)+\phi(t-c\tau)[1-\phi(t)], \qquad \text{a.e. in } \mathbb{R},$$

$$(\text{resp. } c\phi' \leqslant \phi''(t)+\phi(t-c\tau)[1-\phi(t)], \quad \text{a.e. in } \mathbb{R}).$$

Let $X_c=C([-c\tau,0],\mathbb{R})$ and f be the nonlinear term in (2.1), i.e., $f: X_c\to\mathbb{R}$ is defined by $f(\phi)=\phi(-c\tau)[1-\phi(0)]$. Then it is easily seen that

(A1) $f(\hat{0})=f(\hat{1})=0$ and $f(\hat{u})\neq 0$ for $u\in(0,1)$, where $\hat{u}$ denotes the constant function taking value u on $[-c\tau,0]$.
(A2) (quasi-monotonicity) There exists $\beta\geqslant 0$ (say, $\beta=1$) such that

$$f(\phi)-f(\psi)+\beta[\phi(0)-\psi(0)]\geqslant 0$$

for $\phi,\ \psi\in X_c$ with $0\leqslant\psi(s)\leqslant\phi(s)\leqslant 1,\ s\in[-c\tau,0]$.

Therefore, we can apply Wu and Zou [19, Theorem 3.6] to (2.1)–(2.2) to produce the following result:

Lemma 2.1. *Suppose that* (2.1) *has an upper solution* $\bar{\phi}$ *in* Γ *and a lower solution* $\underline{\phi}$ (*which is not necessarily in* Γ) *with* $0 \leqslant \underline{\phi}(t) \leqslant \bar{\phi}(t) \leqslant 1$ *and* $\underline{\phi}(t) \not\equiv 0$ *in* $\mathbb{R}$, *then* (2.1)–(2.2) *has a monotone a solution.*

In what follows, we will construct a pair of upper-lower solutions required in the above lemma, and thus conclude the existence of traveling wave fronts for (1.4). To this end, we linearize (2.1) at 0 to get

$$c\phi'(t) = \phi''(t) + \phi(t - c\tau). \tag{2.4}$$

The characteristic equation of (2.4) is

$$\Delta_c(\lambda) := \mathrm{e}^{-\lambda c\tau} - \lambda[c - \lambda] = 0. \tag{2.5}$$

Lemma 2.2. *Under each of the two conditions in Theorem* 2.2, $\Delta_c(\lambda) = 0$ *has two positive real roots*: $\lambda_1 < \lambda_2$ *and*

$$\Delta_c(\lambda) = \begin{cases} > 0 & \text{for } \lambda > \lambda_2, \\ < 0 & \text{for } \lambda \in (\lambda_1, \lambda_2), \\ > 0 & \text{for } \lambda < \lambda_1. \end{cases} \tag{2.6}$$

Proof. The function $h(\lambda) = \lambda[c - \lambda]$ is concave down with the maximum $c^2/4$ attained at $c/2$. The function $g(\lambda) = \mathrm{e}^{-c\tau\lambda}$ is nonincreasing and $g(c/2) = \mathrm{e}^{-\tau c^2/2}$. The conclusion immediately follows from comparing $c^2/4$ with $\mathrm{e}^{-\tau c^2/2}$. □

In the rest of this section, we will use the two positive real roots $\lambda_1 < \lambda_2$ in Lemma 2.2 to construct the upper and lower solutions required in Lemma 2.1.

Lemma 2.3. $\bar{\phi}(t) = \min\{1, \mathrm{e}^{\lambda_1 t}\}$ *is an upper solution of* (2.1) *and* $\bar{\phi} \in \Gamma$.

Proof. The proof is a direct verification, and is thus omitted here. □

Next, we choose $\varepsilon > 0$ such that $\varepsilon < \lambda_1 < \lambda_1 + \varepsilon < \lambda_2$, and define $\underline{\phi}(t) = \max\{0, (1 - M\mathrm{e}^{\varepsilon t})\mathrm{e}^{\lambda_1 t}\}$, where the constant $M > 1$ is to be determined.

Lemma 2.4. *For sufficiently large* M, $\underline{\phi}(t)$ *is a lower solution of* (2.1).

Proof. Let $t_1 = (1/\varepsilon)\ln 1/M$. Then $t_1 < 0$ for $M > 1$ and

$$\underline{\phi}(t) = \begin{cases} 0 & \text{for } t > t_1, \\ (1 - M\mathrm{e}^{\varepsilon t})\mathrm{e}^{\lambda_1 t} & \text{for } t < t_1. \end{cases}$$

For $t > t_1$, $\underline{\phi}(t) = 0$, and hence,

$$\underline{\phi}''(t) - c\underline{\phi}'(t) + \underline{\phi}(t)[1 - \underline{\phi}(t - c\tau)] = \underline{\phi}(t - c\tau) \geqslant 0.$$

For $t < t_1$,

$$\underline{\phi}(t) = [1 - M e^{\varepsilon t}]e^{\lambda_1 t},$$

$$\underline{\phi}'(t) = [\lambda_1 - M(\varepsilon + \lambda_1]e^{\varepsilon t})e^{\lambda_1 t},$$

$$\underline{\phi}''(t) = [\lambda_1^2 - M(\lambda_1 + \varepsilon)^2 e^{\varepsilon t}]e^{\lambda_1 t}\underline{\phi}(t - c\tau) = [1 - M e^{\varepsilon(t - c\tau)}]e^{\lambda_1(t - c\tau)}.$$

Thus,

$$\begin{aligned}
&\underline{\phi}''(t) - c\underline{\phi}'(t) + \underline{\phi}(t)[1 - \underline{\phi}(t - c\tau)] \\
&\quad = [\lambda_1^2 - M(\lambda_1 + \varepsilon)^2 e^{\varepsilon t}]e^{\lambda_1 t} - c[\lambda_1 - M(\lambda_1 + \varepsilon)e^{\varepsilon t}]e^{\lambda_1 t} \\
&\qquad + [1 - M e^{\varepsilon(t - c\tau)}]e^{\lambda_1(t - c\tau)}[1 - (1 - M e^{\varepsilon t})e^{\lambda_1 t}] \\
&\quad = e^{\lambda_1 t}\{\Delta_c(\lambda_1) - M e^{\varepsilon t}\Delta_c(\lambda_1 + \varepsilon) - [1 - M e^{\varepsilon(t - c\tau)}][1 - M e^{\varepsilon t}]e^{\lambda_1 t}e^{-\lambda_1 c\tau}\} \\
&\quad = e^{\lambda_1 t}\{-M e^{\varepsilon t}\Delta_c(\lambda_1 + \varepsilon) - [1 - M e^{\varepsilon(t - c\tau)}][1 - M e^{\varepsilon t}]e^{\lambda_1 t}e^{-\lambda_1 c\tau}\}.
\end{aligned}$$

Since $t < t_1 < 0$ and $\varepsilon < \lambda_1$, we have $e^{\lambda_1 t}e^{-\lambda_1 c\tau} < e^{\varepsilon t}$ and

$$[1 - M e^{\varepsilon(t - c\tau)}][1 - M e^{\varepsilon t}] \leqslant [1 + M e^{\varepsilon t}]^2 \leqslant [1 + M e^{\varepsilon t_1}]^2 = [1 + 1]^2 = 4.$$

Therefore,

$$\begin{aligned}
&\underline{\phi}''(t) - c\underline{\phi}'(t) + \underline{\phi}(t)[1 - \underline{\phi}(t - c\tau)] \\
&\quad \geqslant e^{\lambda_1 t}\{-M e^{\varepsilon t}\Delta_c(\lambda_1 + \varepsilon) - 4e^{\varepsilon t}\} \\
&\quad = e^{(\lambda_1 + \varepsilon)t}[-\Delta_c(\lambda_1 + \varepsilon)]\left\{M - \frac{4}{-\Delta_c(\lambda_1 + \varepsilon)}\right\}.
\end{aligned}$$

Now, by the choice of λ_1 and ε, we know $\Delta_c(\lambda_1 + \varepsilon) < 0$, and hence, the right-hand side of the above inequality is nonnegative if we choose

$$M > \frac{4}{-\Delta_c(\lambda_1 + \varepsilon)}.$$

For such a M, $\underline{\phi}$ is a lower solution of (2.1), and this completes the proof of Lemma 2.4. □

Remark 2.2. Note that in the proof of Theorem 2.2, the two positive real roots λ_1 and λ_2, whose existence are established in Lemma 2.2, play a crucial role. From the proof of Lemma 2.2, (2.3) is only a convenient and explicit condition on τ that ensures the existence of such two positive roots, and the best possible condition on τ for this purpose can be obtained by solving the following equations:

$$\lambda(c-\lambda)=\mathrm{e}^{-\lambda c\tau},$$
$$c-2\lambda=-c\tau\mathrm{e}^{-\lambda c\tau},$$

which come from the requirement that the two curves of $g(\lambda)$ and $h(\lambda)$ just touch.

Remark 2.3. Existence of solutions of (2.1)–(2.2) is obtained by constructing a proper pair of upper-lower solutions of (2.1) as stated in Lemma 2.1. Indeed, a profile for the wave front of (1.4), that is, a solution of (2.1)–(2.2), can be obtained by the convergence of the following iterations:

$$\phi_{\mathrm{m}}(t)=\frac{1}{\mu_2-\mu_1}\left[\int_{-\infty}^{t}\mathrm{e}^{\mu_1(t-s)}H(\phi_{m-1})(s)\,\mathrm{d}s+\int_{t}^{\infty}\mathrm{e}^{\mu_2(t-s)}H(\phi_{m-1})(s)\,\mathrm{d}s\right],$$
$$\phi_0(t)=\bar{\phi}(t),$$

where $t\in\mathbb{R}$, $m=1,2,\ldots$, and $H:C(\mathbb{R},\mathbb{R})\to C(\mathbb{R},\mathbb{R})$ is defined by

$$H(\phi)(t)=\phi(t-c\tau)[1-\phi(t)]+\phi(t),\quad \phi\in C(\mathbb{R},\mathbb{R}),\ t\in\mathbb{R}$$

and μ_i, $i=1,2$ are given by

$$\mu_1=\frac{c-\sqrt{c^2+4}}{2},\qquad \mu_2=\frac{c+\sqrt{c^2+4}}{2}.$$

For details of the iteration scheme for more general delayed systems, see [19].

3. Traveling waves fronts in the nonlocal model (1.6)

In this section, we give some results on existence of traveling wave fronts for the nonlocal model (1.6), which are parallel to those for (1.4) in Section 2. For convenience of discussion, we assume the following throughout this section:

(H1) $d_{\mathrm{I}}(a)\equiv 0$ (thus $\delta=1$);
(H2) $D_{\mathrm{I}}(a)=D_{\mathrm{I}}$ for all $a\in[0,\tau]$ (thus $\alpha=\tau D_{\mathrm{I}}$) with $D_{\mathrm{I}}<2D_{\mathrm{m}}$;
(H3) $d_{\mathrm{m}}<r\leqslant 2d_{\mathrm{m}}$

and focus on the impact of the delay $\tau\geqslant 0$. Again as usual, by a traveling wave front of (1.6), we mean a solution of (1.6) of the form $w(t,x)=\phi(x+ct)$ with $c>0$ being a constant (wave speed), and the profile function ϕ being monotone and satisfying $\lim_{s\to-\infty}\phi(s)=w_1$ and $\lim_{s\to\infty}\phi(s)=w_2$ where $w_1<w_2$ are two equilibria of (1.6).

When $\tau=0$ (hence $\alpha=0$), (1.6) reduces to

$$\frac{\partial w}{\partial t}=D_{\mathrm{m}}\frac{\partial^2 w}{\partial x^2}+(r-d_{\mathrm{m}})w(t,x)\left[1-\frac{r\beta}{r-d_{\mathrm{m}}}w(t,x)\right]. \tag{3.1}$$

Note that since $\delta=1$ and $\int_{-\infty}^{\infty}f_\alpha(y)\,\mathrm{d}y=1$, (1.6) and (3.1) share the same equilibria 0 and $(r-d_{\mathrm{m}})/r\beta$. For (3.1), the following result is classical (see, e.g., [3,13]).

Theorem 3.1. (i) *For every* $c \geqslant 2\sqrt{D_m(r-d_m)}$, (3.1) *has a traveling wave front with speed c connecting* 0 *and* $(r-d_m)/r\beta$; (ii) *For any* $c \in (0, 2\sqrt{D_m(r-d_m)})$, (3.1) **does not** *admit traveling wave front with speed c connecting* 0 *and* $(r-d_m)/r\beta$.

Allowing $\tau > 0$ in (1.6), we have the following main theorem:

Theorem 3.2. *The following hold*:

(i) *For every* $c \geqslant 2\sqrt{D_m(r-d_m)}$, **regardless** *of the value of* $\tau \geqslant 0$, (1.6) *always has a traveling wave front with speed c connecting* 0 *and* $(r-d_m)/r\beta$.
(ii) *For every* $c \in (0, 2\sqrt{D_m(r-d_m)})$, (1.6) **also** *admits a traveling wave front with speed c connecting* 0 *and* $(r-d_m)/r\beta$, *provided that*

$$\tau \geqslant \frac{2D_m}{c^2[1-D_I/(2D_m)]} \ln \frac{r}{[c^2/(4D_m^2)+d_m}. \tag{3.2}$$

The proof of Theorem 3.2 is similar to that of Theorem 2.2. Thus, in the rest of this section we only give an outline consisting of several lemmas, but omitting the details. We also refer a reader to [15] where the traveling wave fronts of (1.5) are considered for the birth function $b(w) = pwe^{-aw}$ which also has a single hump.

By the definition of traveling wave fronts, we need to look for a monotone function $\phi(t)$ satisfying the following equation:

$$c\phi'(t) = D_m\phi''(t) - d_m\phi(t) + r\int_{-\infty}^{\infty} \phi(t-c\tau-y)[1-\beta\phi(t-c\tau-y)]f_\alpha(y)\,dy \tag{3.3}$$

and subject to the boundary conditions

$$\phi(-\infty) = 0, \qquad \phi(\infty) = \frac{r-d_m}{r\beta} := K. \tag{3.4}$$

Eq. (3.3) is a second-order functional equation of *mixed type* (namely, with both advanced and delayed arguments), and thus (as mentioned in the introduction), the method and theory in [14] are not applicable.

Based on (3.3)–(3.4), we define the **profile set** for traveling wave fronts of (1.6) as

$$\Gamma^* = \left\{ \phi \in C(\mathbb{R};\mathbb{R}): \begin{array}{ll} \text{(i)} & \phi \text{ is nondecreasing in } \mathbb{R}; \\ \text{(ii)} & \lim_{t\to-\infty}\phi(t) = 0, \quad \lim_{t\to\infty}\phi(t) = K. \end{array} \right\}.$$

We also define $H^*: C(\mathbb{R};\mathbb{R}) \to C(\mathbb{R};\mathbb{R})$ by

$$H^*(\phi)(t) = r\int_{-\infty}^{\infty} \phi(t-c\tau-y)[1-\beta\phi(t-c\tau-y)]f_\alpha(y)\,dy, \quad \phi \in C(\mathbb{R},\mathbb{R}),\ t \in \mathbb{R}.$$

The operator H^* has some nice properties as stated below.

Lemma 3.1. *For any* $\phi \in \Gamma^*$, *we have*

(i) $H^*(\phi)(t) \geqslant 0$, *for all* $t \in \mathbb{R}$;
(ii) $H^*(\phi)(t)$ *is nondecreasing in* $t \in \mathbb{R}$.
(iii) $H^*(\psi)(t) \leqslant H^*(\phi)(t)$, *for all* $t \in \mathbb{R}$, *provided* $\psi \in C(\mathbb{R};\mathbb{R})$ *is such that* $0 \leqslant \psi(t) \leqslant \phi(t) \leqslant K$ *for* $t \in \mathbb{R}$.

As in Section 2, we can define upper and lower solutions for (3.3) as follows:

Definition 3.2. A function $\phi \in C(\mathbb{R};\mathbb{R})$ is called an **upper** (resp. **lower**) **solution** of (3.3) if it is differentiable almost everywhere (a.e.) and satisfies the inequality

$$c\phi' \geqslant D_{\mathrm{m}}\phi''(t) - d_{\mathrm{m}}\phi(t) + H^*(\phi)(t), \quad \text{a.e. in } \mathbb{R}$$

$$(\text{resp. } c\phi' \leqslant D_{\mathrm{m}}\phi''(t) - d_{\mathrm{m}}\phi(t) + H^*(\phi)(t), \text{ a.e. in } \mathbb{R}).$$

Now, assume that an upper solution $\bar{\phi} \in \Gamma^*$ and a lower solution of $\underline{\phi}$ (which is not necessarily in Γ^*) of (3.3) are given (we will see how to obtain such a pair later in this section) so that

(P1) $0 \leqslant \underline{\phi}(t) \leqslant \bar{\phi}(t) \leqslant K$ for all $t \in \mathbb{R}$;
(P2) $\underline{\phi}(t) \not\equiv 0$.

Consider the following iteration scheme:

$$cw_n'(t) = D_{\mathrm{m}}w_n''(t) - d_{\mathrm{m}}w_n(t) + H^*(w_{n-1})(t), \quad t \in \mathbb{R}, \ n = 1,2,\ldots \tag{3.5}$$

with the boundary conditions

$$\lim_{t\to-\infty} w_n(t) = 0, \qquad \lim_{t\to\infty} w_n(t) = K, \tag{3.6}$$

where $w_0 = \bar{\phi}$. Solving (3.5)–(3.6) for $n = 1,2,\ldots$, leads to a sequence of functions $\{w_n\}_{n=1}^{\infty}$, given by

$$w_0(t) = \bar{\phi}(t), \quad t \in \mathbb{R},$$

$$w_n(t) = \frac{1}{D_{\mathrm{m}}(\beta_2 - \beta_1)}\left[\int_{-\infty}^{t} e^{\beta_1(t-s)}H^*(w_{n-1})(s)\,ds + \int_t^{\infty} e^{\beta_2(t-s)}H^*(w_{n-1})(s)\,ds\right], \tag{3.7}$$

where $t \in \mathbb{R}$, $n = 1,2,\ldots$, and

$$\beta_1 = \frac{c - \sqrt{c^2 + 4D_{\mathrm{m}}d_{\mathrm{m}}}}{2D_{\mathrm{m}}}, \qquad \beta_2 = \frac{c + \sqrt{c^2 + 4D_{\mathrm{m}}d_{\mathrm{m}}}}{2D_{\mathrm{m}}}. \tag{3.8}$$

Using Lemma 3.1, one can establish the following result (see [19, Lemmas 3.3–3.4 and Proposition 3.5]).

Theorem 3.3. *The sequence of functions* $\{w_n\}_{n=0}^{\infty}$ *satisfies*

(i) $w_n \in \Gamma^*$, *for all* $n = 1,2,\ldots$,
(ii) $\underline{\phi}(t) \leqslant w_n(t) \leqslant w_{n-1}(t) \leqslant \bar{\phi}(t)$, *for all* $n = 1,2,\ldots$ *and* $t \in \mathbb{R}$,

(iii) *each* w_n *is an upper solution of* (3.3), *and,*
(iv) $\phi(t) := \lim_{n\to\infty} w_n(t)$ *is a solution of* (3.3) *and* (3.4).

From Theorem 3.3, we see that the existence of traveling fronts for Eq. (1.6) follows from the existence of a pair of upper and lower solutions of (3.3) satisfying (P1)–(P2). Theorem 3.3 also provides a way to approximate the traveling wave front. In the remainder of this section, we will construct such a pair of upper and lower solutions, under the conditions of Theorem 3.2. To this end, we proceed as in Section 2.

Linearizing (3.3) at 0 gives

$$c\phi'(t) = D_{\mathrm{m}}\phi''(t) - d_{\mathrm{m}}\phi(t) + r\int_{-\infty}^{\infty}\phi(t - cr - y)f_\alpha(y)\,\mathrm{d}y$$

whose characteristic equation is

$$\Delta_c^*(\lambda) := r\mathrm{e}^{\alpha\lambda^2 - \lambda c\tau} - [c\lambda + d_{\mathrm{m}} - D_{\mathrm{m}}\lambda^2].$$

It is easy to show

Lemma 3.4. *Under each of the two conditions in Theorem* 3.2, $\Delta_c^*(\lambda) = 0$ *has two positive real roots* $\lambda_1 < \lambda_2$ *and*

$$\Delta_c^*(\lambda) = \begin{cases} > 0 & \text{for } \lambda < \lambda_1, \\ < 0 & \text{for } \lambda \in (\lambda_1, \lambda_2), \\ > 0 & \text{for } \lambda > \lambda_2. \end{cases}$$

The proof of this lemma is by analyzing the properties of the two functions $h(\lambda) = r\mathrm{e}^{\alpha\lambda^2 - \lambda c\tau}$ and $g(\lambda) = c\lambda + d_{\mathrm{m}} - D_{\mathrm{m}}\lambda^2$ using elementary calculus, and thus is omitted here.

Assuming the conditions of Theorem 3.2, we can choose $\varepsilon > 0$ such that $\varepsilon < \lambda_1 < \lambda_1 + \varepsilon < \lambda_2$. Using these positive constants, we define functions $\bar{\phi}$ and $\underline{\phi}$ by

$$\bar{\phi}(t) = \min\{K, K\mathrm{e}^{\lambda_1 t}\},$$

$$\underline{\phi}(t) = \max\{0, K(1 - M\mathrm{e}^{\varepsilon t})\mathrm{e}^{\lambda_1 t}\},$$

where $M > 1$ is a constant to be determined. Clearly, $\bar{\phi}$ and $\underline{\phi}$ satisfy (P1)–(P2). What is left is to show that $\bar{\phi}$ is an upper solution and $\underline{\phi}$ is a lower solution of (3.3), as claimed in the following lemma:

Lemma 3.5. *The following hold*:

(i) $\bar{\phi}(t)$ *is an upper solution of* (3.3) *and* $\bar{\phi}(t) \in \Gamma^*$;
(ii) $\underline{\phi}(t)$ *is a lower solution of* (3.3) *if* M *is chosen such that*

$$M \geqslant \frac{r\beta KG}{-\Delta_c^*(\lambda_1 + \varepsilon)},$$

where

$$G = \int_{-\infty}^{\infty} \mathrm{e}^{-2\lambda_1(y+cr)}[1 + \mathrm{e}^{-\varepsilon(y+cr)}]^2 f_\alpha(y)\,\mathrm{d}y < \infty.$$

The proof of this lemma is a direct verification and is omitted.

Finally, Theorem 3.2 follows from Theorem 3.3 and Lemmas 3.4–3.5.

Remark 3.1. Theorems 3.1–3.2 shows that even in the nonlocal Eq. (1.6), delay can also induce slow wave fronts.

4. Discussion

Traveling wave fronts are of fundamental importance in mathematical biology as well as in many other fields and the most basic reaction–diffusion model is the KPP–Fisher equation (1.1). In this paper, we have discussed the impact of time delay on the existence of traveling wave fronts of two equations ((1.4) and (1.6)) related to the KPP–Fisher equation, and found that delay can induce (slow) traveling wave fronts. Our approach is by upper–lower solution technique and an iteration method recently developed in [19], and it turns out that this approach is applicable to both (1.4) with only local effect and (1.6) with nonlocal effect. This is mainly due to the fact that the nonlinearities are monotonically increasing with respect to the delayed term within the profile set. When a nonlinearity does not possess the monotonicity, the problem becomes much harder. To see this, let us compare the nonlinearities of (1.3) and (1.4). As is seen in [14], for (1.4), a typical maximum principle still holds and the positive invariance of the interval $[0,1]$ can be easily established. But this is not true for (1.3) for which, the nonlinearity is not monotone with respect to the delayed term. Indeed, a solution of (1.3) with an initial function within $[0,1]$ can go beyond the upper boundary 1, and this constitutes one source for nonmonotone traveling waves. In fact, Gourley [6] considered traveling wave fronts for the following equation:

$$\frac{\partial u}{\partial t}(x,t) = u(x,t)\left(\frac{1 - u(x,t-\tau)}{1 + \gamma u(x,t-\tau)}\right) + \frac{\partial^2 u}{\partial x^2}(x,t), \tag{4.1}$$

which reduces to (1.3) when $\gamma = 0$, and showed that the (monotone) traveling wave fronts persist for every $c > 2$ provided the τ is sufficiently small (this coincides with the conclusion in [19] but obtained only for (1.3)). But as τ increases, the wave fronts will lose their monotonicity. Similarly, when $r > 2d_{\mathrm{m}}$ in (1.6), the nonlinearity will not be monotone in $[0,K]$, and the impact of delay on the existence of the traveling wave fronts remains an open problem.

Our nonlocal model (1.6) is derived based on an age structure. In the absence of delay, Gourley [5] considered traveling wave fronts of the nonlocal Fisher equation

$$u_t = u_{xx} + u(x,t)\left(1 - \int_{-\infty}^{\infty} g(x-y)u(y,t)\,\mathrm{d}y\right), \quad x \in R, \quad t > 0, \tag{4.2}$$

where the kernel $g(y)$ satisfies some biologically reasonable conditions, and the main concern of Gourley [5] is the impact of the kernel on the existence of wave fronts. Also incorporating *both* delay and nonlocal effect, Ashwin et al. [1] studied the traveling wave fronts for the following equation related to KPP–Fisher equation

$$u_t = u_{xx} + u(x,t)[1 - g**u], \quad x \in R, \ t > 0, \tag{4.3}$$

where

$$g**u = \int_{-\infty}^{t}\int_{-\infty}^{\infty} g(x-y,t-s)u(y,s)\,\mathrm{d}y\,\mathrm{d}s$$

and the convolution kernel $g(y,s)$ satisfies some biologically reasonable assumptions (see [2] for a justification of (4.3)). Using a geometric singular perturbation analysis, Ashwin et al. [1] showed that for a class of kernels including the one that reduces (4.3) to (1.3), for every $c > 2$, the (monotone) traveling wave fronts persist provided the delay is sufficiently small. When the delay is increased, numeric simulations by Ashwin et al. [1] show that the wave fronts will lose monotonicity and develop oscillations on the lee-side of the front. Also numerically observed in [1] are periodic traveling waves (wave trains) and orbits connecting 0 and a wave train, when τ in (1.3) is further increased to $\tau > \pi/2$. Theoretically proving these observations provides challenging but interesting and important mathematical problems. We conjecture that (1.6) will demonstrate similar behavior when $r > 2d_{\mathrm{m}}$ and $\tau > 0$ is increased to some extent.

In summary, the impact of delay on the existence of (monotone) traveling wave fronts is complicated. Delay can induce in some cases and can prevent in some other cases traveling wave fronts, depending on the nonlinearities.

Acknowledgements

The author would like to thank the anonymous referees for their comments which have lead to an improvement of this paper.

References

[1] P. Ashwin, M. Bartuccelli, T.J. Bridges, S.A. Gourley, Travelling fronts for the KPP equation with spatio-temporal delay, Z. Angew. Math. Phys. 53 (2002) 103–122.
[2] N.F. Britton, Spatial structure and periodic traveling waves in an integro-differential reaction–diffusion population model, SIAM J. Appl. Math. 50 (1990) 1663–1688.
[3] P.C. Fife, Mathematical Aspects of Reaction and Diffusion Systems, Lecture Notes in Biomathematics, Vol. 28, Springer, Berlin, New York, 1979.
[4] R.A. Fisher, The wave of advance of advantageous genes, Ann. Eugenics 7 (1937) 355–369.
[5] S.A. Gourley, Travelling front solutions of a nonlocal Fisher equation, J. Math. Biol. 41 (2000) 272–284.
[6] S A. Gourley, Wave front solutions of a diffusive model for populations of Daphnia magna, Compt. Math. Appl. 42 (2001) 1421–1430.
[7] J.K. Hale, S.M. Verduyn Lunel, Introduction to Functional Differential Equations, Applied Mathematical Sciences, Vol. 99, Springer, New York, 1993.
[8] G.E. Hutchinson, Circular causal systems in ecology, Ann. New York Acad. Sci. 50 (1948) 221–246.
[9] K. Kobayashi, On the semilinear heat equation with time-lag, Hiroshima Math. J. 7 (1977) 459–472.

[10] A.N. Kolmogorov, I.G. Petrovskii, N.S. Piskunov, Study of a diffusion equation that is related to the growth of a quality of matter, and its application to a biological problem, Byul. Mosk. Gos. Univ. Ser. A, Mat. Mekh. 1 (1937) 1–26.
[11] Y. Kuang, Delay Differential Equations With Applications in Population Dynamics, Academic Press, Boston, 1993.
[12] S. Ma, Traveling wavefronts for delayed reaction–diffusion systems via a fixed point theorem, J. Differential Equations 171 (2001) 294–341.
[13] J.D. Murray, Mathematical Biology, Springer, Berlin, 1993.
[14] K. Schaaf, Asymptotic behavior and traveling wave solutions for parabolic functional differential equations, Trans. Amer. Math. Soc. 302 (1987) 587–615.
[15] J.H.-W. So, J. Wu, X. Zou, A reaction–diffusion model for a single species with age structure. I. Travelling-wave fronts on unbounded domains, Proc. R. Soc. Lond. A 457 (2001) 1841–1853.
[16] J.H.-W. So, J. Yu, Global attractivity for a population model with time delay, Proc. Amer. Math. Soc. 125 (1995) 2687–2694.
[17] J.W.-H. So, X. Zou, Traveling waves for the diffusive Nicholson's blowflies equation, Appl. Math. Comp. 122 (2001) 385–392.
[18] J. Wu, Theory and Applications of Partial Functional Differential Equations, Applied Mathematical Sciences, Vol. 119, Springer, New York, 1996.
[19] J. Wu, X. Zou, Traveling wave fronts of reaction–diffusion systems with delay, J. Dynamics Differential Equations 13 (2001) 651–687.