

PERIODIC SOLUTIONS OF SECOND ORDER SELF-ADJOINT DIFFERENCE EQUATIONS

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ABSTRACT

Using critical point theory, some sufficient conditions are obtained for the existence of nonconstant T -periodic solutions of a class of second order self-adjoint difference equations.

1. Introduction

The problem of periodic solutions for differential equations has been the subject of many investigations. By using various methods and techniques, such as fixed point theory, the Kaplan–Yorke method, critical point theory, coincidence degree theory, bifurcation theory and dynamical system theory etc., a series of existence results for periodic solutions have been obtained in the literature, we refer to [6, 7, 17, 18, 23, 26–29, 33]. However, there are few techniques for studying the existence of periodic solutions of difference equations, and thus, the results in the field are very rare (see [2, 14–16, 30]). On the other hand, difference equations occur widely in numerous settings and forms, both in mathematics itself and in its applications to statistics, computing, electrical circuit analysis, dynamical systems, economics, biology and other fields, see for example [1, 22]. At the same time, we also find that difference equations are closely related to differential equations in the sense that (i) a differential equation model is usually derived from a difference equation, and (ii) numerical solutions of a differential equation have to be obtained by discretizing the differential equation (thus resulting in difference equations). Therefore, it is worthwhile to explore this topic.

In this paper, we consider the second order difference equation

$$\Delta[p(t) \Delta u(t-1)] + q(t)u(t) = f(t, u(t)) \quad (1.1)$$

where $p, q : \mathbf{Z} \rightarrow \mathbf{R}$, $f : \mathbf{Z} \times \mathbf{R} \rightarrow \mathbf{R}$ is continuous in the second variable, Δ is the forward difference operator defined by $\Delta u(t) = u(t+1) - u(t)$ and $p(t)$ is nonzero for each $t \in \mathbf{Z}$. Here and hereafter, \mathbf{Z} and \mathbf{R} denote the set of all integers and real numbers respectively. For $a, b \in \mathbf{Z}$, denote $\mathbf{Z}(a) = \{a, a+1, \dots\}$, $\mathbf{Z}(a, b) = \{a, a+1, \dots, b\}$ when $a \leq b$.

When $f(t, x) \equiv 0$ for $(t, x) \in \mathbf{Z} \times \mathbf{R}$, equation (1.1) becomes the second order linear self-adjoint difference equation

$$Lu(t) \equiv \Delta[p(t) \Delta u(t-1)] + q(t)u(t) = 0. \quad (1.2)$$

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Equation (1.2) may arise from various fields such as electrical circuit analysis, matrix theory, control theory and discrete variational theory etc. Many authors have extensively studied its disconjugacy, disfocality, boundary value problem, oscillation and asymptotic behaviour, for example see [3–5, 8–13, 19, 20, 24, 25, 31, 32, 34].

When $p(t) \equiv 1$, equation(1.1) was discussed in two recent papers [15, 16].

Equation (1.1) can be considered as a discrete analogue of the following second order differential equation

$$(p(t)y')' + q(t)y = f(t, y) \tag{1.3}$$

which has also been investigated by many authors, for example see [6, 17, 23, 28, 29] and the references therein.

Our aim in this paper is to use critical point theory to establish the existence of periodic solutions of (1.1). The main idea is to transfer the existence of periodic solutions of (1.1) into the existence of critical points of some functional, which is called the variational framework of (1.1). To this end, we shall recall some basic notations and known results from critical point theory.

Let H be a real Hilbert space, $J \in C^1(H, \mathbf{R})$, which means that J is a continuously Fréchet-differentiable functional defined on H . J is said to satisfy the Palais–Smale condition if any sequence $\{x_n\} \subset H$ for which $\{J(x_n)\}$ is bounded and $\{J'(x_n)\} \rightarrow 0$ as $n \rightarrow \infty$ possesses a convergent subsequence in H .

Let B_r be the open ball in H with radius r and centered at 0 and let ∂B_r denote its boundary. The following lemmas were taken from [33], and will be useful in the proofs of our main results.

LEMMA 1.1 (mountain pass lemma). *Let H be a real Hilbert space, and assume that $J \in C^1(H, \mathbf{R})$ satisfies the Palais–Smale condition and the following conditions.*

(J₁) *There exist constants $\rho > 0$ and $a > 0$ such that $J(x) \geq a$, for all $x \in \partial B_\rho$, where $B_\rho = \{x \in H : \|x\|_H < \rho\}$.*

(J₂) *$J(0) \leq 0$ and there exists $x_0 \notin B_\rho$ such that $J(x_0) \leq 0$.*

Then $c = \inf_{h \in \Gamma} \sup_{s \in [0,1]} J(h(s))$ is a positive critical value of J , where

$$\Gamma = \{h \in C([0, 1], H) \mid h(0) = 0, h(1) = x_0\}.$$

LEMMA 1.2 (linking theorem). *Let H be a real Hilbert space, $H = H_1 \oplus H_2$, where H_1 is a finite dimensional subspace of H . Assume that $J \in C^1(H, \mathbf{R})$ satisfies the Palais–Smale condition and the following hold.*

(J₃) *There exist constants $a > 0$ and $\rho > 0$ such that $J|_{\partial B_\rho \cap H_2} \geq a$.*

(J₄) *There is an $e \in \partial B_1 \cap H_2$ and a constant $R_0 > \rho$ such that $J|_{\partial Q} \leq 0$ and $Q \triangleq (\bar{B}_{R_0} \cap H_1) \oplus \{re \mid 0 < r < R_0\}$.*

Then J possesses a critical value $c \geq a$, where

$$c = \inf_{h \in \Gamma} \max_{x \in Q} J(h(x)),$$

and $\Gamma = \{h \in C(\bar{Q}, H) \mid h|_{\partial Q} = \text{id}\}$, where id denotes the identity operator.

LEMMA 1.3 (saddle point theorem). *Let H be a real Hilbert space, $H = H_1 \oplus H_2$, where $H_1 \neq \{0\}$ and is finite dimensional. Suppose that $J \in C^1(H, \mathbf{R})$, satisfies the Palais–Smale condition and the following hold.*

(J₅) There exist constants $\sigma, \rho > 0$ such that $I|_{\partial B_\rho \cap H_1} \leq \sigma$.

(J₆) There is $e \in B_\rho \cap H_1$ and a constant $\omega > \sigma$ such that $J|_{e+H_2} \geq \omega$.

Then J possesses a critical value $c \geq \omega$ and

$$c = \inf_{h \in \Gamma} \max_{u \in B_\rho \cap H_1} J(h(u)),$$

where $\Gamma = \{h \in C(\bar{B}_\rho \cap H_1, H) | h|_{\partial B_\rho \cap H_1} = \text{id}\}$.

Throughout this paper, we shall assume that (1.1) is a T -periodic system, that is, there exists a positive integer T such that for each $t \in \mathbf{Z}$, $p(t + T) = p(t)$, $q(t + T) = q(t)$ and for any $(t, x) \in \mathbf{Z} \times \mathbf{R}$, $f(t + T, x) = f(t, x)$. In Section 2, we will first establish the variational framework of (1.1) and transfer the existence of periodic solutions of (1.1) into the existence of critical points of the corresponding functional. In Section 3, we will obtain sufficient conditions for the existence of T -periodic solutions of (1.1) when f is unbounded. In Section 4, we will consider the case when f is bounded. As a special case of bounded f , that is when f is independent of the second variable, a necessary and sufficient condition for (1.1) to have a *unique* T -periodic solution will be obtained. Finally, in Section 5, as an example, we show how the problem considered in this paper is related to the study of equilibria of discrete reaction-diffusion equations, and apply one of the main results to conclude the existence and uniqueness of periodic equilibrium for a special equation arising from population dynamics on a lattice.

2. Variational framework for equation (1.1)

In this section, we are going to establish the corresponding variational framework for (1.1).

Let S be the vector space of all real sequences of the form

$$u = \{u(t)\}_{t \in \mathbf{Z}} = (\dots, u(-t), u(-t + 1), \dots, u(-1), u(0), u(1), \dots, u(t), \dots),$$

namely

$$S = \{u = \{u(t)\} | u(t) \in \mathbf{R}, t \in \mathbf{Z}\}.$$

Define the subset E of S as

$$E = \{u = \{u(t)\} \in S | u(t + T) = u(t), \forall t \in \mathbf{Z}\}.$$

Clearly, E is isomorphic to \mathbf{R}^T . E can be equipped with the inner product

$$\langle u, v \rangle_E = \sum_{t=1}^T u(t)v(t) \quad \text{for any } u, v \in E, \tag{2.1}$$

by which the norm $\|\cdot\|_E$ can be induced by

$$\|u\|_E = \sqrt{\langle u, u \rangle_E} = \left(\sum_{t=1}^T u^2(t) \right)^{1/2}, \quad u \in E. \tag{2.2}$$

It is obvious that E with the inner product (2.1) is a finite dimensional Hilbert space and is linearly homeomorphic to \mathbf{R}^T .

Now define the functional I on E as

$$I(u) = \sum_{t=1}^T \left[\frac{1}{2} p(t) (\Delta u(t-1))^2 - \frac{1}{2} q(t) u^2(t) + F(t, u(t)) \right], \quad u \in E, \tag{2.3}$$

where $F(t, x) = \int_0^x f(t, s)ds$. Then $I \in C^1(E, R)$, and for any $u \in E$, by using $u(0) = u(T), u(1) = u(T + 1)$ and (2.3), we can compute the Frechet derivative as

$$\frac{\partial I(u)}{\partial u(t)} = -\Delta[p(t)\Delta u(t - 1)] - q(t)u(t) + f(t, u(t)), \quad t \in \mathbf{Z}(1, T).$$

Thus, u is a critical point of I on E (that is, $I'(u) = 0$) if and only if

$$\Delta[p(t)\Delta u(t - 1)] + q(t)u(t) = f(t, u(t)), \quad \forall t \in \mathbf{Z}(1, T),$$

which is precisely equation (1.1). Therefore, we have reduced the existence of the periodic solution of (1.1) to the existence of a critical point of I on E . In other words, the functional I is just the variational framework of (1.1). For convenience, we identify $u \in E$ with $u = (u(1), u(2), \dots, u(T))^T$, and rewrite $I(u)$ as

$$I(u) = \frac{1}{2}((P + Q)u, u) + \sum_{t=1}^T F(t, u(t)) \tag{2.4}$$

where $u = (u(1), u(2), \dots, u(T))^T \in \mathbf{R}^T$ and P, Q are $T \times T$ matrices:

$$P = \begin{pmatrix} p(1) + p(2) & -p(2) & 0 & \dots & 0 & -p(1) \\ -p(2) & p(2) + p(3) & -p(3) & \dots & 0 & 0 \\ 0 & -p(3) & p(3) + p(4) & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & p(T - 1) + p(T) & -p(T) \\ -p(1) & 0 & 0 & \dots & -p(T) & p(T) + p(1) \end{pmatrix}$$

$$Q = \begin{pmatrix} -q(1) & 0 & 0 & \dots & 0 & 0 \\ 0 & -q(2) & 0 & \dots & 0 & 0 \\ 0 & 0 & -q(3) & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & -q(T - 1) & 0 \\ 0 & 0 & 0 & \dots & 0 & -q(T) \end{pmatrix}.$$

3. Unbounded $f(t, x)$

In this section, we will study the case when f is unbounded. We need the following assumptions.

(A₁) For each $t \in \mathbf{Z}$,

$$\lim_{x \rightarrow 0} \frac{f(t, x)}{x} = 0. \tag{3.1}$$

(A₂) There exist constants $a_1 > 0, a_2 > 0$ and $\beta > 2$ such that

$$\int_0^x f(t, s)ds \leq -a_1|x|^\beta + a_2, \quad \forall x \in \mathbf{R}. \tag{3.2}$$

By (A₂),

$$\lim_{|x| \rightarrow +\infty} \frac{f(t, x)}{x} = -\infty.$$

Hence (A₁) together with (A₂) implies that $f(t, x)$ grows superlinearly both at infinity and at zero.

LEMMA 3.1. *Suppose that $f \in C(\mathbf{Z} \times \mathbf{R})$ satisfies (A₂); then I satisfies the Palais–Smale condition.*

Proof. Suppose that the eigenvalues of $P + Q$ are $\lambda_1, \lambda_2, \dots, \lambda_T$. Since $p(t) \neq 0$, $\max\{|\lambda_1|, |\lambda_2|, \dots, |\lambda_T|\} > 0$ and we denote it by $|\lambda_{\max}|$. For any sequence $\{u_n\} \subset E$, with $I(u_n)$ bounded and $I'(u_n) \rightarrow 0$ as $n \rightarrow +\infty$, there exists a positive constant M such that $|I(u_n)| \leq M$. Thus by (A₂),

$$\begin{aligned} -M &\leq I(u_n) = \frac{1}{2}((P + Q)u_n, u_n) + \sum_{t=1}^T F(t, u_n(t)) \\ &\leq \frac{1}{2}|\lambda_{\max}|\|u_n\|_E^2 - a_1 \sum_{t=1}^T |u_n(t)|^\beta + a_2 T. \end{aligned}$$

By

$$\sum_{t=1}^T |u_n(t)|^2 \leq T^{(\beta-2)/\beta} \left[\sum_{t=1}^T |u_n(t)|^\beta \right]^{2/\beta},$$

we know that

$$\sum_{t=1}^T |u_n(t)|^\beta \geq T^{(2-\beta)/2} \|u_n\|_E^\beta.$$

Then we have

$$-M \leq I(u_n) \leq \frac{1}{2}|\lambda_{\max}|\|u_n\|_E^2 - a_1 T^{(2-\beta)/2} \|u_n\|_E^\beta + a_2 T. \quad (3.3)$$

Therefore, for any $n \in \mathbf{N}$,

$$a_1 T^{(2-\beta)/2} \|u_n\|_E^\beta - \frac{1}{2}|\lambda_{\max}|\|u_n\|_E^2 \leq M + a_2 T.$$

Since $\beta > 2$, the above inequality implies that $\{u_n\}$ is a bounded sequence in E . Thus $\{u_n\}$ possesses a convergent subsequence. \square

THEOREM 3.1. *Suppose that f satisfies (A₁) and (A₂). In addition, assume that the following hold.*

(p) $p(t) > 0$ for all $t \in \mathbf{Z}(1, T)$.

(q) $q(t) \leq 0$ for all $t \in \mathbf{Z}(1, T)$ and there exists at least one $t_0 \in \mathbf{Z}(1, T)$ such that $q(t_0) < 0$.

Then there exist at least two nontrivial T -periodic solutions for (1.1).

Proof. We will use Lemma 1.1 to prove Theorem 3.1. We need to verify that all the assumptions of the mountain pass theorem hold. The Palais–Smale condition is verified in Lemma 3.1. Next, we will check assumptions (J₁) and (J₂) of Lemma 1.1. By matrix theory, it can be easily checked that $P + Q$ is positive definite. We denote its eigenvalues by $\lambda_1, \lambda_2, \dots, \lambda_T$ with $0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_T$. By (A₁), there exists $\rho > 0$ such that for any $|x| < \rho$ and $t \in \mathbf{Z}(1, T)$, $F(t, x) \leq \frac{1}{4}\lambda_1 x^2$. Thus, for any $u \in E$, $\|u\| \leq \rho$, $|u(t)| \leq \rho$, for all $t \in \mathbf{Z}(1, T)$, and

$$I(u) \geq \frac{1}{2}\lambda_1 \|u\|^2 - \frac{1}{4}\lambda_1 \|u\|^2 = \frac{1}{4}\lambda_1 \|u\|^2.$$

Taking $a = \frac{1}{4}\lambda_1\rho^2 > 0$, we have

$$I(u)|_{\partial B_\rho} \geq a$$

and assumption (\mathbf{J}_1) is verified.

Clearly, $I(0) = 0$. For any given $w \in E$ with $\|w\| = 1$ and a constant $\alpha > 0$,

$$\begin{aligned} I(\alpha w) &= \frac{1}{2}((P + Q)\alpha w, \alpha w) + \sum_{t=1}^T F(t, \alpha w(t)) \\ &\leq \frac{1}{2}\alpha^2\lambda_T - a_1\alpha^\beta \sum_{t=1}^T |w(t)|^\beta + a_2T \\ &\leq \frac{1}{2}\alpha^2\lambda_T - a_1T^{(2-\beta)/2}\alpha^\beta + a_2T \rightarrow -\infty \quad \text{as } \alpha \rightarrow +\infty. \end{aligned}$$

Thus we can easily choose a sufficiently large α such that $\alpha > \rho$ and for $u_0 = \alpha w \in E$, $I(u_0) < 0$. Therefore, by Lemma 1.1, there exists at least one critical value $c \geq a > 0$. We suppose that \bar{u} is a critical point corresponding to c , that is, $I(\bar{u}) = c$, and $I'(\bar{u}) = 0$. By a similar argument to the proof of Lemma 3.1,

$$I(u) \leq \frac{1}{2}|\lambda_{\max}|\|u\|_E^2 - a_1T^{(2-\beta)/2}\|u\|_E^\beta + a_2T, \quad \forall u \in E. \quad (3.4)$$

Thus I is bounded from above. We denote by c_{\max} the supremum of $\{I(u), u \in E\}$. Since (3.4) implies that

$$\lim_{\|u\| \rightarrow +\infty} I(u) = -\infty,$$

$-I$ is coercive and I attains its maximum at some point \tilde{u} , that is, $I(\tilde{u}) = c_{\max}$. Clearly, $\tilde{u} \neq 0$. If $\bar{u} \neq \tilde{u}$, then the proof of Theorem 3.1 is complete; otherwise, $\bar{u} = \tilde{u}$ and $c = c_{\max}$. By Lemma 1.1,

$$c = \inf_{h \in \Gamma} \sup_{s \in [0,1]} I(h(s)),$$

where

$$\Gamma = \{h \in C([0, 1], E) \mid h(0) = 0, h(1) = u_0\}.$$

Then for any $h \in \Gamma$, $c_{\max} = \max_{s \in [0,1]} I(h(s))$. The continuity of $I(h(s))$ in s , $I(0) \leq 0$ and $I(u_0) < 0$ show that there exists some $s_0 \in (0, 1)$, such that $I(h(s_0)) = c_{\max}$. If we choose $h_1, h_2 \in \Gamma$ such that the intersection $\{h_1(s) \mid s \in (0, 1)\} \cap \{h_2(s) \mid s \in (0, 1)\}$ is empty, then there exist $s_1, s_2 \in (0, 1)$ such that $I(h_1(s_1)) = I(h_2(s_2)) = c_{\max}$. Thus, we obtain two different critical points $u_1 = h_1(s_1)$, $u_2 = h_2(s_2)$ of I in E . In this case, in fact we may obtain infinitely many nontrivial critical points which correspond to the critical value c_{\max} . The proof of Theorem 3.1 is complete. \square

REMARK 3.1. The periodic solutions we have obtained in Theorem 3.1 are nontrivial, but they may be nonzero constant. If we want to obtain nonconstant

periodic solutions, we only need to exclude nonzero constant solutions. Thus, we immediately have the following corollary.

COROLLARY 3.1. *Suppose that f satisfies (A_1) , (A_2) , (p) and (q) and (A_3) $f(t, x) = 0$ for all $t \in \mathbf{Z}(1, T)$ if and only if $x = 0$.*

Then there exist at least two nonconstant T -periodic solutions for (1.1).

In Theorem 3.1 and Corollary 3.1, (q) requires $q(t) \neq 0$. We will see below that when $q(t) \equiv 0$, the same conclusion remains true, but the proof needs a different treatment since in this case the matrix $P + Q$ becomes P which is, instead of positive definite, only positive *semi-definite*.

THEOREM 3.2. *Suppose that f satisfies (A_1) , (A_2) and (p) and (q') $q(t) = 0$ for all $t \in \mathbf{Z}(1, T)$.*

Then there exist at least two nonconstant T -periodic solutions for (1.1).

Proof. In this case, $Q = 0$ and

$$I(u) = \frac{1}{2}(Pu, u) + \sum_{t=1}^T F(t, u(t)). \quad (3.5)$$

It is easy to see that P is positive semi-definite, and $\text{rank}(P) = T - 1$. We denote its eigenvalues by $\lambda_1, \lambda_2, \dots, \lambda_T$ and assume they are ordered as $0 = \lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots \leq \lambda_T$. Set $E_1 = \{(v, v, \dots, v)^T \in E | v \in \mathbf{R}\}$, $E_2 = (E_1)^\perp$. Then E has the following decomposition of direct sum

$$E = E_1 \oplus E_2,$$

where E_1 and E_2 are invariant subspaces of E with respect to P . Also we have

$$\begin{aligned} Pu &= 0, & \forall u \in E_1, \\ \lambda_2 \|u\|_E^2 &\leq (Pu, u) \leq \lambda_T \|u\|_E^2, & \forall u \in E_2. \end{aligned}$$

In view of Lemma 3.1, I satisfies the Palais–Smale condition. Next, we will prove that conditions (J_3) and (J_4) hold. In fact, by (A_1) , there exists a constant $\rho > 0$ such that $F(t, x) \leq \frac{1}{4}\lambda_2 x^2$, for $t \in \mathbf{Z}(1, T)$, $x \in \{x \in \mathbf{R} | |x| < \rho\} \triangleq B_\rho$. Therefore, for any $u \in B_\rho \cap E_2$, $t \in \mathbf{Z}(1, T)$, we have

$$\begin{aligned} I(u) &= \frac{1}{2}(Pu, u) + \sum_{t=1}^T F(t, u(t)) \\ &\geq \frac{1}{2}\lambda_2 \|u\|^2 - \frac{1}{4}\lambda_2 \|u\|^2 = \frac{1}{4}\lambda_2 \|u\|^2 = \frac{1}{4}\lambda_2 \rho^2 \end{aligned}$$

which shows that (J_3) is satisfied.

Take $e \in \partial B_1 \cap E_2$. For any $w \in E_1$ and $r \in \mathbf{R}$, let $u = re + w$. Then

$$\begin{aligned} I(u) &= \frac{1}{2}(P(re + w), re + w) + \sum_{t=1}^T F(t, u(t)) \\ &= \frac{1}{2}(Pre, re) + \sum_{t=1}^T F(t, re(t) + w(t)) \\ &\leq \frac{1}{2}\lambda_T r^2 - a_1 \sum_{t=1}^T |re(t) + w(t)|^\beta + a_2 T \\ &\leq \frac{1}{2}\lambda_T r^2 - a_1 T^{(2-\beta)/2} \left(\sum_{t=1}^T |re(t) + w(t)|^2 \right)^{\beta/2} + a_2 T \\ &= \frac{1}{2}\lambda_T r^2 - a_1 T^{(2-\beta)/2} \left(\sum_{t=1}^T (r^2 e^2(t) + w^2(t)) \right)^{\beta/2} + a_2 T \\ &= \frac{1}{2}\lambda_T r^2 - a_1 T^{(2-\beta)/2} (r^2 + \|w\|_2^2)^{\beta/2} + a_2 T \\ &\leq \frac{1}{2}\lambda_T r^2 - a_1 T^{(2-\beta)/2} r^\beta - a_1 T^{(2-\beta)/2} \|w\|_2^\beta + a_2 T. \end{aligned}$$

Set

$$g_1(r) = \frac{1}{2}\lambda_T r^2 - a_1 T^{(2-\beta)/2} r^\beta, \quad g_2(\tau) = -a_1 T^{(2-\beta)/2} \tau^\beta + a_2 T.$$

Then, $\lim_{r \rightarrow +\infty} g_1(r) = -\infty$, $\lim_{\tau \rightarrow +\infty} g_2(\tau) = -\infty$. Furthermore, $g_1(r)$ and $g_2(\tau)$ are bounded from above. Accordingly, there is some $R_0 > \rho$, such that for any $u \in \partial K$, $I(u) \leq 0$, where $K \triangleq (\bar{B}_{R_0} \cap E_1) \oplus \{re \mid 0 < r < R_0\}$ and ρ is chosen as in the proof of Theorem 3.1. This verifies (J₄). By the linking theorem, I possesses a critical value $c \geq a > 0$, where

$$c = \inf_{h \in \Gamma} \max_{u \in Q} I(h(u))$$

and $\Gamma = \{h \in C(\bar{K}, E) \mid h|_{\partial K} = \text{id}\}$.

Let $\bar{u} \in E$ be a critical point associated to the critical value c of I , that is, $I(\bar{u}) = c$. If $\bar{u} \neq \tilde{u}$, then the conclusion of Theorem 3.2 holds. Otherwise, $\bar{u} = \tilde{u}$, then $c_{\max} = I(\bar{u}) = I(\tilde{u}) = c$, that is, $\sup_{u \in E} I(u) = \inf_{h \in \Gamma} \sup_{u \in K} I(h(u))$. Choosing $h = \text{id}$, we have $\sup_{u \in K} I(u) = c_{\max}$. Since the choice of $e \in \partial B_1 \cap E_2$ is arbitrary, we can take $-e \in \partial B_1 \cap E_2$. By a similar argument, there exists $R_1 > \rho$, such that for any $u \in \partial K_1$, $I(u) \leq 0$, where $K_1 \triangleq (\bar{B}_{R_1} \cap E_1) \oplus \{-re \mid 0 < r < R_1\}$. Again by the linking theorem, I possesses a critical value $c' \geq a > 0$, and

$$c' = \inf_{h \in \Gamma_1} \max_{u \in K_1} I(h(u)),$$

where $\Gamma_1 = \{h \in C(\bar{K}_1, E) \mid h|_{\partial K_1} = \text{id}\}$.

Similarly, by (3.4) we find that I is bounded from above and $\lim_{\|u\| \rightarrow \infty} I(u) = -\infty$. Thus, I may take the supremum of I on E at some point $\tilde{u} \in E$. That is,

$$I(\tilde{u}) = c_{\max} = \sup_{u \in E} I(u).$$

Clearly, \tilde{u} is a nonzero critical point of I .

If $c' \neq c_{\max}$, then the proof is complete; otherwise, $c' = c_{\max}$, then for any $h \in \Gamma_1$, $\max_{u \in K_1} I(h(u)) = c_{\max}$. In particular, letting $h = \text{id}$, we have $\max_{u \in K_1} I(u) = c_{\max}$. Due to the fact that $I|_{\partial K} \leq 0$ and $I|_{\partial K_1} \leq 0$, I attains

its maximum at some points in the interior of the set K and K_1 . On the other hand, $K \cap K_1 \subset E_1$ and for any $u \in E_1$, $I(u) \leq 0$. This shows that there must be a point $\hat{u} \in E$, $\hat{u} \neq \tilde{u}$ and $I(\hat{u}) = c' = c_{\max}$.

The above argument implies that, if $c < c_{\max}$, equation (1.1) possesses at least two nontrivial periodic solutions with period T ; and if $c = c_{\max}$, equation (1.1) possesses infinitely many nontrivial T -periodic solutions. This completes the proof. \square

Parallel to Corollary 3.1, we have the following.

COROLLARY 3.2. *Suppose that f satisfies (A_1) – (A_3) , (p) and (q); then there exist at least two nonconstant T -periodic solutions for (1.1).*

Combining Corollary 3.1 and Corollary 3.2, we obtain the following.

COROLLARY 3.3. *Suppose that f satisfies (A_1) – (A_3) and (p) and (q'') $q(t) \leq 0$, for all $t \in \mathbf{Z}(1, T)$.*

Then there exist at least two nonconstant T -periodic solutions for (1.1).

Generally, $p(t)$ and $q(t)$ may not satisfy (p) and/or (q'') . In this case, we have the following.

THEOREM 3.3. *Suppose that f satisfies (A_1) – (A_3) and (A_4) ; then at least one of the eigenvalues of $P+Q$ is positive. Then (1.1) possesses at least two nonconstant T -periodic solutions.*

Proof. Suppose that $0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_l$ and $0 > -\mu_1 \geq -\mu_2 \geq \dots \geq -\mu_k$ are the positive and negative eigenvalues of $P+Q$, We also suppose that $\eta_i, 1 \leq i \leq l$ and $\xi_j, 1 \leq j \leq k$ are the eigenvectors of $P+Q$ corresponding to eigenvalues $\lambda_i, 1 \leq i \leq l$ and $\mu_j, 1 \leq j \leq k$ satisfying

$$(\eta_i, \eta_j) = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j, \end{cases} \quad (\xi_i, \xi_j) = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j, \end{cases}$$

and

$$(\eta_i, \xi_j) = 0 \quad \text{for any } 1 \leq i \leq l, 1 \leq j \leq k.$$

Then E has the direct sum decomposition

$$E = E^- \oplus E^0 \oplus E^+,$$

where

$$E^+ = \text{span}\{\eta_i, 1 \leq i \leq l\}, \quad E^- = \text{span}\{\xi_j, 1 \leq j \leq k\} \tag{3.6}$$

and

$$E^0 = (E^+ \oplus E^-)^\perp.$$

For any $u \in E$, u can be decomposed as

$$u = u^+ + u^0 + u^-,$$

where $u^+ \in E^+$, $u^- \in E^-$ and $u^0 \in E^0$. Clearly,

$$\begin{aligned} \lambda_1 \|u^+\|^2 &\leq ((P + Q)u^+, u^+) \leq \lambda_l \|u^+\|^2, \\ -\mu_k \|u^-\|^2 &\leq ((P + Q)u^-, u^-) \leq -\mu_1 \|u^-\|^2. \end{aligned}$$

Thus

$$\frac{1}{2}((P + Q)u, u) \leq \frac{1}{2}(\lambda_l \|u^+\|^2 - \mu_1 \|u^-\|^2).$$

Let $H_1 = E^- \oplus E^0$, $H_2 = E^+$; then $E = H_1 \oplus H_2$. We still need to apply Lemma 1.2 to find the critical points of I . Verification of assumption (\mathbf{J}_3) is similar to that in the proof of Theorem 3.2. In what follows, we only verify assumption (\mathbf{J}_4) for the linking theorem.

Take $e \in \partial B_1 \cap H_2$. For any $w \in H_1$ and $r \in \mathbf{R}$, let $u = re + w$. Since $w = w^0 + w^-$, where $w^0 \in E^0$, $w^- \in E^-$, then

$$\begin{aligned} I(u) &= \frac{1}{2}((P + Q)(re + w), re + w) + \sum_{t=1}^T F(t, u(t)) \\ &= \frac{1}{2}((P + Q)re, re) + \frac{1}{2}((P + Q)w^-, w^-) + \sum_{t=1}^T F(t, re(t) + w(t)) \\ &\leq \frac{1}{2}\lambda_l r^2 - \frac{1}{2}\mu_1 \|w^-\|^2 - a_1 \sum_{t=1}^T |re(t) + w(t)|^\beta + a_2 T \\ &\leq \frac{1}{2}\lambda_l r^2 - a_1 T^{(2-\beta)/2} r^\beta - a_1 T^{(2-\beta)/2} \|w\|_2^\beta + a_2 T. \end{aligned}$$

Thus by a similar argument as in the proof of Theorem 3.2, we can conclude that Theorem 3.3 holds. \square

REMARK 3.2. When (A_4) does not hold, then $P + Q$ is negative semi-definite (or negative definite). If we further assume that

$$(A_5) \quad xf(t, x) < 0, \text{ for all } x \neq 0;$$

then there exist no nontrivial T -periodic solutions of (1.1). In fact, $\{u(t)\}$ is a T -periodic solution of (1.1) if and only if $u = \{u(t)\} \in E$ is a critical point of I , namely,

$$(P + Q)u + f(u) = 0,$$

where $f(u) = (f(1, u(1)), f(2, u(2)), \dots, f(T, u(T)))^T$. Thus

$$((P + Q)u, u) + \sum_{t=1}^T u(t)f(t, u(t)) = 0.$$

On the other hand, $((P + Q)u, u) \leq 0$ and by (A_5) , $u(t)f(t, u(t)) \leq 0$, for all $t \in \mathbf{Z}(1, T)$. Then, $u(t) \equiv 0$, for all $t \in \mathbf{Z}(1, T)$.

4. Bounded $f(t, x)$

In this section, we study the case when f is bounded. In this case, we have the following theorem.

THEOREM 4.1. *Suppose that f, P, Q satisfy the following assumptions.*

(A₆) *f is bounded, that is there exists a positive constant $M > 0$ such that for any $(t, x) \in \mathbf{Z}(0, T) \times \mathbf{R}$, $|f(t, x)| \leq M$.*

(A₇) *The matrix $P + Q$ is non-singular.*

Then there exists at least one T -periodic solution of (1.1).

Proof. We will apply Lemma 1.3, the saddle point theorem, to prove our theorem. Consider the functional

$$I(u) = \frac{1}{2}((P + Q)u, u) + \sum_{t=1}^T F(t, u(t)), \quad u \in E,$$

where P, Q, F and E are defined in Section 2. We only need to find a critical point of the functional I on E .

Firstly, we claim that, under assumptions (A₆) and (A₇), the functional I satisfies the Palais–Smale condition on E . In fact, for any sequence $\{u_n\} \subset E$, with $I(u_n)$ bounded and $I'(u_n) \rightarrow 0$ as $n \rightarrow +\infty$, there is a positive constant $M_1 > 0$ such that for any $n \in \mathbf{N}$, $\|I'(u_n)\| \leq M_1$. Since $I'(u_n) = (P + Q)u_n + f(u_n)$, where $f(u) = (f(1, u(1)), f(2, u(2)), \dots, f(T, u(T)))^T$, we have

$$\|(P + Q)u_n\| \leq M_1 + \|f(u_n)\|.$$

By assumption (A₆),

$$\|f(u_n)\| = \sqrt{f^2(1, u_n(1)) + f^2(2, u_n(2)) + \dots + f^2(T, u_n(T))} \leq \sqrt{TM}.$$

Thus, for any $n \in \mathbf{N}$,

$$\|(P + Q)u_n\| \leq M_2,$$

where $M_2 = M_1 + M\sqrt{T}$.

On the other hand, by (A₇), any eigenvalue of $P + Q$ is not zero. Thus we can suppose that $0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_l$ and $0 > -\mu_1 \geq -\mu_2 \geq \dots \geq -\mu_k$ are the positive and negative eigenvalues of $P + Q$ respectively and $l + k = T$. Denote $\Delta = \min\{|\lambda_1|, |\mu_1|\}$. Then E has the direct sum decomposition

$$E = E^- \oplus E^+,$$

where E^+ and E^- are defined as in (3.6). For any $u \in E$, u can be decomposed as

$$u = u^+ + u^-,$$

where $u^+ \in E^+$, $u^- \in E^-$. Then

$$\|(P + Q)u\| = \|(P + Q)u^+\| + \|(P + Q)u^-\| \geq \lambda_1 \|u^+\| - \mu_1 \|u^-\| \geq \Delta \|u\|.$$

Therefore, for any $n \in \mathbf{N}$,

$$\Delta \|u_n\| \leq \|(P + Q)u_n\| \leq M_2$$

and then

$$\|u_n\| \leq \frac{1}{\Delta} M_2.$$

It follows that $\{u_n\}$ is a bounded sequence of E , and thus, possesses a convergent subsequence, so the Palais–Smale condition is verified.

Next we will verify the assumptions of the saddle point theorem. By (A₆), for any $u \in E^+$,

$$\left| \sum_{t=1}^T F(t, u(t)) \right| \leq \sum_{t=1}^T |F(t, u(t))| \leq M \sum_{t=1}^T |u(t)| \leq M\sqrt{T}\|u\|,$$

so

$$I(u) \geq \frac{1}{2}\lambda_1\|u\|^2 - M\sqrt{T}\|u\|.$$

By minimizing the right side of the above inequality, we have

$$I(u) \geq -\frac{M^2T}{2\lambda_1}.$$

Let $\omega = -M^2T/2\lambda_1$, then for any $u \in E^+$, $I(u) \geq \omega$. On the other hand, for any $u \in E^-$,

$$I(u) \leq \frac{1}{2}\mu_1\|u\|^2 + M\sqrt{T}\|u\|.$$

In view of $\mu_1 < 0$, we have

$$I(u) \rightarrow -\infty \text{ as } \|u\| \rightarrow +\infty.$$

This implies that we can choose constants $\sigma < \omega$, and $\rho > 0$ such that $I|_{\partial B_\rho \cap E^-} \leq \sigma$. Consequently, by the saddle point theorem, there exists at least one critical point of I on E . This completes the proof of Theorem 4.1. \square

In the case when $f(t, x)$ is independent of the second variable x , equation (1.1) reduces to the following second-order *linear* difference equation

$$\Delta[p(t) \Delta u(t-1)] + q(t)u(t) = f(t). \tag{4.1}$$

Thus the critical points of I are just the solutions of the linear algebraic system

$$(P + Q)u = f \tag{4.2}$$

where $f = (f(1), f(2), \dots, f(T))^T$. Clearly (A₇) becomes necessary and sufficient for (4.2) to have a unique solution, that is, we have the following result.

COROLLARY 4.1. *A necessary and sufficient condition for the linear equation (4.1) to have a unique T -periodic solution is that the matrix $P + Q$ is non-singular.*

REMARK 4.1. When (A₇) is not satisfied, that is when $P + Q$ is singular, one still can expect the existence of T -periodic solutions of (4.1) (and (1.1) with bounded f). Unfortunately, we cannot carry out the details of such an analysis at present, and thus have to leave it as future work.

5. An example

There have been extensive and intensive studies on infinite dimensional systems of the form

$$u_t(n, t) = d[u(n+1, t) - 2u(n, t) + u(n-1, t)] + g(u(n, t)), \quad n \in \mathbb{Z}. \tag{5.1}$$

Such a system is a result of discretizing the spatial variable of the reaction-diffusion equation

$$\frac{\partial w(x, t)}{\partial t} = D \frac{\partial^2 w(x, t)}{\partial x^2} + g(w(x, t)), \quad x \in R, \quad (5.2)$$

and is referred to as discrete reaction-diffusion equation, or lattice-differential equation. Here $u(n, t) = w(nh, t)$ for $n \in Z$, and h is the step size of the discretization. Recall that (5.2) is derived by applying the conservation law and by assuming that the flux ϕ of w is negatively proportional to the gradient of w with a constant proportional constant α (that is $\phi = -\alpha w_x$), thus giving the diffusion term $\phi_x = -\alpha w_{xx}$ as in (5.2). The more general situation is that the proportional constant α depends on the spatial variable x , reflecting the heterogeneity of the environment. In such a case, (5.2) should be rewritten as

$$\frac{\partial w(x, t)}{\partial t} = (\alpha(x)w_x(x, t))_x + g(w(x, t)), \quad x \in R, \quad (5.3)$$

and therefore, the corresponding discretization would read

$$u_t(n, t) = \Delta_n(\alpha(n)\Delta_n u(n-1, t)) + g(u(n, t)), \quad n \in Z. \quad (5.4)$$

Noting that (5.1) could demonstrate very rich equilibrium structure (see, for example, Keener [21], and Chow and Mallet-Paret [11, 12]), it becomes an interesting and challenging problem to study the equilibria of the more general system (5.4), that is, the solutions of the second order difference equation

$$\Delta(\alpha(n)\Delta u(n-1)) = -g(u(n, t)), \quad n \in Z. \quad (5.5)$$

In what follows, we consider a *periodic environment* by assuming that $\alpha(n)$ is periodic with period $T > 0$ (for example, a ring environment). Then, a periodic solution of (5.5) gives an equilibrium of (5.4) with a *spatially periodic pattern*.

When (5.1)–(5.4) are used to describe the population growth in a patch environment, $g(u)$ usually takes the form $g(u) = -du + b(u)$ where $-du$ (with $d > 0$) is the death term while $b(u)$ is the birth function. Biologically, $b(u)$ is assumed to be continuous, non-negative and bounded. The following birth functions have been widely adopted in the literature.

$$b(u) = \begin{cases} \beta u e^{-\delta u} & u \geq 0 \\ 0 & u < 0, \end{cases} \quad b(u) = \begin{cases} \frac{\beta u}{u + \delta} & u \geq 0 \\ 0 & u < 0. \end{cases}$$

Now, applying Theorem 4.1 to the corresponding population model system on the lattice (patch environment)

$$u_t(n, t) = \Delta_n(\alpha(n)\Delta_n u(n-1, t)) - du(n, t) + b(u(n, t)), \quad n \in Z, \quad (5.6)$$

one immediately knows that this system admits at least one *periodic equilibrium*, provided that $\alpha(n)$ is periodic in n . Moreover, if $d > 0$ is sufficiently large (so that the corresponding $P + Q$ is nonsingular), then this periodic equilibrium is unique.

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